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Abstract. In this paper, we study a function field analogue of the Uniform Boundedness Conjecture on the torsion of abelian varieties. As a result, we prove the $\mathfrak{p}$-primary Uniform Boundedness Conjecture for 1-dimensional families of Drinfeld modules of arbitary rank.

Contents

1. Main Results

Let K be a function field of transcendental degree 1 over a finite field, and fix a place ∞ of K. Let A be the ring of the elements of K which are regular outside ∞. A Drinfeld A-module, which was first defined by Drinfeld in [7], is defined to be the additive group scheme $\mathbb{G}_a$ with a suitable A-action. We can define the group of torsion points (and the Tate module) of a Drinfeld A-module with regard to this A-action. Drinfeld modules have many properties in common with abelian varieties, and many studies building upon this similarity have been established until now.

For example, the Uniform Boundedness Conjecture for abelian varieties (resp. the p-primary Uniform Boundedness Conjecture for abelian varieties) states that for a fixed integer $d > 0$ and a number field L (resp. a fixed integer $d > 0$, a number field L, and a rational prime p), there exists $C := C(L, d) > 0$ (resp. $C := C(L, d, p) > 0$) such that for any d-dimensional abelian variety X over L, $|X_{\mathrm{tors}}(L)| < C$ (resp.$|X[p^\infty](L)| < C$) holds.

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We review some results towards these conjectures. First, Manin [11, ТЕОРЕМА 0] proved the p-primary Uniform Boundedness Conjecture for elliptic curves (i.e. $d = 1$). Later, Merel [12, Corollaire] proved the Uniform Boundedness Conjecture for elliptic curves. More recently, Cadoret-Tamagawa [4, Theorem A] [5, Theorem 1.1] proved an analogue of the p-primary Uniform Boundedness Conjecture for 1-dimensional families of abelian varieties.

Just as in the case of abelian varieties, the Uniform Boundedness Conjecture for Drinfeld modules (resp. the $\mathfrak{p}$-primary Uniform Boundedness Conjecture for Drinfeld modules) states that for a fixed integer $d > 0$ and a finitely generated extension L of K (resp. a fixed integer $d > 0$, a finitely generated extension L of K, and a maximal ideal $\mathfrak{p}$ of A), there exists $C := C(L, d) > 0$ (resp. $C := C(L, d, \mathfrak{p}) > 0$) such that for any rank d Drinfeld module ϕ over L, $|\phi_{\mathrm{tors}}(L)| < C$ (resp. $|\phi[\mathfrak{p}^\infty](L)| < C$) holds.

We review some results towards these conjectures. First, Poonen [18, Theorem 1] proved the Uniform Boundedness Conjecture for Drinfeld modules of rank 1 using explicit class field theory. He also proved the $\mathfrak{p}$-primary Uniform Boundedness Conjecture for Drinfeld modules of rank 2 for $A = \mathbb{F}_q[T]$, using Drinfeld modular curves. Moreover, Cornelissen-Kato-Kool [6, Corollary I] proved a strong form of the $\mathfrak{p}$-primary Uniform Boundedness Conjecture for Drinfeld modules of rank 2. On the other hand, in contrast to the case of elliptic curves, the Uniform Boundedness Conjecture for Drinfeld modules of rank 2 is still widely open except Pál's result for $A = \mathbb{F}_2[T]$ [14] and Armana's results [1]. (For conditional results, see also Ingram [9].)

In this paper, we prove the following analogue of the $\mathfrak{p}$-primary Uniform Boundedness Conjecture for 1-dimensional families of Drinfeld modules over a finitely generated extension of K. This result can be regarded as a Drinfeld module analogue of the Cadoret-Tamagawa's result.

Theorem 1.1. *Let $\mathfrak{p}$ be a maximal ideal of A, L a finitely generated extension of K, S a 1-dimensional scheme which is of finite type over L and ϕ a Drinfeld A-module over S. Then there exists an integer $N := N(\phi, S, L, \mathfrak{p}) \geq 0$ such that $\phi_s[\mathfrak{p}^\infty](L) \subset \phi_s[\mathfrak{p}^N](L)$ holds for every $s \in S(L)$.*

Next, we explain the strategy of the proof of Theorem 1.1. First, we prove the following result.

Theorem 1.2. *Let $\mathfrak{p}$ be a maximal ideal of A, L an algebraically closed field containing K, S a curve over L with a generic point η and ϕ a Drinfeld A-module over S. Assume that ϕ_η is not L-isotrivial. Then, for every $c \geq 0$, there exists an integer $N := N(c, \phi, S, L, \mathfrak{p}) \geq 0$ such that $g_v \geq c$ or $\mathfrak{p}^N v = 0$ holds for every $v \in \phi_\eta[\mathfrak{p}^\infty](\overline{L(S)})$. Here, g_v denotes the genus of (the compactification of) the curve S_v which corresponds to the stabilizer of v.*

This theorem is proved by an analytic method. More precisely, we use Oesterlé's formula [13, Théorème 6] regarding the asymptotic behavior of sizes of images of analytic sets under reduction maps, together with Breuer-Pink's result [3] concerning monodromy representations of non-isotrivial Drinfeld modules.

We give a sketch of the proof of Theorem 1.1. Just like Cadoret-Tamagawa and Poonen, we use a positive characteristic analogue of Mordell's conjecture proved by Samuel (after using Theorem 1.2). However, Samuel's theorem assumes that the curves not only have genus at least 2, but also are non-isotrivial. Here, Poonen and Cornelissen-Kato-Kool resorted to the non-isotriviality of Drinfeld modular curves, but we do not assume non-isotriviality of the base curve, so we cannot conclude the theorem. Instead, we use Samuel's theorem to show the existence of a suitable model of the base field. Then, by using a specialization argument, we reduce the theorem to the case when the base field is finite.

As a corollary, we can prove the $\mathfrak{p}$-primary Uniform Boundedness Conjecture for Drinfeld modules of rank 2 over a finitely generated extension of K (which was already proved by Poonen).

Theorem 1.3. *Let L be a finitely generated extension of K, and $\mathfrak{p}$ a maximal ideal of A. Then there exists an integer $N := N(L, \mathfrak{p}) \geq 0$ such that $\phi[\mathfrak{p}^\infty](L) \subset \phi[\mathfrak{p}^N](L)$ holds for every Drinfeld A-module ϕ of rank 2 over L.*

Finally, we briefly explain the contents of this paper. In section 2, we recall the theory of Drinfeld modules. In section 3, we prove an asymptotic formula (Proposition 3.1) mentioned above, which is needed to prove Theorem 1.2 in section 4. In section 5, we prove Theorem 1.1 using a specialization argument, and in section 6 we present some corollaries of Theorem 1.1.

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Notation

We list some notations used throughout this paper.

- p: a rational prime.
- q: a power of p.
- K: a function field in one variable over $\mathbb{F}_q$ with $K \cap \overline{\mathbb{F}}_q = \mathbb{F}_q$.
- ∞: a place of K.
- A: the ring of elements of K which are regular outside ∞.
- $\mathfrak{p}$: a maximal ideal of A.
- $A_\mathfrak{p}$: the $\mathfrak{p}$-adic completion of A.
- $K_\mathfrak{p}$: the fraction field of $A_\mathfrak{p}$.
- $\mathbb{G}_a$: the additive group scheme.
- A curve over a field L is defined to be a normal, separated, geometrically integral 1-dimensional scheme which is of finite type over L. Note that if L is algebraically closed, all curves over L are automatically smooth.

2. Drinfeld modules

In this section, we recall the theory of Drinfeld modules. First, we define Drinfeld modules over fields, their torsion points, Tate modules, etc. Next, we define Drinfeld modules over general schemes, moduli spaces of Drinfeld modules with certain level structures, and the minimal models of Drinfeld modules. Then we study monodromy representations associated to Drinfeld modules, introduce the results of Breuer-Pink, and using their results we obtain a variant of the Néron-Ogg-Shafarevich criterion (Theorem 2.43).

2.1. **Drinfeld modules over A-fields.**

Definition 2.1 (Drinfeld modules over A-fields)**.** *Let F be an A-field, i.e. a field with $\mathbb{F}_p$-homomorphism $\iota : A \to F$.*

(1) *A Drinfeld A-module ϕ over F is an $\mathbb{F}_p$-homomorphism $A \to \mathrm{End}(\mathbb{G}_{a,F})$ such that $\phi(A) \not\subset F$ and that the composite of $A \to \mathrm{End}(\mathbb{G}_{a,F}) \to F$ equals to ι. Here, $\mathrm{End}(\mathbb{G}_{a,F}) \to F$ denotes the differentiation at the origin of $\mathbb{G}_{a,F}$. If ϕ is a Drinfeld A-module, we write ϕ_a for $\phi(a)$ for $a \in A$.*

(2) *Let L be an A-field extension of F, i.e. an A-field with an A-algebra homomorphism $i : F \to L$ and ϕ a Drinfeld A-module over F. Then the*

composite of $A \xrightarrow{\phi} \mathrm{End}(\mathbb{G}_{a,F}) \to \mathrm{End}(\mathbb{G}_{a,L})$ *induces a Drinfeld* A*-module over* L*, and we denote it by* $\phi \times_F L$*.*

(3) *Let* ϕ *and* ψ *be Drinfeld* A*-modules over* F*. A homomorphism* $f : \phi \to \psi$ *over* F *is an element of* $\mathrm{End}(\mathbb{G}_{a,F})$ *which satisfies* $f \circ \phi = \psi \circ \psi$*. A non-zero homomorphism is called an isogeny.* $\mathrm{Hom}_F(\phi, \psi)$ *denotes the set of all homomorphisms from* ϕ *to* ψ*. It becomes an* A*-module in an obvious manner. Let* $\mathrm{End}_F(\phi) := \mathrm{Hom}_F(\phi, \phi)$*.*

Remark 2.2.

(1) *If we write* $\tau \in \mathrm{End}(\mathbb{G}_{a,F})$ *for the* q*-th power Frobenius,* $\mathrm{End}(\mathbb{G}_{a,F})$ *equals to* $F\{\tau\}$*, where* $F\{\tau\} := \{\sum_i a_i \tau^i$ *(finite sum)* $\mid a_i \in F\}$*. With this identification, the differentiation at the origin of of* $\mathbb{G}_{a,F}$ *is given by* $\sum_i a_i \tau^i \mapsto a_0$*.*

(2) *We define the degree map* $\deg : F\{\tau\} \to \mathbb{Z} \cup \{-\infty\}$ *by* $\deg(0) = -\infty$ *and* $a = \sum_i a_i \tau^i \mapsto n$ *for* $a \neq 0$*, where* $a_n \neq 0$ *and* $a_m = 0$ *for all* $m > n$*. Also, we also define the height map* $\mathrm{ht} : F\{\tau\} \to \mathbb{Z} \cup \{\infty\}$ *by* $\mathrm{ht}(0) = \infty$ *and* $a = \sum_i a_i \tau^i \mapsto n$ *for all* $a \neq 0$*, where* $a_n \neq 0$ *and* $a_m = 0$ *for all* $m < n$*.*

Definition 2.3. *Let* $(F, \iota : A \to F)$ *be an* A*-field, and* ϕ *a Drinfeld* A*-module over* F*.*

(1) *We define the characteristic of* (F, ι) *as* $\ker(\iota)$*, and denote it by* $\mathrm{ch}(F)$ *if no confusion occurs. We say* F *has the generic characteristic if* $\mathrm{ch}(F) = (0)$ *and otherwise we say* F *has a special characteristic.*

(2) *Let* $d_\infty := [\kappa(\infty) : \mathbb{F}_q]$ *and let* $v_\infty : K \to \mathbb{Z} \cup \{\infty\}$ *be the normalized (additive) valuation corresponding to* ∞*. Then it is known that* $\deg \circ \phi : A \to \mathbb{Z} \cup \{-\infty\}$ *induces a valuation on* K *which is equivalent to* v_∞*. More precisely, there exists a unique rational number* $d \in \mathbb{Q}_{>0}$ *such that* $\deg(\phi_a) = -d d_\infty v_\infty(a)$ *holds for every* $a \in A$*. We define the rank of* ϕ *to be* d*. It is known that* d *is in fact a positive integer.*

(3) *Assume that* F *has a special characterstic* $\mathfrak{p}$*. Let* $d_{\mathfrak{p}} := [\kappa(\mathfrak{p}) : \mathbb{F}_q]$*, and* $v_{\mathfrak{p}} : K \to \mathbb{Z} \cup \{\infty\}$ *be the normalized valuation corresponding to* $\mathfrak{p}$*. Then, it is known that* $\mathrm{ht} \circ \phi : A \to \mathbb{Z} \cup \{\infty\}$ *induces a valuation on* K *which is equivalent to* $v_{\mathfrak{p}}$*. More precisely, there exists a unique rational number* $h \in \mathbb{Q}_{>0}$ *such that* $\mathrm{ht}(\phi_a) = h d_{\mathfrak{p}} v_{\mathfrak{p}}(a)$ *holds for every* $a \in A$*. We define the height of* ϕ *to be* h*. It is known that* h *is in fact a non-negative integer.*

Our main interest in this paper is Drifeld modules defined over various extensions of K. In particular, they have the generic characteristic.

Definition 2.4. *Let* $(F, \iota : A \to F)$ *be an* A*-field, and* ϕ *a Drinfeld* A*-module over* F*. Fix an algebraic closure* $\overline{F}$ *of* F*, and let* L *be an arbitary extension of* F*.*

(1) *For an ideal* I *of* A*, we define the* I*-torsion subgroup of* ϕ *over* L *to be* $\phi[I](L) := \{x \in L \mid \phi_a(x) = 0 \mid (\forall a \in I)\}$*.*

(2) *We define the torsion subgroup of* ϕ *over* L *to be* $\phi_{\mathrm{tors}}(L) := \cup_{0 \neq I \subset A} \phi[I](L)$*. Here, the index runs over all non-zero ideals* I *of* A*.*

(3) *We define the* $\mathfrak{p}$*-primary torsion subgroup of* ϕ *over* L *to be* $\phi[\mathfrak{p}^\infty](L) := \cup_{n \geq 1} \phi[\mathfrak{p}^n](L)$*. For an integer* $n > 0$*, we define* $\phi[\mathfrak{p}^n]^*(L) := \phi[\mathfrak{p}^n](L) \setminus \phi[\mathfrak{p}^{n-1}](L)$ $(\phi[\mathfrak{p}^0]^*(L) := \phi[\mathfrak{p}^0](L))$*.*

(4) *We define the* $\mathfrak{p}$*-adic Tate module of* ϕ *to be* $T_{\mathfrak{p}}(\phi) := \mathrm{Hom}_A(K_{\mathfrak{p}}/A_{\mathfrak{p}}, \phi[\mathfrak{p}^\infty](\overline{F}))$*. We define* $T_{\mathfrak{p}}(\phi)^* := T_{\mathfrak{p}}(\phi) \setminus \mathfrak{p} T_{\mathfrak{p}}(\phi)$*. We define the rational* $\mathfrak{p}$*-adic Tate module of* ϕ *to be* $V_{\mathfrak{p}}(\phi) := T_{\mathfrak{p}}(\phi) \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}}$*.*

Remark 2.5. *Let* $(F, \iota : A \to F)$ *be an* A*-field,* ϕ *a Drinfeld* A*-module over* F*, and* I *an ideal of* A*.*

(1) *By definition, the I-torsion subgroup of a Drinfeld A-module has a natural structure of (A/I)-module. In particular, the $\mathfrak{p}$-adic Tate module has a natural structure of $A_{\mathfrak{p}}$-module.*
(2) *Let L be an arbitary extension of F. If I is generated by $a_1, \ldots, a_s \in A$, then $\phi[I](L) = \phi[a_1](L) \cap \cdots \cap \phi[a_s](L)$ where $\phi[a](L) \coloneqq \mathrm{Ker}(\phi_a : L \to L)$. If I_1 and I_2 are ideals of A such that $I_1 + I_2 = A$, there exists a canonical isomorphism $\phi[I_1 I_2](L) \cong \phi[I_1](L) \oplus \phi[I_2](L)$.*
(3) *If $\mathrm{ch}(F) \notin V(I)$, then $\phi[I](\overline{F}) = \phi[I](F^{\mathrm{sep}}) \subset F^{\mathrm{sep}}$ and it becomes an $(A/I)[G_F]$-module.*

The following are well-known results about the structure of torsion subgroups of Drinfeld modules.

Proposition 2.6 (Drinfeld [7, Proposition 2.2])**.** *Let $(F, \iota : A \to F)$ be an A-field, and ϕ a Drinfeld A module of rank d over F. Fix an algebraic closure $\overline{F}$ of F.*
(1) *Let $I \not\subset \mathrm{ch}(F)$ be an ideal of A. Then $\phi[I](\overline{F})$ is a free (A/I)-module of rank d.*
(2) *Assume that F has a special characteristic and let h be the height of ϕ. Let I be a power of $\mathrm{ch}(F)$. Then, $\phi[I](\overline{F})$ is a free (A/I)-module of rank $d - h$.*
(3) *The $\mathfrak{p}$-adic Tate module of ϕ is a free $A_{\mathfrak{p}}$-module of rank d (resp. $d - h$) if $\mathfrak{p} \neq \mathrm{ch}(F)$ (resp. $\mathfrak{p} = \mathrm{ch}(F)$).*

Moreover, the following exact sequences hold for torsion subgroups of Drinfeld modules.

Proposition 2.7. *Let $(F, \iota : A \to F)$ be an A-field, and ϕ a Drinfeld A-module over F. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A, and $a \in A$ an element of A which satisfies $\mathfrak{a} = \mathfrak{ab} + (a)$. Then the following sequence is exact:*

$$0 \to \phi[\mathfrak{a}](\overline{F}) \to \phi[\mathfrak{ab}](\overline{F}) \xrightarrow{\phi_a} \phi[\mathfrak{b}](\overline{F}) \to 0$$

Proof. From the identity $\mathfrak{a} = \mathfrak{ab} + (a)$, we see $\phi[\mathfrak{a}](\overline{F}) = \ker(\phi[\mathfrak{ab}](\overline{F}) \xrightarrow{\phi_a} \phi[\mathfrak{b}](\overline{F}))$ holds. Therefore, we only have to prove $\phi_a : \phi[\mathfrak{ab}](\overline{F}) \to \phi[\mathfrak{b}](\overline{F}))$ is surjective, which follows from Proposition 2.6. □

Corollary 2.8. *Let $(F, \iota : A \to F)$ be an A-field, and ϕ a Drinfeld A-module over F. Fix a generator π of $\mathfrak{p}$. Then, for every non-negative integer n, the following sequence is exact.*

$$0 \to \phi[\mathfrak{p}](\overline{F}) \to \phi[\mathfrak{p}^{n+1}](\overline{F}) \xrightarrow{\phi_\pi} \phi[\mathfrak{p}^n](\overline{F}) \to 0$$

In particular, the projective limit of $\{\phi[\mathfrak{p}^n](\overline{F})\}_{n \geq 0}$ whose transition maps are induced by ϕ_π are isomorphic to the $\mathfrak{p}$-adic Tate module of ϕ.

Proof. This follows from the equality $\mathfrak{p} = \mathfrak{p}^n + (\pi)$ and Proposition 2.7. □

Next, we discuss the structure of homomorphisms between Drinfeld modules.

Proposition 2.9. *Let $(F, \iota : A \to F)$ be an A-field, and ϕ and ψ Drinfeld A-modules over F of rank d and d', respectively. Let $\overline{F}$ be an algebraic closure of F, and F^{sep} the separable closure of F in $\overline{F}$.*
(1) *If $d \neq d'$, $\mathrm{Hom}_F(\phi, \psi)$ is zero. Ohterwise, $\mathrm{Hom}_F(\phi, \psi)$ is a projective A-module of rank $\leq d^2$. In particular, $\mathrm{End}_F(\phi)$ is a projective A-algebra of rank $\leq d^2$.*
(2) *$\mathrm{End}_F(\phi) \otimes_A K$ is a skew field.*
(3) *If F has the generic characteristic, then $\mathrm{End}_F(\phi)$ is commutative and $\mathrm{rank}_A \mathrm{End}_F(\phi) \leq d$.*
(4) *Every homomorphism between $\phi \times_F \overline{F}$ and $\psi \times_F \overline{F}$ is defined over F^{sep}.*

Proof. As for (1), (2), and (3), they follow from [7, Proposition 2.4 の Corollary]. We prove (4). Take any $f \in \mathrm{Hom}_{\overline{F}}(\phi, \psi)$ and $a \in A \setminus \mathbb{F}_q$. Then the assertion follows easily by comparing the coefficients of the identity $f \circ \phi_a = \psi_a \circ f$ starting from their leading terms. □

Proposition 2.10 (Goss [8, Proposition 4.7.1])**.** *Let $(F, \iota : A \to F)$ be an A-field, and ϕ and ψ Drinfeld A-modules over F. Then, if $f \in \mathrm{Hom}_F(\phi, \psi) \subset F\{\tau\}$ is an isomorphism of Drinfeld modules, it is F-linear. That is,* $\deg(f) = 0$.

2.2. **Reduction of Drinfeld modules.** In this section, we briefly explain the theory of reduction of Drinfeld modules. Throughout this section, we fix an A-field $(F, \iota : A \to F)$, and a (nontrivial) discrete valuation v on F, and we assume $\iota(A)$ is contained in the valuation ring $O_v \subset F$ of v. Let $\mathfrak{m}_v$ be the maximal ideal of O_v, and $k_v \coloneqq O_v/\mathfrak{m}_v$ the residue field of O_v.

Definition 2.11 (Goss [8, Definition 4.10.1])**.** *Let ϕ be a Drinfeld A-module over F.*

(1) *We say ϕ has integral coefficients with respect to v if, for every $a \in A$, all coefficients of $\phi_a \in \mathrm{End}(\mathbb{G}_{a,F}) = F\{\tau\}$ is contained in O_v, and the composite of $A \to O_v\{\tau\} \to k_v\{\tau\}$ defines a Drinfeld A-module over k_v.*
(2) *We say ϕ has stable reduction at v if ϕ is isomorphic to a Drinfeld A-module over F which has integral coefficients with respect to v.*
(3) *We say ϕ has good reduction at v if ϕ has stable reduction at v and the rank of the resulting Drinfeld A-module over k_v coincides with the rank of ϕ.*
(4) *We say ϕ has potentially stable (resp. good) reduction at v if there exists a finite extension of discrite valuation fields (G, w) of (F, v) such that $\phi \times_F G$ has stable (resp. good) reduction at w.*

Example 2.12.

(1) *Assume $A = \mathbb{F}_q[T]$. Let ϕ be a Drinfeld $\mathbb{F}_q[T]$-module of rank 2 over F which is defined by $\phi_T = T + g\tau + \Delta\tau^2$. Then, for any $u \in \mathrm{Aut}(\mathbb{G}_{a,F}) = F^*$, $u^{-1}\phi_T u = T + gu^{q-1}\tau + \Delta u^{q^2-1}\tau^2$. So $u^{-1}\phi_T u \in O_v\{\tau\}$ is equivalent to $v(gu^{q-1}) \geq 0$ and $v(\Delta u^{q^2-1}) \geq 0$, i.e. $v(u) \geq -\min(\frac{v(g)}{q-1}, \frac{v(\Delta)}{q^2-1})$. Therefore, we can conclude that ϕ has stable reduction at v if and only if $\min(\frac{v(g)}{q-1}, \frac{v(\Delta)}{q^2-1}) \in \mathbb{Z}$. Moreover, we easily see that ϕ has good reduction at v if and only if $\frac{v(g)}{q-1} \geq \frac{v(\Delta)}{q^2-1}$ and $\frac{v(\Delta)}{q^2-1} \in \mathbb{Z}$.*
(2) *Along the same lines, for a general A, we can prove that every Drinfeld A-module over F has potentially good reduction at v and every Drinfeld A-module of rank 1 has potentially good reduction at v. (For more details, see Goss* [8, Corollary 4.10.4.]*.)*

Takahashi [22, Theorem 1] proved the following Drinfeld module analogue of the Néron–Ogg–Shafarevich criterion of abelian varieties.

Theorem 2.13 (Takahashi [22, Theorem 1])**.** *Let ϕ be a Drinfeld A-module over F and assume that $\mathfrak{p}$ is different from* $\mathrm{ch}(F)$*. Then the following are equivalent.*

(1) *ϕ has good reduction at v.*
(2) *$\phi[\mathfrak{p}^\infty](\overline{F})$ is unramified at v, i.e. the inertia subgroup at v acts trivially on $\phi[\mathfrak{p}^\infty](\overline{F})$.*

Next, we discuss the action of the inertia subgroup on $\mathfrak{p}$-adic Tate modules of Drinfeld modules over a discrete valuation field. In the case of abelian varieties over a discrete valuation field F, it is well-known that the inertia subgroup acts quasi-unipotently on ℓ-adic Tate modules if ℓ is not equal to the characteristic of F. We prove a Drinfeld module analogue of this result (Corollary 2.15).[1]

First, we recall the Tate-Drinfeld uniformization.

Proposition 2.14 (Drinfeld [7, §7, Proposition 7.2])**.** *Let (F, v) be a pair of an A-field F and a complete discrete valuation v on it. Let O_v be the valuation ring of v, and k_v the residue field of O_v. Let ϕ be a Drinfeld A-module of rank d over F which has stable reduction at v, and d_1 the rank of the Drinfeld A-module over k_v obtained from the reduction of ϕ. Then there exists a Drinfeld A-module of rank d_1 over F which has good reduction at v, and $u \in 1 + O_v\{\{\tau\}\}\tau$ which converges entirely on the completion of an algebraic closure of F such that $u\psi_a = \phi_a u$ holds for every $a \in A$.*

Corollary 2.15. *Let (F, v), ϕ, and ψ be as in Propositon 2.14, and assume that $\mathfrak{p}$ is different from $\mathrm{ch}(F)$. Then there exist an $A[G_F]$-module Λ which is finitely generated as an A-module and the following exact sequence of $A[G_F]$-modules.*

$$0 \to T_{\mathfrak{p}}(\psi) \to T_{\mathfrak{p}}(\phi) \to \Lambda \otimes_A A_{\mathfrak{p}} \to 0$$

In particular, the action of the inertia subgroup of G_F on the $\mathfrak{p}$-adic Tate module of every Drinfeld A-module over F is quasi-unipotent.

Proof. Take u as in the above Proposition 2.14, and define $\Lambda := \mathrm{Ker}(u : F^{\mathrm{sep}} \to F^{\mathrm{sep}})$. Then Λ has a natural structure of A-module induced by ψ. Since $u \in O_v\{\{\tau\}\}$, Λ is closed under the natural G_F-action on F^{sep}.

We fix a positive integer h (e.g. the class number of A) so that $\mathfrak{p}^h$ is principal, and a generator a of $\mathfrak{p}^h$. First, for any $n > 0$, we prove that the following sequence of $(A_{\mathfrak{p}}/\mathfrak{p}^{nh})[G_F]$-modules is exact:

$$(*) \qquad 0 \to \psi[\mathfrak{p}^{nh}](F^{\mathrm{sep}}) = \psi_{a^n}^{-1}(0) \to \psi_{a^n}^{-1}(\Lambda)/\Lambda \xrightarrow{\psi_{a^n}} \Lambda/\mathfrak{p}^{nh}\Lambda \to 0$$

For the injectivity on the left side, it suffices to prove $\psi[\mathfrak{p}^{nh}](F^{\mathrm{sep}}) \cap \Lambda = \{0\}$. Since ψ has good reduction at v, the valuations of the elements of $\psi[\mathfrak{p}^{nh}](F^{\mathrm{sep}}) \setminus \{0\}$ are non-negative. On the other hand, using Newton polygons, we see that the valuations of the nonzero roots of u are negative. Thus, the injectivity follows. The exactness in the middle is obvious. The surjectivity on the right side follows from the fact that $\Lambda \subset F^{\mathrm{sep}}$.

Next, we observe that u induces an $(A/\mathfrak{p}^{nh})[G_F]$-module isomorphism:

$$\psi_{a^n}^{-1}(\Lambda)/\Lambda \cong \phi[\mathfrak{p}^{nh}](F^{\mathrm{sep}}) = \phi_{a^n}^{-1}(0)$$

So, taking the projective limit of $(*)$ gives the desired exact sequence.

Finally, we prove the quasi-unipotence of the action of the inertia subgroup. By Theorem 2.13, the action of the inertia subgroup on $T_{\mathfrak{p}}(\psi)$ is trivial. So, it suffices to prove that the action of the inertia subgroup on Λ is finite. As $\Lambda \subset F^{\mathrm{sep}}$, it suffices to prove that Λ is finitely generated as an A-module.

First, note that $\Lambda/\psi_a(\Lambda)$ is finite. Fix a real number $r > 1$ such that the natural map

$$B(0, r) \cap \Lambda \twoheadrightarrow \Lambda/\psi_a(\Lambda)$$

is surjective. Here, $B(0, r)$ denotes $\{x \in F^{\mathrm{sep}} \mid |x| \leq r\}$. The left side $B(0, r) \cap \Lambda$ is finite (which can be proved by using the Newton polygon of u). We will prove that $B(0, r) \cap \Lambda$ generates Λ as an A-module.

Pick any element $x \in \Lambda$. We show that x is in the A-module generated by $B(0, r) \cap \Lambda$. If $|x| \leq r$, then $x \in B(0, r) \cap \Lambda$, so we may assume that $|x| > r$. Then there exists $y \in B(0, r) \cap \Lambda$ and $x_1 \in \Lambda$ such that $x - y = \psi_a(x_1)$. We may assume

[1] This result seems to be well-known to experts, but we include the proof because we cannot find suitable references.

that $|x_1| > r$ otherwise we conclude $x = y + \psi_a(x_1)$ is in the A-module generated by $B(0, r) \cap \Lambda$. Then, since $\psi_a \in O_v\{\tau\}$ has degree $-d_1 d_\infty v_\infty(a) \geq 1$ and its leading coefficient is in O_v^*, it follows that $|\psi_a(x_1)| = |x_1|^{q^{-d_1 d_\infty v_\infty(a)}}$. So, by using the ultrametric triangle inequality, we see that

$$|x_1|^{q^{-d_1 d_\infty v_\infty(a)}} = |\psi_a(x_1)| = |x - y| = |x|$$

holds. In particular, $|x| > r^{q^{-d_1 d_\infty v_\infty(a)}}$ holds. By repeating this argument for x_1, x_2, ..., we conclude that $x_n \in B(0, r) \cap \Lambda$ for sufficienty large n. Thus, x is in the A-module generated by $B(0, r) \cap \Lambda$. This concludes the proof. □

Next, we briefly recall the theory of minimal models of Drinfeld modules by Taguchi [21, 2]. Before that, we define A-module schemes over general A-schemes.

Definition 2.16. *Let S be an A-scheme. An A-module scheme over S is defined to be a pair (G, ϕ), where G is a commutative group scheme over S and $\phi : A \to \mathrm{End}(G/S)$ is a homomorphism such that the A-action on the tangent space of G at the origin induced by the natural map $A \to \Gamma(S, \mathcal{O}_S)$ equals the one induced by ϕ. A homomorphism between two A-module schemes is defined to be a homomorphism of group schemes which is compatible with the given A-actions.*

In the rest of this section, we fix an integral, normal scheme S of finite type over $\mathrm{Spec}(A)$, and let F be the function field of S which we regard as an A-field through the homomorphism associated to the composite of $\mathrm{Spec}(F) \to S \to \mathrm{Spec}(A)$. We are going to discuss S-models of Drinfeld A-modules over F.

In the following, if L is an invertible sheaf on S, we abbreviate the group scheme $\mathrm{Spec}_S(\mathrm{Sym}(L^\vee))$ over S to L. So, L represents a contravariant functor $(\mathrm{Sch}/S) \to (\mathrm{Grp}) : (f : T \to S) \mapsto \Gamma(T, f^*L)$.

Definition 2.17 (Taguchi [21, 2, DEFINITION])**.** *Let ϕ be a Drinfeld A-module over F. A model of ϕ over S is defined to be a triple $E = (L, \varphi, f)$ where L is an invertible sheaf on S, (L, φ) is an A-module scheme over S, and $f : (L, \varphi) \otimes_S F \to (\mathbb{G}_{a,F}, \phi)$ is an isomorphism of A-module schemes. A homomorphism between two models of ϕ over S is defined to be a homomorphism of A-module schemes which is compatible with the given isomorphisms to $(\mathbb{G}_{a,F}, \phi)$.*

Remark 2.18 (Taguchi [21, 2, Remarks])**.** *Let ϕ be a Drinfeld A-module over F, and $E = (L, \varphi, f)$ a model of ϕ over S. For every $g \in F^*$, the multiplication by g^{-1} induces an isomorphism between the following models:*

$$(L, \varphi, f) \xrightarrow{\sim} (L \otimes_S \mathcal{O}_S(\mathrm{div}(g)), g^{-1}\varphi g, fg)$$

Theorem 2.19 (Taguchi [21, 2, PROPOSITION (2.2)])**.** *Let S and F be as above. Assume that the notion of Weil divisor and that of Cartier divisor coincide on S. Then, for every Drinfeld A-module ϕ over F, there exists uniquely (up to isomorphism) a model $E = (L, \varphi, f)$ of ϕ over S which satisfies the following property:*

For every model E' of ϕ over S, there exists a unique homomorphism from E' to E.

Definition 2.20. *A model of ϕ over S which satisfies the property in Theorem 2.19 is called the minimal model of ϕ over S.*

Remark 2.21. *Let L be an A-field, S an integral, normal scheme of finite type over $\mathrm{Spec}(L)$, and F the function field of S. The proof in* [21, 2, PROPOSITION (2.2)] *also works in this case, i.e. if the notion of Weil divisor and that of Cartier divisor coincide on S, the minimal model over S exists for every Drinfeld A-module over F.*

2.3. **Drinfeld modules which have nontrivial endomorphism rings and their rational Tate modules.**

In this section, we discuss a Drinfeld A-module which has a nontrivial endomphism ring, i.e. a Drinfeld A-module whose endomorphism ring E is strictly larger than A. We show that we can roughly regard such a Drinfeld A-module as a "Drinfeld E-module" up to isogeny. We also discuss how the structure of rational Tate module depends on the endomorphism ring of Drinfeld module.

In this section, we fix an A-field $(F, \iota : A \to F)$ which has the generic characteristic and Drinfeld A-module ϕ of rank d over F. By Proposition 2.9, $E := \mathrm{End}_F(\phi)$ is a commutative projective A-algebra of rank $\leq d$.

Lemma 2.22. *There exists a unique place of* $\mathrm{Frac}(E)$ *above* ∞. *Moreover, this place is induced by* $\deg : E = \mathrm{End}_F(\phi) \subset F\{\tau\} \to \mathbb{Z} \cup \{-\infty\}$

Proof. From the proof of Goss [8, Proposition 4.7.17], it follows that $\mathrm{Frac}(E) \otimes_K K_\infty \cong E \otimes_A K_\infty$ is a field. Thus, there is a unique place of $\mathrm{Frac}(E)$ above ∞. As the map $\deg : E = \mathrm{End}_F(\phi) \to \mathbb{Z} \cup \{-\infty\}$ induces a nontrivial valuation of $\mathrm{Frac}(E)$ above ∞, this concludes the proof. □

From this proposition, if E is an integrally closed domain, we can regard ϕ as a Drinfeld E-module over F. (Here, we regard F as an E-field by the differentiation map $E \subset \mathrm{End}(\mathbb{G}_a) \to F$). Indeed, if the constant field $\mathbb{F}_{q'}$ of E equals the coefficient field $\mathbb{F}_q$ of A, this is obvious. In general, we can easily prove $E \subset F\{\tau^{[\mathbb{F}_{q'}:\mathbb{F}_q]}\}$ and regard ϕ as a Drinfeld E-module. The rank of the Drinfeld E-module obtained as above is equal to $d' := \frac{d}{\mathrm{rank}_A(E)}$.

The following proposition show that, up to isogeny, we can regard ϕ as a "Drinfeld E-module" even if E is not integrally closed.

Proposition 2.23 (Goss [8, Proposition 4.7.19][2])**.** *Let $(F, \iota : A \to F)$ be an A-field, and ϕ a Drinfeld A-module over F. Let $O \subset \mathrm{End}_F(\phi)$ be an A-order of a finite extension of K. Let $\tilde{O}$ be the integral closure of O in its fraction field. Then there exists a Drinfeld A-module ψ over F with an injective homomorphism $\tilde{O} \hookrightarrow \mathrm{End}_F(\psi)$ and an isogeny $\pi : \phi \to \psi$ which is compatible with O-actions. If, moreover, F has the generic characteristic and O is an A-order of* $\mathrm{Frac}(\mathrm{End}_F(\phi))$*, we can take $\tilde{O} \hookrightarrow \mathrm{End}_F(\psi)$ to be an isomorphism.*

Let $\tilde{E}$ be the integral closure of E in its fraction field. From Proposition 2.23, we see that there exists a Drinfeld A-module ψ over F which is isogenous to ϕ and whose endomorphism ring equals $\tilde{E}$. We write $\psi_{\tilde{E}}$ for the Drinfeld $\tilde{E}$-module over F which is induced by ψ.

We make some observations. Let $\overline{K}$ and $\overline{F}$ be algebraic closures of K and F, respectively. If the rank of $\psi_{\tilde{E}}$ is equal to 1, then by [8, Proposition 7.4.3], we see that $\psi_{\tilde{E}} \times_F \overline{F}$ comes from a Drinfeld $\tilde{E}$-module defined over $\overline{K}$. In particular, $\psi \times_F \overline{F}$ comes from a Drinfeld A-module defined over $\overline{K}$. Moreover, by [7, Proposition 2.3], the kernel of an isogeny between Drinfeld modules over an A-field which has the generic characteristic is known to be a finite étale torsion A-module scheme. Therefore, we conclude that $\phi \times_F \overline{F}$ descends to a Drinfeld A-module over $\overline{K}$.

To sum it up, we have proved the following corollary.

Corollary 2.24. *Let $(F, \iota : A \to F)$ be an A-field which has the generic characteristic, and ϕ a Drinfeld A-module of rank d over F. Assume that the endomorphism ring of $\phi \times_F \overline{F}$ is a projective A-algebra of rank d. Then, $\phi \times_F \overline{F}$ descends to a Drinfeld A-module over $\overline{K}$.*

[2]The statement of Proposition 4.7.19 in Goss [8] is not correct, but the proof essentially works.

Now we discuss the structure of the rational Tate module. By replacing F with a finite separable extention of F if needed, we may assume that $E = \mathrm{End}_F(\phi) = \mathrm{End}_{\overline{F}}(\phi)$. Take a Drinfeld A-module ψ over F and an isogeny $\pi : \phi \to \psi$ as in Proposition 2.23.

Let π be a generator of $\mathfrak{p}$. Let $\mathfrak{p}\tilde{E} = \prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P}^{e_{\mathfrak{P}}}$ be the prime ideal decomposition of $\mathfrak{p}$ in $\tilde{E}$. Then, by Corollary 2.8, we have the following commutative diagram in which two rows are exact for each $n \geq 1$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \psi[\mathfrak{p}](F^{\mathrm{sep}}) & \longrightarrow & \psi[\mathfrak{p}^n](F^{\mathrm{sep}}) & \xrightarrow{\psi_\pi} & \psi[\mathfrak{p}^{n-1}](F^{\mathrm{sep}}) & \longrightarrow & 0 \\
 & & \wr\uparrow & & \wr\uparrow & & \wr\uparrow & & \\
0 & \longrightarrow & \bigoplus_{\mathfrak{P}|\mathfrak{p}} \psi_{\tilde{E}}[\mathfrak{P}^{e_{\mathfrak{P}}}](F^{\mathrm{sep}}) & \longrightarrow & \bigoplus_{\mathfrak{P}|\mathfrak{p}} \psi_{\tilde{E}}[\mathfrak{P}^{ne_{\mathfrak{P}}}](F^{\mathrm{sep}}) & \xrightarrow{(\psi_{\tilde{E}})_\pi} & \bigoplus_{\mathfrak{P}|\mathfrak{p}} \psi_{\tilde{E}}[\mathfrak{P}^{(n-1)e_{\mathfrak{P}}}](F^{\mathrm{sep}}) & \longrightarrow & 0
\end{array}$$

Here, F^{sep} denotes the separable closure of F in $\overline{F}$. By taking the projective limit, we obtain an isomorphism $\bigoplus_{\mathfrak{P}|\mathfrak{p}} T_{\mathfrak{P}}(\psi_{\tilde{E}}) \xrightarrow{\sim} T_{\mathfrak{p}}(\psi)$. This isomorphism is compatible with $\tilde{E} \otimes_A A_{\mathfrak{p}} (\cong \prod_{\mathfrak{P}|\mathfrak{p}} \tilde{E}_{\mathfrak{P}})$-action and G_F-action. Each $T_{\mathfrak{P}}(\psi_{\tilde{E}})$ is closed under G_F-action.

Since ψ is isogeneous to ϕ, we have an injective $A_{\mathfrak{p}}[G_F]$-module homomorphism $T_{\mathfrak{p}}(\phi) \to T_{\mathfrak{p}}(\psi)$ whose cokernel is finite. Thus, taking tensor products with $K_{\mathfrak{p}}$, we have an isomorphism $V_{\mathfrak{p}}(\phi) \xrightarrow{\sim} V_{\mathfrak{p}}(\psi) \cong \bigoplus_{\mathfrak{P}|\mathfrak{p}} V_{\mathfrak{P}}(\psi_{\tilde{E}})$ of $K_{\mathfrak{p}}[G_F]$-modules. This decomposition is used in the proof of Proposition 2.43 and Theorem 1.2.

2.4. **Drinfeld modules over general A-schemes.**

In this section, we define Drinfeld modules over general A-schemes. In the following, for a scheme S and $s \in S$, $k(s)$ denotes the residue field of S at s.

Definition 2.25 (Drinfeld [7, p.575, Definition])**.** *Let S be an A-scheme.*

1. *We call the image of S in $\mathrm{Spec}(A)$ the characteristic of S. If S consists of the generic point of $\mathrm{Spec}(A)$, we say S has the generic characteristic.*
2. *Let d be a positive integer. A Drinfeld A-module of rank d over S is an A-module scheme (L, ϕ) where L is an invertible sheaf on S such that, for all $s \in S$, the pullback of (L, ϕ) to $s \in S$ defines a Drinfeld A-module of rank d over $k(s)$.*

Remark 2.26.

1. *Let L be an invertible sheaf on an A-scheme S. Then $\mathrm{End}(L/S)$ is identified with $\{\sum_i a_i \tau^i$ (Zariski locally finite sums) $\mid a_i \in \Gamma(S, L^{\otimes(1-q^i)})\}$ where τ denotes the q-th Frobenius map.*
2. *By (1), we can rephrase the definition of Drinfeld modules as follows: A Drinfeld A-module of rank d over S is a pair of an invertible sheaf L over S and a homomorphism $\phi : A \to \mathrm{End}(L/S)$ such that the following conditions hold: If we write $\phi_a = \sum_i a_i(a)\tau^i$ for every $a \in A \setminus \{0\}$, $a_0(a)$ coincides with the image of a under the natural homomorphism $A \to \Gamma(S, \mathcal{O}_S)$. Moreover, $a_{-dd_\infty v_\infty(a)}(a)$ is nonzero in $L^{\otimes(1-q^{-dd_\infty v_\infty(a)})} \otimes_S k(s)$ for all $s \in S$ and $a_i(a)$ becomes zero in $L^{\otimes(1-q^i)} \otimes_S k(s)$ for all $s \in S$ and for all $i > -dd_\infty v_\infty(a)$.*

Definition 2.27. *Let S be an A-scheme, and let d be a positive integer. A standard Drinfeld A-module of rank d over S is a pair of an invertible sheaf L and a homomorphism $\phi : A \to \mathrm{End}(L/S)$ such that the following conditions hold: If we write $\phi_a = \sum_i a_i(a)\tau^i$ for every $a \in A \setminus \{0\}$, $a_0(a)$ coincides with the image of a under the natural homomorphism $A \to \Gamma(S, \mathcal{O}_S)$. Moreover, $a_{-dd_\infty v_\infty(a)}(a)$ is* *nonzero in* $L^{\otimes(1-q^{-dd_\infty v_\infty(a)})} \otimes_S k(s)$ *for all* $s \in S$ *and* $a_i(a)$ *becomes zero for all* $i > -dd_\infty v_\infty(a)$.

If (L, ϕ) is a Drinfeld A-module over an A-scheme S, we shall omit L and simply say ϕ is a Drinfeld A-module over S if no confusion occurs.

The following proposition ensures that every Drinfeld module is isomorphic to a standard Drinfeld module of the same rank.

Proposition 2.28 (Drinfeld [7, §5, B), Remark])**.** *Let* S *be an* A*-scheme. Then every Drinfeld* A*-module over* S *is isomorphic to a standard Drinfeld* A*-module of the same rank over* S*. Moreover, all isomorphisms between standard Drinfeld* A*-modules are* S*-linear.*

Example 2.29. *Let* $A = \mathbb{F}_q[T]$ *and* S *the spectrum of the ring of dual numbers over* K*, i.e.* $S = \mathrm{Spec} K[\epsilon]/(\epsilon^2)$*. Let* ϕ *be a Drinfeld* A*-module over* S *defined by* $\phi_T = T + \tau + \epsilon\tau^2 \in K[\epsilon]/(\epsilon^2)\{\tau\}$*. Note that this Drinfeld module has rank* 1*, not* 2*. If we define another Drinfeld* A*-module over* S *by* $\psi_T = (1 - \epsilon\tau)\dot{\phi}_T\dot{(}1 - \epsilon\tau)^{-1}$*, an easy computation shows that* $\psi = T + (1 + \epsilon(T - T^q))\tau$*. So* ψ *is standard.*

Definition 2.30. *Let* S *be an* A*-scheme and* ϕ *a Drinfeld* A*-module over* S*. Let* I *be an ideal of* A*. The* I*-torsion subgroup* $\phi[I]$ *of* ϕ *is defined to be the* A*-module scheme over* S *defined by* $\phi[I] = \cap_{a \in I} \ker(\phi_a : L \to L)$*. Here,* $\cap$ *denotes the scheme-theoretic intersection.*

Proposition 2.31 (Drinfeld [7, §5, A), Remark])**.** *Let* S *be an* A*-scheme and* ϕ *a Drinfeld* A*-module of rank* d *over* S*. Let* I *be a nonzero ideal of* A*. Further we assume that the characteristic of* S *does not intersect* $V(I) \subset \mathrm{Spec}(A)$*. Then* $\phi[I]$ *is finite étale over* S*. Moreover,* $\phi[I]$ *is étale locally isomorphic to* $(I^{-1}/A)^d$ *as an* A*-module scheme. Here,* $(I^{-1}/A)^d$ *denote the constant* A*-module scheme over* S *with value* $(I^{-1}/A)^d$*.*

Next we discuss the moduli spaces of Drinfeld modules with level structures.

Definition 2.32. *Let* S *be an* A*-scheme and* (L, ϕ) *a Drinfeld* A*-module of rank* d *over* S*. Let* I *be a nonzero ideal of* A*. We assume that the characteristic of* S *does not intersect* $V(I) \subset \mathrm{Spec}(A)$*.*

(1) *A* $\Gamma(I)$*-structure on* (L, ϕ) *is an isomorphism of* A*-module schemes* $\iota : (I^{-1}/A)^d \xrightarrow{\sim} \phi[I]$*.*
(2) *A Drinfeld* A*-module with* $\Gamma(I)$*-structure over* S *is a triple* (L, ϕ, ι) *such that* (L, ϕ) *is a Drinfeld* A*-module over* S *and* ι *is a* $\Gamma(I)$*-structure on* (L, ϕ)*. Let* (L, ϕ, ι) *and* (L', ϕ', ι') *be Drinfeld* A*-modules with* $\Gamma(I)$*-structure over* S*. A homomorphism between* (L, ϕ, ι) *and* (L', ϕ', ι') *is a homomorphism of Drinfeld* A*-modules between* (L, ϕ) *and* (L', ϕ') *which is compatible with their* $\Gamma(I)$*-structures.*

Remark 2.33.

(1) *By Proposition 2.31, every Drinfeld* A*-module over* S *étale locally has a* $\Gamma(I)$*-structure.*
(2) *More generally, Drinfeld defined the notion of* $\Gamma(I)$*-structure even if the characteristic of* S *intersects* $V(I)$*. We omit this because we mainly concentrate on the case when* S *has the generic characteristic.*

Proposition 2.34 (Laumon [10, THEOREM 1.4.1, THEOREM 1.5.1])**.** *Let* $(0) \subsetneq I \subsetneq A$ *be an ideal of* A*. We define the contravariant functor* $\mathcal{M}_I^d : (\mathrm{Sch}/(\mathrm{Spec}(A) \setminus V(I))) \to (\mathrm{Set})$ *as follows:*

$\mathcal{M}_I^d(S) = \{(L, \phi, \iota) \mid (L, \phi, \iota)$ *is a Drinfeld* A*-module with* $\Gamma(I)$*-structure over* $S\}/ \cong$

Here, $\cong$ is the equivalence relation defined by the isomorphisms of Drinfeld A-modules with $\Gamma(I)$-structure over S. Then $\mathcal{M}_I^d$ is represented by an affine $(\mathrm{Spec}(A)\setminus V(I))$*-scheme M_I^d which is of finite type and smooth of relative dimension $d-1$ over* $\mathrm{Spec}(A)\setminus V(I)$.

For our proof of the $\mathfrak{p}$-primary Uniform Boundedness Conjecture of rank 2, we also consider the following level structure.

Definition 2.35. *Let S be an A-scheme and (L,ϕ) a Drinfeld A-module of rank d over S. Let I be a nonzero ideal of A. Further we assume that the characteristic of S does not intersect $V(I)\subset \mathrm{Spec}(A)$.*

1. *A $\Gamma_1(I)$-structure on (L,ϕ) is a closed immersion of A-module schemes $\iota: I^{-1}/A\hookrightarrow \phi[I]$.*
2. *A Drinfeld A-module with $\Gamma_1(I)$-structure over S is a triple (L,ϕ,ι) such that (L,ϕ) is a Drinfeld A-module over S and ι is a $\Gamma_1(I)$-structure on (L,ϕ). Let (L,ϕ,ι) and (L',ϕ',ι') be Drinfeld A-modules with $\Gamma_1(I)$-structure over S. A homomorphism between (L,ϕ,ι) and (L',ϕ',ι') is a homomorphism of Drinfeld A-modules between (L,ϕ) and (L',ϕ') which is compatible with their $\Gamma_1(I)$-structures.*

Proposition 2.36. *Let $(0)\subsetneq I\subsetneq A$ be an ideal of A. We define the contravariant functor $\mathcal{M}_I^{d,1}:(\mathrm{Sch}/(\mathrm{Spec}(A)\setminus V(I)))\to(\mathrm{Set})$ as follows:*

$$\mathcal{M}_I^{d,1}(S)=\{(L,\phi,\iota)\mid (L,\phi,\iota) \text{ is a Drinfeld } A\text{-module with } \Gamma_1(I)\text{-structure over } S\}/\cong$$

Here, $\cong$ is the equivalence relation defined by the isomorphisms of Drinfeld A-modules with $\Gamma_1(I)$-structure over S. Then $\mathcal{M}_I^{d,1}$ is represented by an affine $(\mathrm{Spec}(A)\setminus V(I))$*-scheme $M_I^{d,1}$ which is of finite type and smooth of relative dimension $d-1$ over* $\mathrm{Spec}(A)\setminus V(I)$.

Proof. First, we fix a $(\mathrm{Spec}(A)\setminus V(I))$-scheme S and an isomorphism $I^{-1}/A\cong A/I$. Take an arbitary element $[(L,\phi,\iota)]\in\mathcal{M}_I^{d,1}(S)$. Here, $[(L,\phi,\iota)]$ denotes an element of $\mathcal{M}_I^{d,1}(S)$ which is represented by (L,ϕ,ι). Then, $\iota(1)$ is an everywhere nonzero section of $L\to S$, so it defines an isomorphism $L\cong\mathcal{O}_S$.

We claim that there exists a unique representative of $[(L,\phi,\iota)]$ such that (L,ϕ) is standard, $L=\mathcal{O}_S$, and $\iota(1)=1$. The existence of such a representative is clear, so we prove the uniqueness. For this, it suffices to prove that an automorphism of a standard Drinfeld A-module with $\Gamma_1(I)$-structure over S is trivial. This follows from the fact that such an automorphism is automatically S-linear by Proposition 2.28. In the following, we may assume that (L,ϕ,ι) is standard, $L=\mathcal{O}_S$ and $\iota(1)=1$.

Next we pick a set of generators $a_1,\dots,a_n (a_i\neq 0)$ of A as an $\mathbb{F}_q$-algebra so that $I=(a_1,\dots,a_s)$ holds for some $s\le n$. Fix a set of generators $f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n)$ of the kernel of the surjective homomorphism $\mathbb{F}_q[x_1,\dots,x_n]\to A$ defined by $x_i\mapsto a_i$.

Under this set-up, a standard Drinfeld A-module of rank d with $\Gamma_1(I)$-structure $(\mathcal{O}_S,\phi,\iota)$ which satisfies $\iota(1)=1$ is equivalent to the following data:

1. For all $i=1,\dots,n$, $\phi_i=\sum_{i=0}^{-dd_\infty v_\infty(a_i)}a_{i,j}\tau^j\in\mathrm{End}(\mathbb{G}_{a,S}/S)$. Here, $a_{i,j}\in\Gamma(S,\mathcal{O}_S)$.
2. For all $i=1,\dots,n$, $a_{i,0}=a_i$ and $a_{i,-dd_\infty v_\infty(a_i)}\in\Gamma(S,\mathcal{O}_S^*)$.
3. For all $i,j=1,\dots,n$, $\phi_i\phi_j=\phi_j\phi_i$.
4. From (1) and (2), we obtain a homomorphism $\tilde{\phi}:\mathbb{F}_q[x_1,\dots,x_n]\to\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,S}/S)$ defined by $x_i\mapsto\phi_i$ for every $i=1,\dots,n$. Then $\tilde{\phi}$ factors through A, i.e. $\tilde{\phi}(f_i(x_1,\dots,x_n))=0$ for every $i=1,\dots,m$.

(5) $1 \in \Gamma(S, \mathcal{O}_S)$ has the exact order I. In other words, $\phi_i(1) = 0$ for every $i = 1, \ldots, s$ and, if we fix a complete representative $T \subset A$ of $(A/I) \setminus \{0\}$, then $\phi_t(1) \in \Gamma(S, \mathcal{O}_S^*)$ for every $t \in T$.

Now we construct a finitely generated A-algebra R_I whose spectrum represents $\mathcal{M}_I^{d,1}$. First, we define a set of indeterminates as follows:

(a) $a_{i,j}$ for every $i = 1, \ldots, n$ and $j = 0, \ldots, -dd_\infty v_\infty(a_i)$.
(b) b_i for every $i = 1, \ldots, n$.
(c) c_t for every $t \in T$.

Then, we define a finitely generated A-algebra R_I as the quotient of $A[\{a_{i,j}\}, \{b_i\}, \{c_t\}]$ divided by the following relations:

(d) $a_{i,0} = a_i$ and $a_{i,-dd_\infty v_\infty(a_i)} b_i = 1$ for every $i = 1, \ldots, n$.
(e) The relations obtained by comparing the coefficients of both sides of (3) for every pair $i, j = 1, \ldots, n$.
(f) The relations obtained by comparing the coefficients of both sides of (4) for every $i = 1, \ldots, m$.
(g) $\sum_{i=0}^{-dd_\infty v_\infty(a_i)} a_{i,j} = 0$ for every $i = 1, \ldots, s$ and $\phi_t(1)c_t = 1$ for every $t \in T$. Here, ϕ_t denotes the image of t under a homomorphism $A \to \mathrm{End}(\mathbb{G}_{a,S}/S)$ obtained from (e) and (f).

Then the above argument shows that there is a natural bijection between $\mathrm{Spec}(R_I)(S) \to \mathcal{M}_I^{d,1}(S)$. Therefore, $\mathcal{M}_I^{d,1}$ is represented by $\mathrm{Spec}(R_I)$. Next we prove that $\mathrm{Spec}(R_I)$ is smooth of relative dimension $d-1$ over $\mathrm{Spec}(A) \setminus V(I)$. It suffices to prove that a morphism $M_I^d \to M_I^{d,1}$ defined by $I^{-1}/A \hookrightarrow (I^{-1}/A)^d$ is a finite étale covering. The surjectivity is clear, and the finiteness follows from the finiteness part of Proposition 2.31. As for the étaleness, we need to prove that a natural map $\mathrm{Hom}_{M_I^{d,1}}(\mathrm{Spec}(R), M_I^d) \to \mathrm{Hom}_{M_I^{d,1}}(\mathrm{Spec}(R/J), M_I^d)$ is bijective for all pairs which consist of affine R_I-scheme R and an ideal J of R satisfying $J^2 = 0$. Using the modular interpretation of M_I^d and $M_I^{d,1}$, it suffices to prove that $\phi[I](R) \to \phi[I](R/I)$ is bijective for all standard Drinfeld A-modules over R and this follows from the étaleness part of Proposition 2.31. □

Example 2.37.

(1) *Let $A = \mathbb{F}_q[T]$ and $I = (T)$. Then a Drinfeld A-module ϕ over an A-field F is given by a degree d polynomial $\phi_T = T + a_1\tau + \cdots + a_d\tau^d$ ($a_i \in K, a_d \neq 0$). If ϕ has a nonzero I-torsion point over F, then by replacing ϕ if necessary, we may assume that $\phi_T(1) = 0$ which is equivalent to $T + \sum_{i=1}^d a_i = 0$. Therefore, we conclude that $M_I^{d,1} \times K \cong \mathbb{A}_K^{d-2} \times \mathbb{G}_{m,K}$.*
(2) *Let $(0) \subsetneq I \subsetneq A$ be an ideal of A. Then $M_I^{2,1} \times_{\mathrm{Spec}(A)\setminus V(I)} K$ is a smooth 1-dimensinal scheme over K. We denote it by $Y_1(I)$.*

We close this section by discussing the results of Breuer-Pink [3] concerning images of monodromy representations associated to Drinfeld modules. Our goal is to prove a variant of Takahashi's result (Theorem 2.43).

First, we define the isotriviality of Drinfeld modules.

Definition 2.38. *Let $K = \mathrm{Frac}(A) \subset M \subset L$ be a tower of fields, $\overline{M}$ and $\overline{L}$ algebraic closures of M and L respectively, and ϕ a Drinfeld A-module over L. We say that ϕ is M-isotrivial if there exists a Drinfeld A-module ψ over $\overline{M}$ such that $\phi \times_L \overline{L}$ and $\psi \times_{\overline{M}} \overline{L}$ are isomorphic.*

We fix some notations. Let $\hat{A} := \prod_{\mathfrak{p} \in \mathrm{Spec}(A)\setminus\{(0)\}} A_\mathfrak{p}$ be the profinite completion of A, and $\mathbb{A}_K^f := \hat{A} \otimes_A K$ the ring of finite adeles of A. Then we have natural inclusions $\hat{A} \subset \mathbb{A}_K^f$, $\mathrm{GL}_d(\hat{A}) \subset \mathrm{GL}_d(\mathbb{A}_K^f)$ and $\mathrm{SL}_d(\hat{A}) \subset \mathrm{SL}_d(\mathbb{A}_K^f)$, in each of which the former is open in the latter.

Breuer and Pink [3] studied monodromy representations associated to (universal) Drinfeld modules over subvarieties of Drinfeld modular varieties, and proved the following result.

Proposition 2.39 (Breuer-Pink [3, Theorem 3])**.** *Let F be a finitely generated extension of K and F^{sep} a separable closure of F. Let ϕ be a Drinfeld A-module of rank d over F which is not K-isotrivial and satisfies $A = \mathrm{End}_{F^{\mathrm{sep}}}(\phi)$. Then the image of the Galois representation associated to the adelic Tate module $\hat{T}(\phi) := \prod_{\mathfrak{p} \in \mathrm{Spec}(A) \setminus \{(0)\}} T_{\mathfrak{p}}(\phi)$ of ϕ*

$$G_F \to \mathrm{GL}_d(\hat{A}) \subset \mathrm{GL}_d(\mathbb{A}_K^f)$$

is open.

More generally, Breuer and Pink proved the following result:

Proposition 2.40 (Breuer-Pink [3, Theorem 8])**.** *Let F be a finitely generated extension of K and F^{sep} a separable closure of F. Let ϕ be a Drinfeld A-module of rank d over F which is not K-isotrivial. Let $E := \mathrm{End}_{F^{\mathrm{sep}}}(\phi)$ and define $d' := \frac{d}{\mathrm{rank}_A(E)}$. Then the image of the Galois representation associated to the adelic Tate module of ϕ*

$$G_F \to \mathrm{GL}_d(\hat{A}) \subset \mathrm{GL}_d(\mathbb{A}_K^f)$$

is commensurable with $\mathrm{GL}_{d'}(E \otimes_A \mathbb{A}_K^f)$. In other words, their intersection is open in the images of G_F and $\mathrm{GL}_{d'}(E \otimes_A \mathbb{A}_K^f)$.

First, using Breuer-Pink's result, we prove the following proposition:

Proposition 2.41. *Let F be a finitely generated extension of K and L a finitely generated extension of F. Let ϕ be a Drinfeld A-module of rank d over L which is not F-isotrivial and satisfies $A = \mathrm{End}_{L^{\mathrm{sep}}}(\phi)$. Then the image of the Galois representation associated to the adelic Tate module of ϕ*

$$G_{F^{\mathrm{sep}}L} \to \mathrm{GL}_d(\hat{A}) \subset \mathrm{GL}_d(\mathbb{A}_K^f)$$

is commensurable with $\mathrm{SL}_d(\mathbb{A}_K^f)$.

Proof. The proof is essentially the same as Breuer-Pink's result [3, Theorem 3].

First, by replacing L with a finite separable extension if necessary, we may assume that ϕ has a sufficiently high level structure Γ. Therefore, if we denote the moduli space over F of Drinfeld modules with Γ-level structure by M_F, ϕ defines a morphism $\mathrm{Spec}(L) \to M_F$. Let $\tilde{L}$ be the residue field of the image of this morphism. Then the F-nonisotriviality of ϕ ensures that $\mathrm{trdeg}_F \tilde{L} \geq 1$. Since $G_{F^{\mathrm{sep}}L} \to \mathrm{GL}_d(\mathbb{A}_K^f)$ factors through $G_{F^{\mathrm{sep}}L} \to G_{F^{\mathrm{sep}}\tilde{L}}$, which is an open map as $F^{\mathrm{sep}}L$ is finitely generated over $F^{\mathrm{sep}}\tilde{L}$, we may replace L by $\tilde{L}$.

Let k be the algebraic closure of F in L. The morphism $\mathrm{Spec}(L) \to M_F$ naturally extends to $\mathrm{Spec}(L) \to M_k$. Let η be the image of this morphism. Then η is the generic point of a subvariety X of M_k. By shrinking X if necessary, we may assume that X is smooth. Then since the Galois representation $G_L \to \mathrm{GL}_d(\mathbb{A}_K^f)$ factors through $G_L \twoheadrightarrow \pi_1(X, \overline{\eta})$, the homomorphism $G_{F^{\mathrm{sep}}L} \to \mathrm{GL}_d(\mathbb{A}_K^f)$ factors through $G_{F^{\mathrm{sep}}L} \twoheadrightarrow \pi_1(X \times_k F^{\mathrm{sep}}, \overline{\eta})$. Now, by a result of Breuer-Pink [3, Theorem 2.(1)], the image of $\pi_1(X \times_k F^{\mathrm{sep}}, \overline{\eta}) \to \mathrm{GL}_d(\mathbb{A}_K^f)$ is an open subgroup of $\mathrm{SL}_d(\mathbb{A}_K^f)$. □

More generally, we can prove the following:

Proposition 2.42. *Let F be a finitely generated extension of K and L a finitely generated extension of F. Let ϕ be a Drinfeld A-module of rank d over L which is not F-isotrivial. Let $E := \mathrm{End}_{L^{\mathrm{sep}}}(\phi)$ and define $d' := \frac{d}{\mathrm{rank}_A(E)}$. Then the image of the Galois representation associated to the adelic Tate module of ϕ*

$$G_{F^{\mathrm{sep}}L} \to \mathrm{GL}_d(\mathbb{A}_K^f)$$

is commensurable with $\mathrm{SL}_{d'}(E \otimes_A \mathbb{A}_K^f)$.

Proof. We may replace L by a finite separable extension and ϕ by a Drinfeld A-module which is isogenous to ϕ. Therefore, by Proposition 2.23, we may assume that $\mathrm{End}_{L^{\mathrm{sep}}}(\phi) = \mathrm{End}_L(\phi) = \tilde{E}$. Here, $\tilde{E}$ denotes the integral closure of E in its fraction field. Let $\phi_{\tilde{E}}$ be the Drinfeld $\tilde{E}$-module of rank d' over L defined by the tautological homomorphism $\tilde{E} \to \mathrm{End}_L(\phi)$.

Let $\hat{\tilde{E}} := \prod_{\mathfrak{P} \in \mathrm{Spec}(\tilde{E}) \setminus \{(0)\}} \tilde{E}_{\mathfrak{P}}$ be the profinite completion of $\tilde{E}$ and $\mathbb{A}^f_{\mathrm{Frac}(\tilde{E})} := \tilde{E} \otimes_A \mathbb{A}_K^f (= E \otimes_A \mathbb{A}_K^f)$. Then discussions at the end of section 2.3 imply that there exists an $\hat{\tilde{E}}[G_L]$-isomorphism $\hat{T}(\phi)$ and $\hat{T}(\phi_{\tilde{E}})$ (the adelic Tate module of $\phi_{\tilde{E}}$).

Then the following commutative diagram exists:

$$\begin{array}{ccc} G_L & \longrightarrow & \mathrm{GL}_{\hat{A}}(\hat{T}(\phi)) \\ & \searrow & \uparrow \\ & & \mathrm{GL}_{\hat{\tilde{E}}}(\hat{T}(\phi_{\tilde{E}})) \end{array}$$

Then by Proposition 2.41 the image of $G_{F^{\mathrm{sep}}L} \to \mathrm{GL}_{\hat{\tilde{E}}}(\hat{T}(\phi_{\tilde{E}}))$ is commensurable with $\mathrm{SL}_{d'}(\mathbb{A}^f_{\mathrm{Frac}(\tilde{E})})$. This concludes the proof. □

Now we prove the following variant of Theorem 2.13.

Theorem 2.43. *Let $K = \mathrm{Frac}(A) \subset M \subset L$ be a tower of finitely generated extensions of fields, M^{sep} and L^{sep} separable closures of M and L respectively, and ϕ a Drinfeld A-module over L. Let v be a discrete valuation on L which is trivial on M, $\overline{v}$ an extension of v to L^{sep}, and I the inertia subgroup of G_L at $\overline{v}$.*

Then, if there exists a nonzero $A_{\mathfrak{p}}[G_{M^{\mathrm{sep}}L}]$-submodule $T \subset T_{\mathfrak{p}}(\phi)$ on which I acts trivially, ϕ has good reduction at v.

Proof. By Theorem 2.13, it suffices to prove that I acts trivially on $T_{\mathfrak{p}}(\phi)$.

Let $E := \mathrm{End}_{L^{\mathrm{sep}}}(\phi)$. By Proposition 2.23, there exists a Drinfeld A-module ψ over L^{sep} such that $\tilde{E} = \mathrm{End}_{L^{\mathrm{sep}}}(\psi)$ and ψ is isogenous to $\phi \times_L L^{\mathrm{sep}}$. Here, $\tilde{E}$ denotes the integral closure of E in its fraction field. First, we prove that ϕ has potentially good reduction at v. By taking a suitable finite separable extension $\tilde{L}$ of L, we may assume that ψ descends to a Drinfeld A-module over $\tilde{L}$ (which we also denote by ψ), $\tilde{E} = \mathrm{End}_{\tilde{L}}(\psi)$ and ψ is isogenous to $\phi \times_L \tilde{L}$. Then it suffices to prove that ψ has potentially good reduction at v. If ψ is M-isotrivial, then by Theorem 2.13 we conclude that ψ has potentially good reduction. Therefore, in the following, we may assume that ψ is not M-isotrivial.

Let $\psi_{\tilde{E}}$ be the Drinfeld $\tilde{E}$-module over $\tilde{L}$ defined by the tautological homomorphism $\tilde{E} \to \mathrm{End}_{\tilde{L}}(\psi)$.Then, by the discussions at the end of section 2.3, we have the following decomposition of the rational $\mathfrak{p}$-adic Tate module

$$V_{\mathfrak{p}}(\psi) \cong \bigoplus_{\mathfrak{P} | \mathfrak{p}} V_{\mathfrak{P}}(\psi_{\tilde{E}})$$

which is compatible with E-action and $G_{\tilde{L}}$-action. Each direct summand $V_{\mathfrak{P}}(\psi_{\tilde{E}})$ of the right hand side is a $\mathrm{Frac}(E)_{\mathfrak{P}}[G_{\tilde{L}}]$-module. Since $\psi_{\tilde{E}}$ is not L-isotrivial, by

Proposition 2.42 we see that $V_{\mathfrak{P}}(\psi)$ is simple as a $\mathrm{Frac}(E)_{\mathfrak{P}}[G_{\tilde{L}}]$-module. Therefore, by taking a prime ideal $\mathfrak{P}$ above $\mathfrak{p}$ such that the composite of $T \hookrightarrow \bigoplus_{\mathfrak{P}|\mathfrak{p}} V_{\mathfrak{P}}(\psi_{\tilde{E}}) \twoheadrightarrow V_{\mathfrak{P}}(\psi_{\tilde{E}})$ is nonzero, we see that $V_{\mathfrak{P}}(\psi_{\tilde{E}}) = (V_{\mathfrak{P}}(\psi_{\tilde{E}}))^{\tilde{I}}$ where $\tilde{I} := I \cap G_{\tilde{L}}$ is the inertia subgroup of $\tilde{L}$ corresponding to $\tilde{v} := \overline{v}|_{\tilde{L}}$. By Theorem 2.13 we conclude that $\psi \times_L \tilde{L}$ has good reduction at $\tilde{v}$, hence also $\phi \times_L \tilde{L}$ does.

Pick an element $u \in \tilde{L} \setminus \{0\}$ such that all coefficients of $u\phi_a u^{-1}$ $(a \in A)$ are in $O_{\tilde{v}}$ and highest coefficients of $u\phi_a u^{-1}$ $(a \in A \setminus \{0\})$ are units of $O_{\tilde{v}}$. By considering Newton polygons, we see that the valuations of all nonzero roots of $u\phi_a u^{-1}$ $(a \in A\setminus\{0\})$ equal zero. On the other hand, if we take a nonzero element $t \in \phi[\mathfrak{p}^\infty](M^{\mathrm{sep}})$ in the image of a suitable projection from T to $\phi[\mathfrak{p}^\infty](L^{\mathrm{sep}})$, the valuation of t is an integer. By combining $\overline{v}(u^{-1}t) = 0$ and $\overline{v}(t) \in \mathbb{Z}$, we conclude that $\overline{v}(u) \in \mathbb{Z}$. Hence u can be chosen in $L \setminus \{0\}$, and this concludes that ϕ has good reduction at v. □

3. Asymptotic behaviors of reductions of analytic manifolds

Our goal in this section is to prove Proposition 3.1. The method used in this section is almost verbatim as Cadoret-Tamagawa's one used in [5], except that we have to work with Lie groups over equal-characteristic local fields.

We fix some notations. Let M be a free $A_{\mathfrak{p}}$-module of rank $m \geq 1$ and set $W := M \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}}$. For each positive integer n, we define $M_n := M/\mathfrak{p}^n M$. For $v \in M$, we denote the image of v under the natural map $M \twoheadrightarrow M_n$ by v_n.

Let $\mathrm{GL}(M) := \mathrm{Aut}_{A_{\mathfrak{p}}}(M)$ and $\mathrm{GL}(M_n) := \mathrm{Aut}_{A/\mathfrak{p}^n}(M_n)$. Let G be a closed analytic subgroup of $\mathrm{GL}(M)$. For each positive integer n, G_n denotes the image of G under the modulo $\mathfrak{p}^n$ map $\mathrm{GL}(M) \twoheadrightarrow \mathrm{GL}(M_n)$, and for $v \in M$ (resp. $v_n \in M_n$), Gv (resp. $G_n v_n$) denotes the orbit of v (resp. v_n) under the action of G (resp. G_n).

In the following, we assume that G and a fixed element $v \in M$ satisfy the following conditions:

(∗) (1) The map $G \to M, g \mapsto gv$ is a subimmersion.
(2) For every open subgroup $H \subset G$, $W = K_{\mathfrak{p}}[H]v$ holds.

The first condition of (∗) ensures that Gv has an analytic manifold structure which makes it be a closed analytic manifold of $M \cong A_{\mathfrak{p}}^m$, and the natural bijection $G/\mathrm{stab}_G(v) \xrightarrow{\sim} Gv$ becomes an isomorphism of analytic manifolds [20, Part II, section IV, §5, Theorem 4].

Proposition 3.1. *Let I be a closed subgroup of G. Then* $\lim_{n\to\infty} \dfrac{|I\backslash G_n v_n|}{|G_n v_n|} = \dfrac{1}{|I|}$ *holds.*

Proof. First, we prove the statement when I is infinite, assuming that the statement holds when I is finite.

We consider the case when I is infinite and quasi-unipotent. By replacing I with an open subgroup if necessary, we may assume that I is unipotent. Then all elements of I are torsion. We claim that I has finite subgroups of arbitary large order. Indeed, if the commutator subgroup $[I, I]$ is finite, the abelianization I^{ab} of I is infinite abelian torsion pro-p group, hence it has finite subgroups of arbitary large order. By taking inverse images in I of such subgroups, we obtain the desired result. So we may assume that $[I, I]$ is infinite and then we can replace I with $[I, I]$. Since I is unipotent, by repeating this argument we complete the proof of the claim. Hence we settle this case by using the statement for various finite subgroups of I.

Next we consider the case when I is infinite and not quasi-unipotent. Then I have a closed subgroup which is isomorphic to $\mathbb{Z}_p$. Indeed, by replacing I with an open subgroup, we may assume I is pro-p. Then the subgroup $\overline{\langle u\rangle}$ of I which is topologically generated by a non-unipotent element $u \in I$ is isomorphic to $\mathbb{Z}_p$. Hence we may assume that I is isomorphic to $\mathbb{Z}_p$. Fix an integer $N \geq 0$ and we set $J := I^{p^N}$. For an element $x_n \in G_n v_n$, I_{x_n} denotes the stabilizer of x_n under the action of I. Then $x_n \notin (G_n v_n)^J$ if and only if $J \not\subset I_{x_n}$, and the latter is equivalent to $I_{x_n} \subsetneq J = I^{p^N}$, or $I_{x_n} \subset I^{p^{N+1}}$. Hence $x_n \notin (G_n v_n)^J$ implies $|Ix_n| = [I : I_{x_n}] \geq [I : I^{p^N+1}] \geq p^{N+1}$. Therefore,

$$0 \leq \frac{|I\backslash G_n v_n|}{|G_n v_n|} \leq \frac{|I\backslash (G_n v_n \setminus (G_n v_n)^J)|}{|G_n v_n|} + \frac{|(G_n v_n)^J|}{|G_n v_n|} \leq \frac{1}{p^{N+1}} + \frac{|(G_n v_n)^J|}{|G_n v_n|}$$

holds. By Proposition 3.2 below, we conclude that

$$0 \leq \lim_{n\to\infty} \frac{|I\backslash G_n v_n|}{|G_n v_n|} \leq \frac{1}{p^{N+1}}$$

holds. Since N is arbitary, this concludes the proof.

Second, we prove the statement assuming that I is finite. Note that the proof similarly as in [5, Theorem 3.1 (1)] also works under our assumption. More specifically, by using Oesterlé's result [13, Théorème 6], we can prove the following proposition (which is an analogue of [5, Theorem 3.2 (1)]).

Proposition 3.2. *For any closed subgroup $J \subset G$, $\lim_{n\to\infty} \frac{|(G_n v_n)^J|}{|G_n v_n|} = 0$ holds unless J is trivial.*

Then the proof in [5, Theorem 3.1 (1)] works as it is. □

We give an example of G which satisfies the first condition of $(*)$. This example is used to prove Theorem 1.2.

Lemma 3.3. *Let r be a positive integer, and $B_1/A_{\mathfrak{p}}, \ldots, B_r/A_{\mathfrak{p}}$ finite extensions of discrete valuation rings with fraction fields $L_1, \ldots, L_r$, respectively. For each $i = 1, \ldots, r$, let M_i be a free B_i-module of finite rank. Let $M \subset \bigoplus_{i=1}^r M_i$ be an open $A_{\mathfrak{p}}$-submodule. Set $W := M \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}} = \bigoplus_{i=1}^r M_i \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}}$. Let $G \subset \mathrm{GL}_{K_{\mathfrak{p}}}(W)$ be a closed subgroup which satisfies the following condition:*

G is commensurable with $\prod_{i=1}^r \mathrm{SL}_{L_i}(M_i \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}}) \subset \mathrm{GL}_{K_{\mathfrak{p}}}(W)$. Moreover, M is closed under the natural G-action.

Then the map $G \to M, g \mapsto gv$ is a subimmersion for every $v \in M$.

Proof. We note that such a G has a structure of closed analytic Lie subgroups of $\mathrm{GL}_{K_{\mathfrak{p}}}(W)$ since $\prod_{i=1}^r \mathrm{SL}_{L_i}(M_i \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}})$ does.

It suffices to prove that the map $\mathrm{SL}_{L_i}(M_i \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}}) \to M_i \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}}, g \mapsto gv_i$ is a subimmersion for every $v_i \in M_i \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}}$. If $v_i = 0$ it is obvious, so we may assume that $v_i \neq 0$. Then the map is a submersion if $\dim_{L_i}(M_i) = 1$ and is an immersion otherwise. Indeed, by taking a suitable basis, we may assume $M_i \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}} \cong K_{\mathfrak{p}}^n$ for some $n \geq 1$ and $v_i = (1, 0, \ldots, 0)$. Then we can explicitly compute the induced map between tangent spaces. Therefore, $\mathrm{SL}_{L_i}(M_i \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}}) \to M_i \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}}, g \mapsto gv_i$ is a subimmersion. □

4. The proof of Theorem 1.2

4.1. **The reduction step.** Now assume L is an algebraically closed field which contains K. Let S be a curve over L, and $\eta : \mathrm{Spec}(L(S)) \to S$ the generic point of

S. We fix an algebraic closure $\overline{L(S)}$ of $L(S)$, and let $\overline{\eta} : \mathrm{Spec}(\overline{L(S)}) \to S$ be the corresponding geometric generic point.

Let ϕ be a Drinfeld A-module over S. For every integer $n \geq 0$, $\phi[\mathfrak{p}^n]$ is a finite étale A-module scheme over S (Proposition 2.31). Therefore, the action of the absolute Galois group $G_{L(S)}$ of $L(S)$ on $\phi_\eta[\mathfrak{p}^n](\overline{L(S)})$ factors through the surjection $G_{L(S)} \twoheadrightarrow \Pi := \pi_1(S, \overline{\eta})$.

For every $v \in \phi_\eta[\mathfrak{p}^n](\overline{L(S)})$, let $\Pi_v \subset \Pi$ be the stabilizer of v with regard to the above Π-action on $\phi_\eta[\mathfrak{p}^n](\overline{L(S)})$, and S_v a connected finite étale cover of S which corresponds to Π_v. We denote the genus of the compactification of S_v by g_v.

First, we prove that the following theorem implies theorem 1.2:

Theorem 4.1. *With the notations and assumptions as in Theorem 1.2, let $v \in T_{\mathfrak{p}}(\phi_\eta)^*$ and $v_n := v \bmod \mathfrak{p}^n T_{\mathfrak{p}}(\phi_\eta) \in \phi_\eta[\mathfrak{p}^n](\overline{L(S)})$ for $n \geq 0$. Then $g_{v_n} \to \infty$ $(n \to \infty)$ holds.*

Proof of Theorem 4.1 $\Rightarrow$ Theorem 1.2. Assume that Theorem 4.1 holds but Theorem 1.2 does not hold. Then there exists $c \geq 0$ such that for every $n \geq 0$, there exists $v_n \in \phi_\eta[\mathfrak{p}^\infty](\overline{L(S)})$ so that $g_{v_n} \leq c$ and $\mathfrak{p}^n v_n \neq 0$. Take such v_n and let $m > n$ be the least integer such that $\mathfrak{p}^m v_n = 0$ holds. If $\pi \in A$ is a generator of $\mathfrak{p}A_{\mathfrak{p}}$, then we see that $w_n := \pi^{m-n} v_n \in \phi_\eta[\mathfrak{p}^n]^*(\overline{L(S)})$. By definition of w_n, it holds that $\Pi_{v_n} \subset \Pi_{w_n} \subset \Pi$ and hence $g_{w_n} \leq g_{v_n} \leq c$.

Therefore, for any $n \geq 0$,

$$\phi_\eta[\mathfrak{p}^n]^*_c(\overline{L(S)}) := \{v \in \phi_\eta[\mathfrak{p}^n]^*(\overline{L(S)}) \mid g_v \leq c\} \neq \emptyset$$

holds. By observing $\Pi_{\pi v} \subset \Pi_v$, we see that $\{\phi_\eta[\mathfrak{p}^n]^*_c(\overline{L(S)})\}_{n \geq 0}$ is a projective subsystem of $\phi_\eta[\mathfrak{p}^n](\overline{L(S)})$ which consists of nonempty finite sets. Hence $\varprojlim \phi_\eta[\mathfrak{p}^n]^*_c(\overline{L(S)})$ is not empty, which contradicts Theorem 4.1. $\square$

Next, we prove that, to prove Theorem 4.1, we may assume that $W := K_{\mathfrak{p}}[\Pi]v$ satisfies the second condition of $(*)$ in section 3. In other words, we prove that, by replacing S with S_{v_N} for sufficiently large N, the following holds.

$(*)$ (2) For every open subgroup $H \subset G$, $W = K_{\mathfrak{p}}[H]v$ hold.

Let $W_\infty(v) := \bigcap_{H \subset \Pi} K_{\mathfrak{p}}[H]v$. Here, the index runs through all open subgroups H of Π. As $v \in W_\infty(v)$, $W_\infty(v)$ is a nonzero $K_{\mathfrak{p}}$-vector subspace of W. If we take an open subgroup $H_0 \subset \Pi$ such that $\dim_{K_{\mathfrak{p}}} K_{\mathfrak{p}}[H_0]v = \min_{H \subset \Pi}(\dim_{K_{\mathfrak{p}}} K_{\mathfrak{p}}[H]v)$, it holds that $W_\infty(v) = K_{\mathfrak{p}}[H_0]$. In particular, as the subgroup

$$\Pi_\infty(v) := \{\gamma \in \Pi \mid \gamma \cdot W_\infty(v) = W_\infty(v)\} \subset \Pi$$

contains H_0, it is open.

Let $\Pi_v \subset \Pi$ be the stabilizer of v. By definition, $\bigcap_{n \geq 0} \Pi_{v_n} = \Pi_v$ holds. We claim that $\Pi_{v_n} \subset \Pi_\infty(v)$ holds for sufficiently large n. First, we observe that

$$\bigcap_{n \geq 0} \Pi_{v_n} = \Pi_v \subset \Pi_\infty(v)$$

holds. Therefore,

$$\Pi = \Pi_\infty(v) \cup \bigcup_{n \geq 0} (\Pi \setminus \Pi_{v_n})$$

holds. By compactness of Π, there exists $N \geq 0$ such that

$$\Pi = \Pi_\infty(v) \cup \bigcup_{n \leq N} (\Pi \setminus \Pi_{v_n})$$

holds. In particular, $\Pi_{v_N} = \bigcap_{n \leq N} \Pi_{v_n} \subset \Pi_\infty(v)$ holds.

The statement of Theorem 4.1 only concerns the asymptotic behavior of genus g_{v_n} with $n \to \infty$, so it is harmless to replace Π with Π_{v_N}. In the following, we assume that the second condition of $(*)$ in section 3 holds for Π and $v \in W$.

4.2. **Proof of Theorem 4.1.**

Let $M := W \cap T_{\mathfrak{p}}(\phi_\eta)$, $\rho_M : \Pi \to \mathrm{GL}(M)$ the representation associated to M, and G the image of ρ_M. For every $n \geq 0$, let $M_n := M \otimes_{A_{\mathfrak{p}}} (A/\mathfrak{p}^n)$.

First, we verify that G and v satisfies the first condition of $(*)$ in section 3, i.e. $G \to M, g \mapsto gv$ is a subimmersion. For this, we recall what we proved at the end of section 2.3. Let $\tilde{E}$ be the integral closure of $E := \mathrm{End}_{L(S)^{\mathrm{sep}}}(\phi_\eta)$ in its fration field, ψ a Drinfeld A-module over $L(S)^{\mathrm{sep}}$ which is isogenous to $\phi_\eta \times_{L(S)} L(S)^{\mathrm{sep}}$ and such that $\tilde{E} = \mathrm{End}_{L(S)^{\mathrm{sep}}}(\psi)$. By taking a suitable finite separable extension $\tilde{L(S)}$ of $L(S)$, we may assume that ψ descends to a Drinfeld A-module over $\tilde{L(S)}$ (which we also denote by ψ), $\tilde{E} = \mathrm{End}_{\tilde{L(S)}^{\mathrm{sep}}}(\psi)$ and ψ is isogenous to $\phi_\eta \times_{L(S)} \tilde{L(S)}$. We define $\psi_{\tilde{E}}$ to be a Drinfeld $\tilde{E}$-module over $\tilde{L(S)}$ which is induced by $\tilde{E} = \mathrm{End}_{\tilde{L(S)}}(\psi)$. Finally, let $V_{\mathfrak{p}}(\phi_\eta \times_{L(S)} \tilde{L(S)}) = \bigoplus_{\mathfrak{P}|\mathfrak{p}} V_{\mathfrak{P}}(\psi_{\tilde{E}})$ be the direct decomposition obtained at the end of section 2.3. We claim the following assertion:

Claim $W \subset V_{\mathfrak{p}}(\phi_\eta \times_{L(S)} \tilde{L(S)})$ is a direct sum of some $V_{\mathfrak{P}}(\psi_{\tilde{E}})$'s.

Indeed, since we assume that ϕ_η is not L-isotrivial, $\psi_{\tilde{E}}$ is also not L-isotrivial. In particular, the rank of $\psi_{\tilde{E}}$ is greater than 1 (cf. Corollary 2.24). By applying proposition 2.42 to $\psi_{\tilde{E}}$, we see that each $V_{\mathfrak{P}}(\psi_{\tilde{E}})$ is simple as a $K_{\mathfrak{p}}[G_{\tilde{L(S)}}]$-module and that for any $\mathfrak{P} \neq \mathfrak{P}'$, $V_{\mathfrak{P}}(\psi_{\tilde{E}})$ is not isomorphic to $V_{\mathfrak{P}'}(\psi_{\tilde{E}})$ as a $K_{\mathfrak{p}}[G_{\tilde{L(S)}}]$-module. Thus the claim follows.

From this claim, Proposition 2.42 and Lemma 3.3, we conclude that M and G satisfy the first condition of $(*)$.

Now, let us add some more notations. Let g be the genus of the smooth compactification of S. For each $n \geq 0$, let G_n be the image of the composite of $\Pi \xrightarrow{\rho} \mathrm{GL}(M) \to \mathrm{GL}(M_n)$. Moreover, for each $n \geq 0$ and $v \in \phi_\eta[\mathfrak{p}^n](\overline{L(S)})$, S_v denotes a connected finite étale cover of S which corresponds to the stabilizer $\Pi_v \subset \Pi$ of v. We define g_v to be the genus of the smooth compactification of S_v. Set $\lambda_{v_n} := \frac{2g_{v_n}-2}{\deg(S_{v_n} \to S)} (= \frac{2g_{v_n}-2}{|G_n v_n|})$. From the Riemann-Hurwitz formula we see that $\lambda_{v_{n+1}} \geq \lambda_{v_n}$ holds for every $n \geq 0$. From Proposition 2.42 and the non-isotriviality of ϕ_η, we see that $|G_n v_n| \to \infty$ $(n \to \infty)$. Thus, to prove $g_{v_n} \to \infty$, it suffices to show that $\lambda_{v_n} > 0$ for sufficiently large n.

Let $P_1, \ldots, P_r$ be the cusps of S. For each $i = 1, \ldots, r$ and each $n \geq 0$, write $I_{i,n} \subset G_n$ for the image of the inertia subgroup $I_i \subset G$ at P_i (which is determined up to conjugacy). Let $d_n(P)$ (resp. $e_n(P)$) be the exponent of different (resp. ramification index) at a given cusp P of S_{v_n} in $S_{v_n} \to S$.

Using the Riemann-Hurwitz formula, we can rewrite λ_{v_n} as follows:

$$\lambda_{v_n} = 2g - 2 + \frac{1}{|G_n v_n|} \sum_{1 \leq i \leq r} \sum_{P \in S_{v_n}, P | P_i} d_n(P)$$

Then by using $d_n(P) \geq e_n(P) - 1$, $\sum_{P \in S_{v_n}, P|P_i} e_n(P) = |G_n v_n|$ and $|\{P \in S_{v_n}^{\mathrm{cpt}} \mid P|P_i\}| = |I_{i,n} \backslash G_n v_n|$ $(i = 1, \dots, r)$ we obtain:

$$\begin{aligned} \lambda_{v_n} &\geq -2 + \frac{1}{|G_n v_n|} \sum_{1 \leq i \leq r} \sum_{P \in S_{v_n}, P|P_i} (e_n(P) - 1) \\ &= -2 + \sum_{1 \leq i \leq r} \left(1 - \frac{|I_{i,n} \backslash G_n v_n|}{|G_n v_n|}\right) \end{aligned}$$

By Proposition 3.1, we see that $1 - \frac{|I_{i,n} \backslash G_n v_n|}{|G_n v_n|}$ tends to $1 - \frac{1}{|I_i|}$ if n goes to infinity. We claim that at least one of $|I_i|$ is infinite. Indeed, assume that all of $|I_i|$ are finite. Then we can replace S with a finite connected étale cover T so that $\phi \times_S T$ has an appropriate level structure and has good reduction at every point of T^{cpt}. Then the corresponding classifying map $T \to M_L$ where M_L denotes the moduli space over L of Drinfeld A-modules with an appropriate level structure, is extended to a morphism $T^{\mathrm{cpt}} \to M_L$ by Theorem 2.13 and by the separatedness of M_L. However, since M_L is affine by Proposition 2.34, the map $T^{\mathrm{cpt}} \to M_L$ is constant. In other words, we see that ϕ_η is L-isotrivial, which contradicts our assumption.

Moreover, we claim that, if we replace S with S_{v_n} for $n \gg 0$, we may assume that $|I_i|$ is infinite for at least three i's. To prove this claim, note that the number of cusps of S_{v_n} above P_i is equal to $|I_{i,n} \backslash G_n v_n|$. Hence it suffices to prove that $|I_i \backslash Gv| = \infty$. Indeed, I_i acts quasi-unipotently on every $V_{\mathfrak{P}}(\psi_{\tilde{E}})$ by Corollary 2.15 whereas the image of Gv in some $V_{\mathfrak{P}}(\psi_{\tilde{E}})$ under the natural projection is open by Proposition 2.42 and the non-isotriviality of ϕ_η. Take such a $\mathfrak{P}$ and choose a basis $(w_1, \dots, w_d)$ of $V_{\mathfrak{P}}(\psi_{\tilde{E}})$ so that an open subgroup of I_i becomes upper triangular with respect to this basis. Then the image of $I_i gv$ under the projection to the w_1-component $V_{\mathfrak{P}}(\psi_{\tilde{E}}) \to \mathrm{Frac}(\tilde{E}_{\mathfrak{P}})$ is finite for every $g \in G$, whereas the image of Gv is open in $\mathrm{Frac}(\tilde{E}_{\mathfrak{P}})$. So it is impossible to cover Gv by using only a finite number of $I_i gv$'s $(g \in G)$. Hence the claim follows.

Therefore, we conclude that $\lambda_{v_n} > 0$ for $n \gg 0$, as desired.

5. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. If ϕ_η is not L-isotrivial, then, based on the results obtained so far, almost the same proof of Cadoret-Tamagawa works. If ϕ_η is L-isotrivial, we prove a stronger result (Theorem 5.4).

First, we reduce to the case where we can apply Theorem 1.2. Note that, by an analogue of the Mordell-Weil theorem for Drinfeld modules proved by Poonen [17, THEOREM 1], we are free to replace S with a dense open subset. By replacing S with S^{red}, we may assume that S is reduced. By shrinking S if necessary, we may assume that S is separated and regular. By decomposing S into connected components, we may assume S is connected. If $S(L) = \emptyset$ there is nothing to prove, so we may assume that $S(L) \neq \emptyset$. Then since S is a 1-dimensional regular scheme, S is smooth at every point of $S(L)$. Hence the smooth locus of S is non-empty, so by shrinking S we may assume that S is smooth and geometrically connected since $S(L) \neq \emptyset$. To sum it up, we reduce to the case when S is separated, geometrically integral and smooth.

5.1. **The non-isotrivial case.**

Assume that ϕ_η is not L-isotrivial. Let $s \in S(L)$. First, we observe that $\phi_\eta[\mathfrak{p}^n](\overline{L(S)})$ is defined over the integral closure of S in the maximal extension of $L(S)$ unramified over S. Therefore there exists a specialization isomorphism

$\mathrm{sp}_s : \phi_\eta[\mathfrak{p}^n](\overline{L(S)}) \xrightarrow{\sim} \phi_s[\mathfrak{p}^n](\overline{L})$. Here, the decomposition group at $s \in S(L)$ acts on the left-hand side, and hence on the right-hand side through sp_s. This action is compatible with the natural surjection from the decomposition group to G_L.

Lemma 5.1 (cf. Cadoret-Tamagawa [4, Claim 4.2(ii)]). *For each $v_n \in \phi_\eta[\mathfrak{p}^n](\overline{L(S)})$, $\mathrm{sp}_s(v_n) \in \phi_s[\mathfrak{p}^n](L)$ if and only if s lifts to an L-rational point of S_{v_n}.*

Proof. $s \in S(L)$ induces a homomorphism (which is defined up to conjugacy) between fundamental groups $s : G_L \to \pi_1(S)$. Then, s lifts to an L-rational point of S_{v_n} if and only if $s(G_L) \subset \pi_1(S_{v_n})$. The latter condition holds if and only if $s(\sigma)v_n = v_n$ for every $\sigma \in G_L$. By using the Galois-equivariance of sp_s, the last condition is equivalent to $\sigma \mathrm{sp}_s(v_n) = \mathrm{sp}_s(v_n)$ for every $\sigma \in G_L$. Since $\phi_s[\mathfrak{p}^n](\overline{L}) = \phi_s[\mathfrak{p}^n](L^{\mathrm{sep}})$ this is equivalent to $\mathrm{sp}_s(v_n) \in \phi_s[\mathfrak{p}^n](L)$. □

Using this lemma, we can rephase Theorem 1.1 as follows.
For each integer $n \geq 0$, We define

$$S_n := \coprod_{v_n \in \phi_\eta[\mathfrak{p}^n]^*(\overline{L(S)})} S_{v_n}.$$

Note that $\{S_{v_n}\}_n$ constitutes a projective system whose transition maps are multiplication by a fixed uniformizer of $\mathfrak{p}A_\mathfrak{p}$ in A.

Lemma 5.2 (cf. Cadoret-Tamagawa [4, Claim 4.3]). *we have:*

(1) $\varprojlim S_n(L) = \emptyset$.
(2) *Theorem 1.1 is equivalent to saying that $S_n(L) = \emptyset$ for $n \gg 0$.*
(3) *Suppose that $S_n(L) \neq \emptyset$ for any $n \geq 0$. Then there exists $(v_n) \in T_\mathfrak{p}(\phi_\eta)^*$ such that $S_{v_n}(L) \neq \emptyset$ for any $n \geq 0$.*

Proof. (1) Assume $\varprojlim S_n(L) \neq \emptyset$ and take $(s_n) \in \varprojlim S_n(L)$. Then by definition there exists $(v_n) \in \varprojlim \phi_\eta[\mathfrak{p}^n]^*(\overline{L(S)})$ such that $(s_n) \in \varprojlim S_{v_n}(L)$. If we define $s := s_0 \in S_0 = S$, then by Lemma 5.1 it holds that $\mathrm{sp}_s(v_n) \in \phi_s[\mathfrak{p}^n]^*(L)$ for any $n \geq 0$. This contradicts the finiteness of $(\phi_s)_{\mathrm{tors}}(L)$ (Poonen [17, THEOREM 1]).

(2) By using Lemma 5.1 again, we see that the assertion of Theorem 1.1 is equivalent to saying that there exists an integer N such that, for every $v_n \in \phi_\eta[\mathfrak{p}^n]^*(\overline{L(S)})$, $S_{v_n}(L) \neq \emptyset$ implies $n \leq N$.

(3) This follows from the fact that $\{v_n \in \phi_\eta[\mathfrak{p}^n]^*(\overline{L(S)}) \mid S_{v_n}(L) \neq \emptyset\}$ forms a projective system of finite sets. □

Lemma 5.3 (cf. Cadoret-Tamagawa [4, Proposition 3.7]). *Let $\mathbf{F}$ be the prime field of arbitary characteristic, and k a field finitely generated over $\mathbf{F}$. Let C be a proper curve over k, and assume that the normalization of $C \times_k \overline{k}$ is of genus ≥ 2. Let S be a nonempry open subscheme of C (which is a curve over k). When $S(k)$ is infinite, put the extra assumption that S is $\mathbf{F}$-isotrivial. (Note that C is automatically $\mathbf{F}$-isotrivial by* [19, Théorème]*.) Then there exists an $\mathbf{F}$-morphism $f : \mathcal{S} \to T$ between separated, normal, integral schemes of finite type over $\mathbf{F}$, such that the following hold:* (a) *the function field $\mathbf{F}(T)$ of T is $\mathbf{F}$-isomorphic to k;* (b) *under the identification $\mathbf{F}(T) = k$, S is isomorphic to the generic fiber $\mathcal{S}_k$ of $\mathcal{S}$; and* (c) *under the identification $S = \mathcal{S}_k$, we have $S(k) = \mathcal{S}(T)$, i.e., each element of $S(k) = \mathcal{S}_k(k)$ uniquely extends to an element of $\mathcal{S}(T)$.*

Now we are ready to prove Theorem 1.1 assuming that ϕ_η is not L-isotrivial.

Proof of Theorem 1.1 assuming that ϕ_η is not L-isotrivial. Assume that the assertion of Theorem 1.1 does not hold. Then by Lemma 5.2 (2) and (3), there exists $v = (v_n) \in \varprojlim \phi_\eta[\mathfrak{p}^n]^*(\overline{L(S)})$ such that $S_{v_n}(L) \neq \emptyset$ holds for any $n \geq 0$. Since S_{v_n} is connected by definition, $S_{v_n}(L) \neq \emptyset$ implies S_{v_n} is geometrically connected.

First, let C be the normal compactification of S, and $\tilde{\phi}$ the minimal model of ϕ over C (Remark 2.21). If we write $\tilde{S} \coloneqq \{s \in C \mid \phi_\eta \text{ has good reduction at } s\}$, $\tilde{S}$ is a non-empty open subset of C because $S \subset \tilde{S}$. Then $\tilde{\phi} \mid_{\tilde{S}}$ defines a Drinfeld A-module over $\tilde{S}$. Moreover, $S_{v_n} \subset \tilde{S}_{v_n}$ holds for any $n \geq 0$. We may replace S with $\tilde{S}$ and ϕ with $\tilde{\phi}$. If $S_{v_n}(L) \neq \emptyset$ is finite for any $n \geq 0$, $\varprojlim S_{v_n}(L)$ is non-empty which contradicts Lemma 5.2 (1). Therefore, in the following, we may assume that $S_{v_n}(L) \neq \emptyset$ is infinite for any $n \geq 0$.

We claim that S is $\mathbb{F}_p$-isotrivial. Let C_n be the normal compactification of S_{v_n}. Then $\{C_n\}_n$ naturally forms a projective system. By Samuel [19, Théorème] each C_n is $\mathbb{F}_p$-isotrivial, so there exists a curve $C_{n,\overline{\mathbb{F}}_p}$ over $\overline{\mathbb{F}}_p$ and an isomorphism $C_n \times_L \overline{L} \cong C_{n,\overline{\mathbb{F}}_p} \times_{\overline{\mathbb{F}}_p} \overline{L}$. Under this identification, by [23, Lemma 1.32], the $\overline{L}$-morphism $C_{n+1} \times_L \overline{L} \to C_n \times_L \overline{L}$ (uniquely) descends to an $\overline{\mathbb{F}}_p$-morphism $C_{n+1,\overline{\mathbb{F}}_p} \to C_{n,\overline{\mathbb{F}}_p}$. Hence $\{C_{n,\overline{\mathbb{F}}_p}\}_n$ also forms a projective system.

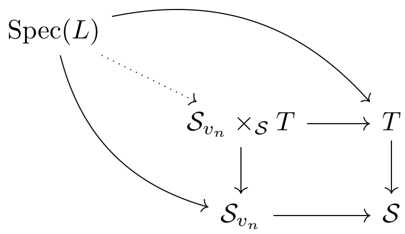
Now we define $S_{n,\overline{\mathbb{F}}_p}$ to be the image of $S_{v_n} \times_L \overline{L}$ in $C_{n,\overline{\mathbb{F}}_p}$ for each $n \geq 0$. In particular, $S_{\overline{\mathbb{F}}_p}$ denotes the image of $S \times_L \overline{L}$ in $C_{\overline{\mathbb{F}}_p}$. Then the same proof as in [4, Claim 4.5] shows that each $S_{n,\overline{\mathbb{F}}_p}$ is open in $C_{n,\overline{\mathbb{F}}_p}$ and $\{S_{n,\overline{\mathbb{F}}_p}\}_n$ forms a projective system of open subschemes of $\{C_{n,\overline{\mathbb{F}}_p}\}_n$ whose transition maps are finite étale. This observation can be rephrased as follows: Write ω_0 for the natural morphism $S \times_L \overline{L} \to S_{\overline{\mathbb{F}}_\mathfrak{p}}$ and let $\pi_1(\omega_0) : \pi_1(S \times_L \overline{L}) \twoheadrightarrow \pi_1(S_{\overline{\mathbb{F}}_p})$ be the homomorphism induced by ω_0. Then for each $n \geq 0$, there exists an open subgroup $H_n \subset \pi_1(S_{\overline{\mathbb{F}}_p})$ such that $\pi_1(\omega_0)^{-1}(H_n)$ equals to the stabilizer $\pi_1(S \times_L \overline{L})_{v_n}$ of v_n in $\pi_1(S \times_L \overline{L})$. Now for each $g \in \pi_1(S \times_L \overline{L})$, observe the following equality:

$$\pi_1(S \times_L \overline{L})_{gv_n} = g\pi_1(S \times_L \overline{L})_{v_n} g^{-1} = g\pi_1(\omega_0)^{-1}(H_n)g^{-1} = \pi_1(\omega_0)^{-1}(\overline{g}H_n\overline{g}^{-1})$$

Here, we define $\overline{g} \coloneqq \pi_1(\omega_0)(g)$. From this, we see that the $\pi_1(S \times_L \overline{L})$-action on the set $\pi_1(S \times_L \overline{L})v_n$ factors through $\pi_1(\omega_0) : \pi_1(S \times_L \overline{L}) \twoheadrightarrow \pi_1(S_{\overline{\mathbb{F}}_p})$. Therefore the $\pi_1(S \times_L \overline{L})$-action on the $A_\mathfrak{p}$-module $< \pi_1(S \times_L \overline{L})v_n >$ also factors through $\pi_1(\omega_0) : \pi_1(S \times_L \overline{L}) \twoheadrightarrow \pi_1(S_{\overline{\mathbb{F}}_p})$. In particular, it follows that, for every $x \in S_{\overline{\mathbb{F}}_p} \times_{\overline{\mathbb{F}}_p} \overline{L}$, the corresponding inertia subgroup I_x acts trivially on a non-zero $A_\mathfrak{p}[\pi_1(S \times_L \overline{L})]$-submodule of the $\mathfrak{p}$-adic Tate module of ϕ_η. By applying Proposition 2.43, we conclude that ϕ_η has good reduction at every point x of $S_{\overline{\mathbb{F}}_p} \times_{\overline{\mathbb{F}}_p} \overline{L}$. However, since S consists of the points in C at which ϕ_η has good reduction, it follows that $S \times_L \overline{L} \cong S_{\overline{\mathbb{F}}_p} \times_{\overline{\mathbb{F}}_p} \overline{L}$. Hence the claim is proved.

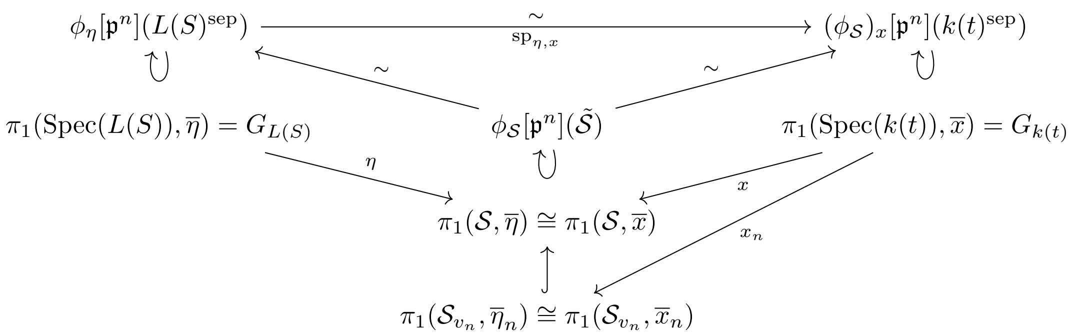
Now we apply Lemma 5.3 to $S \to \mathrm{Spec}(L)$ and obtain an $\mathbb{F}_p$-morphism $f : \mathcal{S} \to T$. Note that the assertions (a), (b) and (c) in Lemma 5.3 are still valid if we replace T with a non-empty open subscheme and $\mathcal{S}$ with its inverse image. Since $\mathcal{S}_L$ is a projective limit of $f^{-1}(U)$ where U runs through non-empty open subschemes of T, it follows that there exists a non-empty open subscheme U of T and an $\mathbb{F}_p$-morphism $U \to \mathrm{Spec}(A)$ which is compatible with $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ such that ϕ extends to a Drinfeld A-module over $\mathcal{U} \coloneqq f^{-1}(U)$. By replacing $\mathcal{S}$ and T with $\mathcal{U}$ and U respectively, we may assume that ϕ extends to a Drinfeld A-module $\phi_\mathcal{S}$ over $\mathcal{S}$. Moreover, note that we may assume that the image of $T \to \mathrm{Spec}(A)$ does not contain $\mathfrak{p}$. It follows that $\phi_\mathcal{S}[\mathfrak{p}^n] (n \geq 0)$ is finite étale over $\mathcal{S}$. In particular, the $\pi_1(S)$-action on the $\mathfrak{p}$-adic Tate module of ϕ_η factors through $\pi_1(S) \to \pi_1(\mathcal{S})$. Let $\pi_1(\mathcal{S})_{v_n}$ be the stabilizer of v_n and $\mathcal{S}_{v_n}$ the corresponding finite étale connected cover of $\mathcal{S}$.

We claim that the natural map $\mathcal{S}_{v_n}(T) \to S_{v_n}(L)$ is bijective for any $n \geq 0$. The injectivity is clear. For a given element of $S_{v_n}(L)$, it follows from the definition of $\mathcal{S}$ that the composite of $\mathrm{Spec}(L) \to S_{v_n} \to S$ extends to a morphism $T \to \mathcal{S}$. Hence there exists a morphism $\mathrm{Spec}(L) \to \mathcal{S}_{v_n} \times_{\mathcal{S}} T$ which makes the following diagram commutative.



If we denote the scheme-theoretic image of $\mathrm{Spec}(L) \to \mathcal{S}_{v_n} \times_{\mathcal{S}} T$ by Z, then the induced map $Z \to T$ is a finite birational morphism. Since T is normal, we conclude that $Z \to T$ is an isomorphism by Zariski's main theorem. The claim is now proved.

Fix a closed point $t \in T$. Since $\mathcal{S}_{v_n}(T) \neq \emptyset$, it follows that $\mathcal{S}_{v_n}(k(t))$ is non-empty and finite. Therefore $\varprojlim \mathcal{S}_{v_n}(k(t)) \neq \emptyset$. Take $(x_n) \in \varprojlim \mathcal{S}_{v_n}(k(t))$ and set $x := x_0 \in \mathcal{S}(k(t))$. We also denote the geometric generic point of $\mathcal{S}$ induced by $\overline{\eta}$ by $\overline{\eta}$. Denote a universal étale covering of $(\mathcal{S}, \overline{\eta})$ by $\tilde{\mathcal{S}}$, and let $\tilde{\eta}$ be a lift of $\overline{\eta}$ to $\tilde{\mathcal{S}}$. Take an algebraic closure $\overline{k(x)}$ of $k(x)$, which induces a geometric point $\overline{x} : \mathrm{Spec}(\overline{k(x)}) \to \mathcal{S}$. Let $\tilde{x}$ be a lift of $\overline{x}$ to $\tilde{\mathcal{S}}$. Note that $\tilde{\eta}$ and $\tilde{x}$ induce geometric points of $\mathcal{S}_{v_n}$ which we denote by $\overline{\eta}_n$ and $\overline{x}_n$, respectively. Finally, take an étale path between $\tilde{\eta}$ and $\tilde{x}$. Under this situation, we have the following commutative diagram:



Here, $\phi_{\mathcal{S}}[\mathfrak{p}^n](\tilde{\mathcal{S}}) \xrightarrow{\sim} \phi_\eta[\mathfrak{p}^n](L(S)^{\mathrm{sep}})$ and $\phi_{\mathcal{S}}[\mathfrak{p}^n](\tilde{\mathcal{S}}) \xrightarrow{\sim} (\phi_{\mathcal{S}})_x[\mathfrak{p}^n](k(t)^{\mathrm{sep}})$ denotes isomorphisms induced by $\tilde{\eta}$ and $\tilde{x}$, respectively. Now the above commutative diagram shows that $\mathrm{sp}_{\eta,x}(v_n) \in (\phi_{\mathcal{S}})_x[\mathfrak{p}^n]^*(k(t))$. In particular, this shows that $(\phi_{\mathcal{S}})_x$ (which is a Drinfeld A-module over a finite field) has infinitely many torsion points over $k(t)$. This is absurd. Hence we conclude the proof of Theorem 1.1. □

5.2. **The isotrivial case.**

In this section, we **do not** assume that S is a curve. Instead, we assume that S is a normal integral scheme which is of finite type over L. We denote the generic point of S by η.

Our goal is to prove the following theorem which asserts the existence of a uniform bound of the order of torsion submodules for a given Drinfeld module over S which is L-isorivial over the generic point of S.

Theorem 5.4. *Let ϕ be a Drinfeld A-module over S. Assume that ϕ_η is L-isotrivial. Then for every $n \geq 1$ there exists $C > 0$ which satisfies the following condition: For every finite extension L' of L with $[L' : L] \leq n$ and for every $s \in S(L')$, $|(\phi_s)_{\mathrm{tors}}(L')| < C$ holds.*

First, we begin with the following lemma.

Lemma 5.5. *Let ψ be a Drinfeld A-module over L. Then for every $n \geq 1$ there exists $C > 0$ which satisfies the following condition: For every finite extension L' of L with $[L' : L] \leq n$, $|\psi_{\mathrm{tors}}(L')| < C$ holds.*

Remark 5.6. *In fact, a result stronger and more explicit than Lemma 5.5 is known. Breuer* [2, Theorem 1.1] *proved that, the notations as in Lemma 5.5, there exists $C > 0$ which depends only on ψ such that $|\psi_{\mathrm{tors}}(L')| \leq C([L' : L] \log\log([L' : L]))^{\gamma}$ holds for every finite extension L' of L. Here, we define $\gamma := \frac{\mathrm{rank}_A \mathrm{End}_{\bar{L}}(\psi \times_L \bar{L})}{d}$ where d is the rank of ψ. His proof uses a result proved by Pink and Rütsche* [16] *concerning adelic representations associated to Tate modules of Drinfeld modules, which we do not use in the proof of Lemma 5.5.*

Proof of Lemma 5.5. If $\mathrm{trdeg}_K(L) = 0$, then the results are proved in [17, THEOREM 1]. So we may assume that $\mathrm{trdeg}_K(L) > 0$.

Take a normal, integral and affine model T of L over K. By shrinking T if necessary, we may assume that ψ is extended to a Drinfeld A-module Ψ over T. Let T' be the normalization of T in L', fix a closed point $t \in T$ and $t' \in T'$ which lies above t, and define $\psi' := \psi \times_{\mathrm{Spec}(L)} \mathrm{Spec}(L')$ and $\Psi' := \Psi \times_T T'$.

We claim that the natural map $\Psi'[\mathfrak{a}](T') \to \psi'[\mathfrak{a}](L')$ is bijective. The injectivity is clear because $\mathrm{Spec}(L') \to T'$ is the generic point of T'. For the surjectivity, for a given element of $\psi'[\mathfrak{a}](L')$, we denote its scheme-theoretic image in $\Psi'[\mathfrak{a}]$ by Z. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Spec}(L') & \longrightarrow & Z & \longrightarrow & \Psi'[\mathfrak{a}] \\ & \searrow & & \searrow & \downarrow \\ & & & & T' \end{array}$$

Here, the morphism $Z \to T'$ is finite and birational by the definition of Z. Since T' is normal, by using the Zariski main theorem we conclude $Z \to T'$ is an isomorphism. Hence we obtain the desired morphism $T' \to Z \to \Psi'[\mathfrak{a}]$.

From this claim, we conclude that there exists a natural isomorphism $\psi[\mathfrak{a}](L') \cong \psi'[\mathfrak{a}](L') \cong \Psi'[\mathfrak{a}](T')$. Moreover, since $\Psi'[\mathfrak{a}]$ is finite étale over T', the reduction map $\Psi'[\mathfrak{a}](T') \to \Psi'_{t'}[\mathfrak{a}](k(t'))$ is injective. So we obtain an injective homomorphism $\psi'[\mathfrak{a}](L') \to \Psi'_{t'}[\mathfrak{a}](k(t'))$. Since $[k(t') : k(t)] \leq n$, we reduced to the case when $\mathrm{trdeg}_K(L) = 0$. □

Proof of Theorem 5.4. Since ϕ is L-isotrivial, there exists a finite extension M of L, a finite extension N of $L(S)$ which contains M, and a Drinfeld A-module ϕ_0 over M such that $\phi_\eta \times_{L(S)} N \cong \phi_0 \times_M N$. If we denote the normalization of S in N by S_N, then by [15, Proposition 3.7] the isomorphism $\phi_\eta \times_{L(S)} N \cong \phi_0 \times_M N$ extends to an isomorphism between $\phi \times_S S_N$ and $\phi_0 \times_M S_N$.

Fix a finte extension L' of L with $[L' : L] \leq n$ and let $s \in S(L')$. Take a closed point $s_N \in S_N$ above s, and let M' be the residue field of S_N at s_N. Then M' is a finite extension of L' and $[M' : M] \leq [M' : L] = [M' : L'][L' : L] \leq [N : L(S)][L' : L] \leq n[N : L(S)]$ holds. Since $\phi \times_S S_N$ is isomorphic to $\phi_0 \times_M S_N$, we obtain an injection $(\phi_s)_{\mathrm{tors}}(L') \hookrightarrow ((\phi \times_S M')_{s_N})_{\mathrm{tors}}(M') \cong (\phi_0)_{\mathrm{tors}}(M')$. As $[M' : M] \leq n[N : L(S)]$, we conclude the proof by applying Lemma 5.5 to ϕ_0. □

6. Applications of Theorem 1.1

In this section, we briefly explain some applications of Theorem 1.1.

Corollary 6.1 (=Theorem 1.3). *Let L be a finitely generated extension of K. Then there exists an integer $N := N(L, \mathfrak{p}) \geq 0$ such that $\phi[\mathfrak{p}^\infty](L) \subset \phi[\mathfrak{p}^N](L)$ holds for every Drinfeld A-module ϕ of rank 2 over L.*

Proof. Apply Theorem 1.1 to $Y_1(\mathfrak{p})$ (cf. Example 2.37 (2)). □

Moreover, we prove the following result concerning the set of rational points of the moduli spaces of Drinfeld modules with Γ_1-structure.

Corollary 6.2. *Let L be a finitely generated extension of K. For each n, let $M^{3,1}_{K,\mathfrak{p}^n}$ be the moduli space over K of Drinfeld modules of rank 3 with $\Gamma_1(\mathfrak{p}^n)$-structure (cf. Proposition 2.36). Then one of the following occurs.*

(1) *The image of $M^{3,1}_{K,\mathfrak{p}^n}(L)$ is dense in $M^{3,1}_{K,\mathfrak{p}^n}$ for each $n > 0$.*
(2) *$M^{3,1}_{K,\mathfrak{p}^n}(L) = \emptyset$ for each $n \gg 0$.*

Proof. It is known that the moduli space of Drinfeld modules with $\Gamma(\mathfrak{p}^n)$-structure over K is irreducible (see Pink [15, 1]). So we conclude that $M^{3,1}_{K,\mathfrak{p}^n}$ is also irreducible. Assume that (1) does not hold. In other words, assume that there exists $n > 0$ such that the Zariski closure S of the image of $M^{3,1}_{K,\mathfrak{p}^n}(L)$ in $M^{3,1}_{K,\mathfrak{p}^n}$ is a proper closed subset. Then we obtain the desired results by applying Theorem 1.1 to S. □

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