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Stability of Algebraic Solitons for Nonlinear Schrödinger Equations of Derivative Type: Variational Approach

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STABILITY OF ALGEBRAIC SOLITONS FOR NONLINEAR SCHRÖDINGER EQUATIONS OF DERIVATIVE TYPE: VARIATIONAL APPROACH

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Abstract. We consider the following nonlinear Schrödinger equation of derivative type:

$$i\partial_t u + \partial_x^2 u + i|u|^2\partial_x u + b|u|^4 u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R},\ b \in \mathbb{R}. \tag{1}$$

If $b = 0$, this equation is a gauge equivalent form of well-known derivative nonlinear Schrödinger (DNLS) equation. The equation (1) can be considered as a generalized equation of (DNLS) while preserving both L^2-criticality and Hamiltonian structure. If $b > -3/16$, the equation (1) has algebraically decaying solitons, which we call *algebraic solitons*, as well as exponentially decaying solitons. In this paper we study stability properties of solitons for (1) by variational approach and prove that if $b < 0$, all solitons including algebraic solitons are stable in the energy space. The stability of algebraic solitons gives the counterpart of the previous instability result for the case $b > 0$.

Contents

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1. Introduction

1.1. **Setting of the problem.** In this paper we consider the following nonlinear Schrödinger equation of derivative type:

$$i\partial_t u + \partial_x^2 u + i|u|^2\partial_x u + b|u|^4 u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R},\ b\in\mathbb{R}. \tag{1.1}$$

This equation has the following conserved quantities:

(Energy) $$E(u) = \frac{1}{2}\|\partial_x u\|_{L^2}^2 - \frac{1}{4}\left(i|u|^2\partial_x u, u\right) - \frac{b}{6}\|u\|_{L^6}^6,$$

(Mass) $$M(u) = \|u\|_{L^2}^2,$$

(Momentum) $$P(u) = (i\partial_x u, u),$$

where $(\cdot,\cdot)$ is an inner product defined by

$$(v,w) = \mathrm{Re}\int_{\mathbb{R}} v(x)\overline{w(x)}dx \quad \text{for } v,w\in L^2(\mathbb{R}).$$

We note that (1.1) can be rewritten as

$$i\partial_t u = E'(u). \tag{1.2}$$

The equation (1.1) is L^2-critical in the sense that the equation and L^2-norm are invariant under the scaling transformation

$$u_\lambda(t,x) = \lambda^{\frac{1}{2}} u(\lambda^2 t, \lambda x), \quad \lambda > 0. \tag{1.3}$$

It is well known (see [16, 30]) that (1.1) is locally well-posed in the energy space $H^1(\mathbb{R})$ and that the energy, mass and momentum of the $H^1(\mathbb{R})$-solution are conserved by the flow.

When $b = 0$, the equation is a gauge equivalent form[1] of well-known derivative nonlinear Schrödinger (DNLS) equation:

(DNLS) $$i\partial_t\psi + \partial_x^2\psi + i\partial_x(|\psi|^2\psi) = 0, \quad (t,x)\in\mathbb{R}\times\mathbb{R},$$

which originally appeared in plasma physics as a model for the propagation of Alfvén waves in magnetized plasma (see [25, 26]). Kaup and Newell [18] showed that (DNLS) is completely integrable.

There is a large literature on the Cauchy problem for (DNLS). Here we briefly review the results which are closely related to this paper (see [14, 17] and references therein for further information). In [33] it was proved that if the initial data $u_0 \in H^1(\mathbb{R})$ satisfies $M(u_0) < 4\pi$, then the corresponding $H^1(\mathbb{R})$-solution is global and bounded. We note that the value 4π corresponds to the mass of algebraic solitons which correspond to the threshold case in the existence of solitons. Later, Fukaya, the author and Inui [8] recovered Wu's result by variational approach and moreover established the global results for $M(u_0) = 4\pi$ and $P(u_0) < 0$ and for the oscillating data containing arbitrarily large mass. On the other hand, in [31, 17] it was proved by using completely integrability that (DNLS) is globally well-posed in weighted Sobolev spaces without the size restriction of the mass (but the spaces are strictly narrower than $H^1(\mathbb{R})$). These results imply at first glance that 4π-mass condition is not necessary for yielding global results.

[1](1.1) for $b = 0$ and (DNLS) are equivalent under the following transformation:

$$\psi(t,x) = u(t,x)\exp\left(-\frac{i}{2}\int_{-\infty}^x |u(t,y)|^2 dy\right).$$

However, in the recent paper [14] the author showed that algebraic solitons and 4π-mass threshold give a certain turning point in variational properties of (DNLS). This result suggests that PDE dynamics of (DNLS) will change at the mass of 4π. We note that the algebraic solitons do not belong to weighted spaces in [31, 17], but they belong to $H^1(\mathbb{R})$, so this difference of function spaces give a delicate issue for (DNLS).

One of the important remaining problems on (DNLS) is to discover the dynamics around algebraic solitons. We note that stability/instability for algebraic solitons of (DNLS) in the energy space $H^1(\mathbb{R})$ remains an open problem. The equation (1.1) can be considered as a generalized equation of (DNLS) while preserving both L^2-criticality and Hamiltonian structure (1.2), so the study of this equation is important to investigate further insight on mathematical structure, especially L^2-critical structure of (DNLS).[2] The aim of this paper is to study stability properties of solitons for (1.1) by variational approach. In this paper we prove that if $b < 0$, all solitons *including algebraic solitons* are stable in $H^1(\mathbb{R})$.

1.2. **Solitons.** It is known (see [29, 14]) that the equation (1.1) has a two-parameter family of solitons. Consider solutions of (1.1) of the form

$$u_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x-ct), \tag{1.4}$$

where $(\omega, c) \in \mathbb{R}^2$. It is clear that $\phi_{\omega,c}$ must satisfy the following equation:

$$-\phi'' + \omega\phi + ic\phi' - i|\phi|^2\phi' - b|\phi|^4\phi = 0, \quad x \in \mathbb{R}. \tag{1.5}$$

Applying the following gauge transformation to $\phi_{\omega,c}$

$$\phi_{\omega,c}(x) = \Phi_{\omega,c}(x)\exp\left(\frac{i}{2}cx - \frac{i}{4}\int_{-\infty}^{x} |\Phi_{\omega,c}(y)|^2\, dy\right), \tag{1.6}$$

then $\Phi_{\omega,c}$ satisfies the following equation

$$-\Phi'' + \left(\omega - \frac{c^2}{4}\right)\Phi + \frac{c}{2}|\Phi|^2\Phi - \frac{3}{16}\gamma|\Phi|^4\Phi = 0, \quad x \in \mathbb{R}, \tag{1.7}$$

where $\gamma := 1 + \frac{16}{3}b$. The positive radial (even) solution of (1.7) is explicitly obtained as follows; if $\gamma > 0$ or equivalently $b > -3/16$,

$$\Phi^2_{\omega,c}(x) = \begin{cases} \dfrac{2(4\omega - c^2)}{\sqrt{c^2 + \gamma(4\omega - c^2)}\cosh(\sqrt{4\omega - c^2}x) - c} & \text{if } -2\sqrt{\omega} < c < 2\sqrt{\omega}, \\[2ex] \dfrac{4c}{(cx)^2 + \gamma} & \text{if } c = 2\sqrt{\omega}, \end{cases} \tag{1.8}$$

if $\gamma \le 0$ or equivalently $b \le -3/16$,

$$\Phi^2_{\omega,c}(x) = \frac{2(4\omega - c^2)}{\sqrt{c^2 + \gamma(4\omega - c^2)}\cosh(\sqrt{4\omega - c^2}x) - c} \quad \text{if } -2\sqrt{\omega} < c < -2s_*\sqrt{\omega}, \tag{1.9}$$

where $s_* = s_*(\gamma) = \sqrt{-\gamma/(1-\gamma)}$. Through the explicit formula of $\Phi_{\omega,c}$, the soliton of (1.1) is represented as

$$u_{\omega,c}(t,x) = e^{i\omega t + \frac{i}{2}c(x-ct) - \frac{i}{4}\int_{-\infty}^{x-ct}|\Phi_{\omega,c}(y)|^2 dy}\Phi_{\omega,c}(x-ct).$$

[2] The Hamiltonian form (1.2) is useful when one studies dynamics around the soliton.

We note that the condition of two parameters (ω, c):

$$\text{if } \gamma > 0 \Leftrightarrow b > -3/16, \quad -2\sqrt{\omega} < c \le 2\sqrt{\omega}, \tag{1.10}$$
$$\text{if } \gamma \le 0 \Leftrightarrow b \le -3/16, \quad -2\sqrt{\omega} < c < -2s_*\sqrt{\omega}$$

is a necessary and sufficient condition for the existence of non-trivial solutions of (1.7) vanishing at infinity. For (ω, c) satisfying (1.10), one can rewrite $(\omega, c) = (\omega, 2s\sqrt{\omega})$, where the parameter s satisfies

$$\text{if } b > -3/16, \quad -1 < s \le 1, \tag{1.11}$$
$$\text{if } b \le -3/16, \quad -1 < s < -s_*.$$

We note that the following curve

$$\mathbb{R}^+ \ni \omega \mapsto (\omega, 2s\sqrt{\omega}) \in \mathbb{R}^2 \tag{1.12}$$

gives the scaling of the soliton, i.e., we have

$$\phi_{\omega,2s\sqrt{\omega}}(x) = \omega^{1/4}\phi_{1,2s}(\sqrt{\omega}x) \quad \text{for } x \in \mathbb{R}. \tag{1.13}$$

We note that the value $b = -3/16$ gives the turning point where the structure of the solitons of (1.1) changes. In particular algebraic solitons exist only for the case $b > -3/16$, which is the main interest in this paper.

We now give the precise definition of stability of solitons in the energy space.

Definition 1.1. We say that the soliton $u_{\omega,c}$ of (1.1) is (orbitally) *stable* in $H^1(\mathbb{R})$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R})$ satisfies $\|u_0 - \phi_{\omega,c}\|_{H^1} < \delta$, then the maximal solution $u(t)$ of (1.1) with $u(0) = u_0$ exists globally in time and satisfies

$$\sup_{t\in\mathbb{R}} \inf_{(\theta,y)\in\mathbb{R}^2} \|u(t) - e^{i\theta}\phi_{\omega,c}(\cdot - y)\|_{H^1} < \varepsilon.$$

Otherwise, we say that the soliton is (orbitally) *unstable* in $H^1(\mathbb{R})$.

When $b = 0$, Colin and Ohta [5] proved that if $\omega > c^2/4$, the soliton $u_{\omega,c}$ is stable. Their proof depends on variational methods related to the argument in [32] (see Section 1.4 for more details). Liu, Simpson and Sulem [24] calculated linearized operators of a generalized derivative nonlinear Schrödinger (gDNLS) equation:

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \ \sigma > 0, \tag{gDNLS}$$

and studied stability of solitons by applying the abstract theory of Grillakis, Shatah and Strauss [9, 10]. In particular they gave an alternative proof of the stability result in [5] (see also [11] for partial results in this direction). We note that the abstract theory [9, 10] is not applicable for the case $c = 2\sqrt{\omega}$ due to the lack of coercivity property of the linearized operator. The case $c = 2\sqrt{\omega}$ was discussed in [19, 20],[3] while the stability or instability for this case is still an open problem.

When $b > 0$, the situation becomes different due to the focusing effect from the quintic term. Ohta [29] extended the work of [5] and proved that for each $b > 0$ there exists a unique $s^* = s^*(b) \in (0, 1)$ such that the soliton $u_{\omega,c}$ is stable if $-2\sqrt{\omega} < c < 2s^*\sqrt{\omega}$, and unstable if $2s^*\sqrt{\omega} < c < 2\sqrt{\omega}$ (see Figure 1). In [27] it was proved that algebraic soliton $u_{\omega,2\sqrt{\omega}}$ is unstable for small $b > 0$, where the assumption of smallness is used for construction of the unstable direction. If we observe momentum of solitons, the momentum is positive in the stable region, and negative in the unstable region. This implies that momentum of solitons has an essential effect on stability properties. In the

[3] We note that the key proposition in [20] is false (see [14, Appendix A]).

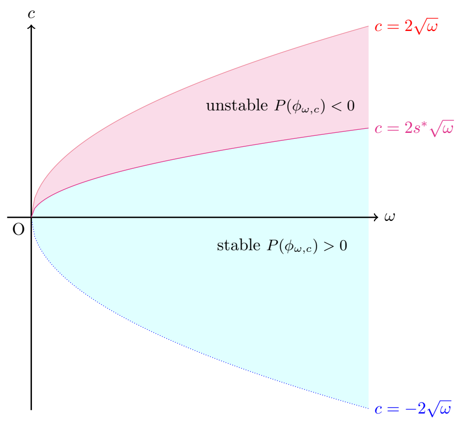

FIGURE 1. The stable/unstable region of solitons in the case $b > 0$.

borderline case $c = 2s^*\sqrt{\omega}$, momentum of the soliton is zero, which corresponds to the degenerate case. Recently, in [28] instability for this case was proved for small $b > 0$.

Stability properties of solitons for the case $b < 0$ seem to have been less studied. In this case momentum of all solitons is positive, which suggests that they are stable. Indeed, this is true as we show in this paper.

1.3. **Statement of the results.** Our first theorem gives the connection between two types of solitons, which would be of independent interest. To state the result, we introduce the set Ω defined by

$$\Omega = \left\{(\omega, c) \in \mathbb{R}^2 : -2\sqrt{\omega} < c < 2\sqrt{\omega}\right\}.$$

Then we have the following result.

Theorem 1.2. *Let $b > -3/16$. Suppose that (ω_0, c_0) satisfies $c_0 = 2\sqrt{\omega_0}$. Then, we have*

$$\lim_{\substack{(\omega,c)\to(\omega_0,c_0)\\ (\omega,c)\in\Omega}} \|\phi_{\omega,c} - \phi_{\omega_0,c_0}\|_{H^m(\mathbb{R})} = 0$$

for any $m \in \mathbb{Z}_{\geq 0}$.

Remark 1.3. By Theorem 1.2 and Sobolev's embedding theorem, we obtain that

$$\lim_{\substack{(\omega,c)\to(\omega_0,c_0)\\ (\omega,c)\in\Omega}} \|\phi_{\omega,c} - \phi_{\omega_0,c_0}\|_{W^{m,\infty}(\mathbb{R})} = 0$$

for any $m \in \mathbb{Z}_{\geq 0}$.

Theorem 1.2 shows that algebraic solitons and exponentially decaying solitons are connected in strong topology. This relation may be useful for further study on algebraic solitons. Here we adapt the approach in [13] and give a simple proof by using explicit formulae of solitons. Recently in [7] a similar statement of Theorem 1.2 was proved in the context of a double power nonlinear Schrödinger equation. The argument in [7] depends on variational characterization of ground states, where explicit formulae of solitons are not necessary.

Now we state our main result. The main result in this paper is the following stability result on two types of solitons.

Theorem 1.4. *Let $-3/16 < b < 0$ and let (ω, c) satisfy $-2\sqrt{\omega} < c \le 2\sqrt{\omega}$. Then the soliton $u_{\omega,c}$ of* (1.1) *is stable. In particular the algebraic soliton is stable.*

1.4. **Comments on the main result.** The stability result of algebraic solitons gives the counterpart of the previous instability result for the case $b > 0$. As pointed out before, the case $c = 2\sqrt{\omega}$ cannot be treated by the abstract theory [9, 10]. It is difficult to study stability properties for this case, based on the study of the linearized operator $S''_{\omega,c}(\phi_{\omega,c})$ (see below for the definition of $S_{\omega,c}$), because of the lack of coercivity property of $S''_{\omega,c}(\phi_{\omega,c})$.[4] For the proof of Theorem 1.4 we use variational approach inspired from the work in [32, 5, 29], which enables us to treat the case $c = 2\sqrt{\omega}$.

First we review the stability theory in the papers [5, 29]. We define the action functional $S_{\omega,c}$ by

$$S_{\omega,c}(\phi) = E(\phi) + \frac{\omega}{2}M(\phi) + \frac{c}{2}P(\phi), \tag{1.14}$$

and we set $d(\omega, c) = S_{\omega,c}(\phi_{\omega,c})$. We note that (1.5) can be rewritten as $S'_{\omega,c}(\phi) = 0$ and $\phi_{\omega,c}$ is a critical point of $S_{\omega,c}$. When $b \ge 0$ the following stability result is known.

Proposition 1.5 ([5, 29])**.** *Let $b \ge 0$ and let (ω, c) satisfy $\omega > c^2/4$. If there exists $\xi \in \mathbb{R}^2$ such that*

$$\left\langle d'(\omega, c), \xi \right\rangle \neq 0, \ \left\langle d''(\omega, c)\xi, \xi \right\rangle > 0, \tag{1.15}$$

then the soliton $u_{\omega,c}$ of (1.1) *is stable.*

Proposition 1.5 is proved in the following variational argument.[5] First we prove that the profile of the soliton $\phi_{\omega,c}$ is a minimizer on the Nehari manifold:

$$\left\{\varphi \in H^1(\mathbb{R}) \setminus \{0\} : K_{\omega,c}(\varphi) = 0\right\},$$

where $K_{\omega,c}(\varphi) := \frac{d}{d\lambda}S_{\omega,c}(\lambda\varphi)\big|_{\lambda=1}$. Next we consider the following potential wells:

$$\mathscr{K}^+_{\omega,c} = \left\{u \in H^1(\mathbb{R}) \setminus \{0\} : S_{\omega,c}(u) < d(\omega, c), K_{\omega,c}(u) > 0\right\},$$
$$\mathscr{K}^-_{\omega,c} = \left\{v \in H^1(\mathbb{R}) \setminus \{0\} : S_{\omega,c}(u) < d(\omega, c), K_{\omega,c}(u) < 0\right\}.$$

By using the variational characterization on the Nehari manifold, we see that $\mathscr{K}^+_{\omega,c}$ and $\mathscr{K}^-_{\omega,c}$ are invariant under the flow of (1.1). Then, under the condition (1.15), one can control the flow around the soliton, based on the calculation of the function $\tau \mapsto d((\omega, c) + \tau\xi)$ and properties of potential wells.

By computing $d''(\omega, c)$ we have the following identity (see Lemma 1 in [29]):

$$\det[d''(\omega, c)] = \frac{-2P(\phi_{\omega,c})}{\sqrt{4\omega - c^2}\left\{c^2 + \gamma(4\omega - c^2)\right\}}. \tag{1.16}$$

Here we note that $P(\phi_{\omega,c})$ is positive if (ω, c) satisfies that

$$\begin{aligned} &\text{if } b > 0, \quad -2\sqrt{\omega} < c < 2s^*\sqrt{\omega}, \\ &\text{if } b = 0, \quad -2\sqrt{\omega} < c < 2\sqrt{\omega}. \end{aligned} \tag{1.17}$$

Therefore, we deduce that $d''(\omega, c) < 0$ under the condition (1.17). This yields the existence of $\xi \in \mathbb{R}^2$ satisfying (1.15) because $d''(\omega, c)$ has one positive eigenvalue. Hence,

[4] The essential spectrum of $S''_{\omega,c}(\phi_{\omega,c})$ is given by $\sigma_{\text{ess}}\left(S''_{\omega,c}(\phi_{\omega,c})\right) = \left[\omega - c^2/4, \infty\right)$, which gives the lack of coercivity property for the case $c = 2\sqrt{\omega}$ (see [24] for more details).

[5] This can be regarded as certain extension of the argument in [32] to a two-parameter family of solitons.

it follows from Proposition 1.5 that if (1.17) holds, the soliton $u_{\omega,c}$ is stable. This is a summary of the stability results in [5, 29].

There are a few difficulties to study stability properties of solitons in the case $b < 0$. When $b < 0$ the defocusing effect from the quintic term $b|u|^4u$ gives an obstacle and then the variational characterization above does not hold. To overcome that, we consider the following gauge equivalent form of (1.1):

$$(1.1') \qquad i\partial_t v + \partial_x^2 v + \frac{i}{2}|v|^2\partial_x v - \frac{i}{2}v^2\partial_x \overline{v} + \frac{3}{16}\gamma|v|^4 v = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}.$$

Considering this form, one can characterize solitons on the Nehari manifold if $b \geq -3/16$. However, the equation (1.1′) does not have the good Hamiltonian structure as in (1.2), so it becomes more delicate to control the flow around the soliton. Another problem arises when we treat algebraic solitons (the case $c = 2\sqrt{\omega}$). We note that $d''(\omega, c)$ does not make sense when $c = 2\sqrt{\omega}$ (see (1.16)) because this case corresponds to the boundary of existence region of solitons. Therefore the stability criteria (1.15) does not make sense for the case $c = 2\sqrt{\omega}$.

In the present paper, we use the scaling curve (1.12) effectively for the control of the flow, based on variational characterization of solitons of (1.1′). This approach enables us to prove the stability for two types of the solitons in a unified way. Also, our variational argument along the scaling curve offers new perspectives to the stability theory of a two-parameter family of solitons (see the end of Section 4.2 for more details).

As a relevant work of this paper, Guo [12] studied stability of algebraic solitons of (gDNLS) for the case $0 < \sigma < 1$ by variational approach. Compared with our setting, stability problems become rather easier because the case $0 < \sigma < 1$ corresponds to L^2-subcritical structure. We note that the well-posedness of (gDNLS) in $H^1(\mathbb{R})$ (which remains an open problem in the case $0 < \sigma < 1$) is assumed in [12]. For well-posedness theory for (gDNLS) we refer to [15, 8, 22] and references therein.

Algebraic solitons also appear in the following double power nonlinear Schrödinger equation:

$$(1.18) \qquad i\partial_t u + \Delta u - |u|^{p-1}u + |u|^{q-1}u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N,$$

where $1 < p < q < 1 + 4/(N-2)_+$. If we consider the standing wave solution $e^{i\omega t}\phi_\omega(x)$, then ϕ_ω satisfies the following elliptic equation:

$$(1.19) \qquad -\Delta\phi + \omega\phi + |\phi|^{p-1}\phi - |\phi|^{q-1}\phi = 0, \quad x \in \mathbb{R}^N.$$

We note that the equation (1.7) for $0 < c \leq 2\sqrt{\omega}$ and $\gamma > 0$ corresponds to (1.19) for $p = 3$, $q = 5$ and $N = 1$. Due to the defocusing effect from the lower power order nonlinearity, (1.19) has algebraically decaying ground states with $\omega = 0$ as well as usual ground states decaying exponentially with $\omega > 0$. Instability and strong instability of two types of ground states were studied in [7], where variational characterization of ground states plays a key role in the proof.

Stability of solitions are closely related to the mass condition yielding global solutions of (1.1) in the energy space. We define the mass threshold value as

$$(1.20) \qquad M^*(b) = \begin{cases} M(\phi_{1,2s^*(b)}) & \text{if } b > 0, \\ \frac{4\pi}{\gamma^{3/2}} & \text{if } -3/16 < b \leq 0 (\Leftrightarrow 0 < \gamma \leq 1). \end{cases}$$

In [14] the author obtained the new mass condition for (1.1) such that if the initial data $u_0 \in H^1(\mathbb{R})$ of (1.1) satisfies $M(u_0) < M^*(b)$, then the corresponding $H^1(\mathbb{R})$-solution is global and bounded.[6] Moreover, it was also shown that $M^*(b)$ gives a turning point in the structure of potential wells generated by solitons. In this sense $M^*(b)$-mass condition of (1.1) corresponds to 4π-mass condition of (DNLS). We note that when $b > 0$, $M^*(b)$ is the mass of the solitons corresponding to the borderline in the stable/unstable region. On the other hand, when $-3/16 < b < 0 (\Leftrightarrow 0 < \gamma < 1)$ we have the following relation:

$$M(\phi_{1,2}) = \frac{4\pi}{\sqrt{\gamma}} < \frac{4\pi}{\gamma^{3/2}} = M(\phi_{1,2}) + P(\phi_{1,2}),$$

which indicates that positive momentum of algebraic solitons boosts the threshold value. This fact and the global result above are compatible with the stability of algebraic solitons because the stability implies that the flow around algebraic solitons is global and bounded.

1.5. **Stability results for the case $b \le -3/16$.** The proof of Theorem 1.4 is not applicable to the case $b < -3/16$ because the argument depends on variational characterization on the Nehari manifold, which does not hold for this case.[7] However, by using another variational approach inspired from Cazenave and Lions [4], we obtain the following result.

Theorem 1.6. *Let $b \le -3/16$ and let (ω, c) satisfy $-2\sqrt{\omega} < c < -2s_*\sqrt{\omega}$. Then the soliton $u_{\omega,c}$ of (1.1) is stable.*

It may be somewhat new to apply the approach of [4] to a two-parameter family of solitons. The key point in the proof is to solve certain variational problem with mass constraint. To this end we consider the gauge equivalent form (1.1′) again. If velocity of the soliton of (1.1′) is negative, one can prove that the soliton is a solution of certain minimization problem with mass constraint. Since velocity of all solitons for the case $b \le -3/16$ is negative, we can apply this variational argument to prove stability of these solitons. We note that the proof of Theorem 1.6 still works for the case $b > -3/16$ and $-2\sqrt{\omega} < c < 0$.

One can also apply the abstract theory of [9, 10] to exponentially decaying solitons, based on spectral analysis of linearized operators. However, as can be seen in [24], the calculation of linearized operators for (1.1) is complex because the nonlinearity contains derivative. We note that our variational proofs of Theorem 1.4 and Theorem 1.6 do not need any calculation of linearized operators.

1.6. **Organization of the paper.** The rest of this paper is organized as follows. In Section 2 we recall the fundamental properties of a two-parameter family of solitons of (1.1) which are used throughout the paper. In Section 3 we study the connection between algebraic solitons and exponentially decaying solitons and prove Theorem 1.2. In Section 4 we study stability of two types of solitons for the case $-3/16 < b < 0$ and prove Theorem 1.4. The key claim in the proof is Proposition 4.5, where we control the flow around the solitons by using the scaling curve (1.12) effectively. Finally, in Section 5 we study stability of solitons with negative velocity and prove Theorem 1.6.

[6] If $b \le -3/16$, for any initial data $u_0 \in H^1(\mathbb{R})$ of (1.1) the corresponding $H^1(\mathbb{R})$-solution is global and bounded.

[7] For the case $b = -3/16$ the proof of Theorem 1.4 works in the same way (see Remark 4.8).

2. **Preliminaries**

In this section we organize the fundamental properties of solitons of (1.1). We refer to [14] for the proofs of the results in this section.

In next sections, we mainly use the equation (1.1′) which is a gauge equivalent form of (1.1). Therefore we state the properties of solitons of (1.1′), which also yield the properties of solitons of (1.1) through the gauge transformation. We first note that (1.1′) is transformed from (1.1) through the following gauge transformation

$$v(t,x) = \mathcal{G}(u)(t,x) := u(t,x)\exp\left(\frac{i}{4}\int_{-\infty}^{x}|u(t,y)|^2dy\right).$$

The equation (1.1′) has the following conserved quantities:

$$\text{(Energy)}\qquad \mathcal{E}(v) = \frac{1}{2}\|\partial_x v\|_{L^2}^2 - \frac{\gamma}{32}\|v\|_{L^6}^6,$$

$$\text{(Mass)}\qquad \mathcal{M}(v) = \|v\|_{L^2}^2,$$

$$\text{(Momentum)}\qquad \mathcal{P}(v) = (i\partial_x v, v) + \frac{1}{4}\|v\|_{L^4}^4.$$

We note that the well-posedness in $H^1(\mathbb{R})$ for each of (1.1) and (1.1′) is equivalent because $u \mapsto \mathcal{G}(u)$ is locally Lipschitz continuous on $H^1(\mathbb{R})$.

Let (ω, c) satisfy (1.10). A two-parameter family of solitons of (1.1′) is given by

$$v_{\omega,c}(t,x) = \mathcal{G}(u_{\omega,c})(t,x) = e^{i\omega t}\varphi_{\omega,c}(x-ct), \tag{2.1}$$

where $\varphi_{\omega,c}$ is represented as

$$\varphi_{\omega,c}(x) = e^{\frac{i}{2}cx}\Phi_{\omega,c}(x). \tag{2.2}$$

We note that $\varphi_{\omega,c}$ satisfies the equation

$$-\varphi'' + \omega\varphi + ic\varphi' + \frac{c}{2}|\varphi|^2\varphi - \frac{3}{16}\gamma|\varphi|^4\varphi = 0, \quad x \in \mathbb{R}. \tag{2.3}$$

We define the action functional with respect to (1.1′) by

$$\mathcal{S}_{\omega,c}(\varphi) = \mathcal{E}(\varphi) + \frac{\omega}{2}\mathcal{M}(\varphi) + \frac{c}{2}\mathcal{P}(\varphi).$$

We note that (2.3) can be rewritten as $\mathcal{S}'_{\omega,c}(\varphi) = 0$ and $\varphi_{\omega,c}$ is a critical point of $\mathcal{S}_{\omega,c}$. Concerning the conserved quantities we have the following relation:

$$\mathcal{E}(\mathcal{G}(u)) = E(u),\ \mathcal{M}(\mathcal{G}(u)) = M(u),\ \mathcal{P}(\mathcal{G}(u)) = P(u),$$

which yields that

$$\mathcal{S}_{\omega,c}(\varphi_{\omega,c}) = \mathcal{S}_{\omega,c}(\mathcal{G}(\phi_{\omega,c})) = S_{\omega,c}(\phi_{\omega,c}) = d(\omega,c). \tag{2.4}$$

In the same way as (1.13), for the parameter s satisfying (1.11) we have

$$\varphi_{\omega,2s\sqrt{\omega}}(x) = \omega^{1/4}\varphi_{1,2s}(\sqrt{\omega}x) \quad \text{for } x \in \mathbb{R},$$

which implies that

$$\mathcal{E}(\varphi_{\omega,2s\sqrt{\omega}}) = \omega\mathcal{E}(\varphi_{1,2s}),\ \mathcal{M}(\varphi_{\omega,2s\sqrt{\omega}}) = \mathcal{M}(\varphi_{1,2s}),\ \mathcal{P}(\varphi_{\omega,2s\sqrt{\omega}}) = \sqrt{\omega}\mathcal{P}(\varphi_{1,2s}).$$

In particular we have

$$d(\omega, 2s\sqrt{\omega}) = \omega d(1,2s). \tag{2.5}$$

Concerning mass of the solitons we have the following result.

Lemma 2.1. *Let* (ω, c) *satisfy* (1.10). *Then we have*

$$\mathcal{M}\left(\varphi_{\omega,c}\right)=\begin{cases}\dfrac{8}{\sqrt{\gamma}}\tan^{-1}\sqrt{\dfrac{1+\alpha}{1-\alpha}} & \text{if } \gamma>0,\\ \dfrac{4\sqrt{4\omega-c^2}}{-c} & \text{if } \gamma=0,\\ \dfrac{4}{\sqrt{-\gamma}}\log\left(-\alpha+\sqrt{\alpha^2-1}\right) & \text{if } \gamma<0,\end{cases}$$

where $\alpha:=c\left(c^2+\gamma(4\omega-c^2)\right)^{-1/2}$. *Furthermore, each of the functions*

$$(-1,1]\ni s\mapsto\mathcal{M}\left(\varphi_{1,2s}\right)\in\left(0,\frac{4\pi}{\sqrt{\gamma}}\right] \quad \text{if } \gamma>0$$

and

$$(-1,-s_*)\ni s\mapsto\mathcal{M}\left(\varphi_{1,2s}\right)\in(0,\infty) \quad \text{if } \gamma\le 0$$

is continuous, strictly increasing and surjective.

By Lemma 2.1 and elementary calculations we have the following claim which is useful to study stability of the soliton with $c<0$.

Lemma 2.2. *Let* (ω, c) *satisfy* (1.10) *and* $\omega > c^2/4$. *Then we have*

$$\partial_\omega\mathcal{M}(\varphi_{\omega,c})=\frac{-8c}{\sqrt{4\omega-c^2}\left\{c^2+\gamma(4\omega-c^2)\right\}}.$$

The momentum of the solitons is represented as follows.

Lemma 2.3. *Let* (ω, c) *satisfy* (1.10). *Then we have*

$$\mathcal{P}(\varphi_{\omega,c})=\begin{cases}\dfrac{c}{2}\left(-1+\dfrac{1}{\gamma}\right)\mathcal{M}(\varphi_{\omega,c})+\dfrac{2}{\gamma}\sqrt{4\omega-c^2} & \text{if } \gamma\gtrless 0,\\ -\dfrac{2\omega+c^2}{3c}\mathcal{M}(\varphi_{\omega,c}) & \text{if } \gamma=0.\end{cases} \tag{2.6}$$

Positivity of momentum of the solitons plays an essential role in the stability theory. Concerning the sign of the momentum we have the following result.

Proposition 2.4. *Let* s *satisfy* (1.11). *Then the following properties hold:*

(i) *If* $b>0$, *there exists a unique* $s^*{=}s^*(b){\in}(0,1)$ *such that* $\mathcal{P}(\varphi_{1,2s^*}){=}0$. *Moreover, we have* $\mathcal{P}(\varphi_{1,2s})>0$ *for* $s\in(-1,s^*)$ *and* $\mathcal{P}(\varphi_{1,2s})<0$ *for* $s\in(s^*,1]$.
(ii) *If* $b=0$, $\mathcal{P}(\varphi_{1,2s})>0$ *for* $s\in(-1,1)$ *and* $\mathcal{P}(\varphi_{1,2})=0$.
(iii) *If* $b<0$, $\mathcal{P}(\varphi_{1,2s})>0$ *for any* s.

Finally we state the energy of the solitons. The following claim is an immediate consequence of the Pohozaev identity.

Lemma 2.5. *Let* s *satisfy* (1.11). *Then we have*

$$\mathcal{E}(\varphi_{1,2s})=-\frac{s}{2}\mathcal{P}(\varphi_{1,2s}).$$

3. Connection between two types of the solitons

In this section we study connection between two types of the solitons and prove Theorem 1.2. From the scaling relation (1.13), it is enough to discuss the convergence of $\phi_{1,2s}$ as $s \to 1$. First we prove the pointwise convergence.

Proposition 3.1. *Let $b > -3/16$. For any $x \in \mathbb{R}$ we have*

$$\lim_{s\to 1-0} \phi_{1,2s}(x) = \phi_{1,2}(x).$$

Proof. Fix any $x \in \mathbb{R}$. From the relation (1.6), it is enough to prove that

$$\lim_{s\to 1-0} \Phi_{1,2s}(x) = \Phi_{1,2}(x). \tag{3.1}$$

From the explicit formula (1.8), we have

$$\Phi_{1,2s}^2(x) = \frac{4(1-s^2)}{\sqrt{s^2+\gamma(1-s^2)}\cosh\left(2\sqrt{1-s^2}x\right) - s} \tag{3.2}$$

for $s \in (-1,1)$. By the Taylor expansion of $x \mapsto \cosh x$ around zero, the denominator is rewritten as

$$\sqrt{s^2+\gamma(1-s^2)}\left(1+2(1-s^2)x^2 + O\left((1-s^2)^2\right)\right) - s. \tag{3.3}$$

By the Taylor expansion of the function $h \mapsto \sqrt{s^2+h}$ around zero, we have

$$\sqrt{s^2+\gamma(1-s^2)} - s = \frac{\gamma}{2s}(1-s^2) + O\left((1-s^2)^2\right),$$

which is valid for $s \in (0,1)$. Thus we have

$$\begin{aligned}(3.3) &= \frac{\gamma}{2s}(1-s^2) + 2(1-s^2)\sqrt{s^2+\gamma(1-s^2)}x^2 + O\left((1-s^2)^2\right)\\ &= (1-s^2)\left(\frac{\gamma}{2s} + 2\sqrt{s^2+\gamma(1-s^2)}x^2 + O\left(1-s^2\right)\right).\end{aligned}$$

We note that the numerator and denominator share a common factor $1-s^2$. Therefore we deduce that

$$\begin{aligned}\Phi_{1,2s}^2(x) &= \frac{4}{\frac{\gamma}{2s} + 2\sqrt{s^2+\gamma(1-s^2)}x^2 + O\left(1-s^2\right)}\\ &\underset{s\to 1-0}{\longrightarrow} \frac{8}{\gamma+4x^2} = \Phi_{1,2}^2(x),\end{aligned}$$

which proves (3.1). ∎

To complete the proof of Theorem 1.2, we effectively use the Brézis–Lieb lemma:

Lemma 3.2 ([1]). *Let $1 \le p < \infty$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(\mathbb{R})$ and $f_n \to f$ a.e. in $\mathbb{R}$ as $n \to \infty$. Then we have*

$$\|f_n\|_{L^p}^p - \|f_n - f\|_{L^p}^p - \|f\|_{L^p}^p \to 0$$

as $n \to \infty$.

Proof of Theorem 1.2. From Lemma 2.1 and Proposition 3.1, we have

$$\lim_{s\to 1-0} \phi_{1,2s}(x) = \phi_{1,2}(x) \text{ for all } x \in \mathbb{R},$$

$$\lim_{s\to 1-0} \|\phi_{1,2s}\|_{L^2}^2 = \|\phi_{1,2}\|_{L^2}^2 .$$

Applying Lemma 3.2, we have

$$\lim_{s\to 1-0}\|\phi_{1,2s}-\phi_{1,2}\|_{L^2}^2=0. \tag{3.4}$$

In the same way we also have

$$\lim_{s\to 1-0}\|\Phi_{1,2s}-\Phi_{1,2}\|_{L^2}^2=0. \tag{3.5}$$

Here we recall that $\Phi_{1,2s}$ is the solution of the equation

$$-\Phi''+(1-s^2)\Phi+s|\Phi|^2\Phi-\frac{3}{16}\gamma|\Phi|^4\Phi=0,\quad x\in\mathbb{R}. \tag{3.6}$$

We note that

$$\begin{aligned}\|\Phi_{1,2s}\|_{L^\infty}^2&=\Phi_{1,2s}^2(0)\\&=\frac{4(1-s^2)}{\sqrt{s^2+\gamma(1-s^2)}-s}\\&=\frac{4}{\gamma}\left(\sqrt{s^2+\gamma(1-s^2)}+s\right).\end{aligned}$$

This formula yields that the function $(-1,1)\ni s\mapsto\|\Phi_{1,2s}\|_{L^\infty}$ is strictly increasing and

$$\lim_{s\to 1-0}\|\Phi_{1,2s}\|_{L^\infty}^2=\frac{8}{\gamma}=\|\Phi_{1,2}\|_{L^\infty}^2.$$

In particular we have

$$\max_{s\in(-1,1]}\|\Phi_{1,2s}\|_{L^\infty}=\|\Phi_{1,2}\|_{L^\infty}. \tag{3.7}$$

By (3.5) and (3.7) we obtain that

$$\begin{aligned}\|s\Phi_{1,2s}^3-\Phi_{1,2}^3\|_{L^2}&\le(1-s)\|\Phi_{1,2s}^3\|_{L^2}+\|\Phi_{1,2s}^3-\Phi_{1,2}^3\|_{L^2}\\&\le(1-s)\|\Phi_{1,2}\|_{L^\infty}^2\|\Phi_{1,2}\|_{L^2}+3\|\Phi_{1,2}\|_{L^\infty}^2\|\Phi_{1,2s}-\Phi_{1,2}\|_{L^2}\\&\underset{s\to 1-0}{\longrightarrow}0.\end{aligned}$$

Similarly, we have

$$\|\Phi_{1,2s}^5-\Phi_{1,2}^5\|_{L^2}\le 4\|\Phi_{1,2}\|_{L^\infty}^4\|\Phi_{1,2s}-\Phi_{1,2}\|_{L^2}\underset{s\to 1-0}{\longrightarrow}0.$$

Therefore, by using the equation (3.6), we deduce that

$$\begin{aligned}\|\Phi_{1,2s}''-\Phi_{1,2}''\|_{L^2}\le{}&(1-s^2)\|\Phi_{1,2s}\|_{L^2}^2+\|s\Phi_{1,2s}^3-\Phi_{1,2}^3\|_{L^2}\\&+\frac{3}{16}\gamma\|\Phi_{1,2s}^5-\Phi_{1,2}^5\|_{L^2}\underset{s\to 1-0}{\longrightarrow}0.\end{aligned}$$

Combined with (3.5) we have

$$\lim_{s\to 1-0}\|\Phi_{1,2s}-\Phi_{1,2}\|_{H^2}=0.$$

By using the formula (1.6), we deduce that

$$\lim_{s\to 1-0}\|\phi_{1,2s}-\phi_{1,2}\|_{H^2}=0.$$

The rest of the proof is done by using the equation (1.5) and a standard bootstrap argument. □

4. **Stability of two types of solitary waves**

In this section we study stability of two types of solitary waves for the case $-3/16 < b < 0$ and prove Theorem 1.4.

4.1. **Variational characterization.** In this subsection we recall variational properties of the solitons of (1.1′). Here we assume that b and (ω, c) satisfy

$$b > -3/16\,(\Leftrightarrow \gamma > 0)\,,\ \ -2\sqrt{\omega} < c \le 2\sqrt{\omega}. \tag{4.1}$$

First we define the function space by

$$\varphi \in X_{\omega,c} \iff \begin{cases} \varphi \in H^1(\mathbb{R}) & \text{if } \omega > c^2/4, \\ e^{-\frac{i}{2}cx}\varphi \in \dot{H}^1(\mathbb{R}) \cap L^4(\mathbb{R}) & \text{if } c = 2\sqrt{\omega}, \end{cases}$$

where the norm of $X_{c^2/4,c}$ is defined by

$$\|\varphi\|_{X_{c^2/4,c}} = \|e^{-\frac{i}{2}c\cdot}\varphi\|_{\dot{H}^1\cap L^4}.$$

We note that $H^1(\mathbb{R}) \subset X_{c^2/4,c}$. We define the functional $\mathcal{K}_{\omega,c}$ by

$$\begin{aligned}\mathcal{K}_{\omega,c}(\varphi) &= \frac{d}{d\lambda}\mathcal{S}_{\omega,c}(\lambda\varphi)\bigg|_{\lambda=1} \\ &= \|\partial_x\varphi\|_{L^2}^2 + \omega\|\varphi\|_{L^2}^2 + c\,(\partial_x\varphi, \varphi) + \frac{c}{2}\|\varphi\|_{L^4}^4 - \frac{3}{16}\gamma\|\varphi\|_{L^6}^6.\end{aligned}$$

Now we consider the following minimization problem:

$$\begin{aligned}\mu(\omega, c) &= \inf\left\{\mathcal{S}_{\omega,c}(\varphi) : \varphi \in X_{\omega,c}\setminus\{0\}, \mathcal{K}_{\omega,c}(\varphi) = 0\right\}, \\ \mathscr{M}_{\omega,c} &= \left\{\varphi \in X_{\omega,c}\setminus\{0\} : \mathcal{S}_{\omega,c}(\varphi) = \mu(\omega,c), \mathcal{K}_{\omega,c}(\varphi) = 0\right\}.\end{aligned}$$

We note that $\mathscr{M}_{\omega,c}$ is the set of minimizers of $\mathcal{S}_{\omega,c}$ on the Nehari manifold. The following result gives a variational characterization of the solitons on the Nehari manifold.

Proposition 4.1 ([14])**.** *Assume* (4.1)*. Then we have*

$$\mathscr{M}_{\omega,c} = \left\{e^{i\theta_0}\varphi_{\omega,c}(\cdot - x_0) : \theta_0 \in [0, 2\pi), x_0 \in \mathbb{R}\right\},$$

and $d(\omega, c) = \mu(\omega, c)$*, where* $d(\omega, c) = \mathcal{S}_{\omega,c}(\varphi_{\omega,c})$ *(see* (2.4)*).*

Here we introduce the following potential wells in the energy space:

$$\begin{aligned}\mathscr{A}^+_{\omega,c} &= \left\{v \in H^1(\mathbb{R})\setminus\{0\} : \mathcal{S}_{\omega,c}(v) < d(\omega,c), \mathcal{K}_{\omega,c}(v) > 0\right\}, \\ \mathscr{B}^+_{\omega,c} &= \left\{v \in H^1(\mathbb{R})\setminus\{0\} : \mathcal{S}_{\omega,c}(v) < d(\omega,c), \mathcal{J}_c(v) < d(\omega,c)\right\}, \\ \mathscr{A}^-_{\omega,c} &= \left\{v \in H^1(\mathbb{R})\setminus\{0\} : \mathcal{S}_{\omega,c}(v) < d(\omega,c), \mathcal{K}_{\omega,c}(v) < 0\right\}, \\ \mathscr{B}^-_{\omega,c} &= \left\{v \in H^1(\mathbb{R})\setminus\{0\} : \mathcal{S}_{\omega,c}(v) < d(\omega,c), \mathcal{J}_c(v) > d(\omega,c)\right\},\end{aligned}$$

where the functional $\mathcal{J}_c$ is defined by

$$\mathcal{J}_c(v) = -\frac{c}{8}\|v\|_{L^4}^4 + \frac{\gamma}{16}\|v\|_{L^6}^6.$$

We note that the functional $\mathcal{S}_{\omega,c}$ is rewritten as

$$\mathcal{S}_{\omega,c}(v) = \frac{1}{2}\mathcal{K}_{\omega,c}(v) + \mathcal{J}_c(v). \tag{4.2}$$

From Proposition 4.1 we obtain the following result.

Proposition 4.2. *Assume* (4.1). *Then* $\mathscr{A}^+_{\omega,c}$ *and* $\mathscr{A}^-_{\omega,c}$ *are invariant under the flow of* (1.1′). *Moreover, we have* $\mathscr{A}^\pm_{\omega,c} = \mathscr{B}^\pm_{\omega,c}$.

Proof. The proof is done in the similar way as [5, Lemma 11]. □

Remark 4.3. One can also prove that if the initial data of (1.1′) belongs to $\mathscr{A}^+_{\omega,c}$, then the corresponding $H^1(\mathbb{R})$-solution is global and bounded.

Finally we prepare the compactness result on minimizers on the Nehari manifold which is important for the proof of stability.

Proposition 4.4 ([14]). *Assume* (4.1). *If a sequence* $\{\varphi_n\} \subset X_{\omega,c}$ *satisfies*

$$\mathcal{S}_{\omega,c}(\varphi_n) \to \mu(\omega,c) \text{ and } \mathcal{K}_{\omega,c}(\varphi_n) \to 0 \text{ as } n \to \infty,$$

then there exist a sequence $\{y_n\} \subset \mathbb{R}$ *and* $v \in \mathscr{M}_{\omega,c}$ *such that* $\{\varphi_n(\cdot - y_n)\}$ *has a subsequence that converges to* v *strongly in* $X_{\omega,c}$.

4.2. **Stability theory with potential wells.** Here we assume that b and (ω, c) satisfy

$$-3/16 < b < 0 \, (\Leftrightarrow 0 < \gamma < 1), \; -2\sqrt{\omega} < c \le 2\sqrt{\omega}. \tag{4.3}$$

We note that $\mathcal{P}(\varphi_{\omega,c}) > 0$ by Proposition 2.4. To prove stability of the soliton, we need to control the flow around the soliton. By taking advantage of potential wells, we obtain the following claim which plays a key role for the proof of stability.

Proposition 4.5. *Assume* (4.3). *Then, for any* $\varepsilon \in (0, \varepsilon_0)$ *there exists* $\delta > 0$ *such that if* $v_0 \in H^1(\mathbb{R})$ *satisfies* $\|v_0 - \varphi_{\omega,c}\|_{H^1} < \delta$, *then the solution* $v(t)$ *of* (1.1′) *with* $v(0) = v_0$ *exists globally in time and satisfies that*

(i) *if* $c = 2s\mu$ *for* $s \in (0,1]$ ($\mu = \sqrt{\omega}$),

$$\begin{aligned} d\big((\mu-\varepsilon)^2, 2s(\mu-\varepsilon)\big) - \frac{s\varepsilon}{4}\|v(t)\|^4_{L^4} \\ < \mathcal{J}_c(v(t)) < d\big((\mu+\varepsilon)^2, 2s(\mu+\varepsilon)\big) + \frac{s\varepsilon}{4}\|v(t)\|^4_{L^4}, \end{aligned} \tag{4.4}$$

(ii) *if* $c = 0$,

$$d(\omega, -\varepsilon) - \frac{\varepsilon}{8}\|v(t)\|^4_{L^4} < \mathcal{J}_0(v(t)) < d(\omega, \varepsilon) + \frac{\varepsilon}{8}\|v(t)\|^4_{L^4}, \tag{4.5}$$

(iii) *if* $c < 0$,

$$d(\omega - \varepsilon, c) < \mathcal{J}_c(v(t)) < d(\omega + \varepsilon, c), \tag{4.6}$$

for all $t \in \mathbb{R}$ *in* (i)-(iii).

Remark 4.6. Compared with the corresponding result [5, Lemma 12], the L^4-norm appears in (4.4) and (4.5), which comes from the lack of the "good" Hamiltonian structure in (1.1′).

Proof. We mainly prove the most difficult case $0 < c \le 2\sqrt{\omega}$.

(i) Let $\varepsilon_0 > 0$ be sufficiently small. For $\varepsilon \in (0, \varepsilon_0)$ we define the function g by

$$g(\tau) = d\big((\mu+\tau)^2, 2s(\mu+\tau)\big) \quad \text{for } \tau \in (-\varepsilon, \varepsilon).$$

From the relation (2.5) we have

$$g(\tau) = (\mu+\tau)^2 d(1, 2s) \quad \text{for } \tau \in (-\varepsilon, \varepsilon),$$

which yields that

$$g(0) = \mu^2 d(1, 2s), \; g'(0) = 2\mu d(1, 2s), \; g''(0) = 2d(1, 2s). \tag{4.7}$$

From Lemma 2.5 we have

$$2d(1,2s)=\mathcal{M}(\varphi_{1,2s})+s\mathcal{P}(\varphi_{1,2s}). \tag{4.8}$$

Assume that $v_0 \in H^1(\mathbb{R})$ satisfies $\|v_0-\varphi_{\mu^2,2s\mu}\|_{H^1}<\delta$, where $\delta>0$ is determined later. First we prove that

$$v_0 \in \mathscr{B}^+_{(\mu+\varepsilon)^2,2s(\mu+\varepsilon)} \cap \mathscr{B}^-_{(\mu-\varepsilon)^2,2s(\mu-\varepsilon)}. \tag{4.9}$$

From (4.8) and (4.7) we have

$$\begin{aligned}\mathcal{S}_{(\mu\pm\varepsilon)^2,2s(\mu\pm\varepsilon)}(v_0) &= \mathcal{S}_{(\mu\pm\varepsilon)^2,2s(\mu\pm\varepsilon)}(\varphi_{\mu^2,2s\mu})+O(\delta)\\ &= \mathcal{E}(\varphi_{\mu^2,2s\mu})+\frac{(\mu\pm\varepsilon)^2}{2}\mathcal{M}(\varphi_{\mu^2,2s\mu})\\ &\qquad + s(\mu\pm\varepsilon)\mathcal{P}(\varphi_{\mu^2,2s\mu})+O(\delta)\\ &= \mu^2 d(1,2s)\pm\varepsilon\mu\left(\mathcal{M}(\varphi_{1,2s})+s\mathcal{P}(\varphi_{1,2s})\right)\\ &\qquad +\frac{\varepsilon^2}{2}\mathcal{M}(\varphi_{1,2s})+O(\delta)\\ &= g(0)\pm\varepsilon g'(0)+\frac{\varepsilon^2}{2}\mathcal{M}(\varphi_{1,2s})+O(\delta).\end{aligned}$$

By using the Taylor expansion,[8] we have

$$g(\pm\varepsilon)=g(0)\pm\varepsilon g'(0)+\frac{\varepsilon^2}{2}g''(0).$$

We note that

$$g''(0)=2d(1,2s)=\mathcal{M}(\varphi_{1,2s})+s\mathcal{P}(\varphi_{1,2s})$$

and $s\mathcal{P}(\varphi_{1,2s})>0$. Therefore, by taking small $\delta>0$ we obtain that

$$\mathcal{S}_{(\mu\pm\varepsilon)^2,2s(\mu\pm\varepsilon)}(v_0)<g(\pm\varepsilon). \tag{4.10}$$

On the other hand, by (4.2) and $\mathcal{K}_{\omega,c}(\varphi_{\omega,c})=0$ we have

$$\begin{aligned}\mathcal{J}_{c+2s\varepsilon}(\varphi_{\omega,c}) &= -\frac{c+2s\varepsilon}{8}\|\varphi_{\omega,c}\|_{L^4}^4+\frac{\gamma}{16}\|\varphi_{\omega,c}\|_{L^6}^6\\ &< \mathcal{J}_c(\varphi_{\omega,c})=g(0)<g(\varepsilon).\end{aligned}$$

By taking smaller $\delta>0$ again, we obtain that $\mathcal{J}_{c+2s\varepsilon}(v_0)<g(\varepsilon)$. Similarly, we have $g(-\varepsilon)<\mathcal{J}_{c-2s\varepsilon}(v_0)$. Combined with (4.10), we deduce that (4.9) holds.

We now prove (4.4). By Proposition 4.2 we have

$$v(t)\in \mathscr{B}^+_{(\mu+\varepsilon)^2,2s(\mu+\varepsilon)} \cap \mathscr{B}^-_{(\mu-\varepsilon)^2,2s(\mu-\varepsilon)} \tag{4.11}$$

for all $t\in\mathbb{R}$. Therefore, we deduce that

$$\begin{aligned}g(\varepsilon)>\mathcal{J}_{c+2s\varepsilon}(v(t)) &= -\frac{c+2s\varepsilon}{8}\|v(t)\|_{L^4}^4+\frac{\gamma}{16}\|v(t)\|_{L^6}^6\\ &= \mathcal{J}_c(v(t))-\frac{s\varepsilon}{4}\|v(t)\|_{L^4}^4.\end{aligned}$$

Similarly, we have

$$g(-\varepsilon)<\mathcal{J}_c(v(t))+\frac{s\varepsilon}{4}\|v(t)\|_{L^4}^4.$$

[8] One can also show this formula without using the Taylor expansion since the function g is the quadratic function.

This completes the proof of (4.4).

(ii) When $c = 0$, by Lemma 2.3 we have

$$\partial_c \mathcal{P}(\varphi_{\omega,c})\Big|_{c=0} = \frac{1}{2}\left(-1 + \frac{1}{\gamma}\right) M(\varphi_{\omega,0}) > 0,$$

which yields that $\partial_c^2 d(\omega, c)\Big|_{c=0} > 0$. From this fact and the calculation based on the function $(-\varepsilon, \varepsilon) \ni \tau \mapsto d(\omega, \tau)$, one can prove that

$$v_0 \in \mathscr{B}^+_{\omega,\varepsilon} \cap \mathscr{B}^-_{\omega,-\varepsilon}. \tag{4.12}$$

In the same way as (i), we see that (4.12) implies (4.5).

(iii) When $c < 0$, by Lemma 2.2 we have

$$\partial_\omega^2 d(\omega, c) = \frac{1}{2} \partial_\omega \mathcal{M}(\varphi_{\omega,c}) > 0.$$

From this fact and the calculation based on the function $(-\varepsilon, \varepsilon) \ni \tau \mapsto d(\omega + \tau, c)$, one can prove that

$$v_0 \in \mathscr{B}^+_{\omega+\varepsilon,c} \cap \mathscr{B}^-_{\omega-\varepsilon,c},$$

which yields (4.6). □

Combined with Proposition 4.4, one can prove the following stability result.

Theorem 4.7. *Assume* (4.3). *Then the soliton* $v_{\omega,c}$ *of* (1.1′) *is stable.*

Proof. The claim is proved by contradiction. Assume that there exist $\varepsilon_1 > 0$, a sequence of the maximal solutions $\{v_n\}$ to (1.1′) and a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$\|v_n(0) - \varphi_{\omega,c}\|_{H^1} \underset{n\to\infty}{\longrightarrow} 0, \tag{4.13}$$

$$\inf_{(\theta,y)\in\mathbb{R}^2} \|v_n(t_n) - e^{i\theta}\varphi_{\omega,c}(\cdot - y)\|_{H^1} \geq \varepsilon_1. \tag{4.14}$$

Since $\mathcal{S}_{\omega,c}(\cdot)$ is a conserved quantity, by (4.13) we have

$$\mathcal{S}_{\omega,c}(v_n(t_n)) = \mathcal{S}_{\omega,c}(v_n(0)) \underset{n\to\infty}{\longrightarrow} \mathcal{S}_{\omega,c}(\varphi_{\omega,c}) = d(\omega, c). \tag{4.15}$$

By (4.13), (4.14) and the continuity $t \mapsto v(t) \in H^1(\mathbb{R})$, one can pick up t_n (still denoted by the same letter) such that

$$\inf_{(\theta,y)\in\mathbb{R}^2} \|v_n(t_n) - e^{i\theta}\varphi_{\omega,c}(\cdot - y)\|_{H^1} = \varepsilon_1. \tag{4.16}$$

This equality yields the boundedness of $\{v_n(t_n)\}$ in $H^1(\mathbb{R})$, i.e.,

$$\sup_{n\in\mathbb{N}} \|v_n(t_n)\|_{H^1} \leq C, \tag{4.17}$$

where C only depends on $\|\varphi_{\omega,c}\|_{H^1}$ and ε_1. From Proposition 4.5 and (4.17) we obtain that

$$\mathcal{J}_c(v_n(t_n)) \underset{n\to\infty}{\longrightarrow} d(\omega, c).$$

Combined with (4.2), we have

$$\mathcal{K}_{\omega,c}(v_n(t_n)) \underset{n\to\infty}{\longrightarrow} 0. \tag{4.18}$$

Therefore, by (4.15), (4.18) and Proposition 4.4, there exist a sequence $\{y_n\}$ and $\theta_0, y_0 \in \mathbb{R}$ such that $\{v_n(t_n, \cdot + y_n)\}$ has a subsequence (still denoted by the same letter) that converges to $e^{i\theta_0}\varphi_{\omega,c}(\cdot - y_0)$ in $X_{\omega,c}$. If $\omega > c^2/4$, this yields that

$$\|v_n(t_n) - e^{i\theta_0}\varphi_{\omega,c}(\cdot - y_0 - y_n)\|_{H^1} \underset{n\to\infty}{\longrightarrow} 0, \tag{4.19}$$

which contradicts (4.16).

When $c = 2\sqrt{\omega}$, we need to modify the argument slightly. From the definition of $X_{c^2/4,c}$, we have

$$e^{-\frac{i}{2}c\cdot}v_n(t_n, \cdot + y_n) \to e^{-\frac{i}{2}c\cdot}e^{i\theta_0}\varphi_{\omega,c}(\cdot - y_0) \text{ in } \dot{H}^1(\mathbb{R}). \tag{4.20}$$

By using this convergence one can easily prove that

$$e^{-\frac{i}{2}c\cdot}v_n(t_n, \cdot + y_n) \rightharpoonup e^{-\frac{i}{2}c\cdot}e^{i\theta_0}\varphi_{\omega,c}(\cdot - y_0) \text{ weakly in } L^2(\mathbb{R}). \tag{4.21}$$

From (4.13) and mass conservation we obtain that

$$\mathcal{M}(v_n(t_n)) = \mathcal{M}(v_n(0)) \to \mathcal{M}(\varphi_{\omega,c}). \tag{4.22}$$

Therefore, it follows from (4.21) and (4.22) that

$$e^{-\frac{i}{2}c\cdot}v_n(t_n, \cdot + y_n) \to e^{-\frac{i}{2}c\cdot}e^{i\theta_0}\varphi_{\omega,c}(\cdot - y_0) \text{ in } L^2(\mathbb{R}). \tag{4.23}$$

Hence (4.19) follows from (4.20) and (4.23), which contradicts (4.16). This completes the proof. □

Proof of Theorem 1.4. We note that $v_{\omega,c} = \mathcal{G}(u_{\omega,c})$ and

$$\mathcal{G}(e^{i\theta}u(\cdot - y))(x) = e^{i\theta}\mathcal{G}(u)(x - y)$$

for $u \in H^1(\mathbb{R})$ and $x, y, \theta \in \mathbb{R}$. We also note that the gauge transformation $u \mapsto \mathcal{G}(u)$ is Lipschitz continuous on bounded subsets of $H^1(\mathbb{R})$. Hence the result follows from Theorem 4.7 and these properties of the gauge transformation. □

Remark 4.8. The stability of the solitons for the case $b = -3/16$ is proved in the same way. Indeed, the results in Section 4.1 still hold in this case, and Proposition 4.5 (iii) holds since velocity of the solitons is negative. Hence the claim follows.

We note that the formula (1.16) still holds including the case $b < 0$, i.e., we have

$$\det[d''(\omega, c)] = \frac{-2P(\phi_{\omega,c})}{\sqrt{4\omega - c^2}\,\{c^2 + \gamma(4\omega - c^2)\}} \quad \text{for } \omega > c^2/4. \tag{4.24}$$

By Proposition 2.4, the momentum $P(\phi_{\omega,c})$ is always positive when $b < 0$, which yields that $d''(\omega, c)$ has one positive eigenvalue. Therefore there exists $\xi \in \mathbb{R}$ such that

$$\langle d'(\omega, c), \xi\rangle \neq 0, \ \langle d''(\omega, c)\xi, \xi\rangle > 0.$$

As in the proof of Proposition 4.5, the calculation of the function $\tau \mapsto d((\omega, c) + \tau\xi)$ and variational characterization yields the control of the flow around the soliton. This is an adaptation of the argument in [5] to our setting, but one cannot treat algebraic solitons in this approach.

Our variational approach offers a new perspective to the stability theory of a two-parameter family of solitons. We note that Proposition 4.5 is obtained without calculating the Hessian matrix $d''(\omega, c)$. The calculation along the scaling curve gives a simpler argument on the stability theory, and also enables us to treat two types of solitons in a unified way. This indicates that the curve (1.12) gives not only the scaling of the soliton but also "good" measure of the stability.

5. Stability of solitons with negative velocity

In this section we study stability of the solitons for the case $b \le -3/16$ and prove Theorem 1.6. For the proof we apply variational arguments introduced by Cazenave and Lions [4]. Here we assume that b and (ω, c) satisfy

$$b \le -3/16\,(\Leftrightarrow \gamma \le 0)\,,\ -2\sqrt{\omega} < c < -2s_*\sqrt{\omega}. \tag{5.1}$$

We remark that our proof in this section still works for the case $b > -3/16$ and $-2\sqrt{\omega} < c < 0$ (see the end of this section).

First we note that

$$\begin{aligned}\mathcal{S}_{\omega,c}(e^{\frac{i}{2}cx}\psi) =& \frac{1}{2}\|\partial_x\psi\|_{L^2}^2 + \frac{1}{2}\left(\omega - \frac{c^2}{4}\right)\|\psi\|_{L^2}^2 + \frac{c}{8}\|\psi\|_{L^4}^4 - \frac{\gamma}{32}\|\psi\|_{L^6}^6 \\ =& \mathcal{E}_c(\psi) + \frac{1}{2}\left(\omega - \frac{c^2}{4}\right)\|\psi\|_{L^2}^2,\end{aligned} \tag{5.2}$$

where $\mathcal{E}_c$ is defined by

$$\mathcal{E}_c(\psi) = \frac{1}{2}\|\partial_x\psi\|_{L^2}^2 + \frac{c}{8}\|\psi\|_{L^4}^4 - \frac{\gamma}{32}\|\psi\|_{L^6}^6.$$

We note that $\mathcal{S}'_{\omega,c}(e^{\frac{i}{2}cx}\psi)$ is equivalent that

$$-\psi'' + \left(\omega - \frac{c^2}{4}\right)\psi + \frac{c}{2}|\psi|^2\psi - \frac{3}{16}\gamma|\psi|^4\psi = 0, \quad x \in \mathbb{R},$$

which is nothing but (1.7).

Now we consider a variational problem with mass constraint:

$$\begin{aligned}\mathscr{A}_m &= \left\{\psi \in H^1(\mathbb{R}) : \|\psi\|_{L^2}^2 = m\right\}, \\ -\nu(c,m) &= \inf\left\{\mathcal{E}_c(\psi) : \psi \in \mathscr{A}_m\right\}, \\ \mathscr{M}_{c,m} &= \left\{\psi \in \mathscr{A}_m : \mathcal{E}_c(\psi) = -\nu(c,m)\right\}\end{aligned}$$

for $m > 0$. We begin with the following lemma.

Lemma 5.1. *Assume $\gamma \le 0$, $c < 0$ and $m > 0$. Then $-\infty < -\nu(c,m) < 0$.*

Proof. From the assumption, $\mathcal{E}_c$ is rewritten as

$$\mathcal{E}_c(\psi) = \frac{1}{2}\|\partial_x\psi\|_{L^2}^2 - \frac{|c|}{8}\|\psi\|_{L^4}^4 + \frac{|\gamma|}{32}\|\psi\|_{L^6}^6.$$

For $\psi \in \mathscr{A}_m$ we set $\psi_\lambda = \lambda^{1/2}\psi(\lambda x)$. Then, $\psi_\lambda \in \mathscr{A}_m$ and

$$\begin{aligned}\mathcal{E}_c(\psi_\lambda) &= \lambda^2\left(\frac{1}{2}\|\partial_x\psi\|_{L^2}^2 + \frac{|\gamma|}{32}\|\psi\|_{L^6}^6\right) - \frac{|c|}{8}\lambda\|\psi\|_{L^4}^4 \\ &= \lambda^2\left(\mathcal{E}(\psi) - \lambda^{-1}\frac{|c|}{8}\|\psi\|_{L^4}^4\right).\end{aligned}$$

One can see $\mathcal{E}_c(\psi_\lambda) < 0$ for sufficiently small $\lambda > 0$, which yields that $-\nu(c,m) < 0$.

By using the following Gagliardo–Nirenberg's inequality

$$\|f\|_{L^4} \le C_1\|\partial_x f\|_{L^2}^{1/4}\|f\|_{L^2}^{3/4},$$

we obtain that

$$\mathcal{E}_c(\psi) \ge \frac{1}{2}\|\partial_x\psi\|_{L^2}^2 - C_1\frac{|c|}{8}\|\partial_x\psi\|_{L^2}\|f\|_{L^2}^3 \ge \frac{1}{4}\|\partial_x\psi\|_{L^2}^2 - C_2c^2\|\psi\|_{L^2}^6 \tag{5.3}$$

for some constant $C_2 > 0$. Therefore we deduce that

$$-\nu(c,m) = \inf_{\psi\in\mathscr{A}_m} \mathcal{E}_c(\psi) \geq -C_2 c^2 m^3 > -\infty.$$

This completes the proof. □

The following claim on sequence compactness plays a key role for the proof of stability in this section.

Proposition 5.2. *Assume $\gamma \leq 0$, $c < 0$ and $m > 0$. If a sequence $\{\psi_n\} \subset H^1(\mathbb{R}) \setminus \{0\}$ satisfies $\|\psi_n\|_{L^2}^2 \to m$ and $\mathcal{E}_c(\psi_n) \to -\nu$, then there exist $\varphi \in \mathscr{M}_{c,m}$ and a sequence $\{y_n\} \subset \mathbb{R}$, such that $\{\psi_n(\cdot - y_n)\}$ has a subsequence that converges to φ strongly in $H^1(\mathbb{R})$.*

For the proof of Theorem 5.2 we use the following Lieb's compactness lemma and Brezis–Lieb's lemma (Lemma 3.2). We note that the original argument in [4] (see also [3, Chapter 8]) relies on the concentration compactness method by Lions [23].

Lemma 5.3 ([21])**.** *Let $\{f_n\}$ be a bounded sequence in $H^1(\mathbb{R})$. Assume that there exists $q \in (2,\infty)$ such that $\limsup_{n\to\infty} \|f_n\|_{L^q} > 0$. Then, there exist $\{y_n\} \subset \mathbb{R}$ and $f \in H^1(\mathbb{R}) \setminus \{0\}$ such that $\{f_n(\cdot - y_n)\}$ has a subsequence that converges to f weakly in $H^1(\mathbb{R})$.*

Proof of Proposition 5.2. We proceed in three steps.

Step 1: Boundedness of $\{\psi_n\}$. From (5.3) we obtain that

$$-\frac{\nu}{2} > \mathcal{E}_c(\psi_n) \geq \frac{1}{4}\|\partial_x\psi_n\|_{L^2}^2 - C_2 c^2 m^3$$

for large n. Since $\|\psi_n\|_{L^2}^2 \to m$, this yields that $\{\psi_n\}$ is bounded in $H^1(\mathbb{R})$. From the definition of $\mathcal{E}_c$ and $\mathcal{E} \geq 0$, we have

$$-\frac{\nu}{2} > \mathcal{E}_c(\psi_n) = \mathcal{E}(\psi_n) - \frac{|c|}{8}\|\psi_n\|_{L^4}^4 \geq -\frac{|c|}{8}\|\psi_n\|_{L^4}^4,$$

which implies that

$$0 < \frac{4\nu}{|c|} < \|\psi_n\|_{L^4}^4 \quad \text{for large } n.$$

To summarize we have obtained that

$$\sup_{n\in\mathbb{N}} \|\psi_n\|_{H^1} < \infty, \ \inf_{n\in\mathbb{N}} \|\psi_n\|_{L^4}^4 > 0. \tag{5.4}$$

Step 2: Limits. From (5.4) one can apply Lemma 5.3 to the sequence $\{\psi_n\}$. Then there exist $\{y_n\} \subset \mathbb{R}$ and $\varphi \in H^1(\mathbb{R}) \setminus \{0\}$ such that a subsequence of $\{\psi_n(\cdot - y_n)\}$ (we denote it by $\{\varphi_n\}$) converges to φ weakly in $H^1(\mathbb{R})$. By the weak lower semicontinuity of the L^2 norm we have

$$\|\varphi\|_{L^2}^2 \leq \liminf_{n\to\infty} \|\varphi_n\|_{L^2}^2 = \lim_{n\to\infty} \|\psi_n\|_{L^2}^2 = m. \tag{5.5}$$

Taking a subsequence (still denoted by the same letter), we have $\varphi_n \to \varphi$ a.e. in $\mathbb{R}$. Applying Lemma 3.2 we obtain that

$$\mathcal{E}_c(\varphi_n) - \mathcal{E}_c(\varphi_n - \varphi) - \mathcal{E}_c(\varphi) \to 0, \tag{5.6}$$

$$\|\varphi_n\|_{L^2}^2 - \|\varphi_n - \varphi\|_{L^2}^2 - \|\varphi\|_{L^2}^2 \to 0. \tag{5.7}$$

Step 3: Strong convergence. We prove by contradiction. Assume that $\|\varphi\|_{L^2}^2 < m$. Then, combined with (5.7) and $\varphi \neq 0$, we have

$$0 < \lim_{n\to\infty} \|\varphi_n - \varphi\|_{L^2}^2 < m.$$

We set $\xi_n = \varphi_n - \varphi$. Following an idea from [6, 2], we modify $\{\xi_n\}$ and φ by using the scaling transformation as

$$\widetilde{\xi}_n(x) = \xi_n(\lambda_n^{-1}x), \ \widetilde{\varphi}(x) = \varphi(\lambda^{-1}x),$$

where

$$\lambda_n = \frac{m}{\|\xi_n\|_{L^2}^2}, \ \lambda = \frac{m}{\|\varphi\|_{L^2}^2}.$$

We note that $\lambda, \lambda_n > 1$ and $\widetilde{\xi}_n, \widetilde{\varphi} \in \mathscr{A}_m$. By a direct calculation we have

$$\begin{aligned} \mathcal{E}_c(\varphi) &= \frac{1-\lambda^{-2}}{2}\|\partial_x\varphi\|_{L^2}^2 + \lambda^{-1}\mathcal{E}_c(\widetilde{\varphi}), \\ \mathcal{E}_c(\xi_n) &= \frac{1-\lambda_n^{-2}}{2}\|\partial_x\xi_n\|_{L^2}^2 + \lambda_n^{-1}\mathcal{E}_c(\widetilde{\xi}_n). \end{aligned} \tag{5.8}$$

Then it follows from (5.6), (5.8) and (5.7) that

$$\begin{aligned} -\nu = \lim_{n\to\infty}\mathcal{E}_c(\varphi_n) &= \lim_{n\to\infty}\mathcal{E}_c(\xi_n) + \mathcal{E}_c(\varphi) \\ &= \lim_{n\to\infty}\left[\frac{1-\lambda_n^{-2}}{2}\|\partial_x\xi_n\|_{L^2}^2 + \frac{1-\lambda^{-2}}{2}\|\partial_x\varphi\|_{L^2}^2 + \lambda_n^{-1}\mathcal{E}_c(\widetilde{\xi}_n) + \lambda^{-1}\mathcal{E}_c(\widetilde{\varphi})\right] \\ &\geq \frac{1-\lambda^{-2}}{2}\|\partial_x\varphi\|_{L^2}^2 - \nu\lim_{n\to\infty}\left(\lambda_n^{-1} + \lambda^{-1}\right) \\ &= \frac{1-\lambda^{-2}}{2}\|\partial_x\varphi\|_{L^2}^2 - \nu > -\nu, \end{aligned}$$

which gives a contradiction. Therefore we deduce that $m = \|\varphi\|_{L^2}^2$.

Since we have the following relation

$$\lim_{n\to\infty}\|\varphi_n\|_{L^2}^2 = m = \|\varphi\|_{L^2}^2,$$

we deduce that

$$\varphi_n \to \varphi \quad \text{in } L^2(\mathbb{R}).$$

From boundedness of $\{\varphi_n\}$ in $H^1(\mathbb{R})$ and elementary interpolation estimates, we have

$$\varphi_n \to \varphi \quad \text{in } L^r(\mathbb{R}) \quad \text{for all } r \in [2,\infty]. \tag{5.9}$$

Combined with the lower semicontinuity of the H^1-norm, we deduce that

$$\mathcal{E}_c(\varphi) \leq \liminf_{n\to\infty}\mathcal{E}_c(\varphi_n) = -\nu.$$

On the other hand, it follows from $\varphi \in \mathscr{A}_m$ that $-\nu(c,m) \leq \mathcal{E}_c(\varphi)$, which yields that $-\nu = \mathcal{E}_c(\varphi)$. Hence $\varphi \in \mathcal{M}_{c,m}$. By (5.6) we have $\mathcal{E}(\varphi_n - \varphi) \to 0$. Combined with (5.9) we obtain that

$$\frac{1}{2}\|\partial_x\varphi_n - \partial_x\varphi\|_{L^2}^2 = \mathcal{E}_c(\varphi_n - \varphi) - \frac{c}{8}\|\varphi_n - \varphi\|_{L^4}^4 + \frac{\gamma}{32}\|\varphi_n - \varphi\|_{L^6}^6 \to 0,$$

which yields that

$$\varphi_n \to \varphi \quad \text{strongly in } H^1(\mathbb{R}).$$

This completes the proof. □

The set $\mathscr{M}_{c,m}$ is characterized as follows.

Lemma 5.4. *Assume* (5.1). *Suppose further that*

$$m = \|\varphi_{\omega,c}\|_{L^2}^2 = \|\Phi_{\omega,c}\|_{L^2}^2. \tag{5.10}$$

Then we have

$$\mathscr{M}_{c,m} = \left\{ e^{i\theta}\Phi_{\omega,c}(\cdot - y) : \theta, y \in \mathbb{R} \right\} \; \textit{and} \; -\nu(c,m) = \mathcal{E}_c(\Phi_{\omega,c}). \tag{5.11}$$

Proof. By Proposition 5.2 we note that $\mathcal{M}_{c,m} \neq \emptyset$. Let $\psi \in \mathscr{M}_{c,m}$. Then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$\mathcal{E}_c'(\psi) + \lambda \mathcal{M}'(\psi) = 0 \iff -\psi'' + \lambda\psi + \frac{c}{2}|\psi|^2\psi - \frac{3}{16}\gamma|\psi|^4\psi = 0.$$

Since $\psi \neq 0$, one can easily prove that $\lambda > 0$. If we set $\tilde{\omega} = \lambda + \frac{c^2}{4} > 0$, then ψ satisfies the equation

$$-\psi'' + \left(\tilde{\omega} - \frac{c^2}{4}\right)\psi + \frac{c}{2}|\psi|^2\psi - \frac{3}{16}\gamma|\psi|^4\psi = 0, \quad x \in \mathbb{R}. \tag{5.12}$$

By uniqueness of the solution of (5.12), there exist $\theta, y \in \mathbb{R}$ such that $\psi = e^{i\theta}\Phi_{\tilde{\omega},c}(\cdot - y)$. From the assumption we have

$$\|\Phi_{\tilde{\omega},c}\|_{L^2}^2 = \|\psi\|_{L^2}^2 = m = \|\Phi_{\omega,c}\|_{L^2}^2.$$

Since $c < 0$, it follows from Lemma 2.2 that the function

$$\left(\frac{c^2}{4}, \infty\right) \ni \mu \mapsto \|\Phi_{\mu,c}\|_{L^2}^2 \in (0, \infty)$$

is strictly increasing, especially which implies that $\tilde{\omega} = \omega$. Hence we have $\psi = e^{i\theta}\Phi_{\omega,c}(\cdot - y)$. We also obtain that

$$-\nu(c,m) = \mathcal{E}_c(\psi) = \mathcal{E}_c(\Phi_{\omega,c}). \tag{5.13}$$

Conversely, if $\psi = e^{i\theta}\Phi_{\omega,c}(\cdot - y)$ for some $\theta, y \in \mathbb{R}$, then it follows from (5.10) and (5.13) that $\psi \in \mathscr{M}_{c,m}$. This completes the proof. $\square$

Next we prove the following claim on sequence compactness.

Proposition 5.5. *Assume* (5.1). *Suppose further that* m *is defined by* (5.10). *If a sequence* $\{\varphi_n\} \subset H^1(\mathbb{R})$ *satisfies*

$$\mathcal{E}(\varphi_n) \to \mathcal{E}(\varphi_{\omega,c}), \; \mathcal{P}(\varphi_n) \to \mathcal{P}(\varphi_{\omega,c}), \; \mathcal{M}(\varphi_n) \to \mathcal{M}(\varphi_{\omega,c}),$$

then there exist a subsequence of $\{\varphi_n\}$ *(still denoted by the same letter) and* $\{\theta_n\}, \{y_n\} \subset \mathbb{R}$ *such that*

$$e^{i\theta_n}\varphi_n(\cdot - y_n) \to \varphi_{\omega,c} \quad \textit{strongly in } H^1(\mathbb{R}).$$

Proof. We first note that

$$\mathcal{E}_c(e^{-\frac{i}{2}cx}\psi) = \mathcal{S}_{\omega,c}(\psi) - \frac{1}{2}\left(\omega - \frac{c^2}{4}\right)\|\psi\|_{L^2}^2 \quad \text{for } \psi \in H^1(\mathbb{R}), \tag{5.14}$$

which follows from (5.2). If we set $\varphi = \varphi_{\omega,c}$, we have

$$\mathcal{E}_c(e^{-\frac{i}{2}cx}\varphi_{\omega,c}) = d(\omega,c) - \frac{1}{2}\left(\omega - \frac{c^2}{4}\right)\|\varphi_{\omega,c}\|_{L^2}^2. \tag{5.15}$$

From the assumption we have

$$\mathcal{S}_{\omega,c}(\varphi_n) \to \mathcal{S}_{\omega,c}(\varphi_{\omega,c}) = d(\omega, c).$$

Combined with (5.14) and (5.15), we have

$$\mathcal{E}_c(e^{-\frac{i}{2}cx}\varphi_n) \to \mathcal{E}_c(e^{-\frac{i}{2}cx}\varphi_{\omega,c}) = \mathcal{E}_c(\Phi_{\omega,c}) = -\nu(c, m),$$

where we used (2.2) and (5.11). Therefore, by Proposition 5.2 and Lemma 5.4, there exist a subsequence of $\{\varphi_n\}$ (still denoted by the same letter), $\{z_n\} \subset \mathbb{R}$ and $\theta, y \in \mathbb{R}$ such that

$$e^{-\frac{i}{2}c(\cdot - z_n)}\varphi_n(\cdot - z_n) \to e^{i\theta}\Phi_{\omega,c}(\cdot - y) \quad \text{strongly in } H^1(\mathbb{R}),$$

which yields that

$$e^{-\frac{i}{2}c(y-z_n)-i\theta}\varphi_n(\cdot + y - z_n) \to e^{\frac{i}{2}c\cdot}\Phi_{\omega,c} = \varphi_{\omega,c} \quad \text{strongly in } H^1(\mathbb{R}).$$

Therefore if we set

$$\theta_n = \frac{c}{2}(z_n - y) - \theta, \ y_n = z_n - y,$$

then the conclusion follows. □

We are now in a position to prove the following stability result.

Theorem 5.6. *Assume* (5.1). *Then the soliton* $v_{\omega,c}$ *of* (1.1′) *is stable.*

Proof. For completeness we give a proof. Assume by contradiction that there exist $\varepsilon > 0$, a sequence of the maximal solutions $\{v_n\}$ to (1.1′) and a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$\|v_n(0) - \varphi_{\omega,c}\|_{H^1} \underset{n\to\infty}{\longrightarrow} 0, \tag{5.16}$$

$$\inf_{(\theta,y)\in\mathbb{R}^2} \|v_n(t_n) - e^{i\theta}\varphi_{\omega,c}(\cdot - y)\|_{H^1} \geq \varepsilon. \tag{5.17}$$

From conservation laws and (5.16), we have

$$\mathcal{E}(v_n(t_n)) = \mathcal{E}(v_n(0)) \to \mathcal{E}(\varphi_{\omega,c}),$$
$$\mathcal{M}(v_n(t_n)) = \mathcal{M}(v_n(0)) \to \mathcal{M}(\varphi_{\omega,c}),$$
$$\mathcal{P}(v_n(t_n)) = \mathcal{P}(v_n(0)) \to \mathcal{P}(\varphi_{\omega,c}).$$

Therefore, by Proposition 5.5, there exist a subsequence of $\{v_n(t_n)\}$ (still denoted by the same letter) and $\{\theta_n\}, \{y_n\} \subset \mathbb{R}$ such that

$$v_n(t_n) - e^{i\theta_n}\varphi_{\omega,c}(\cdot - y_n) \to 0 \quad \text{strongly in } H^1(\mathbb{R}),$$

which contradicts (5.17). □

Proof of Theorem 1.6. Similarly as in the proof of Theorem 1.4, the result follows from Theorem 5.6 and the properties of the gauge transformation $u \mapsto \mathcal{G}(u)$. □

Our proof in this section still works for the case $b > -3/16$ and $-2\sqrt{\omega} < c < 0$. For this case we note that

$$0 < \|\varphi_{\omega,c}\|_{L^2}^2 < \|\varphi_{\omega,0}\|_{L^2}^2 = \frac{2\pi}{\sqrt{\gamma}},$$

which follows from Lemma 2.1. By the following sharp Gagliardo–Nirenberg inequality

$$\frac{\gamma}{32}\|f\|_{L^6}^6 \leq \frac{1}{2}\|\partial_x f\|_{L^2}^2 \cdot \left(\frac{\sqrt{\gamma}}{2\pi}\|f\|_{L^2}^2\right)^2,$$

one can prove that $-\infty < -\nu(c,m) < 0$ for $m \in (0, \frac{2\pi}{\sqrt{\gamma}})$. Other parts in the proof work without any changes. We note that the condition $m \in (0, \frac{2\pi}{\sqrt{\gamma}})$ is essential to prove $-\infty < -\nu(c,m)$, so that we need to restrict our approach to the case of negative velocity.

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