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# A Universal Dynamic Auction for Unimodular Demand Types: An Efficient Auction Design for Various Kinds of Indivisible Commodities

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# A Universal Dynamic Auction for Unimodular Demand Types: An Efficient Auction Design for Various Kinds of Indivisible Commodities*

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**Abstract:** We propose a new and general dynamic design for efficiently auctioning multiple heterogeneous indivisible items. The auction applies to all unimodular demand types of Baldwin and Klemperer (2019) which are a necessary and sufficient condition for the existence of competitive equilibrium in economies with indivisible goods and accommodate a variety of substitutes, complements, gross substitutes and complements, strong substitutes, and other kinds. Every bidder has private valuation on each of his interested bundles of items and the seller has a reserve price for every bundle of items. The auctioneer announces the current prices for all items, bidders respond by reporting their demands at these prices, and then the auctioneer adjusts the prices of items. The trading rules are simple, transparent, and detail-free. Although bidders are not assumed to be price-takers so they can strategically exercise their market power, this auction induces bidders to bid truthfully and yields an efficient outcome. Bidding sincerely is an ex post perfect Nash equilibrium. The auction is also privacy-preserving and independent of any probability distribution assumption.

**Keywords:** Dynamic Auction, Incentive-Compatibility, Competitive Equilibrium, Unimodular Demand Types, Substitute, Complement, Indivisibility, Dynamic Auction Game of Incomplete Information.
**JEL classification:** D44, C78.

## 1 Introduction

This paper offers a new and general dynamic design for auctioning all kinds of heterogeneous indivisible goods to many bidders and obtains both efficient and incentive-compatible

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outcomes and preserves all agents' privacy. Our model deals with a general situation where multiple items are to be sold and each bidder may demand any number of items. Each bidder has a private valuation on each of his interested bundles and may bid strategically. The seller has a reserve price for every bundle below which the bundle will not be sold.

The major impetus to study dynamic auctions for multiple items comes from the sale of radio spectrum licenses in the United States, the United Kingdom and elsewhere in the 1990s and 2000s. The first radio spectrum auction adopted by the US Federal Communications Commission in 1994 was designed by Paul Milgrom, Robert Wilson, and Preston McAfee. It was hailed as "The greatest auction ever," New York Times, March 16, 1995, p.A17. The UK 3G mobile spectrum auction in 2000 designed by Ken Binmore and Paul Klemperer raised over 34 billion dollars, almost 600 dollars per head of the total UK population and set a record in the history of auction; see, e.g., Klemperer (2004). Both US and UK spectrum auctions are simultaneous ascending multi-item auctions.

The advent of the Internet together with the information technology advance has significantly increased and virtually exploded the use of auctions since the 1990s. Nowadays auctions can be conducted both online and off-line and have been widely explored by private and public sectors to carry out a broad range of and vast volumes of economic activities. For instance, at the heart of every stock market lie the double auctions. Auctions are used by governments to sell treasury bills, timber rights, electricity, off-shore oil and gas leases, mineral rights and pollution permits, and to procure public projects including goods and services, and to privatize state companies (in the former Soviet Unions and other eastern European socialist states), and by firms and individuals to sell all sorts of commodities and services ranging from antiques, artworks, flowers and fish, to airline routes, takeoff and landing slots, and keywords; see McAfee and McMillan (1987), Klemperer and Meyer (1989), Green and Newbery (1992), Maskin (2000), Ausubel and Cramton (2004A), Janssen (2004), Klemperer (2000, 2004), Milgrom (2004), Wolinsky (2005), Edelman et al. (2007), and Varian (2007) among others. Ausubel and Cramton (2008), and Klemperer (2008, 2010, 2018) examined practical auction design for banks in financial crises.

In this paper, we develop a new and general dynamic auction mechanism for multiple heterogeneous indivisible items. The auction applies to all unimodular demand types of Baldwin and Klemperer (2019) that are a necessary and sufficient condition for the existence of competitive equilibrium in economies with indivisible goods and accommodate a variety of substitutes, complements, gross substitutes and complements, strong substitutes, and other kinds. It unifies and extends the existing auction designs into uncharted territory.

Efficient dynamic auctions include Crawford and Knoer (1981), Kelso and Crawford (1982), Demange et al. (1986), Gul and Stacchetti (2000), Milgrom (2000), Perry and

Reny (2005), Ausubel (2004, 2006),[1] Hatfield and Milgrom (2005), Milgrom and Strulovici (2009), and Sun and Yang (2009, 2014).[2] All papers except the last two are concerned with the case that all indivisible goods are substitutes, while Sun and Yang (2009)[3] incorporate complements into the model of Kelso and Crawford (1982) and Sun and Yang (2014) study the case of multiple complements.

In dynamic auction design, prices play an instrumental rule in guiding the market toward a competitive equilibrium. A key building block of our auction is Baldwin and Klemperer (2019)'s unimodular demand types: a necessary and sufficient condition for competitive equilibrium existence. They explored a geometric structure of utility functions via prices and demands and classified them by unimodular matrices, called *unimodular demand types*. These demand types capture the essential and natural attributes of the goods under consideration but do not reveal their values. For instance, the physical property of tables is the same to all consumers but they can each have different valuations on tables. Unimodular demand types are rich, unify all previous sufficient conditions but also can easily identify previously-unknown environments in which a competitive equilibrium still exists. Baldwin and Klemperer (2014, 2019) have proved that every unimodular demand type with $N$ goods is a unimodular basis change of a unimodular complements demand type contained in $\{0, 1\}^N$, meaning that there are far more classes of purely-complements than of purely-substitutes for equilibrium existence. They also provided various examples of unimodular demand types and ways of constructing them.

While Baldwin and Klemperer (2019) have established an important but also elegant result of equilibrium existence for economies with indivisible goods via a nonconstructive method, our current article goes further by addressing an equally fundamental problem of how competitive equilibrium prices are formed and efficient allocations can be achieved in an incomplete information environment with strategic bidders. The current study is closely linked with and motivated by several strands of the literature. The starting point for our auction design goes back to many classic studies of general equilibrium theory. The competitive or Walrasian equilibrium is the cornerstone of the general equilibrium theory, offering an efficient distribution of goods and its supporting price for every good. This equilibrium exists in markets with divisible goods under very general conditions (see Debreu 1959, Arrow and Hahn 1971). Adam Smith (1776) famously narrated how the invisible hand would miraculously work to reach an equilibrium. Leon Walras (1874) formulated

[1] Ausubel (2006) considers both general divisible goods and substitutable indivisible goods.

[2] In Crawford and Knoer (1981) and Demange et al. (1986), every bidder is allowed to demand at most one item. These are the so-called unit-demand assignment market of Koopmans and Beckmann (1957) and Shapley and Shubik (1971). Other models permit bidders to demand multiple items.

[3] In Sun and Yang (2009), there are two sets of items and goods in the same set are substitutes but goods across the two sets are complements, like workers and machines.

a tâtonnement adjustment process. Samuelson (1941), Arrow and Hurwicz (1958), Arrow et al. (1959) established the convergence of certain tâtonnement process in the case of substitutable goods. Scarf (1973) introduced a convergent procedure for general economies as described by Debreu (1959) and Arrow and Hahn (1971). While all these works concern economies with perfectly divisible goods, Kelso and Crawford (1982) proposed an auction-like convergent process for a job-matching market with indivisibility where every firm may hire several workers but views workers as gross substitutes.

In the studies discussed in the previous paragraph and other traditional analyses, it has been essential to assume that agents are price-takers or have no market power at all (see Debreu and Scarf 1963, and Aumann 1964). Unfortunately, this assumption can hardly be satisfied in any reasonable auction model, because in reality the number of bidders is usually small and bidders do possess considerable market power so it is inconceivable that they would not bid strategically if it should be in their interests to do so. To address the incentive issue, Vickrey (1961), Clark (1972), and Grove (1972) proposed a class of sealed-bid auction mechanisms, now known as the Vickrey-Clarke-Grove (VCG) mechanism. Despite its impressive theoretical virtue, the VCG mechanism has rarely been utilized in practice due to several serious drawbacks; see, e.g., Rothkopf et al. (1990), Yokoo et al. (2004), Ausubel and Milgrom (2005), Perry and Reny (2005), Milgrom (2007, 2017), and Rothkopf (2007). To name but a few, although, in theory, it is optimal for every bidder to reveal his true valuation on every bundle of items, in practice it is extremely difficult for any businessman to disclose such valuable confidential information. Also in any sale of large public assets, the exposure of the big gap between the highest bid (say 100 million dollars) and the second-highest bid (1 million dollars) could cause outrage; see McMillan (1994). So the VCG mechanism lacks privacy protection. Secondly, the VCG mechanism is vulnerable to false identities created by cheating bidders. Thirdly, it yields low or zero revenues for the seller. Fourthly, because the VCG mechanism is static like other sealed-bid auctions, requiring all valuation information of all bidders at once with no consideration for its cost or necessity (see Perry and Reny 2005), it ignores any competitive price system and does not offer any chance for bidders to learn nor allows them to use information economically. The importance of the competitive price system cannot be overstated. The price system is acclaimed by Hayek (1954) as a marvelous mechanism to coordinate the activities of different agents in an economy where agents each possess private and incomplete information. Hurwicz (1973) also stressed the efficient and effective use of dispersed private information.

Every auction has its rules that guide and coordinate the activities of all bidders. The rules will affect the behaviors of bidders and shape the outcome of the market. So the design of these rules matters. The current article offers a novel and general dynamic auction design which does not only yield an efficient and competitive outcome but also overcomes the weaknesses of the VCG mechanism and exhibits several attractive features. Our auction model consists of finitely many bidders and several heterogeneous indivisible items. The seller has a reserve price for every bundle of items and tries to gain a maximal revenue by selling her goods. All bidders have their private valuations on bundles of items and may act strategically. Our dynamic auction is built upon a competitive price system and works as follows: In each round of the auction, the auctioneer announces the current prices for all items, bidders respond by reporting their demands at these prices, and then the auctioneer adjusts the prices of items accordingly. We establish that although bidders are not assumed to be price-takers so they can strategically exercise their market power, this auction always induces bidders to bid truthfully and yields a competitive equilibrium and a generalized VCG payment for every bidder.

Firstly, our auction works for all unimodular demand types, including several well-known cases such as gross substitutes, strong substitutes, gross substitutes and complements, particularly complements and others recently identified by Baldwin and Klemperer (2014, 2019). The case of substitutes has been well studied by Kelso and Crawford (1982), Gul and Stacchetti (2000), Milgrom (2000), Ausubel (2004, 2006), Hatfield and Milgrom (2005), and Milgrom and Strulovici (2009), etc. As our auction applies to all types of indivisible objects, it works naturally for complements and thus resolves the well-known exposure problem and threshold problem in auction design involving complementarity between items as pointed out by Milgrom (2000, 2004), Jehiel and Moldovanu (2003), Noussair (2003), Porter et al. (2003), and Maskin (2005). Our auction achieves also a strong, appealing incentive-compatibility result for all unimodular demand types that sincere bidding by every bidder is an ex post perfect Nash equilibrium in the underlying dynamic auction game of incomplete information.[4] In his ingenious design, Ausubel (2004, 2006) obtained a similar result in the cases of identical items and substitutes respectively.

Secondly, because the current auction permits the seller to have a reserve price for every bundle of items below which the bundle will not be sold, it can easily avoid low or zero revenues and thus reduce payoff uncertainty for the seller. The current auction is privacy-preserving, as bidders only need to report their chosen bundles at a number of price vectors along the path from the starting prices to market-clearing prices; see also Ausubel (2004, 2006). The current auction can tolerate various dishonest behaviors and mistakes made by bidders and allows them to adjust and correct but offenders may have to pay a price. Unlike the conventional approach of a huge penalty for violation, we adopt a lenient policy and show that no bidder will end up with a negative payoff as long as he can differentiate a positive number from a negative one, no matter how his competitors bid. The current auction

[4]By a result of Leonard (1983) we know that in the face of the dynamic auction of Demange et al. (1986) sincerely bidding is an ex post Nash equilibrium.

is independent of any probability distribution assumption, detail-free, and robust against any regret and needs only a minimal common knowledge assumption that the unimodular demand type of the goods in the market is known. This is desirable and important; see Wilson (1987). Our auction rules are simple and transparent and can help reduce the payoff uncertainty and strategic uncertainty for bidders; see Bergemann and Morris (2007).

Thirdly, we prove that the set of competitive equilibrium price vectors in the market is an integrally convex polytope as long as all agents have integer valuations. This result holds for all unimodular demand types and generalizes and also improves several previously known results. An *integrally convex polytope* is a convex hull of finitely many integer vectors with a striking geometry that the distance between any two contiguous integer points in the polytope is 1. Shapley and Shubik (1971) established the lattice structure of equilibrium price vectors for the assignment market and showed that the two extreme points of the lattice are integral. Gul and Stacchetti (1999) and Ausubel (2006) proved that the same conclusion holds for more general markets with gross substitutes. By our current result, we can say more about their lattice and reveal a deep structure. Integral convexity is an important concept studied in discrete optimization (see Favati and Tardella 1990 and Murota 2003).

Fourthly, the current auction works for all unimodular demand types and converges globally from any starting point to a competitive equilibrium of the market. Our approach is very general, employing only convexity and unimodularity. As our approach has to deal with all unimodular demand types, it does not and cannot use submodularity, which has often been used in the literature; see Gul and Stacchetti (2000) and Ausubel (2006). Submodularity indeed holds for substitutes but, in general, does not hold for other demand types. Besides substitutes, there are so many other different unimodular demand types. Another noteworthy difference from the previous designs is that our current design is not an ad hoc but a universal approach based on a new concept "a search set". A search set is a finite set of integer vectors in $\mathbb{R}^N$ defined for every given unimodular demand type for any market with $N$ indivisible items. It is used by the auctioneer to update prices in a neighborhood of the current prices in the auction process. The search set is also served as a test set for the optimality of a constrained nonlinear optimization problem.

The article is organized as follows. The auction model is introduced in Section 2. The structure of the set of competitive equilibria and other properties of the model are explored in Section 3. The basic dynamic auction design and convergence are discussed in Section 4. The incentive-compatible dynamic auction built upon the basic dynamic auction and its strategic properties are examined in Section 5. Section 6 concludes. Several proofs are given in the appendix.

## 2 The Model

An auctioneer or a seller wants to sell a set $N = \{1, 2, \cdots, n\}$ of $n$ indivisible items to a finite group $B$ of $m$ potential bidders. Some of the items can be heterogeneous and the other can be identical. Identical items will be labelled differently. This way of treating indivisible items in a competitive equilibrium model causes no loss of generality as identical units of the same good can be treated as different goods but will have the same equilibrium price. In the paper the symbols $\mathbb{R}$ and $\mathbb{Z}$ mean the sets of all real and integer numbers respectively. $\mathbb{R}^N$ denotes the $n$-dimensional Euclidean space where each coordinate is indexed by a number from the set $N$. Let $\mathbb{Z}^N$ stand for the family of all integer vectors in $\mathbb{R}^N$. For every $i \in N$, let $e(i)$ denote the $i$th unit vector in $\mathbb{R}^N$. A subset $S$ of $N$ represents a bundle of items in $S$. For easy exposition, we regard a set $S$ and the corresponding vector $\sum_{i \in S} e(i)$ as the same bundle. Every bidder (he) $j \in B$ has a utility function $u^j : \{0,1\}^N \to \mathbb{Z}_+ \cup \{-\infty\}$ specifying his valuation $u^j(x)$ (in units of money, say, in dollars) on each bundle $x$ with $u^j(\mathbf{0}) = 0$, where $\{0,1\}^N$ denotes the family of all bundles of items. The set $\text{dom}(u^j) = \{x \in \{0,1\}^N \mid u^j(x) > -\infty\}$ is called the *effective domain* of $u^j$, which we assume to contain the dummy bundle $\mathbf{0}$ and at least one nonzero vector for all $j \in B$. Bidders have quasi-linear utilities in money and face no budget constraints. The seller (she) denoted by 0 has a weakly increasing reserve function $u^0 : \{0,1\}^N \to \mathbb{Z}_+$ with $u^0(\mathbf{0}) = 0$ and $\text{dom}(u^0) = \{0,1\}^N$. So the seller will not sell any bundle if the total price for the bundle is less than her reserve value. Let $B_0 = B \cup \{0\}$ stand for the set of all market participants (all bidders and the seller). We use $\mathcal{M} = (u^j, j \in B_0, N)$ or simply $\mathcal{M}$ to represent the market.

A submarket is what is left in the market $\mathcal{M}$ by deleting a number of its bidders and a number of its items. In the paper when we talk about a generic agent which can be a bidder or the seller, we treat the agent as female. Note that in our model if a bundle $x$ is unacceptable to any bidder $j \in B$, then the utility of that bundle is $-\infty$, i.e., $u^j(x) = -\infty$. That the effective domain of every bidder $j \in B$ contains the dummy bundle $\mathbf{0}$ and at least one nonzero vector means that every bidder has the option of buying nothing and is interested in buying some goods. The seller has the entire set $\{0,1\}^N$ as her effective domain and is happy to retain any bundle if it is not sold. Observe that the seller's utility function $u^0$ is weakly increasing. Free disposal is sufficient for this weak monotonicity. All agents have nonnegative valuations on their interested bundles. So our model treats all items in $N$ as goods and our analysis will focus on this most important case. Indivisible "bads" can be analyzed analogously. In the rest of the paper the effective domain of any utility function is assumed to contain the dummy bundle and at least one nonzero bundle.

A price vector $p = (p_1, \cdots, p_n) \in \mathbb{R}^N$ specifies a price $p_i$ for each item $i \in N$ and is the same for all bidders. This is a linear and anonymous pricing rule and has long and widely being used in theory and practice.[5] Every bidder $j \in B$ tries to maximize his profit and his demand correspondence $D^j(p)$ is given by

$$D^j(p) \quad = \quad \arg\max_{x\in\{0,1\}^N}\{u^j(x) - p\cdot x\}, \tag{1}$$

where $p \cdot x = \sum_{i\in N} p_i x_i$. At a price $p \in \mathbb{R}^N$, the seller chooses bundles to maximize her revenues and her demand correspondence $D^0(p)$ is given by

$$\begin{aligned} D^0(p) &= \arg\max_{x\in\{0,1\}^N}\{u^0(x) + p\cdot(\textstyle\sum_{i\in N} e(i) - x)\} \\ &= \arg\max_{x\in\{0,1\}^N}\{u^0(x) - p\cdot x + \textstyle\sum_{i\in N} p_i\} \\ &= \arg\max_{x\in\{0,1\}^N}\{u^0(x) - p\cdot x\}. \end{aligned}$$

The set $D^0(p)$ contains those bundles that the seller wishes to keep in hand and give her the highest revenues. Although the seller has a totally different objective from the bidders, her revenue-maximizing behavior is similar to a bidder's profit-maximizing behavior. Observe that if $x \in D^0(p)$ at price $p$, the seller will retain the bundle $x$ and sell all other items by receiving the payment of $p \cdot (\sum_{i\in N} e(i) - x) = \sum_{i\in N} p_i - p\cdot x$.

An *allocation* of items in $N$ is a *redistribution* $X = (x^j, j \in B_0)$ of items among all market participants in $B_0$ such that $\sum_{j\in B_0} x^j = \sum_{i\in N} e(i)$ and $x^j \in \{0,1\}^N$ for all $j \in B_0$. Note that $x^j = \mathbf{0}$ is allowed. At allocation $X$, agent $j$ receives bundle $x^j$. An allocation $X = (x^j, j \in B_0)$ is *efficient* if $\sum_{j\in B_0} u^j(x^j) \geq \sum_{j\in B_0} u^j(y^j)$ for every allocation $Y = (y^j, j \in B_0)$. Given an efficient allocation $X$, let $R(N) = \sum_{j\in B_0} u^j(x^j)$. We call $R(N)$ *the market value* of the items which is the same for all efficient allocations.

**Definition 1** A *competitive or Walrasian equilibrium* $(p, X)$ consists of a price vector $p \in \mathbb{R}^N_+$ and an allocation $X$ such that $x^j \in D^j(p)$ for every $j \in B_0$.

If $(p, X)$ is a competitive equilibrium, then we call $p$ the *equilibrium price vector* and $X$ the *equilibrium allocation.* We say that $X$ is supported by $p$. It is well-known that every equilibrium allocation is efficient, but an equilibrium may not always exist.

We like to stress once more that in the current model restricting every agent's utility on the basic domain $\{0,1\}^N$ does not cause any loss of generality. This fact is already known in the literature. We quote the following passage from Bikhchandani and Mamer (1997, p. 391): "The assumption of one unit supply of each object is without loss of generality. In case there are multiple units of some objects, one can expand the commodity space by treating each unit of an object as a different commodity. It may be verified that market clearing prices exist in the original economy with multiple units per object if and only if

[5] Ausubel and Milgrom (2002) and Milgrom (2007) discussed nonlinear and discriminatory or personalized pricing rules in package auctions. Wilson (1993) examined nonlinear and anonymous pricing rules in some utilities sector and Sun and Yang (2014) explored the same pricing rule for auctioning multiple complementary indivisible items.

market clearing prices exist in the new economy with one unit per object. Moreover, the sets of market allocations supported by equilibrium prices in the two economies (which may be empty sets) are identical, except for relabelling."

It is also worth pointing out that our model can cover and accommodate a variety of situations and environments by simply adjusting the effective domain of each bidder's utility function. We illustrate this by the celebrated assignment market as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1971), Crawford and Knoer (1981), and Demange et al. (1986). In this market every buyer demands at most one item. In this case, the effective domain of every buyer $j \in B$ is the set of $\{e(i), i \in N\} \cup \{\mathbf{0}\}$.

Economies with indivisible goods may not always have a competitive equilibrium. Baldwin and Klemperer (2019) have recently proposed a remarkably elegant necessary and sufficient condition for the existence of competitive equilibrium in an exchange economy with indivisible goods, [6] which will be the condition to be used in our auction. Their condition covers and generalizes many previous conditions including the widely-used Gross Substitutes condition of Kelso and Crawford (1982).

For any finite set $A$, $\sharp A$ denotes the number of elements in $A$. The dimension of any given set $S \subset \mathbb{R}^N$ is understood as the dimension of the affine span of $S$. With respect to any given utility function $u : S \to \mathbb{R} \cup \{-\infty\}$ with a finite set $S \subset \mathbb{Z}^N$ and $\sharp\text{dom}(u) > 1$, let the demand set at a price vector $p \in \mathbb{R}^N$ be given by

$$D_u(p) = \arg\max_{x \in S}\{u(x) - p \cdot x\}.$$

Following Baldwin and Klemperer (2019), we say that the set

$$\mathcal{T}_u = \{p \in \mathbb{R}^N \mid \sharp D_u(p) > 1\}$$

is *the locus of indifference prices* (LIP) of the demand set $D_u$. This set $\mathcal{T}_u$ concerns those price vectors $p$ at which there are at least two optimal bundles for any agent who has the utility function $u$. It is known that LIP contains the only prices at which demand can change in response to a price change, and is the union of $(n-1)$-dimensional polyhedral pieces called facets. These facets separate the unique demand regions, in each of which some bundle is the unique demand. A *facet* of $\mathcal{T}_u$ is an $(n-1)$-dimensional subset $F$ of the set $\mathcal{T}_u$ such that there exist $x, y \in D_u(p)$ with $x \neq y$ for some $p \in F$. A facet of a polytope of dimension $n$ is a face that has dimension $n-1$. The *normal vector* to a facet $F$ is a vector which is perpendicular to $F$ at a point in its relative interior. A non-zero integer vector is *primitive*

[6] Earlier existence results include Koopmans and Beckmann (1957), Shapley and Shubik (1971), Kelso and Crawford (1982), Bikhchandani and Mamer (1997), van der Laan et al. (1997), Ma (1998), Bevia et al. (1999), Gul and Stacchetti (1999), Yang (2000, 2003), Danilov et al. (2001), Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield et al. (2013), Shioura and Yang (2015).

if the greatest common divisor of its coordinates is one. An *edge* of a polyhedron is a face of dimension one and an *edge-vector* is a direction vector of an edge, which is any non-zero scalar multiple of the difference of any two distinct points on the edge. By definition, if $v$ is an edge vector of a polyhedron $P$, then $\alpha v$ for any $\alpha \neq 0$ is also an edge-vector of $P$; in particular, so is $-v$.

**Definition 2** A set $\mathcal{D}_u \subseteq \mathbb{Z}^N$ is a *demand edge-set* of a utility function $u$ if $\mathcal{D}_u$ is the family of the primitive edge-vectors of the convex hull of every demand set $D_u(p)$ with $\sharp D_u(p) > 1$.

It should be noted that we have $v \in \mathcal{D}_u$ if and only if $v$ is normal to some facet of the LIP $\mathcal{T}_u$. Every agent $j \in B_0$ has a demand edge-set $\mathcal{D}_{u^j}$, which can be different from any other agent's demand edge-set. In particular, the demand edge-set $\mathcal{D}_{u^0}$ of the seller contains every unit vector $e(i)$, $i \in N$, and spans $\mathbb{R}^N$. Clearly, the union of the demand edge-sets $\mathcal{D}_{u^j}$ for all $j \in B_0$ also spans $\mathbb{R}^N$.

**Definition 3** Suppose that a finite set $\mathcal{D} \subseteq \mathbb{Z}^N$ consists of primitive vectors satisfying that $v \in \mathcal{D}$ implies $-v \in \mathcal{D}$. For each $j \in B_0$ we say that the demand edge-set $\mathcal{D}_{u^j}$ is *of type* $\mathcal{D}$ if we have $\mathcal{D}_{u^j} \subseteq \mathcal{D}$. We also call $\mathcal{D}$ a *demand type*.

A square matrix is *unimodular* if all its elements are integral and its determinant is $+1$ or $-1$. A matrix $M$ is *totally unimodular* if every minor of $M$ is 0 or $\pm 1$. A set of $n$ integer vectors in $\mathbb{R}^N$ is *a unimodular basis* for $\mathbb{R}^N$ if the $n \times n$ matrix which has the $n$ integer vectors as its columns is unimodular. The following definition of a unimodular demand type is introduced by Baldwin and Klemperer (2019).

**Definition 4** A demand type $\mathcal{D}$ is *unimodular* if every linearly independent subset of $\mathcal{D}$ can be extended to a unimodular basis for $\mathbb{R}^N$.

Note that in the above definition of a unimodular demand type $\mathcal{D}$ additional vectors required to form a unimodular basis are possibly chosen from outside $\mathcal{D}$. As the union of the demand edge-set $\mathcal{D}_{u^j}$ for all $j \in B_0$ spans $\mathbb{R}^N$, when $\mathcal{D}_{u^j} \subseteq \mathcal{D}$ for every $j \in B_0$, the demand type $\mathcal{D}$ also spans $\mathbb{R}^N$.

The following two conditions will be imposed on our auction market $\mathcal{M}$:

(A1) *Integer Private Values*: Every agent $j \in B_0$ knows her own utility function $u^j : \{0,1\}^N \to \mathbb{Z}_+ \cup \{-\infty\}$ privately.

(A2) *Common Unimodular Demand Type*: All agents $j \in B_0$ have the same unimodular demand type $\mathcal{D}$ for their utility functions $u^j$.

When agent $j$'s utility function $u^j$ satisfies Condition (A2), we say that agent $j$ has a *UDT $\mathcal{D}$ utility function $u^j$*. The integer-valued assumption is a standard and natural assumption, as people valuate the bundles of goods in units of currency, say, in dollars, which cannot be closer to the nearest penny. Assumption (A1) means that every agent treats her valuation as her private, personal information. Assumption (A2) says that agents may have quite different valuations on the same goods but they all have the same demand type, which captures the quintessence of the goods.

Observe that the definition of a unimodular demand type is the same as Definition 4.2 of Baldwin and Klemperer (2019, p. 888), while we consider each demand edge-set $\mathcal{D}_{u^j}$ ($j \in B_0$) and assume the existence of a unimodular demand type $\mathcal{D}$ such that $\mathcal{D}_{u^j} \subseteq \mathcal{D}$ for every $j \in B_0$. Such a demand type $\mathcal{D}$ is not unique, but if at least one such unimodular demand type exists, then the union of $\mathcal{D}_{u^j}$ for all $j \in B_0$ is the unique minimal such unimodular demand type in the sense of Baldwin and Klemperer (2019) and we can simply take the union as the unimodular demand type $\mathcal{D}$ in Assumption (A2).

Baldwin and Klemperer (2019, Theorem 4.3) have shown by tropical geometry and convex analysis that a market together with all its submarkets has a competitive equilibrium if and only if Assumption (A2) holds; see Baldwin et al. (2020) for their extension to include income effects. Tran and Yu (2019) have given an alternative proof for this result through the linear programming approach. Assumption (A2) is a test condition imposed upon every individual agent. This condition is neat and easy to check compared with the earlier necessary and sufficient conditions introduced by Bikhchandani and Mamer (1997), Ma (1998) and Yang (2003) which are given as aggregated conditions on the entire market.

# 3 On the Structure of Competitive Equilibria

In this section we present several basic results on the structure of the set of competitive equilibrium price vectors. Some of these results will play an indispensable role in our auction design and others are interesting on their own right.

For every agent $j \in B_0$ we can define her indirect utility function by

$$V^j(p) = \max_{x \in \{0,1\}^N} \{u^j(x) - p \cdot x\} \qquad (p \in \mathbb{R}^N). \tag{2}$$

It is known that for any utility function $u^j : \{0,1\}^N \to \mathbb{R} \cup \{-\infty\}$, the indirect utility function $V^j$ is a decreasing, continuous and convex function.

For the market model, define the Lyapunov function $\mathcal{L} : \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{L}(p) = \sum_{i \in N} p_i + \sum_{j \in B_0} V^j(p) \tag{3}$$

where $V^j$ is the indirect utility function of agent $j \in B_0$. This type of function is well-known in the literature for economies with divisible goods (see e.g., Arrow and Hahn (1971) and Varian (1981)) and has been recently explored by Ausubel (2006) and Sun and Yang (2009) for auction markets with indivisible goods.

Let us first introduce several basic concepts before presenting our results on the structure of the set of competitive equilibrium price vectors.

A set $S \subseteq \mathbb{R}^N$ is a *polyhedron* if $S = \{x \in \mathbb{R}^N \mid Ax \leq b\}$ for some $m \times n$ matrix $A$ and an $m$-vector $b$. A polyhedron $S \subseteq \mathbb{R}^N$ is *integral* if all its vertices are integral. A polyhedron $S$ is a *polytope* if it is bounded, or equivalently, if it is a convex hull of finitely many vectors in $\mathbb{R}^N$. The *Minkowski sum* of any two sets $S$ and $T$ in $\mathbb{R}^N$ is defined as $S + T = \{x + y \mid x \in S,\ y \in T\}$. Given any $x, y \in \mathbb{R}^N$, define their meet $x \wedge y$ as the componentwise minimum of $x$ and $y$ and join $x \vee y$ as the componentwise maximum of $x$ and $y$. A set $S \subset \mathbb{R}^N$ is *a lattice* if $x \wedge y \in S$ and $x \vee y \in S$ for any $x, y \in S$. A polyhedron is called *a polyhedron with a lattice structure* if it is also a lattice. It is known that a lattice is not necessarily a polyhedron.

For any set $T \subseteq \mathbb{R}^N$, $\text{Conv}(T)$ denotes its convex hull. For any two points $x, y \in \mathbb{R}^N$, we use the maximum norm to measure their distance, i.e., $||x - y|| = \max_{i \in N} |x_i - y_i|$. For any given $x \in \mathbb{R}^N$ we define the integral neighbor of $x$ as

$$\mathbf{N}(x) = \{y \in \mathbb{Z}^N \mid ||y - x|| < 1\}$$

A set $D \subseteq \mathbb{R}^N$ is *integrally convex* if $D = \text{Conv}(D)$ and $x \in D$ implies $x \in \text{Conv}(D \cap \mathbf{N}(x))$, i.e., every point $x \in D$ can be represented as a convex combination of integral points in $\mathbf{N}(x) \cap D$. Favati and Tardella (1990) originally introduced this concept for discrete subsets of $\mathbb{Z}^n$ and called a discrete set $D \subseteq \mathbb{Z}^n$ an *integrally convex set* if every $x \in \text{Conv}(D)$ satisfies $x \in \text{Conv}(\text{Conv}(D) \cap \mathbf{N}(x))$. We see that the convex hull of every integrally convex set $D \subseteq \mathbb{Z}^N$ is integrally convex in $\mathbb{R}^N$. Conversely, any integrally convex set $D$ in $\mathbb{R}^N$ is the convex hull of $D \cap \mathbb{Z}^N$ that is integrally convex in $\mathbb{Z}^N$.[7] We will call an integrally convex set in $\mathbb{R}^N$ an *integrally convex polyhedron.*

An integrally convex polyhedron is an extremely well-behaved integral polyhedron with a distance of one between any two contiguous integer points it may contain. For instance, the integral polyhedron $S_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 2x_1,\ x_2 \geq 0\}$ is not an integrally convex polyhedron nor is the integral polytope $S_2 = \text{Conv}(\{(0,0), (1,3), (2,1), (2,2)\})$. The integral polyhedron $S_3 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq -x_2,\ x_2 \geq 0\}$ is an integrally convex polyhedron; so is the polytope $S_4 = \text{Conv}(\{(0,1), (0,2), (1,0), (1,3), (4,3)\})$.

[7]For, let $D$ be any integrally convex set in $\mathbb{R}^N$ and put $D' = D \cap \mathbb{Z}^N$. For any $x \in \text{Conv}(D')$, we have $\text{Conv}(D \cap \mathbf{N}(x)) = \text{Conv}(D' \cap \mathbf{N}(x))$, so that $x \in \text{Conv}(D' \cap \mathbf{N}(x))$ (by the assumption and since $x \in D$). Hence $D'$ is integrally convex in $\mathbb{Z}^N$. Moreover, for any $x \in D$ we have $x \in \text{Conv}(D \cap \mathbf{N}(x)) = \text{Conv}(D' \cap \mathbf{N}(x)) \subseteq \text{Conv}(D')$, which implies $D \subseteq \text{Conv}(D')(\subseteq D)$. Consequently, $D = \text{Conv}(D')$.

Given a function $f : S \to \mathbb{R}$ with a polyhedral convex set $S \subseteq \mathbb{R}^N$, the set

$$\{(x, \alpha) \mid x \in S,\ \alpha \in \mathbb{R},\ \alpha \geq f(x)\}$$

is called *the epigraph of* $f$. A function $f$ with a polyhedral effective domain $S$ in $\mathbb{R}^N$ is called *a polyhedral convex function* if it is given as

$$f(x) = \max\{B_j \cdot x + c_j \mid j = 1, \cdots, m\} \qquad (x \in S),$$

where $B_j$ is an $n$-vector and $c_j$ is a constant, $j = 1, \cdots, m$ for a given positive integer $m$. A polyhedral convex function $f$ defined on $\mathbb{R}^N$ is called *conical* if its epigraph is a translation of a polyhedral convex cone.

A set $S \subseteq \mathbb{Z}^N$ is *discrete convex* if for arbitrary $x^1, \cdots, x^m \in S$ and for arbitrary $\lambda_i \geq 0, \cdots, \lambda_m \geq 0$ with $\sum_{j=1}^m \lambda_j = 1$, $x = \sum_{j=1}^m \lambda_j x^j \in \mathbb{Z}^N$ implies $x \in S$. A function $f : \mathbb{Z}^N \to \mathbb{R}$ is *discrete concave* if for arbitrary finite number of $\lambda_j \geq 0$, $j = 1, \cdots, t$ and arbitrary $x^j \in \mathbb{Z}^N$ for $j = 1, \cdots, t$ with $\sum_{j=1}^t \lambda_j = 1$ and $\sum_{j=1}^t \lambda_j x^j \in \mathbb{Z}^N$ we have

$$f(\lambda_1 x^1 + \lambda_2 x^2 + \cdots + \lambda_t x^t) \geq \sum_{j=1} \lambda_j f(x^j).$$

Given a function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}(f) \neq \emptyset$, we define *the local concave extension* of $f$ by

$$\bar{f}(x) = \inf_{p \in \mathbf{R}^N,\, \alpha \in \mathbf{R}} \{p \cdot x + \alpha \geq f(y) \text{ for all } \ y \in \mathbf{N}(x)\}$$

for every $x \in \mathbb{R}^N$. If $\bar{f}$ is concave on $\mathbb{R}^N$, then the function $f$ is said to be *integrally concave.* The class of integrally concave functions is also due to Favati and Tardella (1990).

Given a lattice $S \subseteq \mathbb{Z}^N$, a function $f : S \to \mathbb{R} \cup \{-\infty\}$ is *submodular* if $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$ for any $x, y \in S$. When a utility function of items is submodular, it has decreasing marginal returns over any item. This means that items exhibits substitutability. A function $f : S \to \mathbb{R}$ is *subadditive* if $f(x+y) \leq f(x)+f(y)$ for any $x, y \in S$. Subadditivity reflects a more general substitutability. A function $f$ is *supermodular* if $-f$ is submodular. If a utility function of items is supermodular, then these items have increasing marginal returns and show complementarity. A function $f$ is *superadditive* if $-f$ is subadditive. Superadditivity is more general than supermodularity.

We have the following two basic results.

**Lemma 1** *For any given function $f : S \to \mathbb{R}$ with a finite set $S \subset \mathbb{Z}^N$, the function $g : \mathbb{R}^N \to \mathbb{R}$ defined by*

$$g(p) = \max_{x \in S} \{f(x) - p \cdot x\}$$

*for every $p \in \mathbb{R}^N$ is a decreasing polyhedral convex function.*

**Lemma 2** *For the market model, the Lyapunov function $\mathcal{L}$ defined by* (3) *is a polyhedral convex function bounded from below.*

The above two lemmas are very general and do not depend on any particular conditions such as Assumptions (A1) and (A2). Proposition 1 of Ausubel (2006) and Lemma 1 of Sun and Yang (2009) imply that $p \in \mathbb{R}^N$ is an equilibrium price vector if and only if it is a minimizer of the Lyapunov function $\mathcal{L}$ provided that the market has an equilibrium.

The following result reveals an important property concerning the unimodular demand type and will be invoked later.

**Lemma 3** (Tran and Yu 2019, Lemma 4.3) *Let $\mathcal{D}$ be a unimodular demand type. If $P_1$ and $P_2$ are integral polytopes in $\mathbb{R}^N$ with edge directions in $\mathcal{D}$, then* $\mathrm{Conv}(P_1 + P_2) \cap \mathbb{Z}^N = P_1 \cap \mathbb{Z}^N + P_2 \cap \mathbb{Z}^N$ *holds true.*

The following lemma shows when a full-dimensional demand set occurs at integer prices.

**Lemma 4** *For any integer-valued UDT $\mathcal{D}$ utility function $u : S \to \mathbb{Z} \cup \{-\infty\}$ with a finite set $S \subset \mathbb{Z}^N$ and $\sharp\mathrm{dom}(u) > 1$, if the convex hull of the demand set $D_u(p)$ for a price $p$ is full-dimensional, then the price vector $p$ must be integral and unique.*

Consider the *convolution* $u$ of $u^j$ for all agents $j \in B_0$ given by

$$u(x) = \max\{u^0(y^0)+u^1(y^1)+\cdots+u^m(y^m) \mid x = y^0+y^1+\cdots+y^m, \forall j \in B_0 : y^j \in \{0,1\}^N\} \quad (4)$$

for all $x \in \{0, 1, \cdots, m\}^N$. For every $p \in \mathbb{R}^N$ and every $x^j \in D^j(p)$ of every $j \in B_0$, we define

$$g(x) = \sum_{j \in B_0} u^j(x^j) \text{ where } x = \sum_{j \in B_0} x^j. \quad (5)$$

From all demand sets $D^j(p)$ for $j \in B_0$ we obtain the following demand set of Minkowski sum

$$D^{Ms}(p) = D^0(p) + D^1(p) + \cdots + D^m(p) \quad (6)$$

It is worth pointing out that the following general properties on discrete convexity might be known in some circle as folklore.

**Proposition 1** *Let $f : \mathbb{Z}^N \to \mathbb{Z} \cup \{-\infty\}$ be any function with a nonempty bounded effective domain.*

(a) *If for every $p \in \mathbb{R}^N$ the set* $\arg\max\{f(x) - p \cdot x \mid x \in \mathbb{Z}^N\}$ *is discrete convex, then $f$ is a discrete concave function.*

(b) *If $f$ is a discrete concave function and for every $p \in \mathbb{R}^N$ the convex hull of the set* $\arg\max\{f(x) - p \cdot x \mid x \in \mathbb{Z}^N\}$ *has a* $\{0, \pm 1\}$*-valued edge-direction vector for its every edge, then $f$ is an integrally concave function.*

The following lemma shows some desirable properties of the function $g$ of (5) and the Minkowski sum $D^{Ms}$ of (6).

**Lemma 5** *Assume that the market model satisfies Assumptions* (A1) *and* (A2)*. Then, the function $g$ of* (5) *is well-defined, coinciding with the convolution function $u$ of* (4) *and being integrally concave with the unimodular demand type $\mathcal{D}$. Moreover, the Minkowski sum $D^{Ms}$ of* (6) *has the same unimodular demand type $\mathcal{D}$.*

**Proof.** Take any $x^j \in D^j(p)$ for all $j \in B_0$. Then $g(x) = \sum_{j \in B_0} u^j(x^j)$ with $x = \sum_{j \in B_0} x^j$. By definition for all $j \in B_0$ we have

$$u^j(x^j) - p \cdot x^j \geq u^j(y^j) - p \cdot y^j, \text{ for all } y^j \in \mathrm{dom}(g). \tag{7}$$

Clearly, for all $y^j \in \mathrm{dom}(g)$ $(j \in B_0)$ satisfying $\sum_{j \in B_0} x^j = \sum_{j \in B_0} y^j$ we have

$$u^j(x^j) - p \cdot x^j \geq u^j(y^j) - p \cdot y^j.$$

Now adding all inequalities up yields

$$\sum_{j \in B_0} u^j(x^j) \geq \sum_{j \in B_0} u^j(y^j)$$

for all $y^j \in \mathrm{dom}(g)$ $(j \in B_0)$ satisfying $\sum_{j \in B_0} x^j = \sum_{j \in B_0} y^j$. By definition $u(x) = \sum_{j \in B_0} u^j(x^j)$. Observe that the above relationship still holds true if $g(z) = \sum_{j \in B_0} u^j(z^j)$ with $x = z = \sum_{j \in B_0} z^j$ and $z^j \in D^j(q)$ for $j \in B_0$ and $q \neq p$. This shows

(i) $g(x) = u(x)$ and $g$ is well-defined.

For any edge $F^{Ms}$ (a one-dimensional face) of $\mathrm{Conv}(D^{Ms}(p))$ there exists a vector $w \in \mathbb{R}^N$ such that

$$F^{Ms} = \arg\max\{w \cdot x \mid x \in \mathrm{Conv}(D^{Ms}(p))\}. \tag{8}$$

Then we see from (6) that by defining $F^j = \arg\max\{w \cdot x \mid x \in \mathrm{Conv}(D^j(p))\}$ for each $j \in B_0$, we have

$$F^{Ms} = F^0 + F^1 + \cdots + F^m. \tag{9}$$

Since $F^{Ms}$ is an edge, it follows from (9) that each $F^j$ $(j \in B_0)$ is either a vertex or an edge of $\mathrm{Conv}(D^j(p))$ and at least one $F^j$ must be an edge, and furthermore, if $F^j$ and $F^{j'}$ are edges of $\mathrm{Conv}(D^j(p))$ and $\mathrm{Conv}(D^{j'}(p))$, respectively, for distinct $j, j' \in B_0$, then the two must have the same direction vector $d \in \mathcal{D}$. Hence, because of (9), $F^{Ms}$ also has the same direction vector $d \in \mathcal{D}$. This means the following:

(ii) $D^{Ms}(p)$ is of demand type $\mathcal{D}$.

Also note that from (i) shown above and from (4) and Lemma 3

(iii) $D^{Ms}(p) = \arg\max\{u(x) - p \cdot x \mid x \in \mathbb{Z}^N\}$.

It follows from Lemma 3 and the above (i), (ii), and (iii) that for every $p \in \mathbb{R}^N$ the demand correspondence $D^{Ms}(p)$ of $u : \mathbb{Z}^N \to \mathbb{Z} \cup \{-\infty\}$ is a discrete convex set. Because every set $D^{Ms}(p)$ is a discrete convex set, this implies that $u$ is a discrete concave function. Note that the convex hull of any set of unimodular demand type $\mathcal{D}$ has a $\{0, \pm 1\}$-valued edge-direction vector for its every edge. Hence the function $u$ (or $g$) here is actually an integrally concave function, due to Proposition 1. □

We are ready to establish our first major result on the set of competitive equilibrium price vectors, which exhibits a remarkably well-behaved geometric structure.

**Theorem 1** *Assume that the market model satisfies Assumptions* (A1) *and* (A2). *Then the set of competitive equilibrium price vectors forms a nonempty integrally convex polytope.*

In the rest of this section we discuss three typical and important cases of demand type identified by Baldwin and Klemperer (2019): Gross Substitutes, complements, and Gross Substitutes and Complements.

**Definition 5** A demand type $\mathcal{D}$ is said to be *Gross Substitutes* if every vector $x \in \mathcal{D}$ has at most one 1 entry and at most one $-1$ entry and no other nonzero entries.

This definition captures the following well-known Gross Substitutes condition of Kelso and Crawford (1982) which describes the demand behavior in terms of prices change.

**Definition 6** A demand correspondence $D_u(\cdot)$ satisfies the *Gross Substitutes* condition if for every two price vectors $p$ and $q$ in $\mathbb{R}^N$ with $p \leq q$ and for every $A \in D_u(p)$, there exists $B \in D_u(q)$ with $\{k \in A \mid p_k = q_k\} \subseteq B$.

Gul and Stacchetti (1999, 2000) have proposed the single improvement (SI) property and no complementarities property, two different equivalent forms of the Gross Substitutes condition. Fujishige and Yang (2003) proved that any utility function satisfies the Gross Substitutes condition if and only if it is an $M^{\natural}$-concave function of Murota and Shioura (1999). The SI property plays an important role in the auction design of Gul and Stacchetti (2000) and Ausubel (2006).

Gul and Stacchetti (2000) and Ausubel (2006) have shown that when every bidder's demand set satisfies the Gross Substitutes condition, the set of competitive equilibrium price vectors is a nonempty lattice. Their results generalize an earlier lattice theorem of Shapley and Shubik (1972) for the assignment market in which every buyer demands only one item. The following result strengthens and refines all these results.

**Corollary 1** *Assume that the market model satisfies Assumption* (A1) *and all agents* $j \in B_0$ *have the same Gross Substitutes demand type* $\mathcal{D}$ *for their utility functions* $u^j$*. Then the set of competitive equilibrium price vectors forms a nonempty integrally convex polytope with a lattice structure.*

The following is a generalization of Gross Substitutes to accommodate complementarities in some way.

**Definition 7** Assume that $S_1$ and $S_2$ are disjoint subsets of $N$ and their union equals $N$. A demand type $\mathcal{D}$ is said to be *Gross Substitutes and Complements* (GSC) if every vector $x \in \mathcal{D}$ has at most two nonzero entries so that if two nonzero entries of $x$ have the same sign, then one nonzero component must be indexed by an element in $S_1$ and the other must be indexed by an element in $S_2$.

In general GSC says that items in either $S_1$ or $S_2$ are substitutes but items across the two sets are complementary. Observe when either $S_1$ or $S_2$ becomes empty, GSC coincides with GS and thus generalizes GS. GSC condition is introduced in Sun and Yang (2006, 2009) as a generalization of Definition 6. Similar to Corollary 1, we have

**Corollary 2** *Assume that the market model satisfies Assumption* (A1) *and all agents* $j \in B_0$ *have the same Gross Substitutes and Complements demand type* $\mathcal{D}$ *for their utility functions* $u^j$*. Then the set of competitive equilibrium price vectors forms a nonempty integrally convex polytope.*

Finally, we introduce a very rich class of complements demand types concerning complementary goods.

**Definition 8** A demand type $\mathcal{D}$ is said to be *complements demand type* if $x \in \mathcal{D}$ implies either $x \in \{0,1\}^N$ or $x \in \{0,-1\}^N$.

In other words, all nonzero coordinates of every element in a complements demand type $\mathcal{D}$ have the same sign. In this case we write $\mathcal{D} \subset \pm\{0,1\}^N$ for brevity. Baldwin and Klemperer (2014) have shown that this corresponds to the traditional definition of complementary goods. That is, items are complements if, for any price vectors $q \geq p$ satisfying $\sharp D_u(p) = \sharp D_u(q) = 1$, $\{k \in B \mid p_k = q_k\}$ is a subset of $A$, where $A \in D_u(p)$ and $B \in D_u(q)$.

**Corollary 3** *Assume that the market model satisfies Assumption* (A1) *and all agents* $j \in B_0$ *have the same unimodular complements demand type* $\mathcal{D}$*. Then the set of competitive equilibrium price vectors forms a nonempty integrally convex polytope.*

A basis change is called *a unimodular transformation* if we have $y = Ax$ for every $x \in \mathbb{R}^N$ and $A$ is a unimodular matrix of order $n$. The following result shows that unimodular complements demand types are so rich that any other unimodular demand types can be obtained from them. Unimodular complements demand types are far more pervasive than substitutes. In fact, Gross Substitutes condition is the most general condition for equilibrium existence in the presence of substitutability; see Gul and Stacchetti (1999, Theorem 2, p. 103). But we cannot have a similar statement for complements, because unimodular complements demand types are numerous and varied.

**Theorem 2** (Baldwin and Klemperer 2019, Proposition 6.2; 2014, Theorem 5.27) *Every unimodular demand type is a unimodular transformation of a unimodular complements demand type contained in* $\pm\{0,1\}^N$.

# 4 A Universally Convergent Dynamic Auction

In this section we consider the basic case that bidders bid sincerely, and propose an auction to be called a universally convergent dynamic auction which applies to all unimodular demand types. In the next section we deal with the case that bidders have market power and may bid strategically rather than straightforwardly as price-takers. In a dynamic auction, at each time $t \in \mathbb{Z}_+$ the auctioneer publicly announces a price for every good and then every bidder chooses a bid. A bidder is said to bid sincerely if he always reports his true demand correspondence. Formally,

**Definition 9** (Sincere Bidding) Agent $j \in B_0$ *bids sincerely or straightforwardly* with respect to her utility function $u^j$ if she always submits a bid $B^j(t)$ equal to her demand set $D^j(p(t)) = \arg\max_{x\in\{0,1\}^N}\{u^j(x) - p(t)\cdot x\}$ at every time $t \in \mathbb{Z}_+$ and any price vector $p(t) \in \mathbb{R}^N$.

Roughly speaking, our universally convergent dynamic auction works as follows: At each time $t \in \mathbb{Z}_+$, the auctioneer announces the current prices $p(t) \in \mathbb{Z}^n$ and asks every bidder $j$ to report his demand $D^j(p(t))$. Then she uses every bidder's reported demand $D^j(p(t))$ to search for a price adjustment $\delta$ in an appropriate neighborhood of prices $p(t)$ in order to update the current prices. To do so, the auctioneer tries to reduce the value of the Lyapunov function $\mathcal{L}(p(t) + \delta)$ as much as possible, until a minimizer of the Lyapunov function, i.e., a competitive equilibrium price vector, is found.

We first introduce the concept of a search set which is a key building block of our auction design and gives an appropriate neighborhood of the current prices. The search set is defined with respect to any given demand type $\mathcal{D}$, which is the one given in Assumption (A2) in our auction design.

**Definition 10** For any given demand set $\mathcal{D}$, its *search set* is the collection of the zero vector and all nonzero primitive integer vectors $\delta \in \mathbb{Z}^N$ such that we have $\delta \cdot d_j = 0$ for some $n-1$ linearly independent vectors $d_1, \cdots, d_{n-1} \in \mathcal{D}$. The search set is denoted by $\mathcal{SD}$.

Geometrically, one may view the search set as a family of the zero vector and all nonzero primitive integer vectors $\delta \in \mathbb{Z}^N$ such that $\delta$ is a normal vector of a facet of a full-dimensional demand set at some price vector $p$.

**Lemma 6** *Let $\mathcal{SD}$ be the search set of a unimodular demand type $\mathcal{D}$ and $\delta \in \mathcal{SD}$ be a primitive normal vector of an $(n-1)$-dimensional space spanned by $d_1, \cdots, d_{n-1} \in \mathcal{D}$. If $d_1, \cdots, d_{n-1}, d_n \in \mathcal{D}$ form a basis, we have $\alpha|\delta \cdot d_n| = 1$ for some $\alpha \geq 1$.*

As mentioned above, the underlying principle of our auction is to find a minimizer of the nonlinear Lyapunov function $\mathcal{L}$, although we will not be able to use the function $\mathcal{L}$ directly, because the utility function of every bidder is private information. The following lemma concerning the Lyapunov function will play an important role in our auction design, showing that the nonlinear optimization problem (10) over the convex hull of the discrete search set is equivalent to the nonlinear optimization problem (10) over the discrete search set. This implies that when the auctioneer tries to adjust prices, she just needs to focus on the few choices in the search set $\mathcal{SD}$ rather than gropes around the entire convex hull of the search set $\mathcal{SD}$.

**Lemma 7** *Under Assumptions* (A1) *and* (A2) *we have*

$$\max_{\delta \in \mathrm{Conv}(\mathcal{SD})} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\} = \max_{\delta \in \mathcal{SD}} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\}. \tag{10}$$

**Proof.** We only need to consider the case that $p(t)$ is not a Walrasian equilibrium price vector. Choose any $\delta \in \mathcal{SD}$. Let $\delta$ be a primitive normal vector of an $(n-1)$-dimensional space spanned by $d_1, \cdots, d_{n-1} \in \mathcal{D}$.

Regarding $\mathcal{L}(p(t)+\varepsilon\delta)$ as a function in $\varepsilon \geq 0$, we have a function that changes linearly as $\varepsilon$ increases from 0 up to the point $\varepsilon = \varepsilon^* > 0$ where $D^{Ms}(p(t)+\varepsilon\delta) \setminus D^{Ms}(p(t)) \neq \emptyset$.[8] (If such a point $\varepsilon^*$ does not exist, we consider $\varepsilon^* = +\infty$ and we can choose $\varepsilon = 1 < \varepsilon^*$ in the following argument. Hence we assume such a finite $\varepsilon^*$ exists. Also recall that $u$ is defined by (4).) Then there exist some $d^* \in \mathcal{D}$ and an element (a vertex) $x^*$ of $D^{Ms}(p(t)+\varepsilon\delta)$ for $0 < \varepsilon < \varepsilon^*$ such that $x^* + d^* \in D^{Ms}(p(t)+\varepsilon^*\delta) \setminus D^{Ms}(p(t))$ and for the convolution $u$ of all $u^j$ we have

$$u(x^* + d^*) = (p(t) + \varepsilon^*\delta) \cdot ((x^* + d^*) - x^*) + u(x^*). \tag{11}$$

[8]This is equivalent to that for all $j \in B$ we have $D^j(p(t)+\varepsilon\delta) \subseteq D^j(p(t))$ $(\forall \varepsilon \in [0, \varepsilon^*))$ and for some $j \in B$ we have $D^j(p(t)+\varepsilon^*\delta) \setminus D^j(p(t)) \neq \emptyset$. Also note that $D^{Ms}(p(t)+\varepsilon\delta)$ remains the same for all $\varepsilon \in (0, \varepsilon^*)$.

From this we have

$$\varepsilon^* \delta \cdot d^* = u(x^* + d^*) - u(x^*) - p(t) \cdot d^*. \tag{12}$$

Moreover, we can see that $d^*$ is not spanned by $d_1, \cdots, d_{n-1}$. Hence $d_1, \cdots, d_{n-1}, d^*$ is linearly independent and we have $\delta \cdot d^* > 0$ due to the definition of $d^*$. It follows from Lemma 6 that we have

$$0 < \delta \cdot d^* \leq 1. \tag{13}$$

Since the right-hand side of (12) is a non-zero integer, we see from (12) and (13) that $\varepsilon^* \geq 1$.

Since $\delta \in \mathcal{SD}$ is chosen arbitrarily in the above argument, we see that for each $\delta \in \mathcal{SD}$ the function $\mathcal{L}(p(t) + \varepsilon\delta)$ in $\varepsilon$ is linear on the interval $[0, 1]$. Hence $\mathcal{L}(p(t) + \delta')$ as a function in $\delta'$ is a polyhedral conical convex function restricted on $\text{Conv}(\mathcal{SD})$. This implies that equation (10) holds. □

The following lemma shows that the constrained nonlinear optimization problem (10) always has optimal integer solutions, which correspond to the vertices of the set of all optimal solutions.

**Lemma 8** *If Assumptions* (A1) *and* (A2) *hold for the market model, then the set of solutions to the left-side problem of* (10) *is a nonempty integral polytope.*

The next corollary follows the proof of Lemma 7, saying that for any prices $p \in \mathbb{Z}^N$, any bidder $j$, any price adjustment $\delta$ in the search set $\mathcal{SD}$, and any $\varepsilon \in [0, 1)$, his demand set $D^j(p+\varepsilon\delta)$ is always contained by his demand set $D^j(p)$. This result substantially generalizes Proposition 2 of Ausubel (2006) for Gross Substitutes to all unimodular demand types.

**Corollary 4** *If Assumptions* (A1) *and* (A2) *hold for the market model, then for any* $j \in B_0$*, any* $p \in \mathbb{Z}^N$*, and any* $\delta \in \mathcal{SD}$*, we have* $D^j(p + \varepsilon\delta) \subseteq D^j(p)$ *for all* $\varepsilon \in [0, 1)$ *and* $x^j \in \arg\min_{x \in D^j(p)} x \cdot \delta$ *lies in* $D^j(p + \varepsilon\delta)$ *for all* $\varepsilon \in [0, 1]$.

The following lemma gives a local characterization of competitive equilibrium price vectors, saying that the search set $\mathcal{SD}$ is a simple and easy test set for verifying whether an integral point is a competitive equilibrium price vector.

**Lemma 9** *Under Assumptions* (A1) *and* (A2), $p^* \in \mathbb{Z}^N$ *is a competitive equilibrium price vector if and only if* $\mathcal{L}(p^*) \leq \mathcal{L}(p^* + \delta)$ *for all* $\delta \in \mathcal{SD}$.

**Proof.** By Theorem 1 the auction market has a competitive equilibrium with an integral equilibrium price vector. Proposition 1 of Ausubel (2006) says that if there is an equilibrium, a minimizer of the Lyapunov function must be an equilibrium price vector.

Obviously, if $p^*$ is an equilibrium vector, then $\mathcal{L}(p^*) \leq \mathcal{L}(p^* + \delta)$ for all $\delta \in \mathcal{SD}$.

Assume now that $\mathcal{L}(p^*) \leq \mathcal{L}(p^* + \delta)$ for all $\delta \in \mathcal{SD}$. We claim that $\mathcal{L}(p) \geq \mathcal{L}(p^*)$ for all $p \in \mathbb{R}^N$. Then $p^*$ is an equilibrium price vector. Suppose to the contrary that there exists some $p \neq p^*$ such that $\mathcal{L}(p) < \mathcal{L}(p^*)$. Then we can choose a sufficiently small $\alpha \in (0,1)$ such that $p' = \alpha p + (1-\alpha)p^* \in \{p^*\} + \text{Conv}(\mathcal{SD})$ ($p'$ is a strictly convex combination of $p$ and $p^*$). Because of the convexity of $\mathcal{L}(\cdot)$, $\alpha > 0$, and $\mathcal{L}(p) - \mathcal{L}(p*) < 0$, we have

$$\mathcal{L}(p') \leq \alpha\mathcal{L}(p) + (1-\alpha)\mathcal{L}(p^*) = \mathcal{L}(p^*) + \alpha(\mathcal{L}(p) - \mathcal{L}(p^*)) < \mathcal{L}(p^*).$$

It follows from Lemma 8 and equation (10) that

$$\min_{\delta \in \text{Conv}(\mathcal{SD})} \mathcal{L}(p^* + \delta) = \min_{\delta \in \mathcal{SD}} \mathcal{L}(p^* + \delta) \leq \mathcal{L}(p') < \mathcal{L}(p^*)$$

contradicting the hypothesis. This shows that $\mathcal{L}(p^*) \leq \mathcal{L}(p)$ holds for all $p \in \mathbb{R}^N$ and so $p^*$ is an equilibrium price vector. □

We can now discuss the universally convergent dynamic auction in detail. Starting with an arbitrarily given current price vector $p(t) \in \mathbb{Z}^N$, the auction tries to solve the following maximization problem with the unobservable Lyapunov function $\mathcal{L}$

$$\max_{\delta \in \text{Conv}(\mathcal{SD})} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta)\} \tag{14}$$

It follows from Lemma 7 that the continuous maximization problem over the entire convex hull of the search set $\mathcal{SD}$ can be considerably reduced to the following discrete optimization problem over the finite set $\mathcal{D}$ of integer vectors:

$$\max_{\delta \in \mathcal{SD}} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta)\} \tag{15}$$

The maximand of (15) can be further written as

$$\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta) = \sum_{j \in B}(V^j(p(t)) - V^j(p(t) + \delta)) - \sum_{i \in N} \delta_i \tag{16}$$

Observe that the above formula involves every bidder's valuation of every bundle of goods, so it involves private information. Apparently, it is impossible for the auctioneer to know such information unless the bidders are willing to tell her. Fortunately, by Corollary 4 above she can immediately infer the difference between $\mathcal{L}(p(t))$ and $\mathcal{L}(p(t) + \delta)$ just from the reported demands $D^j(p(t))$ and the price variation $\delta$ because $D^j(p(t)) \supseteq D^j(p(t) + \varepsilon\delta)$ for all $j \in B$ and all $\varepsilon \in [0, 1)$. In fact, when prices move from $p(t)$ to $p(t) + \delta$, the reduction in indirect utility for bidder $i$ is uniquely given by

$$V^j(p(t)) - V^j(p(t) + \delta) = \min_{x^j \in D^j(p(t))} x^j \cdot \delta. \tag{17}$$

Consequently, the equation (16) becomes the following simple formula whose right side involves only price variation $\delta$ and optimal choices at $p(t)$:

$$\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta) = \sum_{j\in B} \min_{x^j \in D^j(p(t))} x^j \cdot \delta - \sum_{i\in N} \delta_i. \tag{18}$$

In summary, we have the following important relation regarding the problem (14):

$$\max_{\delta\in \text{Conv}(\mathcal{SD})} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\} = \max_{\delta\in\mathcal{SD}} \{\sum_{j\in B_0} \min_{x^j \in D^j(p(t))} x^j \cdot \delta - \sum_{i\in N} \delta_i\}. \tag{19}$$

Notice that the relation above shows a dramatic change from the unobservable Lyapunov function $\mathcal{L}$ to the observable reported demands of bidders and integer price adjustment $\delta$. The right-hand max-min formula admits an intuitive and interesting interpretation: when the auctioneer adjusts the prices from $p(t)$ to $p(t+1) = p(t)+\delta(t)$, she tries to balance two opposing forces by minimizing every bidder's loss for every possible price change $\delta$ in $\mathcal{D}$ and choosing one price change that maximizes the seller's gain from all possible price changes. In the auction process bidders do nothing but report their demand sets $D^j(p(t))$ and the auctioneer adjusts prices according to the right-hand formula of (19). Formally, we can present the detailed steps of the auction as follows:

**The Universally Convergent Dynamic (UCD) Auction**

**Step 1:** The auctioneer announces an (arbitrary) initial price vector $p(0) \in \mathbb{Z}^N$. Let $t := 0$ and go to **Step 2**.

**Step 2:** Every agent $j \in B_0$ reports her demand $D^j(p(t))$ at $p(t)$ to the auctioneer. Then based on reported demands $D^j(p(t))$, the auctioneer finds an integer solution $\delta(t)$ to the right side problem of (19). If $\delta(t) = 0$, the auction stops. Otherwise the auctioneer adjusts prices by setting $p(t+1) := p(t) + \delta(t)$ and $t := t+1$. Return to **Step 2**.

We now present the convergence of the UCD auction given above.

**Theorem 3** *Assume that the market model satisfies Assumptions* (A1) *and* (A2). *Then, starting with any given initial price vector* $p(0) \in \mathbb{Z}^N$*, the UCD auction finds an integer competitive equilibrium vector in a finite number of rounds.*

**Proof.** Because the Lyapunov function $\mathcal{L}(\cdot)$ is convex and bounded from below and has a minimizer, any minimizer of the Lyapunov function is a competitive equilibrium price vector. Since the prices and value functions take only integer values and the GGD auction lowers the value of the Lyapunov function by a positive integer value in each round, the process must terminate in finite rounds, i.e., $\delta(t^*) = 0$ in Step 2 for some $t^* \in \mathbb{Z}_+$. Let $p(0), p(1), \cdots, p(t^*)$ be the generated finite sequence of price vectors. It follows from 9 that $p(t^*)$ is a competitive equilibrium price vector. □

Observe that the above theorem is very general and holds for all unimodular demand types. This means that items can be substitutes, complements, or possess any other possible properties beyond substitutability or complementarity. The proof of the theorem makes use of mainly convexity and unimodularity and does not invoke the familiar submodularity. In the literature, submodularity is commonly used for the convergence of auction; see Gul and Stacchetti (2000) and Ausubel (2006). It is known from Ausubel and Milgrom (2002, p.31, Th.10) that items are (gross) substitutes to a bidder if and only if the bidder's indirect utility function is submodular. Therefore for the Gross Substitutes, the Lyapunov function must be submodular. Substitutes are closely related to submodularity and complements are related to supermodularity. Besides the Gross Substitutes, there are so many other different demand types which may not have a clear cut property like substitutes or complements and thus the corresponding Lyapunov function can be neither submodular nor supermodular. As a result, it is natural and logical that the proof of the above theorem relies mainly on convexity and unimodularity and cannot and do not use submodularity.

We point out that Klemperer (2008, 2010) proposed a sealed-bid auction for substitutes. His auction was used to help the Bank of England deal with the credit crunch and implicitly contains some idea of demand type. Klemperer (2018) has further extended this auction.

Ascending auctions like English auctions or descending ones like Dutch auctions are familiar formats of auction. In the rest of this section we will discuss such formats for the well-known case of the Gross Substitutes. Let $\mathcal{D}$ be the Gross Substitutes demand type given in Definition 5. Then we have its search set $\mathcal{SD} = \{0,1\}^N \cup \{0,-1\}^N$. Let $\Delta = \{0,1\}^N$ and let $\bar{\Delta}$ be the convex hull of the set $\Delta$. We need to address the following continuous maximization problem

$$\max_{\delta \in \bar{\Delta}} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta)\} \tag{20}$$

The next result follows from Corollary 1 and strengthens Proposition 3 of Ausubel (2006).

**Lemma 10** *If Assumption* (A1) *holds for the market model and every agent's demand type is Gross Substitutes, then the set of solutions to Problem* (20) *is a nonempty integrally convex polytope with a lattice structure.*

Similar to the discussion earlier and by Lemma 10, at each round $t \in \mathbb{Z}_+$ of the auction, the decision problem (20) becomes the following discrete optimization problem:

$$\max_{\delta \in \bar{\Delta}} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta)\} = \max_{\delta \in \Delta} \{\sum_{j \in B} \min_{x^j \in D^j(p(t))} x^j \cdot \delta - \sum_{i \in N} \delta_i\} \tag{21}$$

Let $\Delta^* = -\Delta$ and $\bar{\Delta}^* = -\bar{\Delta}$. We can introduce the following special global dynamic auction for Gross Substitutes:

**The Special Global Dynamic (SGD) Auction**

**Step 1:** The auctioneer announces an initial price vector $p(0) \in \mathbb{Z}^N$. Let $t := 0$ and go to **Step 2**.

**Step 2:** Every bidder $i \in B$ reports his demand $D^i(p(t))$ at $p(t)$ to the auctioneer. Then based on reported demands $D^i(p(t))$, the auctioneer finds a solution $\delta(t)$ to the problem (21). If $\delta(t) = 0$, go to **Step 3**. Otherwise the auctioneer adjusts prices by setting $p(t+1) := p(t) + \delta(t)$ and $t := t + 1$. Return to **Step 2**.

**Step 3:** Every bidder $i \in B$ reports his demand $D^i(p(t))$ at $p(t)$ to the auctioneer. Then based on reported demands $D^i(p(t))$, the auctioneer finds a solution $\delta(t)$ to the problem (21) where $\Delta$ is replaced by $\Delta^*$. If $\delta(t) = 0$, the auction stops. If $\delta(t) \neq 0$, the auctioneer adjusts prices by setting $p(t+1) := p(t) + \delta(t)$ and $t := t + 1$. Return to **Step 3**.

Observe that in both Step 2 and Step 3 the auctioneer needs only an arbitrary solution to the problem (21) with respect to $\Delta$ or $\Delta^*$. This is in contrast to the process of Ausubel (2006) which requires to take the smallest or largest solution to similar problems which typically have multiple solutions. This improvement is very useful and important for practical auction design and makes the implementation easy and fast. In Step 2, the auction increases prices and is an ascending auction, while in Step 3, the auction decreases prices and is a descending auction.

The following theorem shows the convergence of the global dynamic auction for the Gross Substitutes.

**Theorem 4** *Assume that Assumption* (A1) *holds and that every bidder's demand type is Gross Substitutes. Starting with any integer price vector, the special global dynamic* (*SGD*) *auction converges to an equilibrium price vector in a finite number of rounds.*

From Corollary 1 we know that the set of competitive equilibrium price vectors has a unique smallest integer equilibrium price vector $\underline{p}$ and a unique largest integer equilibrium price vector $\bar{p}$. The special global dynamic auction has two special cases that the initial prices $p(0)$ are chosen in two particular ways. The first one is to set the initial prices $p(0) \in \mathbb{Z}^N$ below $\underline{p}$. This can be easily achieved by choosing prices $p(0)$ so low that all the items are demanded by every agent, i.e., $\sum_{i \in N} e(i) \in D^j(p(0))$ for every $j \in B_0$. In this case, the auction is an ascending auction. By modifying the proof of Theorem 4, we can show the following two results.

**Corollary 5** *Assume that Assumption* (A1) *holds and that every bidder's demand type is Gross Substitutes. Starting with any integer price vector* $p(0) \in \mathbb{Z}^N$ *below* $\underline{p}$*, the special global dynamic auction is an ascending auction and converges to an equilibrium price vector in a finite number of rounds.*

The second one is to set the initial prices $p(0) \in \mathbb{Z}^N$ above $\bar{p}$. This can be done by choosing prices $p(0)$ so high that none of the items is demanded by any agent, i.e., $D^j(p(0)) = \{\mathbf{0}\}$ for all $j \in B_0$. In this case, the auction is a descending auction.

**Corollary 6** *Assume that Assumption* (A1) *holds and that every bidder's demand type is Gross Substitutes. Starting with any integer price vector* $p(0) \in \mathbb{Z}^N$ *above* $\bar{p}$*, the special global dynamic auction is a descending auction and converges to an equilibrium price vector in a finite number of rounds.*

For Gross Substitutes, Gul and Stacchetti (2000) have developed an ascending auction and Ausubel (2006) has proposed an auction that is global and can be ascending or descending.

# 5 Dynamic Auction Design with Strategic Bidders

In the previous section we have assumed that every agent acts honestly as a price-taker. In reality, there are indeed honest business people, but there are also dishonest or strategic people who will take advantage of every opportunity whenever possible. In this section we drop the honest price-taking behavior assumption by considering a more natural and more realistic environment where bidders are strategic and may therefore act strategically. We will investigate how we should expect such individuals to behave and how to prevent their strategic behavior. More specifically, we will address two basic questions. First, is it possible to design an auction mechanism that induces bidders to act honestly as price-takers? Second, is it possible to devise an auction mechanism that requires just enough or minimal information from bidders so that their privacy can be well preserved? To answer these questions, we will develop an efficient, incentive-compatible dynamic auction mechanism built on the basic auction introduced in the previous section that not only possesses an appealing strategy-proof property but also has the merit of privacy-preservation, transparency and detail-freeness.

We use the following notation. Recall that $\mathcal{M}$ stands for the (original) market with $m$ bidders and the seller with the set $N$ of $n$ items. For every bidder $j \in B$, let $\mathcal{M}_{-j}$ denote the market $\mathcal{M}$ *without the participation of bidder* $j$ and $B_{-j} = B_0 \setminus \{j\}$. For convenience we set $\mathcal{M}_{-0} = \mathcal{M}$ and $B_{-0} = B_0$. So, for every $k \in B_0$, the market $\mathcal{M}_{-k}$ comprises the set $B_{-k}$ of agents and the set $N$ of $n$ items.

The following defines the Vickrey-Clarke-Groves mechanism. The definition is more general than its standard one because here we permit the seller to have her utility function; see Ausubel and Cramton (2004B) on a similar extension for divisible goods. The standard one usually assumes that the seller values everything at zero.

**Definition 11** The *VCG outcome* is the outcome of the following procedure: every agent $j \in B_0$ reports her value function $u^j$. Then the auctioneer computes an efficient allocation $X$ with respect to all reported $u^j$ and assigns bundle $x^j$ to bidder $j \in B$ and charges him a payment of $\beta_j^* = u^j(x^j) - R(N) + R_{-j}(N)$, where $R(N)$ and $R_{-j}(N)$ are the market values of the markets $\mathcal{M}$ and $\mathcal{M}_{-j}$ based on $u^j$ $(j \in B)$, respectively. Bidder $j$'s VCG payoff equals $R(N) - R_{-j}(N)$, $j \in B$.

## 5.1 Incentive Compatible Dynamic Auction Design

We now introduce the following incentive-compatible dynamic auction mechanism which is built on the UCD auction and will induce all bidders to bid truthfully, although they are not assumed to be honest price-takers. The mechanism runs the UCD auction for every market $\mathcal{M}_{-k}$ $(k \in B_0)$ as described in Section 4 with the following modifications: Consider any market $\mathcal{M}_{-k}$. Let $p^k(t)$ denote the prices of the market $\mathcal{M}_{-k}$ at time $t \in \mathbb{Z}_+$. Then at $t \in \mathbb{Z}_+$ and with respect to $p^k(t) \in \mathbb{Z}^N$, every bidder $j \in B_{-k}$ submits his bid $B_k^j(t) \subseteq \{0,1\}^N$ which may differ from his true demand set $D^j(p^k(t))$, but the seller's bid $B_k^0(t)$ always equals her true demand set $D^0(p^k(t))$. The auctioneer solves the following decision problem, i.e., the problem (19)

$$\max_{\delta \in \mathcal{SD}} \{ \sum_{j \in B_{-k}} \min_{x^j \in B_k^j(t)} x^j \cdot \delta - \sum_{i \in N} \delta_i \} \tag{22}$$

When the price adjustment $\delta^k(t)$ being a solution to (22) is equal to the vector of zeros, it means that the auction finds an "equilibrium allocation" $X^k = (x^{k,j}, j \in B_{-k})$ in the market $\mathcal{M}_{-k}$ in the sense that $x^{k,j} \in B_k^j(t)$ for every $j \in B_{-k}$ and $\sum_{j \in B_{-k}} x^{k,j} = \sum_{i \in N} e(i)$. As long as the vector $\delta^k(t)$ is not equal to the vector of zeros, the auctioneer adapts prices by setting $p^k(t+1) = p^k(t) + \delta^k(t)$. However, because bidders may act strategically and therefore their bids may not be their true demand sets, it is possible that the auction may never find an equilibrium allocation in some market $\mathcal{M}_{-k}$. In this case, we say that the auction stops at the time of $\infty$, following Ausubel (2006, p. 613).

**The Incentive Compatible Universal Dynamic (ICUD) Auction**

**Step 1:** At the start, the auctioneer announces a common price vector $p^k(0) = p(0) \in \mathbb{Z}^N$ for all markets $\mathcal{M}_{-k}$, $k \in B_0$. Let $t := 0$ and go to **Step 2**.

**Step 2:** At each time $t \in \mathbb{Z}_+$ and prices $p^k(t) \in \mathbb{Z}^N$, every agent $j \in B_{-k}$ submits her bid $B^j_k(t) \subseteq \{0,1\}^N$. Based on reported bids, if the auctioneer finds an equilibrium allocation $X^k$ in any market $\mathcal{M}_{-k}$ at the current round, she records the current prices as $p^k(T^k) \in \mathbb{Z}^n$ and the current time as $T^k \in \mathbb{Z}_+$. For any market $\mathcal{M}_{-k}$ which is not in equilibrium, the auctioneer calculates a price change $\delta^k(t)$ according to (22) and announces a new price vector $p^k(t+1) = p^k(t) + \delta^k(t)$. The UCD auction goes back to **Step 2** with $t := t+1$. If the auction has found an equilibrium allocation $X^k$ in every market $\mathcal{M}_{-k}$, $k \in B_0$, then go to **Step 3**. Otherwise go to **Step 4**.

**Step 3:** All markets now clear. For every market $k \in B_0$ and every agent $j \in B_{-k}$ at every time $t = 0, 1, \cdots, T^k - 1$, based on her reported bids $B^j_k(t)$ and the price change $\delta^k(t)$, the auctioneer calculates agent $j's$ 'indirect utility reduction' $\Delta^k_j(t)$ when prices are changed from $p^k(t)$ to $p^k(t+1)$ in the market $\mathcal{M}_{-k}$, where

$$\Delta^k_j(t) = \min_{x^j(t) \in B^j_k(t)} x^j(t) \cdot \delta^k(t) \tag{23}$$

Every bidder $j \in B$ will be assigned the bundle $x^{0,j}$ of the allocation $X^0 = (x^{0,j}, j \in B_0)$ found in the original market $\mathcal{M}_{-0} = \mathcal{M}$ and asked to pay $\beta_j$, with the option to decline and walk away, when his payoff becomes negative, where

$$\beta_j = \sum_{h \in B_{-j}} \Big[ \Big( \sum_{t=0}^{T^0-1} \Delta^0_h(t) - \sum_{t=0}^{T^j-1} \Delta^j_h(t) \Big) + x^{j,h} \cdot p^j(T^j) - x^{0,h} \cdot p^0(T^0) \Big] \tag{24}$$

The seller keeps the bundle $x^{0,0}$ of the allocation $X^0$ and receives the total payment $\sum_{j \in B} \beta_j$. The auction stops.

**Step 4:** In this case the auction does not find an allocation in every market $\mathcal{M}_{-k}$, $k \in B_0$. In the end, every bidder $j \in B$ gets nothing and pays nothing. The auction stops.

The payment formula $\beta_j$ above has three terms and can be explained intuitively as follows: The first term is the accumulation of 'indirect utility reduction' of bidder $j's$ all opponents in $B_{-j}$ along the path from $p^j(T^j)$ to $p(0)$ in the market $\mathcal{M}_{-j}$ and along the path from $p(0)$ to $p^0(T^0)$ in the market $\mathcal{M}$; the second term stands for the total equilibrium payment by all bidders in the market $\mathcal{M}_{-j}$, i.e., all opponents of bidder $j$; and the third term represents the total equilibrium payment by all opponents of bidder $j$ in the market $\mathcal{M}$. The final payment $\beta_j$ of bidder $j$ equals the first term by adding the second term and subtracting the third term. This payment formula uses only revealed information and is simple and easy to calculate. Observe that in Step 3 we allow any bidder $j$ to decline any unacceptable assignment and walk away, if accepting the assignment would give him a negative utility of $u^j(x^{0,j}) - \beta_j < 0$.

When the auction goes to Step 4, namely, it does not find an allocation in every market, we adopt a lenient policy of imposing no punishment upon bidders and simply letting them walk out. This rule is plausible in practice and may yield a better incentive for buyers. But it can be a disadvantage to the seller in the sense that the seller will not get the penalty paid by badly behaved bidders in comparison with any auction which may impose the penalty of infinity, when the auction stops at the time of infinity. It is possible to use this lenient policy of no punishment in our auction, because we can use the conventional argument that if honesty can be one of every bidder's optimal policies, he will act truthfully. This lenient policy is different from Ausubel's auction (2006) which imposes a severe penalty of infinity. In his auction because bidders are not given any opportunity to walk away, a bidder may have to pay a huge amount according to his payment formula (7) on p.611 if he has made mistakes before a time $\bar{t}$. In order to incentivize him to act rationally from $\bar{t}$ on it is necessary to impose the penalty of infinity. Otherwise, a single bidder can cause the auction to drag on for ever.

It will be also interesting to note that our incentive compatible auction can tolerate any mistake or manipulation made by bidders and allow them to adjust and correct so that for any time $t^* \in \mathbb{Z}_+$, no matter what has happened before $t^*$, as long as from $t^*$ on every bidder bids truthfully and Assumptions (A1) and (A2) are satisfied, the ICUD auction will find a competitive equilibrium in every market in finite time in Step 3, because the UCD auction always converges to a competitive equilibrium wherever it starts from $\mathbb{Z}^N$.

## 5.2 The Dynamic Auction Game and Its Strategic Properties

In this section we will discuss how the ICUD auction can induce strategic bidders to bid truthfully as price-takers, generating efficient outcomes even when these bidders have market power. To do so, we need to formulate the auction as an extensive-form dynamic game of incomplete information. In this (dynamic) auction game, all bidders are players. Prior to the start of the game, every player $j \in B$ knows privately only his own value function $u^j$ satisfying Assumptions (A1) and (A2). The auctioneer knows that every bidder's utility function satisfies Assumptions (A1) and (A2) but does not know their utility functions.

In the auction, the auctioneer initially announces a common price vector for all markets and every bidder responds by reporting his bid to the auctioneer for every market in which he is involved. Then based on reported bids the auctioneer checks if the aggregated demands equal the aggregated supplies in every market or not. If all markets are cleared, the auction stops. Otherwise, the auctioneer adjusts prices and bidders update their bids. The auction process goes on. In this auction, announced prices in each market are observable to all bidders in the market. Every bidder knows of course his own bids. Whether a bidder can observe bids of other bidders depends on the specification of the auction rule. In the current auction the auctioneer can ask every bidder to either publicly reveal his bids or just submit his bids privately to her. We use $H_j^t$ to denote the part of the information or history of play that player $j$ has observed so far right after prices at time $t \in \mathbb{Z}_+$ have been announced but no players have placed their bids at the current prices. A natural specification is that $H_j^t$ contains his own utility function $u^j$, all observable prices before and at time $t$ in every market in which he takes part, all his own bids and all possibly revealed bids of other players before time $t$. Regarding all possibly revealed bids of other players, as discussed in Ausubel (2004, p.1461), there could be at least three interesting possibilities: Full bid information reveals every bidder's all bids before time $t$; Aggregate bid information contains the aggregate demand of all bidders without revealing who bids what before time $t$; and No bid information means that every bidder knows only his own bids but nothing about those of any other bidder. In open auctions, prices are publicly announced and thus observable to every player. In some auctions, every bidder's bids can be observed by all other bidders, however in other auctions, bidders may not be able to see other's bids.

At every time $t \in \mathbb{Z}_+$, after the auctioneer announces current prices for each market, every bidder will think about how to bid based upon all currently available information to him. The (dynamic) *strategy* $\sigma_j$ of player $j$, $j \in B$, is a set-valued function which specifies his bids $\sigma_j(t, k, H_j^t) = B_k^j(t) \subseteq \{0,1\}^N$ for every market $\mathcal{M}_{-k}$, $k \in B_0 \setminus \{j\}$, at every time $t \in \mathbb{Z}_+$, and for every history $H_j^t$. Let $\Sigma_j$ denote the strategy space of all player $j$'s strategies $\sigma_j$. Obviously, player $j$'s strategy space $\Sigma_j$ contains his sincere bidding strategies as specified in Definition 9 and many other strategies as well. The outcome of the ICUD auction game relies totally upon the auction rules, the histories, and the strategies the bidders may adopt. When every bidder $j \in B$ takes a strategy $\sigma_j$ and the ICUD auction terminates in Step 2, then bidder $j \in B$ receives bundle $x^{0,j}$ and pays $\beta_j$ given by (24), or simply walks away. In this case, his payoff equals $\max\{u^j(x^{0,j}) - \beta_j, 0\}$. Otherwise, the auction fails to find an allocation in every market and stops in Step 3. In this case, no bidder gets anything and pays anything.

In the literature for static auction games of incomplete information, the notion of *ex post equilibrium* has been used by Cremer and McLean (1985) and Krishna (2002). This solution requires that the strategy for every player should remain optimal if the player were to get to know types of his opponents. Ausubel (2004, 2006) has adopted the solution of *ex post perfect equilibrium* to dynamic auction games of incomplete information which requires the same condition for every player at every node of the dynamic auction game. Following Ausubel (2004, p.1461, 2006, pp. 613-614), formally we have the following

**Definition 12** The strategy $m$-tuple $\{\sigma_j\}_{j \in B}$ of the dynamic auction game of incomplete information is said to be *an ex post perfect equilibrium* if for every time $t \in \mathbb{Z}_+$, following any history $\{H_j^t\}_{j \in B}$, and for any realization $\{u^j\}_{j \in B}$ of private information, the continuation strategies $\sigma_j(\cdot, \cdot, \cdot \mid t, k, H_j^t)$ for every player $j \in B$ and for every market $k \in B_0 \setminus \{j\}$ constitute a Nash equilibrium of the game even if the realization $\{u^j\}_{j \in B}$ becomes common knowledge.[9]

As pointed out by Ausubel (2004, 2006), the concept of ex post perfect equilibrium has several additional desirable properties over Bayesian equilibrium or perfect Bayesian equilibrium: it is not only robust against any regret but also independent of any probability distribution. It is very useful in practice as it is extremely difficult to elicit or gauge a probability distribution of a bidder's valuation. Furthermore, in the complete information case, an ex post perfect equilibrium simply reduces to the familiar notion of subgame perfect equilibrium.

We say that a mechanism is *beneficial to every agent* if the payment the seller receives for every sold bundle is at least as big as her reserve value of the bundle or the total utility she receives is at least as good as she does not trade, and if the net profit for every bidder is nonnegative. Finally, we introduce one more desirable property called *ex post individual rationality*, which is important for practical auction design. An auction mechanism is said to be *ex post individually rational*, if, for every bidder, no matter how his opposing bidders act in the auction, as long as he is sufficiently able to judge whether his payoff is negative or nonnegative, he will never end up with a negative payoff.

Now we are ready to derive several appealing properties of the ICUD auction.

**Theorem 5** *Let the market $\mathcal{M}$ satisfy Assumptions* (A1) and (A2). *If every bidder acts truthfully, the ICUD auction converges to a competitive equilibrium, yields a VCG outcome for the market $\mathcal{M}$ in finite time.*

**Proof.** Because every bidder $j$ bids straightforwardly according to his true UTD $\mathcal{D}$ function $u^j$ and Assumptions (A1) and (A2) are satisfied, by Theorem 3 the auction finds a Competitive equilibrium $(p^k(T^k), X^k)$ in every market $\mathcal{M}_{-k}$, $k \in B_0$. As bidders act truthfully, then for every bidder $j \in B_{-k}$ in every market $\mathcal{M}_{-k}$ at any time $t \in \mathbb{Z}_+$ we have $B_k^j(t) = D^j(p^k(t))$. It further follows from (17) in Section 4 that

$$\Delta_j^k(t) = \min_{x^j(t) \in B_k^j(t)} x^j(t) \cdot \delta^k(t) = V^j(p^k(t)) - V^j(p^k(t+1))$$

By the rule in Step 3 of the auction, every bidder $j \in B$ pays $\beta_j$ of (24) for the bundle $x^{0,j}$ assigned to him. It will be shown that $\beta_j$ is actually equal to the VCG payment of bidder

[9]Note that while Ausubel (2004, 2006) treats every player's utility function and history separately, in the current model every player $j$'s history $H_j$ contains also his private utility function $u^j$ and possibly other private information.

$j$ given by

$$\beta_j^* = u^j(x^{j,0}) - R(N) + R_{-j}(N)$$

where $R(N) = \sum_{h\in B} u^j(x^{0,h})$ and $R_{-j}(N) = \sum_{h\in B_{-j}} u^j(x^{j,h})$. Recall that $p^k(0) = p(0)$ for every $k \in B_0$. It follows from (24) that

$$\begin{aligned}
\beta_j &= \sum_{h\in B_{-j}} \Big[\Big(\sum_{t=0}^{T^0-1} \Delta_i^0(t) - \sum_{t=0}^{T^j-1} \Delta_h^j(t)\Big) + x^{j,h}\cdot p^j(T^j) - x^{0,h}\cdot p^0(T^0)\Big] \\
&= \sum_{h\in B_{-j}} \Big(\sum_{t=0}^{T^0-1}(V^h(p^0(t)) - V^h(p^0(t+1))) \\
&\qquad - \sum_{t=0}^{T^j-1}(V^h(p^j(t)) - V^h(p^j(t+1)))\Big) \\
&\qquad + \sum_{h\in B_{-j}} x^{j,h}\cdot p^j(T^j) - \sum_{h\in B_{-j}} x^{0,h}\cdot p^0(T^0) \\
&= \sum_{h\in B_{-j}} \Big((V^h(p^0(0)) - V^h(p^0(T^0))) - (V^h(p^j(0)) - V^h(p^j(T^j)))\Big) \\
&\qquad + \sum_{h\in B_{-j}} x^{j,h}\cdot p^j(T^j) - \sum_{h\in B_{-j}} x^{0,h}\cdot p^0(T^0) \\
&= \sum_{h\in B_{-j}} \Big(V^h(p^j(T^j)) + x^{j,h}\cdot p^j(T^j)\Big) \\
&\qquad - \sum_{h\in B_{-j}} \Big(V^h(p^0(T^0)) + x^{0,h}\cdot p^0(T^0)\Big) \\
&= \sum_{h\in B_{-j}} u^j(x^{j,h}) - \sum_{h\in B_{-j}} u^j(x^{0,h}) \\
&= u^j(x^{0,j}) - R(N) + R_{-j}(N) \\
&= \beta_j^*.
\end{aligned}$$

□

The following theorem shows that sincere bidding is an ex post perfect equilibrium in the dynamic ICUD auction game with incomplete information.

**Theorem 6** *Assume that the market $\mathcal{M}$ satisfies Assumptions* (A1) and (A2). *Then sincere bidding by every bidder is an ex post perfect equilibrium in the ICUD auction.*

**Proof.** Consider any time $\hat{t} \in \mathbb{Z}_+$, any history profile $\{H_h^{\hat{t}}\}_{h\in B}$ (which may be on or off the equilibrium path), and any realization $\{u^h\}_{h\in B}$ of profile of utility functions of private information. Clearly, the outcome of the game depends on the histories $H_h^{\hat{t}}$ for $h \in B$ and actions that bidders will take in the continuation game starting from $\hat{t}$. Note that bidders cannot change histories but can influence the path of the future from $\hat{t}$ on. Take any player $j \in B$. Suppose that in the continuation game from time $\hat{t}$ on, every opponent $h(h \in B_{-j})$ of player $j$ bids sincerely at any $t \in \mathbb{Z}_+(t \geq \hat{t})$ and in every market $\mathcal{M}_{-k}$ for $k \in B_0$, namely,

$$\sigma_h(t,k,H_h^t) = B_k^h(t) = D^h(p^k(t)) = \arg\max_{x\in\{0,1\}^N}\{u^h(x) - x\cdot p^k(t)\}$$

It implies that for every bidder $h \in B_{-j}$ in the markets $\mathcal{M}_{-j}$ and $\mathcal{M}$ at every time $t \geq \hat{t}$

$$\Delta_h^j(t) = \min_{x^h(t)\in B_j^h(t)} x^h(t)\cdot\delta^h(t) = V^h(p^j(t)) - V^h(p^j(t+1))$$

and

$$\Delta_h^0(t) = \min_{x^h(t)\in B_j^h(t)} x^h(t)\cdot\delta^h(t) = V^h(p^0(t)) - V^h(p^0(t+1))$$

However, the above equations do not necessarily hold true for time $t < \hat{t}$.

Clearly, in this continuation game from time $\hat{t}$, when all opponents of player $j$ choose sincere bidding strategies, because of the option of walking away in Step 3, bidder $j$ prefers a strategy which causes the auction to stop at Step 3 and yields a nonnegative payoff to him, to any other strategy which leads the auction to Step 4 and gives him only a zero payoff. Therefore, it sufficient to compare the sincere bidding strategy with any other strategy which leads the auction to Step 3. Suppose that $\sigma'_j(\cdot,\cdot,\cdot \mid \hat{t}, k, H_j^{\hat{t}})$ ($\sigma'_j$ in short) for all $k\in B_0\setminus\{j\}$ is such a continuation strategy of player $j$ resulting in an allocation $(y^{0,h}, h\in B)$ in the market $\mathcal{M}$, and that bidder $j$'s (continuation) sincere bidding strategy results in an allocation $(x^{0,h}, h\in B)$ in the market $\mathcal{M}$. Without any loss of generality, we assume that by the time $\hat{t}$, the auction has not found any allocation in the markets $\mathcal{M}$ and $\mathcal{M}_{-j}$, i.e., $\hat{t} < T^{-0}$ and $\hat{t} < T^{-j}$. When player $j$ chooses the strategy $\sigma'_j$, his payment $\beta'_j$ given by (24) is

$$\begin{aligned}
\beta'_j &= \textstyle\sum_{h\in B_{-j}}\Big[\Big(\sum_{t=0}^{T^0-1}\Delta_h^0(t) - \sum_{t=0}^{T^j-1}\Delta_h^j(t)\Big) + x^{j,h}\cdot p^j(T^j) - x^{0,h}\cdot p^0(T^0)\Big]\\
&= \textstyle\sum_{h\in B_{-j}}\Big(\sum_{t=0}^{\hat{t}-1}\Delta_h^0(t) + \sum_{t=\hat{t}}^{T^0-1}\Delta_h^0(t) - \sum_{t=0}^{\hat{t}-1}\Delta_h^j(t) - \sum_{t=\hat{t}}^{T^j-1}\Delta_h^j(t)\Big)\\
&\quad \textstyle+\sum_{h\in B_{-j}} x^{j,h}\cdot p^j(T^j) - \sum_{h\in B_{-k}} x^{0,h}\cdot p^{0,h}(T^0)\\
&= \textstyle\sum_{h\in B_{-j}}\Big[\sum_{t=0}^{\hat{t}-1}\Delta_h^0(t) + \sum_{t=\hat{t}}^{T^0-1}(V^h(p^0(t)) - V^h(p^0(t+1)))\\
&\qquad \textstyle-\sum_{t=0}^{\hat{t}-1}\Delta_h^j(t) - \sum_{t=\hat{t}}^{T^j-1}(V^h(p^j(t)) - V^h(p^j(t+1)))\Big]\\
&\quad \textstyle+\sum_{h\in B_{-j}} x^{j,h}\cdot p^j(T^j) - \sum_{h\in B_{-j}} x^{0,h}\cdot p^{0,h}(T^0)\\
&= \textstyle\sum_{h\in B_{-j}}\Big(\sum_{t=0}^{\hat{t}-1}[\Delta_h^0(t) - \Delta_h^j(t)] + V^h(p^0(\hat{t})) + V^h(p^j(T^j)) - V^h(p^j(\hat{t}))\Big)\\
&\qquad \textstyle+\sum_{h\in B_{-j}} x^{j,h}\cdot p^j(T^j) - \Big(\sum_{h\in B_{-j}} V^h(p^0(T^0)) + \sum_{h\in B_{-j}} x^{0,h}\cdot p^0(T^0)\Big)\\
&= \textstyle\Gamma_{-j} - \sum_{h\in B_{-j}} u^h(\rho(j)),
\end{aligned}$$

where $\Gamma_{-j}$ is given by

$$\begin{aligned}
\Gamma_{-j} &= \textstyle\sum_{h\in B_{-j}}\Big[\sum_{t=0}^{\hat{t}-1}\Big(\Delta_h^0(t) - \Delta_h^j(t)\Big) + V^h(p^0(\hat{t})) + V^h(p^j(T^j)) - V^h(p^j(\hat{t}))\\
&\quad + x^{j,h}\cdot p^j(T^j)\Big]
\end{aligned}$$

Observe that $\Gamma_{-j}$ is totally determined by the history profile $\{H_h^{\hat{t}}\}_{h\in B}$ and the market $\mathcal{M}_{-j}$ without bidder $j$, and does not depend on player $j$'s strategy $\sigma'_j$. Similarly, we can prove that if bidder $j$ adopts the sincere bidding strategy, his payment $\hat{\beta}_j$ will be

$$\hat{\beta}_j = \Gamma_{-j} - \sum_{h\in B_{-j}} u^h(x^{0,h})$$

where $\Gamma_{-j}$ is the same as the previous one. Moreover it follows from the argument in Section 4 that when bidders bid truthfully according to their utility functions $u^h$, $h\in B$

and Assumptions (A1) and (A2) are satisfied, the allocation $(x^{0,h}, h \in B)$ in the market $\mathcal{M}$ found by the auction will be efficient. That is,

$$u^j(x^{0,j}) + \sum_{h \in B_{-j}} u^h(x^{0,h}) \geq u^j(y^{0,j}) + \sum_{h \in B_{-j}} u^h(y^{0,h}).$$

Taking the option of walking away into every bidder's account together with the above discussion gives the payoff $\hat{\mathcal{P}}_j$ of bidder $j$ in the case of using the sincere bidding strategy and his payoff $\mathcal{P}'_j$ in the case of using the strategy $\sigma'_i$ as follows

$$\begin{aligned} \hat{\mathcal{P}}_j &= \max\{u^j(x^{0,j}) - \hat{\beta}_j, 0\} \\ &= \max\{u^j(x^{0,j}) - (\Gamma_{-j} - \textstyle\sum_{h \in B_{-j}} u^h(x^{0,h})), 0\} \\ &= \max\{u^j(x^{0,j}) + \textstyle\sum_{h \in B_{-j}} u^j(x^{0,j}) - \Gamma_{-j}, 0\} \\ &\geq \max\{u^j(y^{0,j}) + \textstyle\sum_{h \in B_{-j}} u^j(y^{0,h}) - \Gamma_{-j}, 0\} \\ &= \max\{u^j(y^{0,j}) - \beta'_j, 0\} \\ &= \mathcal{P}'_j \end{aligned}$$

This demonstrates that every player's sincere bidding strategy is indeed his ex post perfect strategy. Therefore sincere bidding by every bidder is an ex post perfect equilibrium. □

The following result shows that the ICUD auction mechanism is beneficial to every market participant if the seller has either a submodular utility function or a superadditive utility function.

**Proposition 2** *Let the market* $\mathcal{M}$ *satisfy Assumptions* (A1) and (A2). *If every bidder acts truthfully, the ICUD auction mechanism is beneficial to every agent provided that the seller's utility function* $u^0$ *is either submodular or superadditive.*

**Proof.** It follows from the proof of Theorem 5 that every bidder $j \in B$ receives bundle $x^{0,j}$ and pays $\beta^*_j$ and his net profit equals

$$\begin{aligned} u^j(x^{0,j}) - \beta^*_j &= R(N) - R_{-j}(N) \\ &= \textstyle\sum_{h \in B_0} u^h(x^{0,h}) - \sum_{h \in B_{-j}} u^h(x^{j,h}) \\ &= \textstyle\sum_{h \in B_0} u^h(x^{0,h}) - \sum_{h \in B_0} u^h(x^{j,h}) \\ &\geq 0 \end{aligned}$$

where $x^{j,j} = 0$.

We now prove that the auction is also beneficial to the seller. First, consider the case that $u^0$ is submodular. Recall that for every $j \in B$, $(x^{k,h}, h \in B_{-k})$ is the equilibrium allocation in market $\mathcal{M}_{-k}$ found by the auction. By definition it is easy to see that

$$R_{-j}(N) = \sum_{h \in B_{-j}} u^h(x^{j,h}) \geq \sum_{h \in B \setminus \{j\}} u^h(x^{0,h}) + u^0(x^{0,0} + x^{0,j}).$$

The utility $\tilde{\mathcal{P}}_0$ received by the seller equals

$$\begin{array}{rcl}\tilde{\mathcal{P}}_0 &=& u^0(x^{0,0}) + \sum_{j\in B}\beta_j^* \\ &=& \sum_{j\in B}\Big(u^j(x^{0,j}) - R(N) + R_{-j}(N)\Big) \\ &=& \sum_{j\in B} R_{-j}(N) - (m-1)R(N) \\ &\geq& \sum_{j\in B}\Big(u^0(x^{0,0}+x^{0,j}) + \sum_{h\in B\setminus\{j\}} u^h(x^{0,h})\Big) - (m-1)R(N) \\ &=& \sum_{j\in B} u^0(x^{0,0}+x^{0,j}) - (m-1)u^0(x^{0,0})\end{array}$$

Then submodularity implies that for every $j = 1, 2, \cdots, m-1$ we have

$$u^0(\sum_{h=0}^{j} x^{0,h}) + u^0(x^{0,0}+x^{0,j+1}) \geq u^0(\sum_{h=0}^{j+1} x^{0,h}) + u^0(x^{0,0})$$

Summing up these inequalities leads to

$$\sum_{j\in B} u^0(x^{0,0}+x^{0,j}) \geq u^0(\sum_{j\in B_0} x^{0,j}) + (m-1)u^0(x^{0,0})$$

from which we have

$$\begin{array}{rcl}\tilde{\mathcal{P}}_0 &=& \sum_{j\in B} u^0(x^{0,0}+x^{0,j}) - (m-1)u^0(x^{0,0}) \\ &\geq& u^0(\sum_{j\in B_0} x^{0,j}) = u^0(N).\end{array}$$

So the utility the seller receives from trading is at least as good as she does not trade at all.

Second, consider the case that $u^0$ is superadditive. For every $j \in B$ we have

$$R_{-j}(N) = \sum_{h\in B_{-j}} u^h(x^{j,h}) \geq \sum_{h\in B\setminus\{j\}} u^h(x^{0,h}) + u^0(x^{0,0}+x^{0,j})$$

and $u^0(x^{0,0}+x^{0,j}) \geq u^0(x^{0,0}) + u^0(x^{0,j})$. Then the utility $\tilde{\mathcal{P}}_0$ received by the seller equals

$$\begin{array}{rcl}\tilde{\mathcal{P}}_0 &=& u^0(x^{0,0}) + \sum_{j\in B}\beta_j^* \\ &=& u^0(x^{0,0}) + \sum_{j\in B}[u^j(x^{0,j}) - R(N) + R_{-j}(N)] \\ &=& u^0(x^{0,0}) + \sum_{j\in B}\Big(u^j(x^{0,j}) - (u^0(x^{0,0}) + \sum_{h\in B} u^h(x^{0,h})) + R_{-j}(N)\Big) \\ &=& u^0(x^{0,0}) + \sum_{j\in B}[R_{-j}(N) - (u^0(x^{0,0}) + \sum_{h\in B\setminus\{j\}} u^h(x^{0,h}))] \\ &=& u^0(x^{0,0}) + \sum_{j\in B}\Big(u^0(x^{0,j}) + R_{-j}(N) - (u^0(x^{0,0}) + u^0(x^{0,j}) + \sum_{h\in B\setminus\{j\}} u^h(x^{0,h}))\Big) \\ &\geq& u^0(x^{0,0}) + \sum_{j\in B}[u^0(x^{0,j}) + R_{-j}(N) - (u^0(x^{0,0}+x^{0,j}) + \sum_{h\in B\setminus\{j\}} u^h(x^{0,h}))] \\ &\geq& u^0(x^{0,0}) + \sum_{j\in B}\Big(u^0(x^{0,j}) + R_{-j}(N) - R_{-j}(N)\Big) \\ &=& \sum_{j\in B_0} u^0(x^{0,j})\end{array}$$

This shows that the payment $\beta_j^*$ received by the seller for every sold bundle $x^{0,j}$ is at least as big as its reserve price $u^0(x^{0,j})$. We are done. □

Recall that when agents have substitutes demand type, their utility functions must be submodular, and when agents have complements demand type, their utility functions must be superadditive. These are the two most common and essential cases. Proposition 2 implies immediately

**Corollary 7** *Assume that the market $\mathcal{M}$ satisfies Assumption* (A1) *and that all agents share a common either Gross Substitutes demand type or unimodular complements demand type. If every bidder acts truthfully, the ICUD auction mechanism is beneficial to every agent.*

For any superadditive function, we have the following simple observation.

**Lemma 11** *Let $f : \mathbb{Z}^N \to \mathbb{R}$ be a superadditive function and $A$ a matrix of order $n$. Then we have $f(A(x+y)) = f(Ax+Ay) \geq f(Ax) + f(Ay)$ for any $x, y \in \mathbb{Z}^N$.*

It follows from Theorem 2 and Lemma 11 that the property of Proposition 2 holds for any unimodular demand type, as it is a basis change of a unimodular complements demand type.

The next result demonstrates another interesting property of the ICUD auction: ex post individual rationality.

**Proposition 3** *Assume that the market $\mathcal{M}$ satisfies Assumptions* (A1) and (A2)*. Then the ICUD auction is ex post individually rational.*

# 6 Conclusion

Auctions are one of the most common and most important market institutions with explicit rules determining how to allocate resources and set their prices. They have become not only a very valuable platform to address the fundamental question of how prices are formed in the environments with dispersed, asymmetric and incomplete information and strategic agents, but also a crucial testing-ground for economic theory of competitive markets and game theory with incomplete information. Significant progress has been made especially in the design of dynamic auction for the case of substitutes.

This article has examined a general auction market where all goods are inherently indivisible or traded in discrete quantities. This consideration is both important and practical as indivisible goods constitute a prominent part of modern economies such as houses, employees, cars, trains, airplanes, machines, and artworks, and in reality divisible commodities are also sold in integer/rational quantities like oil being traded in barrels. Indivisibility is an extreme form of nonconvexity, posing a challenge for analysis. Our model can cover and accommodate all kinds of indivisible goods, whether they are substitutes or complements, or any other kinds. It applies to all unimodular demand types of Baldwin and Klemperer (2019), which are a necessary and sufficient condition for the existence of competitive equilibrium. It is known that there are several kinds of substitutes but there are far more variants of complements, plus numerous combinations of substitutes and complements.

We have developed a general dynamic design framework for efficiently allocating goods in the market. Bidders each have their private valuation on each of their interested bundles of goods and may strategically exploit their private information to their own advantage. Taking the complex nature of rational and strategic agents into account, we have proposed an efficient, incentive compatible dynamic auction mechanism. Based on the competitive price system, this auction applies to every unimodular demand type, converging globally to a competitive equilibrium, yielding a generalized VCG outcome, and inducing strategic bidders to bid truthfully. It uses information efficiently by requiring bidders to reveal only their bids on the announced prices nothing else and is independent of any probability distribution of their valuations and robust against their regret. The trading rules are extremely simple, transparent, detail-free, allowing bidders to learn, adapt and correct.

We have shown that in our dynamic auction game of incomplete information sincere bidding by every bidder is an ex post Nash perfect equilibrium, which is an important strategic property and a fundamental solution concept to dynamic non-cooperative games of incomplete information. We allow the seller to have her reserve value for every bundle of goods and have demonstrated that the auction can always guarantee the benefit of trading for all market participants and avoid too low revenues for the seller. As our auction applies to all unimodular demand types or all kinds of indivisible goods, our approach is and has to be very general. We make use of only convexity and unimodularity. We do not and cannot use submodularity, which has frequently been used in the literature. We have also found an important result about the structure of the set of competitive equilibrium price vectors, which forms a convex hull of finitely many integer vectors with an elegant geometry that the distance between any two adjacent integer points in the set equals one.

The current study has left at least two topics to be explored. First, this article and most other papers on auction theory have focused on the private value models. The private value assumption is plausible when the utility of any interested bundle of goods to an agent is derived only from its use or consumption. A natural question is how the results of the current article can be extended to an interdependent value setting. In this more general setting, the valuation of each bidder on every his interested bundle of goods depends both on his own information and other bidders'. There are several important papers on this subject. See Milgrom and Weber (1982) on the single-item auction, Jehiel and Moldovanu (2001), Ausubel (2004), and Perry and Reny (2002, 2005) on multi-item auctions. Second, laboratory experiments and simulations have become important tools for testing new auction mechanisms and can shed new practical insights; see Kagel (1995). It would be therefore of considerable interest to put the current auction mechanism to the test and see how it performs and whether it can be improved, before it may make its way to any practical use.

We hope that the results and insights obtained in this article will provide useful guidance on the design of dynamic auction for allocating various complex resources in the environments with dispersed, incomplete information and strategic agents.

# Appendix

**Proof of Lemma 2:** It follows immediately from the definition of Lyapunov function $\mathcal{L}$ and Lemma 1. □

**Proof of Lemma 4:** By the assumption the demand edge-set $\mathcal{D}_u$ is full-dimensional and so is the demand type $\mathcal{D}(\supseteq \mathcal{D}_u)$. Let $x^*$ be an extreme point of the full-dimensional, convex hull of the set $D_u(p)$. There exists a set of $n$ linearly independent edge-vectors $d_1, \cdots, d_n \in \mathcal{D}_u$ that are extreme vectors of the tangent cone of the convex hull of the set $D_u(p)$ at $x^*$. Let $y = p \cdot (x - x^*) + u(x^*)$ be the hyperplane that supports $u$ at every point of $D_u(p)$. Then we have

$$p \cdot d_i = u(x^* + d_i) - u(x^*) \qquad (\forall i = 1, \cdots, n). \tag{25}$$

Since $d_1, \cdots, d_n \in \mathcal{D}$ form a unimodular matrix and the right-hand side of (25) is an integer for each $i = 1, \cdots, n$ by the assumption, $p$ is the unique integral vector satisfying (25). □

**Proof of Theorem 1:** Let $P$ be the set of competitive equilibrium price vectors. It follows from Baldwin and Klemperer (2018, Theorem 4.3) that there exists at least one competitive equilibrium price vector. Because all $u^j$, $j \in B_0$, are integer-valued and of unimodular demand type $\mathcal{D}$, it follows from Lemma 5 that their convolution $u$ is an integrally concave integer-valued function with the same unimodular demand type $\mathcal{D}$. We know that $p \in \mathbb{R}^N$ is a competitive equilibrium price vector if and only if it is a minimizer of the Lyapunov function $\mathcal{L}$. The convexity of the function $\mathcal{L}$ implies that the set $P$ is a polyhedral convex set since the function $\mathcal{L}$ is polyhedral by Lemma 2. Clearly, it is nonempty and bounded, and hence it is a polytope.

Next we prove that every vertex of $P$ is integral. This can be seen from the fact that the extreme points of the set $P$ are normal vectors $p$ of hyperplanes supporting the convolution $u$ at a full-dimensional demand set $D^{Ms}(p)$ and hence integral because of Assumption (A1) and (A2) by Lemma 4.

It remains to show that $P$ is integrally convex. Note that the demand type $\mathcal{D}$ contains every unit vector $e(i)$, $i \in N$, due to our assumption on $u^0$. Then for any $n$ linearly independent vectors $d_1, \cdots, d_n$ chosen from $\mathcal{D}$ the $n \times n$ matrix $M$ formed by these vectors is totally unimodular, so that its inverse $M^{-1}$ is a $\{0, \pm 1\}$-matrix and the first row, say, of $M^{-1}$ is orthogonal to vectors $d_2, \cdots, d_n$. This means that every minimal, integral edge-vector of $P$ is a $\{0, \pm 1\}$-vector. Recall that every edge-vector of $P$ is a normal vector of a facet of some full-dimensional demand set $D^{Ms}$. Hence $P$ is integrally convex by applying Theorem 2.2 of Fujishige (2019) and must be an integrally convex polytope. □

**Proof of Corollary 1:** Let $P$ be the set of competitive equilibrium price vectors. Note that by definition the Gross Substitutes demand type $\mathcal{D}$ is unimodular and contains all unit vectors $e(i)$, $i \in N$. Clearly, the market has a competitive equilibrium by Kelso and Crawford (1982) or Baldwin and Klemperer (2019). It follows from Theorem 1 that the set $P$ is an integrally convex polytope. From Theorem 3 of Gul and Stacchetti (1999, p. 104) or Corollary to Proposition 1 of Ausubel (2006, p.625) we know that $P$ is a lattice. We can conclude that $P$ is a nonempty integrally convex polytope with a lattice structure. □

**Proof of Corollary 2:** Note that by definition the Gross Substitutes and Complements demand type $\mathcal{D}$ is unimodular and contains all unit vectors $e(i)$, $i \in N$. It follows immediately from Theorem 1. □

**Proof of Lemma 6:** Let $M = [d_1, \cdots, d_{n-1}, d_n]$ be the $n \times n$ matrix and $\delta^*$ be the $n$th row of $M^{-1}$. Then we have $\delta^* \cdot d_j = 0$ for $j = 1, \cdots, n-1$ and $\delta^* \cdot d_n = 1$. Since $M$ is a unimodular matrix, $\delta^*$ is an integral vector. Hence $\delta^* = \alpha\delta$ or $\delta^* = -\alpha\delta$ for some $\alpha \geq 1$ because of the definition of $\delta$. Consequently, we have $\alpha|\delta \cdot d_n| = \delta^* \cdot d_n = 1$. □

**Proof of Lemma 8:** We see from the proof of Lemma 7 that $\mathcal{L}(p(t) + \delta')$ as a function in $\delta'$ is a polyhedral conical convex function restricted on $\mathrm{Conv}(\mathcal{SD})$ and is generated by function values $(\varepsilon, \mathcal{L}(p(t) + \varepsilon\delta)$ for all $\varepsilon \in [0, 1]$ and all $\delta \in \mathcal{SD}$. Hence the set of solutions to the left-side problem of (10) is a nonempty integral polytope. □

**Proof of Corollary 4:** The proof of Lemma 7 implies that $D^j(p+\varepsilon\delta) \subseteq D^j(p)$ and hence

$$x^j \in \arg \min_{x \in D^j(p)} x \cdot \delta$$

lies in $D^j(p + \varepsilon\delta)$ for all $\varepsilon \in [0, 1)$. If $D^j(p + \delta) \not\subseteq D^j(p)$, then we have

$$D^j(p) \cap D^j(p + \delta) = \arg \min_{x \in D^j(p)} \delta \cdot x = \arg \max_{x \in D^j(p+\delta)} \delta \cdot x. \tag{26}$$

Hence $x^j \in \arg\min_{x \in D^j(p)} x \cdot \delta$ lies in $D^j(p + \delta)$. □

**Proof of Theorem 4:** By Kelso and Crawford (1982), the market has a competitive equilibrium. Then by Ausubel and Milgrom (2002, Theorem 10) the Lyapunov function $\mathcal{L}(\cdot)$ is submodular. Moreover, $\mathcal{L}(\cdot)$ is convex and bounded from below and has a minimizer. By Ausubel (2006, Proposition 1) or Sun and Yang (2009, Lemma 1) any minimizer of the Lyapunov function is an equilibrium price vector. Since the prices and value functions take only integer values and the SGD auction decreases the value of the Lyapunov function by a positive integer value in each round, the auction must stop in finite rounds, i.e., $\delta(t^*) = 0$ in Step 3 for some $t^* \in \mathbb{Z}_+$. Assume that the auction generates the sequence of price vectors $p(0), p(1), \cdots, p(t^*)$. Let $\hat{t} \in \mathbb{Z}_+$ be the time when the auction reaches $\delta(\bar{t}) = 0$ at Step 2.

First we prove that $\mathcal{L}(p) \geq \mathcal{L}(p(\hat{t}))$ for all $p \geq p(\hat{t})$. Suppose to the contrary that there exists some $p \geq p(\hat{t})$ such that $\mathcal{L}(p) < \mathcal{L}(p(\hat{t}))$. By the convexity of $\mathcal{L}(\cdot)$, there is a strict convex combination $p'$ of $p$ and $p(\hat{t})$ such that $p' \in \{p(\hat{t})\} + \bar{\Delta}$ and $\mathcal{L}(p') < \mathcal{L}(p(\hat{t}))$. It follows from equation (21) that

$$\mathcal{L}(p(\hat{t}) + \delta(\hat{t})) = \min_{\delta \in \bar{\Delta}} \mathcal{L}(p(\hat{t}) + \delta) = \min_{\delta \in \Delta} \mathcal{L}(p(\hat{t}) + \delta) \leq \mathcal{L}(p') < \mathcal{L}(p(\hat{t})),$$

and so $\delta(\hat{t}) \neq 0$ in Step 2 of the SGD auction, yielding a contradiction. Clearly we also have $\mathcal{L}(p \vee p(\hat{t})) \geq \mathcal{L}(p(\hat{t}))$ for all $p \in \mathbb{R}^n$, because $p \vee p(\hat{t}) \geq p(\hat{t})$ for all $p \in \mathbb{R}^n$ and $\mathcal{L}(p) \geq \mathcal{L}(p(\hat{t}))$ for all $p \geq p(\hat{t})$.

Second, we prove that $\mathcal{L}(p) \geq \mathcal{L}(p(t))$ for all $t = \hat{t}+1, \hat{t}+2, \cdots, t^*$ and all $p \geq p(t)$. Clearly we have $\mathcal{L}(p \vee p(t)) \geq \mathcal{L}(p(t))$ for all $t = \hat{t}+1, \hat{t}+2, \cdots, t^*$ and all $p \in \mathbb{R}^n$. We first consider the case of $t = \hat{t}+1$. Then the other cases follow by induction. Recall that $p(\hat{t}+1) = p(\hat{t}) + \delta(\hat{t})$, where $\delta(\hat{t}) \in \Delta^*$ is found in Step 3 of the SGD auction. Suppose to the contrary that there is some $q \geq p(t)$ such that $\mathcal{L}(q) < \mathcal{L}(p(\hat{t}+1))$. By the convexity of $\mathcal{L}(\cdot)$, there is a strict convex combination $p'$ of $q$ and $p(\hat{t}+1)$ such that $p' \in \{p(\hat{t}+1)\} + \bar{\Delta}$ and $\mathcal{L}(p') < \mathcal{L}(p(\hat{t}+1))$. It follows from equation (21) that

$$\mathcal{L}(p(\hat{t}+1) + \delta^0) = \min_{\delta \in \bar{\Delta}} \mathcal{L}(p(\hat{t}+1) + \delta) = \min_{\delta \in \Delta} \mathcal{L}(p(\hat{t}+1) + \delta) \leq \mathcal{L}(p') < \mathcal{L}(p(\hat{t}+1)),$$

and so $\delta^0 \neq 0$ and $\delta^0 \in \Delta$. Since $\mathcal{L}(\cdot)$ is submodular, we have

$$\mathcal{L}(p(\hat{t}) \vee (p(\hat{t}+1) + \delta^0)) + \mathcal{L}(p(\hat{t}) \wedge (p(\hat{t}+1) + \delta^0)) \leq \mathcal{L}(p(\hat{t}) + \mathcal{L}(p(\hat{t}+1) + \delta^0).$$

Recall that $\mathcal{L}(p(\hat{t}) \vee (p(\hat{t}+1) + \delta^0)) \geq \mathcal{L}(p(\hat{t}))$. It follows that

$$\mathcal{L}(p(\hat{t}) \wedge (p(\hat{t}+1) + \delta^0)) \leq \mathcal{L}(p(\hat{t}+1) + \delta^0) < \mathcal{L}(p(\hat{t}+1)).$$

Let $\delta' = 0 \wedge (\delta(\hat{t}) + \delta^0)$. $\mathcal{L}(p(\hat{t}) > \mathcal{L}(p(\hat{t}+1)$ implies $\delta^0 \neq -\delta(\hat{t})$. Then we have $\delta' \in \Delta^*$ and $p(\hat{t}) \wedge (p(\hat{t}+1) + \delta^0) = p(\hat{t}) + \delta'$. This leads to $\mathcal{L}(p(\hat{t}) + \delta') < \mathcal{L}(p(\hat{t}) + \delta(\hat{t}))$ and $\delta' \neq \delta(\hat{t})$ which contradicts $\delta(\hat{t}) \in \Delta^*$ and the equality

$$\mathcal{L}(p(\hat{t}) + \delta(\hat{t})) = \min_{\delta \in \Delta^*} \mathcal{L}(p(\hat{t}) + \delta).$$

Third, we prove that $\mathcal{L}(p) \geq \mathcal{L}(p(t^*))$ for all $p \leq p(t^*)$. Suppose to the contrary that there exists some $p \leq p(t^*)$ such that $\mathcal{L}(p) < \mathcal{L}(p(t^*))$. By the convexity of $\mathcal{L}(\cdot)$, there is a strict convex combination $p'$ of $p$ and $p(t^*)$ such that $p' \in \{p(t^*)\} + \bar{\Delta}^*$ and $\mathcal{L}(p') < \mathcal{L}(p(t^*))$. Then it follows from Lemma 10 and equation (21) that

$$\mathcal{L}(p(t^*) + \delta(t^*)) = \min_{\delta \in \bar{\Delta}^*} \mathcal{L}(p(t^*) + \delta) = \min_{\delta \in \bar{\Delta}^*} \mathcal{L}(p(t^*) + \delta) \leq \mathcal{L}(p') < \mathcal{L}(p(t^*))$$

and so $\delta(t^*) \neq 0$, contradicting the fact that the SGD auction terminates in Step 3 with $\delta(t^*) = 0$. So we have $\mathcal{L}(p) \geq \mathcal{L}(p(t^*))$ for all $p \leq p(t^*)$. Because $p \wedge p(t^*) \leq p(t^*)$ for all $p \in \mathbb{R}^n$, it follows that $\mathcal{L}(p \wedge p(t^*)) \geq \mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$.

Finally, because $\mathcal{L}(\cdot)$ is submodular, $\mathcal{L}(p \vee p(t^*)) \geq \mathcal{L}(p(t^*))$ and $\mathcal{L}(p \wedge p(t^*)) \geq \mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$, we have $\mathcal{L}(p) + \mathcal{L}(p(t^*)) \geq \mathcal{L}(p \vee p(t^*)) + \mathcal{L}(p \wedge p(t^*)) \geq 2\mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$. It follows that $\mathcal{L}(p(t^*)) \leq \mathcal{L}(p)$ for all $p \in \mathbb{R}^n$. Therefore $p(t^*)$ must be an equilibrium price vector. □

**Proof of Proposition 3:** Because every bidder has the option of walking away in Step 3 and faces no punishment in Step 4, his final payoff cannot be negative if he is able to judge between positive and negative numbers, not necessarily acting optimally. Consequently, the ICUD auction is ex post individually rational. □

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