Combinatorial Construction of the Absolute Galois Group of the Field of Rational Numbers

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Abstract

In this paper, we give a purely combinatorial/group-theoretic construction of the conjugacy class of subgroups of the Grothendieck-Teichmüller group $GT$ determined by the absolute Galois group $G_{\mathbb{Q}} \overset{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ [where $\overline{\mathbb{Q}}$ denotes the field of algebraic numbers] of the field of rational numbers $\mathbb{Q}$. In fact, this construction also yields, as a by-product, a purely combinatorial/group-theoretic characterization of the GT-conjugates of closed subgroups of $G_{\mathbb{Q}}$ that are “sufficiently large” in a certain sense. We then introduce the notions of TKND-fields [i.e., “torally Kummer-nondegenerate fields”] and AVKF-fields [i.e., “abelian variety Kummer-faithful fields”], which generalize, respectively, the notions of “torally Kummer-faithful fields” and “Kummer-faithful fields” [notions that appear in previous work of Mochizuki]. For instance, if we write $\mathbb{Q}^{ab}$ for the maximal abelian extension field of $\mathbb{Q}$, then every finite extension of $\mathbb{Q}^{ab}$ is a TKND-AVKF-field [i.e., both TKND and AVKF]. We then apply the purely combinatorial/group-theoretic characterization referred to above to prove that, if a subfield $K \subseteq \overline{\mathbb{Q}}$ is TKND-AVKF, then the commensurator in GT of the subgroup $G_K \subseteq G_{\mathbb{Q}}$ determined by $K$ is contained in $G_{\mathbb{Q}}$. Finally, we combine this computation of the commensurator with a result of Hoshi-Minamide-Mochizuki concerning GT to prove a semi-absolute version of the Grothendieck Conjecture for higher dimensional [i.e., of dimension $\geq 2$] configuration spaces associated to hyperbolic curves of genus zero over TKND-AVKF-fields.

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Introduction

The present paper builds on the theory of combinatorial Belyi cuspidalization
developed in [Tsjm], §1. The theory of combinatorial Belyi cuspidalization
may be understood as a certain combinatorial version of the theory of Belyi
cuspidalization developed in [AbsTopII], §3.

In the present paper, we apply the theory of combinatorial Belyi cuspi-
dalization to give a purely combinatorial/group-theoretic definition of a certain
class of closed subgroups “BGT” [cf. Definition 3.3, (v)] of the Grothendieck-
Teichmüller group

$$GT \subseteq \text{Out}(\Pi_{\mathbb{Q}}^{\text{pd}})$$

where, for $n \geq 1$, $\Pi_{\mathbb{Q}}^{\text{pd}}$ denotes the étale fundamental group of the n-th con-
figuration space associated to the projective line, minus the three points “0”,
“1”, “∞”, over the field of algebraic numbers $\overline{\mathbb{Q}}$ [cf. [CmbCsp], Definition 1.11,
(i); [CmbCsp], Remark 1.11.1; the first display of [HMM], Corollary C]. In the
following, we shall also write $\Pi_{\mathbb{Q}}^{\text{pd}} \overset{\text{def}}{=} \Pi_{\mathbb{Q}}^{\text{pd}}$. This class of closed subgroups
“BGT” is defined to be the class of closed subgroups of $GT$ that satisfy certain
properties, which may be summarized roughly as follows:

- the COF-property, i.e., “cofiltered property” [cf. Definition 3.3, (ii)]:
  for any pair of arithmetic Belyi diagrams [cf. [Tsjm], Definition 1.4], there
  exists an arithmetic Belyi diagram that dominates [cf. Definition 3.3, (i)]
  both of the given arithmetic Belyi diagrams;

- the RGC-property, i.e., “Relative Grothendieck Conjecture property”
  [cf. Definition 3.3, (iii)]: if there exists a geometric domination between
two arithmetic Belyi diagrams, then it is the unique geometric domination
between the two arithmetic Belyi diagrams.

Our first main result is the following [cf. Theorem 4.4]:
Theorem A (Combinatorial construction of an algebraic closure of the field of rational numbers). Let $BGT \subseteq GT$ be a closed subgroup that satisfies the COF- and RGC-properties [cf. Definition 3.3, (ii), (iii), (v)]. Then one may construct from $BGT$ a field $\mathbb{T}_{BGT}$ isomorphic to the field of algebraic numbers $\mathbb{Q}$ and a natural homomorphism

$$C_{GT}(BGT) \to G_{\mathbb{T}_{BGT}} \overset{\text{def}}{=} \text{Aut}(\mathbb{T}_{BGT})$$

from the commensurator of $BGT$ in $GT$ to the group of automorphisms of the field $\mathbb{T}_{BGT}$. In particular, one may construct a natural outer homomorphism

$$C_{GT}(BGT) \to G_{\mathbb{Q}} \overset{\text{def}}{=} \text{Gal}(\mathbb{T}/\mathbb{Q})$$

from the commensurator of $BGT$ in $GT$ to the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$.

At the time of writing, the authors do not know whether or not the outer homomorphism $C_{GT}(BGT) \to G_{\mathbb{Q}}$ is injective in general. On the other hand, by imposing further purely combinatorial/group-theoretic conditions — i.e., the QAA- and AA-properties [cf. Definition 5.12; the brief description following Theorem C below] — on $BGT$, one may conclude that the following hold [cf. Theorems 5.15, (ii); 5.17, (i), (ii)]:

**Theorem B** (Injectivity of the natural homomorphism $C_{GT}(BGT) \to G_{\mathbb{Q}}$). Let $BGT \subseteq GT$ be a closed subgroup that satisfies the COF- and RGC-properties [cf. Definition 3.3, (ii), (iii), (v)]. Suppose that $BGT$ satisfies the QAA-property [cf. Definition 5.12]. Then the above natural outer homomorphism

$$C_{GT}(BGT) \to G_{\mathbb{Q}}$$

is injective.

**Theorem C** (Combinatorial construction of $G_{\mathbb{Q}}$).

(i) Write $\text{Out}^{[\mathbb{C}]}(\Pi^{\text{pd}}) \subseteq \text{Out}(\Pi^{\text{pd}})$ for the closed subgroup of outer automorphisms that induce the identity automorphisms on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi^{\text{pd}}$. Then the conjugacy class of subgroups of $\text{Out}^{[\mathbb{C}]}(\Pi^{\text{pd}})$ determined by the absolute Galois group of $\mathbb{Q}$ may be constructed from the abstract topological group $\Pi^{\text{pd}}$ [cf. Corollary 4.5, Remark 4.5.1], in a purely combinatorial/group-theoretic way, as the set of maximal elements [relative to the relation of inclusion] in the set of closed subgroups of $\text{Out}^{[\mathbb{C}]}(\Pi^{\text{pd}})$ that arise as $\text{Out}^{[\mathbb{C}]}(\Pi^{\text{pd}})$-conjugates of closed subgroups of $GT$ that satisfy the QAA-property [cf. Definitions 3.3, (v); 5.12].
(ii) The conjugacy class of subgroups of $G_T$ determined by the absolute Galois group of $\mathbb{Q}$ may be constructed from the abstract topological group $\Pi^{\text{pd}}_2$ [cf. Corollary 4.5, Remark 4.5.1], in a purely combinatorial/group-theoretic way, as the set of maximal elements [relative to the relation of inclusion] in the set of closed subgroups of $G_T$ that satisfy the AA-property [cf. Definitions 3.3, (v); 5.12].

The class of closed subgroups “BGT” satisfying the QAA-property [i.e., “quasi-algebraically ample property”] (respectively, the AA-property [i.e., “algebraically ample property”]) is defined to be the class of closed subgroups of $G_T$ that satisfy the COF- and RGC-properties, together with certain properties (i), (ii), (iii) (respectively, (i), (ii), (iii), (iv)), which may be summarized roughly as follows:

(i) The Kummer theory associated to BGT is sufficiently nondegenerate.

(ii) The Kummer theory associated to the various arithmetic Belyi diagrams arising from BGT is sufficiently nondegenerate.

(iii) There exists a family of $\overline{\mathbb{Q}}_{\text{BGT}}$-valued set-theoretic functions on a certain set of cuspidal inertia subgroups associated to the various arithmetic Belyi diagrams arising from BGT that satisfies properties satisfied by the function fields arising from these arithmetic Belyi diagrams.

(iv) The family of set-theoretic functions in (iii) determines a Galois group that satisfies a certain compatibility property involving $\Pi^{\text{pd}}_2$.

Of course, it is by no means the case that the approach of Theorem C to constructing the conjugacy class of subgroups of $G_T$ determined by $G_\mathbb{Q}$ is, in any sense, unique. On the other hand, the approach of Theorem C is an attractive application of the technique of combinatorial Belyi cuspidalization developed in [Tsjm], §1. Moreover, the approach of Theorem C has interesting applications, i.e., Theorems F and G, given below.

The approach of Theorem C to constructing the conjugacy class of subgroups of $G_T$ determined by $G_\mathbb{Q}$ may be thought of as a sort of conditional [cf. the condition of maximality within a certain collection of closed subgroups] surjectivity counterpart of the well-known injectivity result of Belyi, i.e., to the effect that the natural outer homomorphism $G_\mathbb{Q} \to G_T$ is injective.

The idea that there should exist such a conditional surjectivity counterpart of Belyi injectivity that could be proven by applying Belyi maps in some suitable fashion [i.e., just as in the case of Belyi injectivity!] was motivated in part by the proofs given in [CmbCsp], §2, §3, of the injectivity/bijectivity of the natural homomorphism

$$\text{Out}^{\text{FC}}(\Pi_n) \to \text{Out}^{\text{FC}}(\Pi_{n-1})$$
of [CmbCsp], Theorem A, (i). That is to say, these proofs given in [CmbCsp], §2, §3, are in some sense remarkable in the sense that

the conditional surjectivity proven in [CmbCsp], §3, is proven by applying an argument that is entirely similar to the argument applied in the proof of the corresponding injectivity result in [CmbCsp], §2.

In this context, it is of interest to note that this fascinating general phenomenon — i.e., of obtaining [conditional] surjectivity results by means of essentially similar arguments to the arguments used to verify corresponding injectivity results — may also be observed in numerous well-known aspects of algebraic topology, such as the theory of long exact sequences of (co)homology groups and the homotopy theory of CW-complexes.

The proofs of Theorems B and C depend on the following elementary field-theoretic results proven in §1 [cf. Theorem 1.2, Corollary 1.3]:

Theorem D (Non-algebraicity of field automorphisms of algebraically closed fields). Let $K$ be an algebraically closed field. Write $\text{Aut}(K)$ for the group of the field automorphisms of $K$. Let $\alpha \in \text{Aut}(K)$. Write

$$\alpha_\Gamma : K \hookrightarrow K \times K = \mathbb{A}^2(K)$$

for the graph of $\alpha$, i.e., the map $K \ni x \mapsto (x, x^\alpha) \in K \times K$. If $K$ is of characteristic 0 (respectively, $p > 0$), then we shall write $\text{Fr} \in \text{Aut}(K)$ for the identity automorphism (respectively, the Frobenius automorphism [i.e., given by raising to the $p$-th power]) of $K$; $\text{Fr}^\mathbb{Z} \subseteq \text{Aut}(K)$ for the subgroup generated by $\text{Fr}$. Then the image $\text{Im}(\alpha_\Gamma) \subseteq \mathbb{A}^2(K)$ of $\alpha_\Gamma$ is Zariski-dense if and only if $\alpha \notin \text{Fr}^\mathbb{Z}$.

Corollary E (A criterion for the algebraicity of certain set-theoretic automorphisms). In the notation of Theorem D, write $X \overset{\text{def}}{=} \mathbb{P}^1_K$ [i.e., the projective line over $K$]. Let $Y \to X$ be a finite ramified Galois covering of smooth, proper, connected curves over $K$. Write $X(K)$ (respectively, $Y(K)$) for the set of $K$-valued points of $X$ (respectively, $Y$); $\text{Aut}_{X(K)}(Y(K))$ for the group of bijections $Y(K) \to Y(K)$ which preserve the fibers of the natural map $Y(K) \to X(K)$; $\text{Gal}(Y/X)$ for the rational function field of $Y$. For $\tau \in \text{Aut}_{X(K)}(Y(K))$, $f \in \text{Fn}(Y(K), K \cup \{\infty\})$ [where $\text{Fn}(-, -)$ denotes the set of maps from the first argument to the second argument], write

$$f^\tau \overset{\text{def}}{=} f \circ \tau \in \text{Fn}(Y(K), K \cup \{\infty\})$$

We shall regard $K(Y)$ as a subset of $\text{Fn}(Y(K), K \cup \{\infty\})$ by evaluating rational functions at closed points of $Y$ and $\text{Gal}(Y/X)$ as a subgroup of $\text{Aut}_{X(K)}(Y(K))$ by means of the natural action of $\text{Gal}(Y/X)$ on $Y(K)$. Let $k \subseteq K$ be a subfield such that the covering $Y \to X$ descends to a Galois covering $Y_k \to X_k$ defined over $k$, and

$$(\text{Aut}(K) \supseteq) \text{Aut}(K/k) \nsupseteq \text{Fr}^\mathbb{Z} \subseteq \text{Aut}(K),$$
where we write $\text{Aut}(K/k) \subseteq \text{Aut}(K)$ for the subgroup of automorphisms that restrict to the identity automorphism of $k$. Let $\sigma \in \text{Aut}_X(K)(Y(K))$ that satisfies the following property: for each $f \in K(Y)^\times$, there exist

$$\phi_f \in \text{F}n(Y(K), k^\times) \subseteq \text{F}n(Y(K), K \cup \{\infty\}), \quad g_f \in K(Y)^\times$$

such that $f^\sigma = \phi_f \cdot g_f$. Then $\sigma \in \text{Gal}(Y/X)$.

Next, let $K \subseteq \overline{\mathbb{Q}}$ be a subfield. Write $G_K \overset{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/K)$. If $K$ is stably $\times\mu$-indivisible [cf. [Tsjm], Definition 3.3, (v)], then we recall from [Tsjm], Corollary E, that one may construct a natural homomorphism

$$C_{\text{GT}}(G_K) \to G_Q$$

whose restriction to $C_{G_Q}(G_K) \subseteq C_{\text{GT}}(G_K)$ is the natural inclusion.

In the present paper, we shall say that the subfield $K \subseteq \overline{\mathbb{Q}}$ is an AVKF-field [i.e., “abelian variety Kummer-faithful field”] if the following property holds [cf. Definition 6.1, (iii)]:

Let $A$ be an abelian variety over a finite extension $L$ of $K$. Write $A(L)^\infty$ for the group of divisible elements $\in A(L)$. Then $A(L)^\infty = \{1\}$.

Here, we recall in passing that any finite extension of the maximal abelian extension field $\mathbb{Q}^{ab} \subseteq \overline{\mathbb{Q}}$ of $\mathbb{Q}$ is a stably $\times\mu$-indivisible AVKF-field [cf. [Tsjm], Theorem 3.1, and its proof; [Tsjm], Remark 3.4.1]. On the other hand, it is not clear to the authors at the time of writing

- whether or not there exist AVKF-fields that are not stably $\times\mu$-indivisible;
- whether or not there exist stably $\times\mu$-indivisible fields that are not AVKF.

If $K$ is an AVKF-field, then one may also construct a natural homomorphism

$$C_{\text{GT}}(G_K) \to G_Q$$

whose restriction to $C_{G_Q}(G_K) \subseteq C_{\text{GT}}(G_K)$ is the natural inclusion [cf. Corollary 6.5, (iii)].

At the time of writing, the authors do not know whether or not these natural homomorphisms [i.e., of Corollary 6.5, (iii), and [Tsjm], Corollary E] are injective in general. On the other hand, by imposing a further condition on $K$, one may conclude that the natural homomorphism $C_{\text{GT}}(G_K) \to G_Q$ arising from Corollary 6.5, (iii), is injective [cf. Theorem F below]. We shall say that the subfield $K \subseteq \overline{\mathbb{Q}}$ is a TKND-field [i.e., “torally Kummer-nondegenerate field”] if the following property holds [cf. Definition 6.6, (iii)]:

Write

$$K_{\text{div}} \overset{\text{def}}{=} \bigcup_{L/K} L_{\times\infty} \subseteq \overline{\mathbb{Q}},$$

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where $L \subseteq \overline{\mathbb{Q}}$ ranges over the finite extensions of $K$, and we write

$$L^\times \overset{\text{def}}{=} L \setminus \{0\}, \quad L_{\infty} \overset{\text{def}}{=} \bigcap_{m \geq 1} (L^\times)_m, \quad L_{\infty} \overset{\text{def}}{=} \mathbb{Q}(L_{\infty}) \subseteq L.$$

Then $\overline{\mathbb{Q}}$ is an infinite field extension of $K_{\text{div}}$.

We shall say that the subfield $K \subseteq \overline{\mathbb{Q}}$ is a TKND-AVKF-field if $K$ is both TKND and AVKF. Our main result concerning TKND-AVKF-fields is the following [cf. Theorem 6.8]:

**Theorem F (Injectivity of the natural homomorphism $C_{\text{GT}}(G_K) \to G_{\overline{\mathbb{Q}}}$).** Suppose that $K \subseteq \overline{\mathbb{Q}}$ is a TKND-AVKF-field. Then the natural homomorphism $C_{\text{GT}}(G_K) \to G_{\overline{\mathbb{Q}}}$ is injective.

Theorem F is proved by applying the theory developed in §3, §4, §5 of the present paper, i.e., the theory that underlies the proof of Theorem C [cf. the discussion surrounding Theorem C].

Finally, by combining Theorem F with certain combinatorial anabelian results proven in §2 of the present paper and applying a result of Hoshi-Minamide-Mochizuki [cf. the first display of [HMM], Corollary C], we obtain a semi-absolute version of the Grothendieck Conjecture for the higher dimensional (of dimension $\geq 2$) configuration spaces [cf. [MT], Definition 2.1, (i)] associated to hyperbolic curves of genus 0 over $K$ [cf. Theorem 6.10, (ii)]:

**Theorem G (Semi-absolute Grothendieck Conjecture-type result over TKND-AVKF-fields).** Let $(m,n)$ be a pair of positive integers; $K,L \subseteq \overline{\mathbb{Q}}$ TKND-AVKF-fields; $X_K$ (respectively, $Y_L$) a hyperbolic curve over $K$ (respectively, $L$). Write $g_X$ (respectively, $g_Y$) for the genus of $X_K$ (respectively, $Y_L$); $(X_K)_m$ (respectively, $(Y_L)_m$) for the $m$-th (respectively, $n$-th) configuration space associated to $X_K$ (respectively, $Y_L$); $G_K \overset{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/K)$ (respectively, $G_L \overset{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/L)$);

for the set of outer isomorphisms $\Pi_{(X_K)_m} \overset{\sim}{\to} \Pi_{(Y_L)_n}$ that induce outer isomorphisms between $G_K$ and $G_L$. Then the following hold:

(i) Suppose that

- $m \geq 4$ or $n \geq 4$ if $X$ or $Y$ is proper;
- $m \geq 3$ or $n \geq 3$ if $X$ or $Y$ is affine.

Then the outer isomorphism

$$G_K \overset{\sim}{\to} G_L$$

induced by any outer isomorphism $\in \text{Out}(\Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L)$ arises from a field isomorphism $K \overset{\sim}{\to} L$.  

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(ii) Suppose that

- \( m \geq 2 \) or \( n \geq 2 \);
- \( g_X = 0 \) or \( g_Y = 0 \).

Then the natural map

\[
\text{Isom}((X_K)_m, (Y_L)_n) \rightarrow \text{Out}((X_K)_m/G_K, (Y_L)_n/G_L)
\]

is bijective.

In this context, we observe that any finite extension \( K \) of \( \mathbb{Q}^{ab} \) is a TKND-AVKF-field [cf. Proposition 6.3, (i); Remark 6.6.3]. Other interesting examples of TKND-AVKF-fields are given in Proposition 6.3, (ii) [cf. also Remarks 6.3.3, 6.3.4, 6.3.5, 6.6.3, 6.6.4]. In particular, we observe [cf. Remark 6.3.5] that

Theorem G constitutes an interesting example of [semi-absolute] anabelian geometry over fields that cannot be treated by means of well-known techniques of anabelian geometry that require the use of \( p \)-adic Hodge theory or Frobenius elements of absolute Galois groups of finite fields [cf. [Tama], Theorem 0.4; [LocAn], Theorem A; [AnabTop], Theorem 4.12].

Next, suppose that \( K \) is a sub-\( p \)-adic subfield [cf. [LocAn], Definition 15.4, (i)] of \( \mathbb{Q} \), i.e., [as is easily verified] a subfield of \( \mathbb{Q} \) that is isomorphic to a subfield of a finite extension of the field of \( p \)-adic numbers \( \mathbb{Q}_p \), for some prime number \( p \). Then \( K \) is a Kummer-faithful field [cf. [AbsTopIII], Definition 1.5; [AbsTopIII], Remark 1.5.4, (i)], hence, in particular, a TKND-AVKF-field. Thus, Theorem G may be regarded as a sort of partial generalization of [AbsTopIII], Theorem 1.9. On the other hand, let us recall that the proof of [AbsTopIII], Theorem 1.9, depends, in an essential way, on [LocAn], Theorem A, hence, in particular, on Faltings’ \( p \)-adic Hodge theory. By contrast, we observe [cf. Remark 3.3.2] that

the proof of Theorem G [say, in the case where \( K \) and \( L \) are assumed to be sub-\( p \)-adic subfields of \( \mathbb{Q} \)] is based solely on results and techniques from combinatorial anabelian geometry and hence is, in particular, entirely independent of results concerning the Grothendieck Conjecture for hyperbolic curves over sub-\( p \)-adic fields [i.e., [LocAn], Theorem A; [Tama], Theorem 0.4].

Moreover, unlike, for instance, [LocAn], Theorem A; [Tama], Theorem 0.4; [AbsCsp], Theorem 3.2,

the proof of Theorem G [say, in the case where \( K \) and \( L \) are assumed to be sub-\( p \)-adic local subfields of \( \mathbb{Q} \)] does not involve the use of any arguments involving theories of “weights”, i.e., theories such as Faltings’ \( p \)-adic Hodge theory or the Weil Conjectures.
Here, we recall that a somewhat weaker version of Theorem G in the case where \( m = n = 1 \) and \( K \) and \( L \) are assumed to be stably \( p\times\mu_p\times\mu_p\)-indivisible fields of characteristic 0 [cf. [Tsjm], Definition 3.3, (v)], i.e., but not necessarily to be TKND-AVKF, is given in [Tsjm], Theorem F. Also, we recall that a version of Theorem G in the case where \( m = n = 1 \) and \( K \) and \( L \) are assumed to be generalized sub-\( p\)-adic may be found in [Hsh2], Corollary 5.6, (ii), (iii).

This paper is organized as follows. In \( \S 1 \), we prove Theorem D and Corollary E, which will be of use in \( \S 5 \). In \( \S 2 \), we give some preliminaries on combinatorial anabelian geometry which will be of use in the later sections. In \( \S 3 \), we give a purely combinatorial/group-theoretic definition of a certain class of closed subgroups \( BGT \) of \( GT \) [cf. the discussion preceding Theorem A]. In \( \S 4 \), for each such closed subgroup \( BGT \subseteq GT \), we give a purely combinatorial/group-theoretic construction of a field \( \mathbb{Q}_{BGT} \) that is isomorphic to the field of algebraic numbers \( \mathbb{Q} \) and, moreover, equipped with a natural action by \( C_{GT}(BGT) \). In particular, we obtain a natural outer homomorphism \( C_{GT}(BGT) \rightarrow G_{\mathbb{Q}} \) [cf. Theorem A]. In \( \S 5 \), by imposing on \( BGT \) certain further combinatorial/group-theoretic conditions, we obtain a certain class of closed subgroups \( BGT \) [cf. the discussion following Theorem C] — whose definition is purely combinatorial/group-theoretic — for which the natural outer homomorphism \( C_{GT}(BGT) \rightarrow G_{\mathbb{Q}} \) is injective [cf. Theorem B]. Moreover, we obtain Theorem C as a consequence of this injectivity. Finally, in \( \S 6 \), we study various types of fields and apply the theory of \( \S 1, \S 2, \S 3, \S 4, \S 5 \), to prove Theorems F and G.

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Notations and Conventions

**Sets:** Let \( A, B \) be sets. Then we shall write \( \text{Fn}(A, B) \) for the set of maps from \( A \) to \( B \). If \( \text{Fn}(A, B) \ni f : A \rightarrow B \) is held fixed in a discussion, then we shall write \( \text{Aut}_B(A) \) for the group of bijections \( A \rightarrow A \) which preserve the fibers of \( f \) over \( B \).

**Numbers:** The notation \( \text{Primes} \) will be used to denote the set of prime numbers. The notation \( \mathbb{N} \) will be used to denote the set or, by a slight abuse of notation, additive monoid of non-negative integers.

**Fields:** The notation \( \mathbb{Q} \) will be used to denote the field of rational numbers. The notation \( \mathbb{Z} \) will be used to denote the ring of integers of \( \mathbb{Q} \); by a slight abuse
of notation, the notation \( \mathbb{Z} \) will also be used to denote the underlying additive group of this ring. The notation \( \mathbb{C} \) will be used to denote the field of complex numbers. The notation \( \mathbb{Q} \subseteq \mathbb{C} \) will be used to denote the set or field of algebraic numbers \( \mathbb{Q} \subseteq \mathbb{C} \). We shall refer to a finite extension field of \( \mathbb{Q} \) as a number field.

If \( q \) is a power of a prime number, then we shall write \( \mathbb{F}_q \) for the finite field consisting of \( q \) elements.

Let \( F \) be a field, \( p \) a prime number, \( n \) a positive integer. Then we shall write \( \text{Aut}(F) \) for the group of field automorphisms of \( F \);

\[
F^\times \overset{\text{def}}{=} F \setminus \{0\}; \quad F^{\times 0} \overset{\text{def}}{=} F \setminus \{0, 1\}; \quad \mu_n(F) \overset{\text{def}}{=} \{ x \in F^\times \mid x^n = 1 \};
\]

\[
\mu(F) \overset{\text{def}}{=} \bigcup_{m \geq 1} \mu_m(F); \quad F^{\times \infty} \overset{\text{def}}{=} \bigcap_{m \geq 1} (F^\times)^m;
\]

\[
\mu_p(F) \overset{\text{def}}{=} \bigcup_{m \geq 1} \mu_{p^m}(F); \quad F^{\times p^\infty} \overset{\text{def}}{=} \bigcap_{m \geq 1} (F^\times)^{p^m}.
\]

If \( K \) is an extension field of \( F \), then we shall write \( \text{Aut}(K/F) \subseteq \text{Aut}(K) \) for the subgroup of automorphisms that restrict to the identity automorphism of \( F \).

**Topological groups:** Let \( G \) be a topological group and \( H \subseteq G \) a closed subgroup of \( G \). Then we shall denote by \( Z_G(H) \) (respectively, \( N_G(H) \); \( C_G(H) \)) the centralizer (respectively, normalizer; commensurator) of \( H \subseteq G \), i.e.,

\[
Z_G(H) \overset{\text{def}}{=} \{ g \in G \mid ghg^{-1} = h \text{ for any } h \in H \}; \quad N_G(H) \overset{\text{def}}{=} \{ g \in G \mid g \cdot H \cdot g^{-1} = H \}; \quad C_G(H) \overset{\text{def}}{=} \{ g \in G \mid H \cap g \cdot H \cdot g^{-1} \text{ is of finite index in } H \text{ and } g \cdot H \cdot g^{-1} \}.
\]

We shall say that the closed subgroup \( H \) is normally terminal in \( G \) if \( H = N_G(H) \). We shall say that the closed subgroup \( H \) is commensurably terminal in \( G \) if \( H = C_G(H) \). We shall say that \( G \) is slim if \( Z_G(U) = \{1\} \) for any open subgroup \( U \) of \( G \).

Let \( G \) be a topological group. Then we shall write \( G^{\text{ab}} \) for the quotient of \( G \) by the closure of the commutator subgroup \( [G,G] \subseteq G \); \( \text{Aut}(G) \) for the group of (continuous) automorphisms of \( G \); \( \text{Inn}(G) \subseteq \text{Aut}(G) \) for the group of inner automorphisms of \( G \); \( \text{Out}(G) \overset{\text{def}}{=} \text{Aut}(G)/\text{Inn}(G) \). Now suppose that \( G \) is center-free [i.e., \( Z_G(G) = \{1\} \)]. Then we have an exact sequence of groups

\[
1 \longrightarrow G \overset{\sim}{\longrightarrow} \text{Inn}(G) \longrightarrow \text{Aut}(G) \longrightarrow \text{Out}(G) \longrightarrow 1.
\]

If \( J \) is a group, and \( \rho : J \rightarrow \text{Out}(G) \) is a homomorphism, then we shall denote by

\[
G \rtimes J
\]

the group obtained by pulling back the above exact sequence of groups via \( \rho \). Thus, we have a natural exact sequence of groups.
Suppose further that $G$ is profinite and topologically finitely generated. Then one verifies immediately that the topology of $G$ admits a basis of characteristic open subgroups, which thus induces a profinite topology on the groups $\text{Aut}(G)$ and $\text{Out}(G)$ with respect to which the above exact sequence relating $\text{Aut}(G)$ and $\text{Out}(G)$ determines an exact sequence of profinite groups. In particular, one verifies easily that if, moreover, $J$ is profinite, and $\rho : J \to \text{Out}(G)$ is continuous, then the above exact sequence relating $G \rtimes J$ to $G$ and $J$ determines an exact sequence of profinite groups.

Fundamental groups: For a connected Noetherian scheme $S$, we shall write $\Pi_S$ for the étale fundamental group of $S$, relative to a suitable choice of basepoint.

Schemes: For a morphism of scheme $S \to T$, we shall write $\text{Aut}_T(S)$ for the group of automorphisms of the $T$-scheme $S$. If $T = \text{Spec} \mathbb{Z}$, then we shall write $\text{Aut}(S)$ for $\text{Aut}_T(S)$.

Log schemes: We shall, by a slight abuse of notation, regard schemes as log schemes equipped with the trivial log structure. If $S^{\log}$ is a log scheme, then we shall write $S$ for the underlying scheme of $S^{\log}$ and $U_S \subseteq S$ for the interior of $S^{\log}$, i.e., the largest open subscheme of $S$ over which the log structure of $S^{\log}$ is trivial.

Curves: We shall use the terms “hyperbolic curve”, “cusp”, “stable log curve”, “smooth log curve”, and “tripod” as they are defined in [CmbGC], §0; [CmbCsp], §0. We shall use the terms “$n$-th configuration space” and “$n$-th log configuration space” as they are defined in [MT], Definition 2.1, (i).

1 The non-algebraicity of field automorphisms

In this section, we discuss an interesting elementary property of field automorphisms of algebraically closed fields, namely, that, with the exception of integral powers of the Frobenius automorphism, such field automorphisms cannot be expressed algebraically [cf. Theorem 1.2]. We then apply this property to give a criterion for the algebraicity of certain set-theoretic automorphisms of sets of rational points of curves valued in algebraically closed fields [cf. Corollary 1.3]. This criterion will play an important role in the theory to be developed in the present paper.

**Lemma 1.1 (The inversion map on the multiplicative group of a field).** Let $k$ be a field. Write

$$\sigma : k^\times \cup \{0\} \cong k^\times \cup \{0\}$$

for the bijection such that
\( \sigma(x) = x^{-1} \) for each \( x \in k^\times \),
\( \sigma(0) = 0 \).

Then the following hold:

(i) The bijection \( \sigma \) is a field automorphism if and only if \( k \cong F_2, F_3, \) or \( F_4 \).

(ii) If \( k \cong F_2 \) or \( F_3 \) (respectively, \( k \cong F_4 \)), then \( \sigma \) is the identity (respectively, the unique non-trivial) automorphism of \( k \).

Proof. First, we verify assertion (i). Sufficiency is immediate. Next, to verify necessity, we observe that if \( \sigma \) is a field automorphism, then, for \( x \in k \setminus \{0, -1\} \),
\[
1 + \frac{1}{x} = \sigma(1) + \sigma(x) = \sigma(1 + x) = \frac{1}{1 + x} \quad (\iff x^2 + x + 1 = 0).
\]
Since the equation \( x^2 + x + 1 = 0 \) has at most 2 solutions in \( k \), we thus conclude that the cardinality of \( k \) is \( 4 \). Assertion (ii) follows immediately from the definitions. This completes the proof of Lemma 1.1.

**Theorem 1.2 (Non-algebraicity of field automorphisms of algebraically closed fields).** Let \( K \) be an algebraically closed field; \( \alpha \in \text{Aut}(K) \). Write
\[
\alpha \Gamma : K \leftrightarrow K \times K = A^2(K)
\]
for the graph of \( \alpha \), i.e., the map \( K \ni x \mapsto (x, x^\alpha) \in K \times K \). If \( K \) is of characteristic 0 (respectively, \( p > 0 \)), then we shall write \( \text{Fr} \in \text{Aut}(K) \) for the identity automorphism (respectively, the Frobenius automorphism [i.e., given by raising to the \( p \)-th power]) of \( K \); \( \text{Fr}^Z \subseteq \text{Aut}(K) \) for the subgroup generated by \( \text{Fr} \). Then the image \( \text{Im}(\alpha \Gamma) \subseteq A^2(K) \) of \( \alpha \Gamma \) is Zariski-dense if and only if \( \alpha \notin \text{Fr}^Z \).

Proof. Necessity is immediate. Thus, it remains to verify sufficiency. If \( \alpha \Gamma \) is not Zariski-dense, then there exists a nonzero polynomial
\[
0 \neq f = f(X, Y) = \sum a_{i,j} X^i Y^j \in K[X, Y]
\]
such that
\[
\text{Im}(\alpha \Gamma) \subseteq V(f) \subseteq A^2(K),
\]
where \( V(f) \) denotes the zero set of \( f \). In particular, for \( x \in K \), we have
\[
\sum a_{i,j} x^i(x^\alpha)^j = 0.
\]
For \( x \in K^\times \), write \( \rho_{i,j}(x) \overset{\text{def}}{=} x^i(x^\alpha)^j \in K^\times \). Then \( \rho_{i,j} : K^\times \to K^\times \) is a character. Thus, it follows immediately from Artin’s well-known result on the linear independence of characters that there exist pairs of integers \( (i_1, j_1) \neq (i_2, j_2) \in \mathbb{N} \times \mathbb{N} \) such that \( \rho_{i_1,j_1} = \rho_{i_2,j_2} \). In particular, there exists a pair of integers
\[
(i, j) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}
\]
such that
\[ x^i = (x^j)^\alpha \]
for \( x \in K^\times \). Since \( K \) is algebraically closed, it follows that \( i \neq 0, j \neq 0 \). Moreover, since \( K^\times \) is divisible, we may assume without loss of generality that \( i \) and \( j \) are co-prime.

Now suppose that the characteristic of \( K \) is \( p > 0 \). Write \( \phi_i : K^\times \to K^\times \) (respectively, \( \phi_j : K^\times \to K^\times \)) for the surjection determined by \( x \mapsto x^i \) (respectively, \( x \mapsto x^j \)). Since \( x^i = (x^j)^\alpha \) for \( x \in K^\times \), it follows that \( \text{Ker}(\phi_i) = \text{Ker}(\phi_j) \). Since \( i \) and \( j \) are co-prime, we thus conclude that \( i, j \notin \{ \pm p^\ell \} \). Moreover, we may assume without loss of generality that \( j = 1 \). Thus, by applying Lemma 1.1, (i), we conclude that \( \alpha \in \mathbb{F}_p \).

Next, we consider the case where the characteristic of \( K \) is 0. In this case, we have, for example, \( 2^i = 2^j \). This implies that \( i = j \). Thus, since \( i \) and \( j \) are co-prime, we conclude that \( \alpha \in \mathbb{F}_p \). This completes the proof of Theorem 1.2.

\[ \square \]

**Remark 1.2.1.**

(i) Theorem 1.2 was in some sense motivated by the following complex analytic analogue of Theorem 1.2, i.e., the non-holomorphicity of the automorphism of \( \mathbb{C} \) given by complex conjugation. Let \( n \) be a positive integer; \( U \subset \mathbb{C} \) a nonempty relatively compact open subset; \( \{ f_j(z) \}_{1 \leq j \leq n} \) a family of holomorphic functions on \( U \). Write \( \mu \) for the Lebesgue measure on \( \mathbb{C} \); \( z \in \mathbb{C} \) for the complex conjugate of \( z \in \mathbb{C} \). Then
\[ \exists z \in U \text{ such that } \bar{z} \notin \{ f_j(z) \}_{1 \leq j \leq n}. \]

Indeed, suppose that \( \bar{z} \in \{ f_j(z) \}_{1 \leq j \leq n} \) for every \( z \in U \). By enlarging the family of holomorphic functions \( \{ f_j(z) \}_{1 \leq j \leq n} \) if necessary, we may assume without loss of generality that it is stabilized by multiplication by \(-1\). Write
\[ g_j(z) \overset{\text{def}}{=} f_j(z) + z, \quad E_j \overset{\text{def}}{=} \{ z \in U \mid \pm \bar{z} = f_j(z) \}. \]

Then it follows immediately from the definitions that \( E_j \subset U \) is a closed [hence, in particular, Lebesgue measurable] subset, and \( U = \bigcup_{1 \leq j \leq n} E_j \).

Thus, we conclude that
\[ 0 < \mu(U) \leq \sum_{1 \leq j \leq n} \mu(E_j) < \infty. \]

In particular, there exists an element \( j \in \{1, \ldots, n\} \) such that \( \mu(E_j) > 0 \). Fix such an element \( j \). Since the family of holomorphic functions \( \{ f_j(z) \}_{1 \leq j \leq n} \) is stabilized by multiplication by \(-1\), by possibly replacing \( j \) by \( j' \in \{1, \ldots, n\} \) such that \( f_j(z) = -f_{j'}(z) \) for \( z \in U \) [which implies that \( E_j = E_{j'} \)], we may assume without loss of generality that \( g_j(z) \) is
a non-constant holomorphic function. But then \( g_j(E_j) \subseteq \mathbb{R} \cup \sqrt{-1} \cdot \mathbb{R} \), which implies that
\[
0 < \mu(g_j(E_j)) \leq \mu(\mathbb{R} \cup \sqrt{-1} \cdot \mathbb{R}) = 0
\]
— a contradiction!

(ii) Finally, we observe that Theorem 1.2 in the case where \( K = \mathbb{C} \), and \( \alpha \) is the automorphism given by complex conjugation follows immediately from the fact verified in Remark 1.2.1, (i). Indeed, if \( \alpha_T \) is not Zariski-dense, then there exists a nonzero polynomial
\[
0 \neq f = f(X,Y) = \sum a_{i,j} X^i Y^j \in \mathbb{C}[X,Y]
\]
such that
\[
\text{Im}(\alpha_T) \subseteq V(f) \subseteq \mathbb{A}^2(\mathbb{C}),
\]
where \( V(f) \) denotes the zero set of \( f \). Since the map \( V(f) \to \mathbb{C} \) induced by the first projection \( \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) is a nonconstant algebraic map [i.e., corresponds to a dominant morphism between one-dimensional schemes of finite type over \( \mathbb{C} \)], there exists a nonempty relatively compact open subset \( U \subseteq \mathbb{C} \) such that the induced map
\[
V(f)|_U \overset{\text{def}}{=} V(f) \cap (U \times \mathbb{C}) \to U
\]
determines a split finite étale morphism of complex analytic spaces. The finite collection of sections of this induced map thus determines a family of holomorphic functions as in Remark 1.2.1, (i). This yields the desired contradiction.

**Corollary 1.3 (A criterion for the algebraicity of certain set-theoretic automorphisms).** In the notation of Theorem 1.2, write \( X \overset{\text{def}}{=} \mathbb{P}_K^1 \) [i.e., the projective line over \( K \)]. Let \( Y \to X \) be a finite ramified Galois covering of smooth, proper, connected curves over \( K \). Write \( X(K) \) (respectively, \( Y(K) \)) for the set of \( K \)-valued points of \( X \) (respectively, \( Y \)); \( \text{Aut}_{X(K)}(Y(K)) \) for the group of bijections \( Y(K) \to Y(K) \) which preserve the fibers of the natural map \( Y(K) \to X(K) \); \( K(Y) \) for the rational function field of \( Y \). For \( \tau \in \text{Aut}_{X(K)}(Y(K)), f \in \text{Fn}(Y(K), K \cup \{\infty\}) \), write
\[
f^\tau \overset{\text{def}}{=} f \circ \tau \in \text{Fn}(Y(K), K \cup \{\infty\}).
\]
We shall regard \( K(Y) \) as a subset of \( \text{Fn}(Y(K), K \cup \{\infty\}) \) by evaluating rational functions at closed points of \( Y \) and \( \text{Gal}(Y/X) \) as a subgroup of \( \text{Aut}_{X(K)}(Y(K)) \) by means of the natural action of \( \text{Gal}(Y/X) \) on \( Y(K) \). Let \( k \subseteq K \) be a subfield such that the covering \( Y \to X \) descends to a Galois covering \( Y_k \to X_k \) defined over \( k \), and
\[
(\text{Aut}(K) \supseteq) \text{Aut}(K/k) \not\subseteq \text{Fr}^Z (\subseteq \text{Aut}(K)).
\]
Let $\sigma \in \text{Aut}_{\mathcal{X}(K)}(Y(K))$ that satisfies the following property: for each $f \in K(Y)^\times$, there exist
\[ \phi_f \in \text{Fn}(Y(K), k^\times) \left( \subseteq \text{Fn}(Y(K), K \cup \{\infty\}) \right), \quad g_f \in K(Y)^\times \]
such that $f^\sigma = \phi_f \cdot g_f$. Then $\sigma \in \text{Gal}(Y/X)$.

Proof. Write $n$ for the degree of the covering $Y \to X$; $\sigma_1, \ldots, \sigma_n$ for the $n$ distinct elements of $\text{Gal}(Y/X)$. Let $\alpha \in \text{Aut}(K/k) \setminus \mathbb{P}^2$. Write
\[
\begin{align*}
\alpha_{\Gamma, X} : X(K) &\to X(K) \times X(K), \\
\alpha_{\Gamma, Y} : Y(K) &\to Y(K) \times Y(K)
\end{align*}
\]
for the respective graphs of $\alpha$, i.e., the maps $X(K) \ni x \mapsto (x, x^\alpha) \in X(K) \times X(K)$ and $Y(K) \ni y \mapsto (y, y^\alpha) \in Y(K) \times Y(K)$. Then it follows immediately from Theorem 1.2 that the subset $\text{Im}(\alpha_{\Gamma, X}) \subseteq X(K) \times X(K)$ is Zariski-dense in $X(K) \times X(K)$. Next, we observe that
\begin{itemize}
  \item the covering $Y \to X$, hence also the morphism $Y \times Y \to X \times X$ [i.e., the product over $K$ of two copies of the covering $Y \to X$] is finite;
  \item the map $\text{Im}(\alpha_{\Gamma, Y}) \to \text{Im}(\alpha_{\Gamma, X})$ induced by the finite morphism $Y \times Y \to X \times X$ is surjective.
\end{itemize}
Thus, since the Zariski closure of $\text{Im}(\alpha_{\Gamma, Y})$ is an algebraic set in $Y(K) \times Y(K)$, it follows immediately from the above observations that $\text{Im}(\alpha_{\Gamma, Y})$ is Zariski-dense in $Y(K) \times Y(K)$.

Next, we observe that the existence of the Galois covering $Y_k \to X_k$ [i.e., whose base-change over $k$ to $K$ is the covering $Y \to X$] implies that the natural action of $\text{Aut}(K/k)$ on $K$ induces a natural action of $\text{Aut}(K/k)$ on $Y(K)$ that commutes with the natural action of $\text{Gal}(Y/X)$ on $Y(K)$. If, moreover, $\beta \in \text{Aut}(K/k)$, $h \in \text{Fn}(Y(K), K \cup \{\infty\})$, then we shall write
\[ h^\beta \overset{\text{def}}{=} \beta^{-1} \circ h \circ \beta \in \text{Fn}(Y(K), K \cup \{\infty\}). \]

For each pair of integers $(i, j)$ such that $1 \leq i, j \leq n$, write
\[ Y_{i,j} \overset{\text{def}}{=} \{(y_1, y_2) \in Y(K) \times Y(K) \mid y_1^{\sigma_i} = y_1, \ y_2^{\sigma^{\alpha^{-1}\sigma_i}} = y_2^{\alpha^{-1}\sigma_i}\}. \]

Since $\sigma \in \text{Aut}_{\mathcal{X}(K)}(Y(K))$, it follows immediately that
\[ Y(K) \times Y(K) = \bigcup_{1 \leq i, j \leq n} Y_{i,j}. \]

Write
\[ Z_{i,j} \]
for the Zariski closure of $\text{Im}(\alpha_{\Gamma, Y}) \cap Y_{i,j}$ in $Y(K) \times Y(K)$. Since the subset $\text{Im}(\alpha_{\Gamma, Y}) \subseteq Y(K) \times Y(K)$ is Zariski-dense, there exists a pair of integers $(i, j)$ such that
\[ Y(K) \times Y(K) = Z_{i,j}. \]
Fix such a pair of integers \((i, j)\).

Next, we observe that, for each \(f \in K(Y)^\times\),

\[
(\phi_f^\sigma, (\phi_f^{\alpha^{-1}})^\sigma) = (f^{\sigma_\sigma}, (g_f^{-1})^{\sigma_i}, \{(f^\sigma)^{\alpha^{-1}}\}^{\sigma_i}, \{(g_f^{-1})^{\alpha^{-1}}\}^{\sigma_i})
\]

on some subset \(Y_{i,j}^* \subseteq Y_{i,j}\) [i.e., so that all of the values of functions that appear are finite] such that \(Y_{i,j} \setminus Y_{i,j}^*\) is a finite set — which implies that the Zariski closure \(Z_{i,j}^*\) of \(\text{Im}(\alpha_{\Gamma, Y}) \cap Y_{i,j}^*\) is equal to \(Y(K) \times Y(K)\). Now consider the morphism

\[
\psi \overset{\text{def}}{=} (h_f^1, h_f^1) : Y \times_K Y \to \mathbb{P}_K^1 \times_K \mathbb{P}_K^1,
\]
determined by the rational functions \(h_f^1 \overset{\text{def}}{=} f \cdot (g_f^{-1})^{\sigma_\sigma}\) and \(h_f^1 \overset{\text{def}}{=} (f^{\sigma_\sigma})^{\sigma_i}, \{(g_f^{-1})^{\alpha^{-1}}\}^{\sigma_i}\). Write \(\Delta\) for the diagonal divisor of \(\mathbb{P}_K^1 \times_K \mathbb{P}_K^1\). Then it follows immediately from the above observation [i.e., the observation discussed at the beginning of the present paragraph], together with the fact that the natural actions of \(\alpha\) and \(\sigma_i\) on \(Y(K)\) commute, that

\[
\psi(\text{Im}(\alpha_{\Gamma, Y}) \cap Y_{i,j}^*) \subseteq \Delta(k) \subseteq \Delta(K) \subseteq \mathbb{P}_K^1(K) \times \mathbb{P}_K^1(K).
\]

Since \(Y(K) \times Y(K) = Z_{i,j}^*\), we conclude that \(\text{Im}(\psi) \subseteq \Delta(K)\), hence, in particular, that the morphism \(\psi\) is not dominant. On the other hand, if both \(h_f^1\) and \(h_f^1\) are nonconstant rational functions, then the morphism \(\psi\) is easily verified to be dominant. Thus, we conclude that either \(h_f^1\) or \(h_f^1\) is constant, and hence, since \(\text{Im}(\psi) \subseteq \Delta(K)\), that both \(h_f^1\) and \(h_f^1\) are constant. Write \(c_f \in K\) for the unique constant value of \(h_f^1\). Thus,

\[
f^\tau = \phi_f \cdot g_f = c_f^{-1} \cdot \phi_f \cdot f^{\sigma_\sigma^{-1}},
\]

for every \(f \in K(Y)^\times\). In particular, if we write \(\tau \overset{\text{def}}{=} \sigma_\sigma, \phi_f^1 \overset{\text{def}}{=} \phi_f^{\sigma_i}, \text{then}

\[
f^\tau = c_f^{-1} \cdot \phi_f^1 \cdot f,
\]

for every \(f \in K(Y)^\times\). For each \(y \in Y(K)\), let \(f_y \in K(Y)^\times\) be a rational function on \(Y\) such that \(f_y\) has a pole at \(y\) and no pole on \(Y(K) \setminus \{y\}\). [The existence of such rational functions follows immediately from the Riemann-Roch theorem.] Thus, since \(f_y^\tau = c_f^{-1} \cdot \phi_f^1 \cdot f_y\), we conclude that \(y^\tau = y\) for each \(y \in Y(K)\), hence that \(\tau\) is the identity automorphism, i.e., \(\sigma = \sigma_i^{-1} \in \text{Gal}(Y/X)\). This completes the proof of Corollary 1.3. 

\(\Box\)
Remark 1.3.1.

(i) Corollary 1.3 was in some sense motivated by the following complex analytic analogue of Corollary 1.3. Write $S^1 \overset{\text{def}}{=} \{ z \in \mathbb{C} \mid |z| = 1 \} \subseteq \mathbb{C}^\times$. In the notation of Corollary 1.3 in the case where $K \subseteq \mathbb{C}$, let $\zeta \in \text{Aut}_{X(K)}(Y(K))$ that satisfies the following property: for each $f \in K(Y)^\times$, there exist

$$\omega_f \in \text{Fn}(Y(K), S^1), \quad q_f \in K(Y)^\times$$

such that $f^\zeta = \omega_f \cdot q_f$. Then

$$\zeta \in \text{Gal}(Y/X).$$

Indeed, write $\mu$ for the Lebesgue measure on $\mathbb{C}$; $\mu_Y$ for the measure on $Y(\mathbb{C})$ induced by a nowhere-vanishing volume form on the Riemann surface associated to $Y \times_K \mathbb{C}$; $n$ for the degree of the covering $Y \rightarrow X$; $\zeta_1, \ldots, \zeta_n$ for the $n$ distinct elements of $\text{Gal}(Y/X)$. For each $j = 1, \ldots, n$, write

$$E_j \overset{\text{def}}{=} \{ y \in Y(K) \mid y^\zeta_j = y \} \subseteq Y(\mathbb{C});$$

$$F_j \subseteq Y(\mathbb{C})$$

for the closure of $E_j \subseteq Y(\mathbb{C})$ in the complex topology [i.e., the topology induced by the topology of the topological field $\mathbb{C}$]. Thus, $E_j \subseteq Y(\mathbb{C})$ is measurable [i.e., with respect to the measure $\mu_Y$]. Note that, since $\zeta \in \text{Aut}_{X(K)}(Y(K))$,

$$\bigcup_{1 \leq j \leq n} E_j = Y(K).$$

Since the subset $Y(K) \subseteq Y(\mathbb{C})$ is easily verified to be dense in the complex topology, it follows immediately that

$$\bigcup_{1 \leq j \leq n} F_j = Y(\mathbb{C}).$$

Thus, we conclude that

$$0 < \mu_Y(Y(\mathbb{C})) \leq \sum_{1 \leq j \leq n} \mu_Y(F_j) < \infty.$$ 

In particular, there exists an element $j \in \{1, \ldots, n\}$ such that $\mu_Y(F_j) > 0$. Fix such an element $j$. Next, for each $f \in K(Y)^\times$, it follows immediately that

$$\omega_f^\zeta_j = f^\zeta_j \cdot (q_f^\zeta_j)^{-1} = f \cdot (q_f^\zeta_j)^{-1}$$

on some subset $E_j^* \subseteq E_j$ [i.e., so that all of the values of functions that appear are finite] such that $E_j \setminus E_j^*$ is a finite set — which implies that $\mu_Y(F_j^*) > 0$, where $F_j^*$ denotes the closure of $E_j^* \subseteq Y(\mathbb{C})$ in the complex topology. Thus, we conclude that, for $y \in F_j^* (\subseteq Y(\mathbb{C}))$,

$$|f \cdot (q_f^\zeta_j)^{-1})(y)| = 1 \iff (f \cdot (q_f^\zeta_j)^{-1})(y) \in S^1.$$
In particular, since \( \mu(S^1) = 0 \) and \( \mu_\gamma(F_\gamma^*) > 0 \), the meromorphicity of [the function \( Y(\mathbb{C}) \to \mathbb{C} \cup \{ \infty \} \) determined by] \( f \cdot (q_\gamma^*)^{-1} \) implies that \( f \cdot (q_\gamma^*)^{-1} \) is in fact a constant function. Thus, we conclude as in the final portion of the proof of Corollary 1.3 that \( \zeta \in \text{Gal}(Y/X) \).

(ii) Finally, we observe that Corollary 1.3 in the case where

\[ \bullet \quad K = \overline{\mathbb{Q}}, k = \mathbb{Q}^{ab} \triangleq \mathbb{Q}(\mu(\overline{\mathbb{Q}})) (\subseteq \overline{\mathbb{Q}} \subseteq \mathbb{C}); \]

\[ \bullet \quad \text{for each } f \in K(Y)^\times, \]

\[ \phi_f \in \text{Fn}(Y(\overline{\mathbb{Q}}), \mu(\mathbb{Q}^{ab})) (\subseteq \text{Fn}(Y(\overline{\mathbb{Q}}), (\mathbb{Q}^{ab})^\times)), \]

follows immediately [since \( \mu(\mathbb{Q}^{ab}) \subseteq S^1 \)] from the fact verified in Remark 1.3.1, (i).

## 2 Preliminaries on combinatorial anabelian geometry

In this section, we give some preliminaries on combinatorial anabelian geometry which will be of use in the theory developed in the present paper.

**Theorem 2.1 (Outer automorphisms of configuration space groups induced by open immersions).** Let \( n \) be an integer such that \( n \geq 2 \); \( k \) an algebraically closed field of characteristic 0; \( X \) a hyperbolic curve over \( k \) of type \( (g, r_X) \); \( U \) an open subscheme of \( X \) which is a hyperbolic curve over \( k \) of type \( (g, r_U) \), where \( r_U > r_X \) [which implies that \( (g, ru) \notin \{(0, 3), (1, 1)\}; ru > 0 \]. Write \( \mathfrak{S}_n \) for the symmetric group on \( n \) letters; \( X_n \) (respectively, \( U_n \)) for the \( n \)-th configuration space associated to \( X \) (respectively, \( U \)). Let

\[ \alpha \in \text{Out}(\Pi_{U_n}). \]

Recall that there exists a unique permutation \( \sigma \in \mathfrak{S}_n \subseteq \text{Out}(\Pi_{U_n}) \) of the factors of \( U_n \) [cf. [ChTpII], Theorem B] such that

\[ \bullet \quad \alpha \circ \sigma \in \text{Out}(\Pi_{U_n}) \] [cf. [ChTpII], Theorem B, (i)];

\[ \bullet \quad \text{the outer automorphism } \alpha_1 \in \text{Out}(\Pi_U) \text{ induced by } \alpha \circ \sigma \text{ which does not depend on the choice of projection morphisms of co-length } 1 \text{ — cf. [ChTpI], Theorem A, (i)] preserves the set of cuspidal inertia subgroups of } \Pi_U \text{ [cf. [ChTpI], Theorem A, (ii)].} \]

Suppose that

\[ (a) \quad \text{if } n = 2, \text{ then either } r_X > 0 \text{ or } \alpha \circ \sigma \in \text{Out}^\text{FC}(\Pi_{U_n}) \text{ [cf. [CmbCsp], Definition 1.1, (ii)];} \]
(b) $\alpha_1$ stabilizes the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_U$ associated to the cusps of $U$ that arise from the cusps of $X$:

Then $\alpha$ determines an outer automorphism of $\Pi_{X_n}$ via the natural outer surjection $\Pi_{U_n} \rightarrow \Pi_{X_n}$ induced by the natural open immersion $U_n \hookrightarrow X_n$.

Proof. First, since $\mathfrak{S}_n$ acts compatibly on $U_n$ and $X_n$, by replacing $\alpha \circ \sigma$ by $\alpha$, we may assume without loss of generality that

$$\alpha \in \text{Out}^F(\Pi_{U_n}).$$

Next, observe that it follows immediately from condition (b) that, by replacing $\alpha$ by the composite of $\alpha$ with a suitable element $\in \text{Out}^F(\Pi_{U_n})$ that

- arises, via various specialization and generization isomorphisms, from $\log$ scheme theory, and, moreover,
- determines an outer automorphism of $\Pi_{X_n}$ via the natural outer surjection $\Pi_{U_n} \rightarrow \Pi_{X_n}$

[cf. the proof of [CmbCsp], Lemma 2.4], we may also assume without loss of generality that

(c) $\alpha_1$ induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_U$.

Let $V \subseteq U$ be an open subscheme which is a hyperbolic curve over $k$ of type $(g, r_U + 1)$; $\tilde{\alpha} \in \text{Aut}^F(\Pi_{U_n})$ a lifting of $\alpha \in \text{Out}^F(\Pi_{U_n})$. Write

$$\{x\} \overset{\text{def}}{=} U \setminus V, \quad X_x \overset{\text{def}}{=} X \setminus \{x\} \subseteq X.$$

Then, for suitable choices of basepoints, we obtain a commutative diagram of homomorphisms of profinite groups

$$
\begin{array}{cccccc}
1 & \longrightarrow & \Pi_{V_{n-1}} & \longrightarrow & \Pi_{U_n} & \longrightarrow & \Pi_U & \longrightarrow & 1 \\
& & \downarrow & \phi_n & & \downarrow & & \\
1 & \longrightarrow & \Pi_{(X_x)_{n-1}} & \longrightarrow & \Pi_{X_n} & \longrightarrow & \Pi_X & \longrightarrow & 1,
\end{array}
$$

where $V_{n-1}$ (respectively, $(X_x)_{n-1}$) denotes the $(n-1)$-th configuration space of $V$ (respectively, $X_x$); the horizontal sequences denote the homotopy exact sequences induced by the first projections $U_n \rightarrow U$ and $X_n \rightarrow X$; the vertical arrows denote the homomorphisms induced by the natural open immersions $V_{n-1} \hookrightarrow (X_x)_{n-1}$, $U_n \hookrightarrow X_n$, and $U \hookrightarrow X$ [cf. [MT], Proposition 2.2, (i)].

Next, we verify the following assertion:

Claim 2.1.A: Suppose that $n = 2$. Then the automorphism $\tilde{\alpha}|_{\Pi_V} \in \text{Aut}(\Pi_V)$ [induced by $\tilde{\alpha} \in \text{Aut}^F(\Pi_{U_n})$ via the injection $\Pi_V \hookrightarrow \Pi_{U_n}$ in the above commutative diagram] preserves and fixes the conjugacy classes of cuspidal inertia subgroups of $\Pi_V$ that are not associated to $x$. 

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In the case where $\alpha \in \Out^{\text{FC}}(\Pi_{U,n})$, it follows immediately from condition (c) that $\widetilde{\alpha}|_{\Pi_U}$ preserves and fixes the conjugacy classes of cuspidal inertia subgroups of $\Pi_V$ [cf. [CmbCsp], Proposition 1.2, (iii); [CbTpII], Lemma 3.2, (iv)]. Thus, by condition (a), we may assume without loss of generality that $r_X > 0$. Then it follows from our assumption that $r_U > r_X$ that $r_U \geq 2$. Write

- $\text{Cusp}(U)$ for the set of cusps of $U$;
- $\rho_U : \Pi_U \to \Out(\Pi_V)$ for the outer representation determined by the exact sequence in the above commutative diagram

$$1 \to \Pi_V \to \Pi_{U_2} \to \Pi_U \to 1;$$

- $Y^{\log}$ for the [uniquely determined, up to unique isomorphism] smooth log curve over $\Spec k$ such that $U'_Y = U$;
- $Y^2_{\log}$ for the second log configuration space associated to $Y^{\log}$;
- for each $y \in \text{Cusp}(U)$, $y^{\log} \overset{\text{def}}{=} y \times_Y Y^{\log}$ [where the fiber product is determined by the natural morphism $Y^{\log} \to Y$ obtained by forgetting the log structure];
- $Y^2_{\log} \overset{\text{def}}{=} Y^2_{\log} \times_Y Y^{\log}$ [where the fiber product is determined by the first projection $Y^2_{\log} \to Y^{\log}$ and the natural projection $y^{\log} \to Y^{\log}$];
- $G_y$ for the semi-graph of anabeloids of pro-$\mathcal{P}$-Primes PSC-type determined by the stable log curve $Y^{\log}_y$ over $y^{\log}$ [cf. [CmbGC], Definition 1.1, (i)];
- $v^\text{new}_y$ (respectively, $v_y$) for the vertex of $G_y$ associated to the irreducible component that contains (respectively, does not contain) the cusp that arises from the diagonal divisor of $Y^2_{\log}$;
- $\Pi_{G_y}$ for the pro-$\mathcal{P}$-Primes fundamental group of $G_y$ [cf. [CmbGC], Definition 1.1, (ii)].

Thus, for each $y \in \text{Cusp}(U)$, we have a natural $\Im(\rho_U)$-torsor of outer isomorphisms

$$\Pi_V \overset{\sim}{\to} \Pi_{G_y}$$

that induces a bijection between the respective sets of cuspidal inertia subgroups. For each $y \in \text{Cusp}(U)$, let us fix an outer isomorphism

$$\Pi_V \overset{\sim}{\to} \Pi_{G_y}$$

that belongs to this collection. Then, by conjugating by this fixed outer isomorphism, we conclude that $\widetilde{\alpha}|_{\Pi_V}$ determines an outer automorphism $\alpha_y \in \Out(\Pi_{G_y})$ for each $y \in \text{Cusp}(U)$.

Let $y, z \in \text{Cusp}(U)$ such that $y \neq z$. [Recall that $r_U \geq 2$.] Then observe [by varying $y, z \in \text{Cusp}(U)$] that it suffices to prove that $\alpha_y$ preserves and fixes the
Next, we recall that \( \alpha_1 \in \text{Out}(U) \) preserves and fixes the conjugacy class of cuspidal inertia subgroups of \( U \) associated to \( y \) \[cf. condition (c)\]. Thus, it follows from [CbTpII], Theorem 1.9, (ii), that, by replacing \( \tilde{\alpha} \) by the composite of \( \tilde{\alpha} \) with an inner automorphism of \( U \), we may assume without loss of generality that \( \alpha_y \) preserves the set of vertical subgroups of \( G_y \). Since \((g, \hat{r}_U) \notin \{(0,3), (1,1)\}\), it follows [cf. [MT], Remark 1.2.2] that \( \alpha_y \) preserves and fixes the conjugacy classes of vertical subgroups of \( G_y \). Let \( \Pi_{v_y} \subseteq \Pi_{G_y} \) be a vertical subgroup associated to \( v_y \); \( \tilde{\alpha}_y \in \text{Aut}(\Pi_{v_y}) \) a lifting of \( \alpha_y \) such that \( \tilde{\alpha}_y(\Pi_{v_y}) = \Pi_{v_y} \). On the other hand, observe that the composite

\[
\Pi_{v_y} \subseteq \Pi_{G_y} \xrightarrow{\sim} \Pi_{V_y} \xrightarrow{\sim} \Pi_{U_2} \xrightarrow{\sim} \Pi_U
\]

— where the final arrow denotes the natural outer surjection induced by the second projection \( U_2 \rightarrow U \) — determines an outer isomorphism \( \Pi_{v_y} \xrightarrow{\sim} \Pi_U \) that induces a bijection between the respective sets of cuspidal inertia subgroups and is compatible with the respective outer automorphisms \( \alpha_y \) and \( \alpha_1 \). Here, we recall that the cusp \( z \) abuts to the vertex \( v_y \). Thus, by condition (c), we conclude that \( \alpha_y \) preserves and fixes the conjugacy class of cuspidal inertia subgroups of \( G_y \) associated to \( z \). This completes the proof of Claim 2.1.A.

In the remainder of the proof of Theorem 2.1, we proceed by induction on \( n \geq 2 \). Next, we verify the following assertion:

Claim 2.1.B: Suppose that \( n = 2 \). Then Theorem 2.1 holds.

Indeed, let us note that, by condition (c), \( \alpha_1 \) preserves the kernel of the natural surjection \( \Pi_U \twoheadrightarrow \Pi_X \). On the other hand, it follows immediately from Claim 2.1.A that \( \tilde{\alpha}_{|\Pi_{v_y}} \in \text{Aut}(\Pi_{V_y}) \) preserves the kernel of the surjection \( \Pi_{V_y} \twoheadrightarrow \Pi_{X_y} \). Thus, since \( \Pi_{X_y} \) is center-free, we conclude that \( \tilde{\alpha} \) induces an automorphism of \( \Pi_{X_y} \). This completes the proof of Claim 2.1.B.

Next, we verify the following assertion [by a similar argument to the argument used to prove Claim 2.1.B]:

Claim 2.1.C: Let \( m \) be an integer such that \( m \geq 2 \). Suppose that Theorem 2.1 holds in the case where \( n = m \). Then Theorem 2.1 holds in the case where \( n = m + 1 \).

Indeed, let us note that, by condition (c), \( \alpha_1 \) preserves the kernel of the natural surjection \( \Pi_U \twoheadrightarrow \Pi_X \). Moreover, since \( m \geq 2 \), it follows from [CbTpII], Theorem A, (ii) [cf. also condition (c); [CbTpI], Theorem A, (i); [CbTpII], Lemma 3.2, (iv)], that the automorphism \( \tilde{\alpha}_{|\Pi_{v_m}} \in \text{Aut}(\Pi_{v_m}) \) [induced by \( \tilde{\alpha} \in \text{Aut}^F(\Pi_{U_{m+1}}) \) via the injection \( \Pi_{V_m} \hookrightarrow \Pi_{U_{m+1}} \) in the above commutative diagram] induces an automorphism of \( \Pi_{v_m} \) that induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of \( \Pi_{V_y} \). On the other hand, since \( X_z \) is an affine hyperbolic curve, it follows from the induction hypothesis that the automorphism \( \tilde{\alpha}_{|\Pi_{v_m}} \in \text{Aut}(\Pi_{v_m}) \) preserves the kernel of the surjection
\( \Pi_{V_m} \rightarrow \Pi_{(X_m)_m} \). Thus, since \( \Pi_{(X_m)_m} \) is center-free [cf. [MT], Proposition 2.2, (ii)], we conclude that \( \tilde{\alpha} \) induces an automorphism of \( \Pi_{X_{m+1}} = \Pi_{(X_m)_m} \rtimes \Pi_X \). This completes the proof of Claim 2.1.C, hence of Theorem 2.1.

Corollary 2.2 (Group-theoreticity of cuspidal inertia subgroups in configuration space groups of genus 0). In the notation of Theorem 2.1, suppose that \( g = 0 \) [so \( r_U \geq 4 \)]. Then

\[
\text{Out}^{\text{FC}}(\Pi_{U_n}) = \text{Out}^F(\Pi_{U_n})
\]

[cf. [CmbCsp], Definition 1.1, (ii)]. In particular,

\[
\text{Out}(\Pi_{U_n}) = \text{Out}^sF(\Pi_{U_n}) \times S_n
\]

\[
= \text{Out}^F(\Pi_{U_n}) \times S_n
\]

\[
= \text{Out}^{\text{FC}}(\Pi_{U_n}) \times S_n
\]

[cf. [HMM], Corollary B].

Proof. Write

\[ p_1, \ldots, n-1 : \Pi_{U_n} \twoheadrightarrow \Pi_{U_{n-1}} \]

for the surjection induced by the projection \( U_n \to U_{n-1} \) obtained by forgetting the \( n \)-th factor. Let \( Z \) be a hyperbolic curve over \( k \) of genus 0 that arises as a fiber of the projection \( U_{n-1} \to U_{n-2} \) obtained by forgetting the \((n-1)\)-th factor. Write \( Z_2 \) for the second configuration space associated to \( Z \); \( p_Z : \Pi_{Z_2} \twoheadrightarrow \Pi_Z \) for the surjection induced by the first projection \( Z_2 \to Z \). Then, for suitable choices of basepoints, we obtain a commutative diagram of homomorphisms of profinite groups

\[
\begin{array}{cccc}
1 & \longrightarrow & \text{Ker}(p_Z) & \longrightarrow & \Pi_{Z_2} & \longrightarrow & \Pi_Z & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \text{Ker}(p_1, \ldots, n-1) & \longrightarrow & \Pi_{U_n} & \longrightarrow & \Pi_{U_{n-1}} & \longrightarrow & 1.
\end{array}
\]

Thus, by replacing \( U \) by \( Z \) and applying [CbTpI], Theorem A, (ii), we may assume without loss of generality that \( n = 2 \).

Let \( \beta \in \text{Out}^F(\Pi_{U_2}) \). Write \( \beta_1 \in \text{Out}(\Pi_U) \) for the outer automorphism induced by \( \beta \) [cf. [CbTpI], Theorem A, (i)]. Observe that, by replacing \( \beta \) by the composite of \( \beta \) with a suitable element \( \in \text{Out}^{\text{FC}}(\Pi_{U_2}) \) [cf. [CmbCsp], Lemma 2.4], we may also assume without loss of generality that \( \beta_1 \) induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of \( \Pi_U \).

In the remainder of the proof, we use the notation in the proof of Claim 2.1.A in the proof of Theorem 2.1 in the case where \( (g, r_X) = (0, 3) \) and \( \alpha \overset{\text{def}}{=} \beta \).

Observe that it follows from Claim 2.1.A that \( \alpha \in \text{Out}^{\text{FC}}(\Pi_{U_2}) \) [cf. [CbTpII], Definition 2.1, (ii)].

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Suppose that \( y, z \in \text{Cusp}(U) \), where \( y \neq z \), arise from cusps of \( X \). Then our goal is to prove that the outer automorphism \( \alpha_y \in \text{Out}(\Pi_{G_y}) \) [which preserves and fixes the conjugacy classes of vertical subgroups of \( \Pi_{G_y} \)] preserves and fixes the conjugacy class of cuspidal inertia subgroups of \( \Pi_{G_y} \) associated to \( x \), i.e., the cusp associated to the diagonal divisor of \( Y_\log \). Let \( \Pi_{v^{\text{new}}} \subseteq \Pi_{G_y} \) be a vertical subgroup associated to \( v^{\text{new}}_y \); \( \tilde{\alpha}_y^{\text{new}} \in \text{Aut}(\Pi_{G_y}) \) a lifting of \( \alpha_y \) such that \( \tilde{\alpha}_y^{\text{new}}(\Pi_{v^{\text{new}}}) = \Pi_{v^{\text{new}}} \). Write
\[
\tilde{\alpha}_X \in \text{Aut}^{FwC}(\Pi_{X_2})
\]
for the automorphism induced by \( \tilde{\alpha} \in \text{Aut}^{FwC}(\Pi_{U_2}) \) and the natural surjection \( \phi_2 : \Pi_{U_2} \twoheadrightarrow \Pi_{X_2} \) [cf. Theorem 2.1]. Write \( T \supseteq \Pi_{X_2} \) for the tripod over \( k \) obtained by eliminating the cusp \( z \) of \( X_x \). Then it follows immediately from the various definitions involved that the composite
\[
\Pi_{v^{\text{new}}} \subseteq \Pi_{G_y} \overset{\tilde{\alpha}_V}{\twoheadrightarrow} \Pi_{X_2} \twoheadrightarrow \Pi_T
\]
— where \( \Pi_{V} \twoheadrightarrow \Pi_{X_2} \) (respectively, \( \Pi_{X_2} \twoheadrightarrow \Pi_T \)) denotes the natural outer surjection induced by the natural open immersion \( V \hookrightarrow X_x \) (respectively, \( X_x \hookrightarrow T \)) — determines an outer isomorphism \( \Pi_{v^{\text{new}}} \overset{\tilde{\alpha}_V^{\text{new}}}{\twoheadrightarrow} \Pi_T \) that induces a bijection between the respective sets of cuspidal inertia subgroups and is compatible with the outer automorphisms [of \( \Pi_{v^{\text{new}}} \), \( \Pi_T \), respectively] induced by \( \tilde{\alpha}_y^{\text{new}} \) and the restriction \( \tilde{\alpha}_X|_{\Pi_{X_2}} \) of \( \tilde{\alpha}_X \) to \( \Pi_{X_2} \) [cf. Claim 2.1.A]. On the other hand, since \( \tilde{\alpha}_X \in \text{Aut}^{FwC}(\Pi_{X_2}) = \text{Aut}^{FC}(\Pi_{X_2}) \) [cf. [ChTpII], Theorem A, (ii)], it follows that \( \tilde{\alpha}_X \) preserves and fixes the conjugacy classes of the cuspidal inertia subgroups of \( \Pi_{X_2} \) [cf. condition (c); [CmbCsp], Proposition 1.2, (iii); [ChTpII], Lemma 3.2, (iv)], hence of \( \Pi_T \). Thus, we conclude that \( \tilde{\alpha}_y^{\text{new}} \) preserves and fixes the conjugacy classes of cuspidal inertia subgroups of \( \Pi_{G_y} \), hence of \( \Pi_{v^{\text{new}}} \) associated to \( x \). This completes the proof of Corollary 2.2. \( \square \)

**Remark 2.2.1.** One verifies immediately that Theorem 2.1 and Corollary 2.2, as well as their proofs, go through without change when the various “\( \Pi \)”s” are replaced by their respective maximal pro-\( l \) quotients, for some prime number \( l \). We leave the routine details to the reader. On the other hand, in the present paper, we shall not need these pro-\( l \) versions of Theorem 2.1 and Corollary 2.2.

### 3 Various properties of closed subgroups of the Grothendieck-Teichmüller group

In this section, we apply the technique developed in [Tsjm], §1, i.e., combinatorial Belyi cuspidalization, to give a purely combinatorial/group-theoretic
definition of certain classes of closed subgroups of GT [cf. Definition 3.3]. Moreover, we prove a certain relationship between these classes [cf. Corollary 3.7] by applying Theorem 2.1.

Write $X \overset{\text{def}}{=} \mathbb{P} \setminus \{0, 1, \infty\}; X_n$ for the $n$-th configuration space associated to $X$, where $n \geq 2$ denotes a positive integer; GT $\subseteq$ Out$(\Pi_X)$ for the Grothendieck-Teichmüller group [cf. [CmbCsp], Definition 1.11, (i); [CmbCsp], Remark 1.11.1]. Then recall from the first display of [HMM], Corollary C, that we have a natural inclusion $\text{GT} \hookrightarrow \text{Out}(\Pi_X)$. We shall write $\text{GT}_n \subseteq \text{Out}(\Pi_X)$ for the image of this inclusion.

**Corollary 3.1 (Purely combinatorial/group-theoretic reconstruction of the symmetric group).** For each positive integer $m$, write $S_m$ for the symmetric group on $m$ letters; $A_m(\subseteq S_m)$ for the alternating group on $m$ letters. Let us regard $A_{n+3} \subseteq S_{n+3}$ as subgroups of Out$(\Pi_X)$ via the natural injection $S_{n+3} \hookrightarrow \text{Out}(\Pi_X)$ induced by the natural action of $S_{n+3}$ on $X_n$ [cf. [HMM], Remark 2.1.1]. Let

$$\psi_n : \text{Out}(\Pi_X) \rightarrow S_{n+3}$$

be a representative of the outer surjection $\xi_n$ induced by the natural action of Out$(\Pi_X)$ on the set of generalized fiber subgroups of length 1 [cf. [HMM], Theorem A, (ii)]. Then the following hold:

(i) Write

$$F \subseteq \Pi_X$$

for the generalized fiber subgroup of co-length 1 associated to the subset $\{5, \ldots, n+3\} \subseteq \{1, \ldots, n+3\}$ of labels of cardinality $n-1$ [cf. [HMM], Theorem A, (ii); [HMM], Definition 2.1, (ii)]. Let

$$\alpha \in \text{Out}(\Pi_X)$$

be an outer automorphism of $\Pi_X$ such that $\psi_n(\alpha) = (1 \ 2)(3 \ 4)$, and $\alpha$ induces the identity outer automorphism of $\Pi_X/F \overset{\sim}{\rightarrow} \Pi_X$ via the natural surjection $\Pi_X \rightarrow \Pi_X/F$. Then

$$\alpha = (1 \ 2)(3 \ 4) \in A_{n+3} \subseteq S_{n+3} \subseteq \text{Out}(\Pi_X),$$

[cf. [CmbCsp], Corollary 4.2, (ii); the first display of [HMM], Corollary C; [HMM], Definition 2.7], and the subgroup $A_{n+3} \subseteq \text{Out}(\Pi_X)$ may be reconstructed, in a purely combinatorial/group-theoretic way, from $\Pi_X$ as the subgroup of Out$(\Pi_X)$ generated by the Out$(\Pi_X)$-conjugacy class of $\alpha$ [which depends only on the outer surjection $\xi_n$].

(ii) Suppose that $n \geq 3$. Write

$$F \subseteq \Pi_X$$

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for the generalized fiber subgroup of length 2 associated to the subset \( \{1, 2\} \subseteq \{1, \ldots, n + 3\} \) of labels of cardinality 2 [cf. [HMM], Theorem A, (ii); [HMM], Definition 2.1, (ii)]. Let
\[
\alpha \in \text{Out}(\Pi_{X_n})
\]
be an outer automorphism of \( \Pi_{X_n} \) such that \( \psi_n(\alpha) = (1 2) \), and \( \alpha \) induces the identity outer automorphism of \( \Pi_{X_n}/F \) \( \cong \Pi_{X_{n-2}} \) via the natural surjection \( \Pi_{X_n} \twoheadrightarrow \Pi_{X_n}/F \). Then
\[
\alpha = (1 2) \in \mathfrak{S}_{n+3} \subseteq \text{Out}(\Pi_{X_n})
\]
[cf. [CmbCsp], Corollary 4.2, (ii); the first display of [HMM], Corollary C; [HMM], Definition 2.7], and the subgroup \( \mathfrak{S}_{n+3} \subseteq \text{Out}(\Pi_{X_n}) \) may be reconstructed, in a purely combinatorial/group-theoretic way, from \( \Pi_{X_n} \) as the subgroup of \( \text{Out}(\Pi_{X_n}) \) generated by the \( \text{Out}(\Pi_{X_n}) \)-conjugacy class of \( \alpha \) [which depends only on the outer surjection \( \xi_n \)].

**Proof.** Write \( \mathfrak{X} \subseteq \text{Out}(\Pi_{X_n}) \) (respectively, \( \mathcal{S} \subseteq \text{Out}(\Pi_{X_n}) \)) for the subgroup constructed by the algorithm of assertion (i) (respectively, assertion (ii)). Then it follows immediately from the well-known structure of \( \mathfrak{S}_{n+3} \) [where we recall that \( n + 3 \geq 5 \) that \( \mathfrak{X}_{n+3} \subseteq \mathfrak{X} \) (respectively, \( \mathfrak{S}_{n+3} \subseteq \mathfrak{S} \)]. [Here, we recall that the kernel of the unique outer surjection \( \mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_3 \) is normally generated by \( (1 2)(3 4) \).] On the other hand, by applying the first display of [HMM], Corollary C, we conclude that \( \mathfrak{X}_{n+3} = \mathfrak{X} \) (respectively, \( \mathfrak{S}_{n+3} = \mathfrak{S} \)). This completes the proof of Corollary 3.1.

**Remark 3.1.1.** In [HMM], Corollary C, the subgroup \( \mathfrak{S}_{n+3} \subseteq \text{Out}(\Pi_{X_n}) \) is reconstructed by forming the local center \( Z^{\text{loc}}(\text{Out}(\Pi_{X_n})) \) of \( \text{Out}(\Pi_{X_n}) \). This local center is calculated by applying the Grothendieck Conjecture for hyperbolic curves over number fields [cf. [LocAn], Theorem A; [Tama], Theorem 0.4]. On the other hand, if \( n \geq 3 \), then, by applying the algorithm given in Corollary 3.1, (ii), the subgroup \( \mathfrak{S}_{n+3} \subseteq \text{Out}(\Pi_{X_n}) \) may be reconstructed, in a purely combinatorial/group-theoretic way, from \( \Pi_{X_n} \) without applying the Grothendieck Conjecture for hyperbolic curves over number fields.

**Definition 3.2.** Let \( n \) be an integer such that \( n \geq 2 \); \( k \) an algebraically closed field of characteristic 0; \( U \) a hyperbolic curve over \( k \). Write \( U_n \) for the \( n \)-th configuration space associated to \( U \). Recall the subgroup
\[
\text{Out}^{\text{cusp}}(U_n) \subseteq \text{Out}(U_n)
\]
[cf. [HMM], Definition 2.1, (iv)]. Then we shall write
\[
\text{Out}^{\text{cusp}}(U_n) \subseteq \text{Out}(U_n)
\]
for the subgroup of elements that induce outer automorphisms of \( \Pi_U \) that preserve and fix the conjugacy classes of cuspidal inertia subgroups of \( \Pi_U \).
Definition 3.3. Let $J \subseteq \Gamma_T$ be a closed subgroup of $\Gamma_T$; $N$ (respectively, $N^\dagger$) a normal open subgroup of $J$:

\[
\begin{array}{ccc}
\Pi_U \times N & \longrightarrow & \Pi_X \times N \\
\downarrow & & \downarrow \\
\Pi_X \times N & & \\
\end{array}
\]

(respectively,

\[
\begin{array}{ccc}
\Pi_{U^\dagger} \times N^\dagger & \longrightarrow & \Pi_X \times N^\dagger \\
\downarrow & & \downarrow \\
\Pi_X \times N^\dagger & & \\
\end{array}
\]

an arithmetic Belyi diagram \cite{[Tsjm]}, Definition 1.4, where we take “$M$” to be $N$ (respectively, $N^\dagger$), and we note that the “$N$” of loc. cit. does not necessarily coincide with the $N$ of the present discussion; Remark 3.3.2 below], which we denote by $B^\times$ (respectively, $B^\times$).

(i) Write $U_2$ (respectively, $U_2^\dagger$) for the second configuration space associated to $U$ (respectively, $U^\dagger$); $p : \Pi_{U_2} \to \Pi_U$ (respectively, $p^\dagger : \Pi_{U_2^\dagger} \to \Pi_{U^\dagger}$) for the outer surjection induced by the first projection. Note that it follows from Remark 3.3.4 below that there exists a (unique — cf. [CmbCsp], Theorem A, (i); [CmbCsp], Proposition 1.2, (iii)] outer action of $N$ (respectively, $N^\dagger$) on $\Pi_{U_2}$ (respectively, $\Pi_{U_2^\dagger}$) which induces the given outer action of $N$ (respectively, $N^\dagger$) on $\Pi_U$ (respectively, $\Pi_{U^\dagger}$) via the outer surjection $p$ (respectively, $p^\dagger$). Then we shall say that $\uparrow B^\times$ dominates $B^\times$ if there exist a normal open subgroup

\[
M \subseteq N \cap N^\dagger
\]
of $J$ and a $\Pi_U$-outer surjection

\[
\phi : \Pi_{U_2^\dagger} \times M \to \Pi_U \times M
\]
such that the following (a), (b) hold:

(a) There exists a (necessarily unique — cf. [CmbCsp], Theorem A, (i); [CmbCsp], Proposition 1.2, (iii)] $\Pi_{U_2}$-outer surjection

\[
\phi_2 : \Pi_{U_2^\dagger} \times M \to \Pi_{U_2} \times M
\]
such that

- the diagram of $\Pi_{(-)}$-outer homomorphisms

\[
\begin{array}{ccc}
\Pi_{U_2^\dagger} \times M & \phi_2 & \Pi_{U_2} \times M \\
p^\dagger \times \text{id}_M & \downarrow & \text{id}_M \times p^\dagger \\
\Pi_{U^\dagger} \times M & \phi & \Pi_U \times M \\
\end{array}
\]
commutes;
• \( \phi_2 \) maps the fiber subgroups of \( \Pi_{U_2} \) to the fiber subgroups of \( \Pi_{U_2} \);
• the kernel of \( \phi_2 \) is topologically generated by [certain of the] cuspidal inertia subgroups of fiber subgroups of \( \Pi_{U_2} \) of length 1 [which implies, in particular, that the kernel of \( \phi \) is topologically generated by [certain of the] cuspidal inertia subgroups of \( \Pi_{U_1} \)];
• the image via \( \phi_2 \) of any cuspidal inertia subgroup of a fiber subgroup of \( \Pi_{U_2} \) of length 1 is either trivial or a cuspidal inertia subgroup of a fiber subgroup of \( \Pi_{U_2} \) of length 1 [which implies, in particular, that the image via \( \phi \) of any cuspidal inertia subgroup of \( \Pi_{U_1} \) is either trivial or a cuspidal inertia subgroup of \( \Pi_{U_2} \)].

(b) The composite of \( \phi \) with the restriction to \( \Pi_U \times M \) of the \( \Pi_X \)-outer surjection

\[
\Pi_U \times N \twoheadrightarrow \Pi_X \times N
\]

[i.e., the horizontal arrow in \( \mathbb{B}^\times \)] coincides with the restriction to \( \Pi_{U_1} \times M \) of the \( \Pi_X \)-outer surjection

\[
\Pi_{U_1} \times N \twoheadrightarrow \Pi_X \times N^\dagger
\]

[i.e., the horizontal arrow in \( \mathbb{B}^\times \)].

In this situation, we shall refer to \( \phi : \Pi_U \times M \twoheadrightarrow \Pi_U \times M \) as an arithmetic domination of \( \Pi_X \) by \( \Pi_U \times M \) and to the \( \Pi_U \)-outer surjection \( \phi_U : \Pi_U \twoheadrightarrow \Pi_U \) obtained by restricting \( \phi \) to \( \Pi_U \) [a restriction whose image lies in \( \Pi_U \), by condition (b)] as a geometric domination of \( \Pi_U \) by \( \Pi_U \times M \) [a geometric domination of \( \Pi_U \) by \( \Pi_U \times M \)]. [Here, we observe in passing that it follows immediately from the definition of \( \times \) that \( \phi_\Pi \) is uniquely determined by \( \phi_U \), \( \mathbb{B}^\times \), and \( \mathbb{B}^\times \).]

(ii) We shall say that the pair \( (\mathbb{B}^\times, \mathbb{B}^\times) \) satisfies the \( \text{COF-property} \) [i.e., “cofiltered property”] if the pair \( (\mathbb{B}^\times, \mathbb{B}^\times) \) satisfies the following condition:

• there exist a normal open subgroup \( N^\dagger \) of \( J \) and an arithmetic Belyi diagram \( \mathbb{B}^\times \)

\[
\Pi_{U_1} \twoheadrightarrow N^\dagger \longrightarrow \Pi_X \twoheadrightarrow N^\dagger
\]

such that \( \mathbb{B}^\times \) dominates \( \mathbb{B}^\times \) and \( \mathbb{B}^\times \).

(iii) We shall say that the pair \( (\mathbb{B}^\times, \mathbb{B}^\times) \) satisfies the \( \text{RGC-property} \) [i.e., “Relative Grothendieck Conjecture property”] if the pair \( (\mathbb{B}^\times, \mathbb{B}^\times) \) satisfies the following condition:
the cardinality of the set of geometric dominations [cf. (i)] of \( \mathbb{B}^\times \) by \( \mathbb{B}^\times \) is \( \leq 1 \).

(iv) Write \( \text{Cusp}(\Pi_U) \) (respectively, \( \text{Cusp}(\Pi_X) \)) for the set of cusps of \( \Pi_U \) (respectively, \( \Pi_X \)) [cf. [Tsjm], Theorem 1.3, (i)]. Note that the horizontal arrow in \( \mathbb{B}^\times \) induces a natural injection \( \text{Cusp}(\Pi_X) \rightarrow \text{Cusp}(\Pi_U) \); we shall regard \( \text{Cusp}(\Pi_X) \) as a subset of \( \text{Cusp}(\Pi_U) \) via this injection. Let \( T \subseteq \text{Cusp}(\Pi_U) \setminus \text{Cusp}(\Pi_X) \). Write \( I(\Pi_U) \) for the set of cuspidal inertia subgroups of \( U \) [cf. [Tsjm], Theorem 1.3, (i)]. Thus, \( \text{Cusp}(\Pi_U) \) may be identified with \( I(\Pi_U) \). Let \( T \rightarrow C \) for the quotient by the normal closed subgroup topologically generated by the cuspidal inertia subgroups of \( U \) associated to the cusps \( \mathbb{B}^\times \) is \( \rightarrow \) for the quotient induced by the quotient \( U \rightarrow C \). For \( I_c \in I(\Pi_U) \), write \( D_c \equiv N_{\Pi_U \rightarrow N}(I_c) \); \( D_{T,c} \) for the image of \( D_c \) via the quotient \( \Pi_U \rightarrow N \rightarrow \Pi_T \rightarrow N \). Then we shall say that the arithmetic Belyi diagram \( \mathbb{B}^\times \) satisfies the CS-property [i.e., “cuspidal separatedness property”] if \( \mathbb{B}^\times \) satisfies the following condition:

- for \( I_c, I_c' \in I(\Pi_U) \), \( D_{T,c} \) is commensurable to \( D_{T,c'} \) if and only if there exists \( \sigma \in \text{Ker}(\Pi_U \rightarrow \Pi_T) \) such that \( (I_c)^\sigma \equiv \sigma I_c \sigma^{-1} = I_c' \).

One verifies immediately that this condition implies that \( D_{T,c} \subseteq \Pi_T \rightarrow N \) is commensurably terminal, hence normally terminal.

(v) We shall say that \( J \) satisfies the COF-property (respectively, the RGC-property) if every pair of arithmetic Belyi diagrams satisfies the COF-property (respectively, the RGC-property). We shall say that \( J \) satisfies the CS-property if every arithmetic Belyi diagram satisfies the CS-property. We shall say that \( J \) satisfies the BC-property [i.e., “Belyi compatibility property”] if \( J \) satisfies the COF- and the RGC-properties. By a slight abuse of notation, we shall use the notation BGT to denote a closed subgroup of \( \text{GT} \) that satisfies the BC-property.

Remark 3.3.1. Note that it follows immediately from the various definitions involved that:

(a) each notion defined in Definition 3.3, (i), (ii), (iii) (respectively, Definition 3.3, (iv)), concerning \( \mathbb{B}^\times \), \( \mathbb{B}^\times \) (respectively, concerning \( \mathbb{B}^\times \)) is equivalent to the corresponding notion concerning the restrictions of \( \mathbb{B}^\times \), \( \mathbb{B}^\times \) (respectively, the restriction of \( \mathbb{B}^\times \)) to arbitrary open subgroups of \( N, N^\dagger \) (respectively, \( N \)) that are normal in \( J \);

(b) each notion defined in Definition 3.3, (v), concerning \( J \) is equivalent to the corresponding notion concerning an arbitrary open subgroup of \( J \).
Remark 3.3.2. Let us recall that there are precisely two situations in [Tsjm] in which the Grothendieck Conjecture for hyperbolic curves over number fields [cf. [LocAn], Theorem A; [Tama], Theorem 0.4] is applied, namely:

(a) Claim 1.3.A in the proof of [Tsjm], Theorem 1.3, (ii) [which is applied in [Tsjm], Definition 1.4, to define the notion of an arithmetic Belyi diagram];

(b) the proof of [Tsjm], Theorem 1.3, (iii) [which must be applied in order to give a purely combinatorial/group-theoretic construction of the outer isomorphism that is used to identify the two copies of $\Pi_X$ that appear in a Belyi diagram].

On the other hand, in Remark 3.3.3 below, we shall give a purely combinatorial/group-theoretic algorithm for constructing, via the algorithm of Corollary 3.1, (ii), the identifying outer isomorphism between the two copies of $\Pi_X$ that appear in a Belyi diagram.

In particular, in the context of the theory of the present paper, instead of applying [Tsjm], Theorem 1.3, (iii), one may apply the purely combinatorial/group-theoretic algorithm of Remark 3.3.3, which does not require any use of the Grothendieck Conjecture for hyperbolic curves over number fields [cf. Remark 3.1.1]. In addition, [Tsjm], Theorem 1.3, (ii) [i.e., the compatibility of the identifying outer isomorphism between the two copies of $\Pi_X$ with the respective outer actions on the two copies] follows immediately from the functoriality of the purely combinatorial/group-theoretic algorithm given in Remark 3.3.3 below. Thus, in summary, in the argument of the present paper, one may in fact avoid any use of the Grothendieck Conjecture for hyperbolic curves over number fields when applying the theory/results of [Tsjm] in the present paper.

Remark 3.3.3. In the following discussion, we use the notation that appears in the statement and proof of [Tsjm], Theorem 1.3.

(i) In the remainder of the present Remark 3.3.3, we shall reconstruct the identifying outer isomorphism between the copies of $\Pi_X$ that appear in a given Belyi diagram $B$ [cf. Remark 3.3.2] — by means of a purely combinatorial/group-theoretic algorithm — from [the underlying purely combinatorial/group-theoretic structure of] the collection of data

(a) the profinite group $\Pi_{X_3}$;

(b) the outer surjections $pr_{i,j} : \Pi_{X_3} \twoheadrightarrow \Pi_{X_2}$, where $(i,j) \in \{(1,2), (1,3), (2,3)\}$, determined by the natural projection $X_3 \twoheadrightarrow X_2$ to the $i$-th and $j$-th factors, i.e., to be precise, the normal closed subgroups $\text{Ker}(pr_{i,j}) \subseteq \Pi_{X_3}$, together with the composite outer isomorphisms $\Pi_{X_3}/\text{Ker}(pr_{i,j}) \xrightarrow{\sim} \Pi_{X_2} \xrightarrow{\sim} \Pi_{X_3}/\text{Ker}(pr_{i',j'})$, where $(i,j), (i',j') \in \{(1,2), (1,3), (2,3)\}$;
(c) the outer surjections \( p_i : \Pi_{X_2} \to \Pi_X \) \((i \in \{1, 2\})\) determined by the natural projection \( X_2 \to X \) to the \( i \)-th factor, i.e., to be precise, the normal closed subgroups \( \text{Ker}(p_1), \text{Ker}(p_2) \subseteq \Pi_{X_2} \), together with the composite outer isomorphism \( \Pi_{X_2}/\text{Ker}(p_1) \cong \Pi_X \cong \Pi_{X_2}/\text{Ker}(p_2) \);

(d) the profinite groups \( \Pi_{X_2} \) and \( \Pi_X \), i.e., to be precise, the quotients of \( \Pi_{X_3} \) discussed in (b) and (c);

(e) surjections

\[
\begin{align*}
pr_1 : \Pi_{X_3} &\to \Pi_X, \\
pr_2 : \Pi_{X_3} &\to \Pi_X, \\
pr_3 : \Pi_{X_3} &\to \Pi_X,
\end{align*}
\]

that represent the respective outer surjections \( p_1 \circ pr_{1,3}, p_1 \circ pr_{2,3}, p_2 \circ pr_{2,3} \);

(f) the open subgroup \( \Pi_U \subseteq \Pi_X \);

(g) the subset of labeled elements \( \{0, 1, \infty\} \subseteq \text{Cusp}(\Pi_U) \) [cf, [Tsjm], Theorem 1.3, (i)];

(h) the subset of labeled elements \( \{0, 1, \infty\} \subseteq \text{Cusp}(\Pi_X) \) [cf, [Tsjm], Theorem 1.3, (i)]

— i.e., without applying the Grothendieck Conjecture for hyperbolic curves over number fields. Here, the data (f), (g), (h) correspond to the given Belyi diagram \( \mathbb{B} \) [cf. the data “\( C(\Pi_X) \)” of [Tsjm], Theorem 1.3, (iii)]. Also, we note that any two collections of choices of surjections as in (e) are related to one another by composition with a single inner automorphism of \( \Pi_{X_3} \). Moreover, by applying Corollary 3.1, (ii); [HMM], Theorem A, (ii), one may regard the data of (b), (c), (d), (e) as data reconstructed [i.e., by using the action of the symmetric group \( S_6 \subseteq \text{Out}(\Pi_{X_3}) \)], up to unique isomorphism, from the data of (a).

(ii) Next, observe that the identifying outer isomorphism between the copies of \( \Pi_X \) in \( \mathbb{B} \) coincides with the composite

\[
\Pi_X \cong \Pi^{\text{ctpd}} \cong \Pi_U^{\text{ctpd}} \cong \Pi_X^{\text{ctpd}} \cong \Pi_X,
\]

where the first and the final arrows denote the outer isomorphisms arising from the [scheme-theoretic] isomorphisms of tripods determined by the data of (i), (e), (h) [which may be used to rigidify the correspondences between cusps]; the second and the third arrows denote the natural isomorphisms induced by the natural outer surjections \( \Pi_{V_3} \to \Pi_{U_3} \) and \( \Pi_{U_3} \to \Pi_X \). Recall that the open subgroup \( \Pi_{V_3} \subseteq \Pi_{X_3} \) is defined to be the inverse image of the open subgroup \( \Pi_U^{X_3} \subseteq \Pi_X^{X_3} \) [determined by the open subgroup \( \Pi_U \subseteq \Pi_X \)] via the surjection \( \Pi_{X_3} \to \Pi_X^{X_3} \) determined by the surjection \( pr_i : \Pi_{X_3} \to \Pi_X \), where \( i = 1, 2, 3 \). Thus, to reconstruct the above composite in a purely combinatorial/group-theoretic way, it suffices to reconstruct the following data:

(a) the 3-central tripods \( \subseteq \Pi_{X_3} \) [i.e., such as \( \Pi^{\text{ctpd}} \)].
(b) the kernel of the natural outer surjection \( \Pi_{U_5} \rightarrow \Pi_{U_3} \) [which allows us to characterize \( \Pi^{\text{ctpd}} \) [cf. Claim 1.3.C in the proof of [Tsjm], Theorem 1.3, (ii)] and reconstruct \( \Pi^{\text{ctpd}}_U \);]

(c) the outer isomorphism \( \Pi_X \sim \Pi^{\text{ctpd}} \);

(d) the kernel of the natural outer surjection \( \Pi_{U_3} \rightarrow \Pi_{X_3} \) [which allows us to reconstruct \( \Pi^{\text{ctpd}}_X \);]

(e) the outer isomorphism \( \Pi^{\text{ctpd}}_X \sim \Pi_X \), where we regard both \( \Pi^{\text{ctpd}}_X \) and \( \Pi_X \) as subquotients of \( \Pi_3 \triangleq \Pi_{U_5}/\text{Ker}(\Pi_{U_5} \rightarrow \Pi_{X_3}) \) (\( \sim \Pi_{X_3} \)).

(iii) The data of (ii), (a), may be reconstructed by applying the algorithm implicit in the proof of [CbTpII], Theorem 3.16, (v) [cf. also [HMM], Corollary B]. Once the data of (ii), (b) (respectively, (d)), has been reconstructed, the data of (ii), (c) (respectively, (e)), may be reconstructed by using the action of the symmetric group \( \mathfrak{S}_6 \subseteq \text{Out}(\Pi_{X_3}) \) (respectively, \( \mathfrak{S}_6 \subseteq \text{Out}(\Pi_3) \)) [cf. Corollary 3.1, (ii); the construction of the geometric outer isomorphism \( \Pi^{\text{ctpd}} \sim \Pi_X \) in the proof of [CbTpII], Lemma 3.13, (iii)]. Thus, it suffices to reconstruct the data of (ii), (b), (d) [cf. (v), (vi), below].

(iv) Recall the set \( I_{X_3} \) of inertia subgroups \( \subseteq \Pi_{X_3} \) of the discussion immediately following Claim 1.3.B in the proof of [Tsjm], Theorem 1.3, (ii). Write

\[
I_{X_3}^F \subseteq I_{X_3}
\]

for the subset consisting of inertia subgroups \( \subseteq \text{Ker}(\text{pr}_{i,j}) \) for some \((i,j) \in \{(1,2), (1,3), (2,3)\}\). Let \((i,j) \in \{(1,2), (1,3), (2,3)\}\). Recall from [HMM], Theorem A, (ii); the first display of [HMM], Corollary C, that

(a) the image \( \text{GT}_3 \subseteq \text{Out}(\Pi_{X_3}) \)

of the natural inclusion \( \text{GT} \rightarrow \text{Out}(\Pi_{X_3}) \) may be reconstructed from the data of (i), (a). Next, observe that the natural outer action of \( \text{GT}_3 = \text{Out}^{\text{sp}}(\Pi_{X_3}) \) on \( \Pi_{X_3} \) stabilizes \( \text{Ker}(\text{pr}_{i,j}) \subseteq \Pi_{X_3} \), hence determines

(b) an outer representation \( \Pi_{X_3} \times \text{GT}_3 \rightarrow \text{Out}(\text{Ker}(\text{pr}_{i,j})) \),

which is \( l \)-cyclotomically full [cf. [CmbGC], Definition 2.3, (ii)] for any prime number \( l \). In particular, by applying the algorithm implicit in the proof of [CmbGC], Corollary 2.7, (i), we conclude that the cuspidal inertia subgroups of \( \text{Ker}(\text{pr}_{i,j}) \) may be reconstructed group-theoretically from the data of (b). Thus, by varying \((i,j) \in \{(1,2), (1,3), (2,3)\}\), we conclude that

(c) the inertia subgroups \( \in I_{X_3}^F \)
may be reconstructed group-theoretically from the data of (i), (a), (b), (c), (d).

(v) Next, we reconstruct the data of (ii), (b). Let \( I \in I^F_{X_3} \) be such that, for each \( h = 1, 2, 3 \), \( \text{pr}_h(I) = \{1\} \). Then there exists a unique pair \((i, j) \in \{(1, 2), (1, 3), (2, 3)\}\) such that \( \text{pr}_{i,j}(I) \neq \{1\} \). Write

- \( \Pi_W \subseteq \Pi_X \) for the maximal normal open subgroup such that \( \Pi_W \subseteq \Pi_U \);
- \( \Pi_{Z_3} \overset{\text{def}}{=} \Pi_{X_3} \times \Pi_{X_3} \times \Pi_{X_3} \Pi_W \times \Pi_W \times \Pi_W \subseteq \Pi_{X_3} \), i.e., the inverse image via the surjection \( \Pi_{X_3} \twoheadrightarrow \Pi_X \times \Pi_X \times \Pi_X \) induced by \( p_1, p_2, \) and \( p_3 \) of the open subgroup \( \Pi_W \times \Pi_W \times \Pi_W \subseteq \Pi_X \times \Pi_X \times \Pi_X \) [determined by the inclusion \( \Pi_W \subseteq \Pi_X \)];

Note that \( I \subseteq \Pi_{V_3} \subseteq \Pi_{X_3} \). Then it follows from a similar argument to the argument applied in the proof of [CmbCsp], Proposition 1.2, (iii), that \( \text{pr}_i \) and \( \text{pr}_j \) induce natural isomorphisms

\[
\text{pr}_{i,j} : N_{\Pi_{Z_3}}(I) / \text{Ker}(\text{pr}_{i,j}) \cap N_{\Pi_{Z_3}}(I) \cong \Pi_W,
\]

and that the outer automorphism of \( \Pi_W \) determined by \( g \circ g_i^{-1} \) coincides with the outer automorphism determined by \( a(u) \) [unique] element \( g \in \Pi_X / \Pi_W \). Next, for each \((i, j) \in \{(1, 2), (1, 3), (2, 3)\}\) and \( g \in \Pi_X / \Pi_W \), we shall write

\[
I_{i,j;g} \subseteq I^F_{X_3}
\]

for the subset consisting of the elements \( I \in I^F_{X_3} \) such that

- for each \( h = 1, 2, 3 \), \( \text{pr}_h(I) = \{1\} \);
- \( \text{pr}_{i,j}(I) \neq \{1\} \);
- \( g \circ g_i^{-1} \) coincides with the outer automorphism of \( \Pi_W \) determined by \( g \in \Pi_X / \Pi_W \).

Then we may reconstruct the kernel of the natural surjection \( \Pi_{V_3} \twoheadrightarrow \Pi_{U_3} \) as the normal closed subgroup of \( \Pi_{V_3} \) topologically normally generated by the elements of the subset

\[
\bigcup_{i,j;g} I_{i,j;g} \subseteq I^F_{X_3},
\]

(vi) Finally, we reconstruct the data of (ii), (d). Write

- \( I^F_{V_3} \overset{\text{def}}{=} \{ I \cap \Pi_{V_3} (\subseteq \Pi_{X_3}) \mid I \in I^F_{X_3} \} \);
- \( I^F_{U_3} \) for the set of images of elements of \( I^F_{V_3} \) via the natural surjection \( \Pi_{V_3} \twoheadrightarrow \Pi_{U_3} \) [cf. (v)].
On the other hand, for each $i = 1, 2, 3$, $\text{pr}_i$ naturally induces an outer surjection $q_i : \Pi_{U_3} \twoheadrightarrow \Pi_U$. Thus, we may reconstruct the kernel of the natural outer surjection $\Pi_{U_3} \twoheadrightarrow \Pi_X$ as the normal closed subgroup topologically generated by the elements $I \in I_{U_3}$ satisfying the following condition:

there exists $i \in \{1, 2, 3\}$ such that $q_i(I) \subseteq \Pi_U$ is a cuspidal inertia subgroup that is not associated to 0, 1, $\infty$.

Remark 3.3.4. We maintain the notation of Remark 3.3.3. Let $J \subseteq \text{GT}$ be a closed subgroup; $N$ a normal open subgroup of $J$;

$$\begin{align*}
\Pi_U \rtimes N & \twoheadrightarrow \Pi_X \rtimes N \\
\downarrow & \\
\Pi_X \rtimes N
\end{align*}$$

an arithmetic Belyi diagram, which we denote by $\mathbb{B}^a$. Write $U_2$ (respectively, $X_2$) for the second configuration space associated to $U$ (respectively, $X$); $p_U : \Pi_{U_2} \rightarrow \Pi_U$ (respectively, $p_X : \Pi_{X_2} \rightarrow \Pi_X$) for the outer surjection induced by the first projection. Let us recall from [Tsjm], Lemma 1.2, (b) [cf. also [Tsjm], Theorem 1.3, (ii); [Tsjm], Definition 1.4], that the outer action of $N$ on $\Pi_U$ extends uniquely [cf. the slimness of $\Pi_X$] to a $\Pi_U$-outer action on $\Pi_X$ that is compatible with the outer action of $J$ ($\supseteq N$) on $\Pi_X$. Then observe that this $\Pi_U$-outer action of $N$ on $\Pi_X$ allows one to construct

- a natural outer action of $N$ on $\Pi_{X_2}$ that determines an injection $N \hookrightarrow \text{Out}^{FC}(\Pi_{X_2})$,

|together with |

- a compatible natural $\Pi_{U_3}$-outer action of $N$ on $\Pi_{X_3}$ that stabilizes $\Pi_{U_3}$ [cf. the discussion preceding Claim 1.3.B in the proof of [Tsjm], Theorem 1.3, (ii)]. Next, recall from Remark 3.3.3, (ii), (b), (d) [cf. also Remark 3.3.3, (v), (vi)], that the resulting outer action of $N$ on $\Pi_{U_3}$ determines injections

$$\begin{align*}
N & \hookrightarrow \text{Out}^{FC}(\Pi_{U_3}), \\
N & \hookrightarrow \text{Out}^{FC}(\Pi_{X_3})
\end{align*}$$

compatible with the outer surjections $\Pi_{U_3} \twoheadrightarrow \Pi_{U_2} \twoheadrightarrow \Pi_{X_3}$. The F-admissibility of these outer actions implies that these natural outer actions of $N$ on $\Pi_{U_2}$ and $\Pi_{X_2}$ determine injections

$$\begin{align*}
N & \hookrightarrow \text{Out}^{gF}(\Pi_{U_2})^{\text{cusp}} \subseteq \text{Out}^{FC}(\Pi_{U_2}), \\
N & \hookrightarrow \text{Out}^{gF}(\Pi_{X_2})^{\text{cusp}} \subseteq \text{Out}^{FC}(\Pi_{X_2})
\end{align*}$$

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[cf. Corollary 2.2; [CbTpII], Theorem A, (ii)] and a commutative diagram

\[
\begin{array}{ccc}
\Pi_{U_2} \times N & \longrightarrow & \Pi_{X_2} \times N \\
p_U \times \text{id}_N & & p_X \times \text{id}_N \\
\Pi_U \times N & \longrightarrow & \Pi_X \times N,
\end{array}
\]

where the lower horizontal arrow is the horizontal arrow of $B^\times$. Note that the outer action of $N$ on $\Pi_{U_2}$ (respectively, $\Pi_{X_2}$) just constructed is uniquely determined by the following two conditions [cf. Corollary 2.2; [CbTpII], Theorem A, (ii); [CmbCsp], Theorem A, (i)]:

- the outer action of $N$ on $\Pi_{U_2}$ (respectively, $\Pi_{X_2}$) determines an injection $N \hookrightarrow \text{Out}^F(\Pi_{U_2})^{\text{cusp}}$ (respectively, $N \hookrightarrow \text{Out}^F(\Pi_{X_2})^{\text{cusp}}$);

- the outer action of $N$ on $\Pi_{U_2}$ (respectively, $\Pi_{X_2}$) induces the given outer action of $N$ on $\Pi_U$ (respectively, $\Pi_X$) via the outer surjection $p_U$ (respectively, $p_X$).

**Proposition 3.4 (Functoriality of cuspidal inertia subgroups via geometric dominations).** In the situation of Definition 3.3, (i), every conjugacy class of cuspidal inertia subgroups of $\Pi_U$ arises as the image via $\phi$ of a unique conjugacy class of cuspidal inertia subgroups of $\Pi_{U^\dagger}$.

**Proof.** We regard $U, U^\dagger$ as open subschemes of $X$ via the respective natural open immersions $U \hookrightarrow X$, $U^\dagger \hookrightarrow X$. Write $\text{Cusp}(U^\dagger)$ for the set of cusps of $U^\dagger$; $S \subseteq \text{Cusp}(U^\dagger)$ for the subset of cusps $s \in \text{Cusp}(U^\dagger)$ such that some [or equivalently, every] cuspidal inertia subgroup of $\Pi_{U^\dagger}$ associated to $s$ is contained in $\text{Ker}(\phi)$; $U^\dagger \subseteq U_S^\dagger \subseteq X$ for the partial compactification of $U^\dagger$ such that $U^\dagger = U_S^\dagger \setminus S$. Thus, the natural outer surjection $\Pi_{U^\dagger} \twoheadrightarrow \Pi_{U^\dagger}_S$ induces a bijection between the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{U^\dagger}$ associated to cusps in $\text{Cusp}(U^\dagger) \setminus S$ and the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{U^\dagger}_S$. Next, observe that it follows immediately from Definition 3.3, (i), (a), (b), that $\phi$ induces an outer isomorphism

\[\phi_S : \Pi_{U^\dagger}_S \cong \Pi_U\]

such that

(i) $\phi_S$ maps every cuspidal inertia subgroup of $\Pi_{U^\dagger}_S$ to a cuspidal inertia subgroup of $\Pi_U$;

(ii) $\phi_S$ maps every cuspidal inertia subgroup of $\Pi_{U^\dagger}_S$ associated to 0, 1, $\infty$ to a cuspidal inertia subgroup of $\Pi_U$ associated to 0, 1, $\infty$, respectively.

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Thus, to complete the proof of Proposition 3.4, it suffices to verify that $\phi_S$ induces [cf. (i)] a bijection between the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{U^\dagger}$ and the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_U$. To this end, let us first observe that injectivity follows immediately from the fact that $\phi_S$ is an outer isomorphism. On the other hand, surjectivity follows immediately, in light of (ii), from the fact that [since the hyperbolic curves $U^\dagger$ and $U$ are of genus 0] $\Pi_{U^\dagger}$ and $\Pi_U$ are topologically generated by their respective collections of cuspidal inertia subgroups associated to cusps $\neq \infty$. This completes the proof of Proposition 3.4.

Proposition 3.5 (Natural action of $\text{GT}$ on the set of geometric dominations). In the notation of Definition 3.3, (i), one may construct a natural action of $C_{\text{GT}}(J) \subseteq \text{Out}(\Pi_X)$ on the set of geometric dominations between arbitrary arithmetic Belyi diagrams. 

Proof. Let us consider the data of Remark 3.3.3, (i), (a), (b), (c), (d), (e), (f), (g), (h), associated to $\mathbb{B}^\mathbb{N}$ and $\mathfrak{F}^\mathbb{N}$. Then the data of

- “$\Pi_{U^\dagger} \rtimes M$”, “$\Pi_{U_2^\dagger} \rtimes M$”, together with

- the respective fiber subgroups of length 1 and cuspidal inertia subgroups of such fiber subgroups

[cf. Definition 3.3, (i)] may be reconstructed from the data of Remark 3.3.3, (i), (a), (b); Remark 3.3.3, (ii), (b); Remark 3.3.3, (iv), (c). Thus, Proposition 3.5 follows immediately from the various definitions involved.

Theorem 3.6 (Faithfulness via the CS-property for certain outer actions on configuration space groups induced by open immersions). Let $J \subseteq \text{GT}$ be a closed subgroup; $N$ a normal open subgroup of $J$;

$$
\begin{array}{ccc}
\Pi_U \rtimes N & \longrightarrow & \Pi_X \rtimes N \\
\downarrow & & \downarrow \\
\Pi_X \rtimes N
\end{array}
$$

an arithmetic Belyi diagram, which we denote by $\mathbb{B}^\mathbb{N}$. Write $U_2$ (respectively, $X_2$) for the second configuration space associated to $U$ (respectively, $X$); $p_U : \Pi_{U_2} \rightarrow \Pi_U$ (respectively, $p_X : \Pi_{X_2} \rightarrow \Pi_X$) for the outer surjection induced by the first projection. Thus, we have a commutative diagram

$$
\begin{array}{ccc}
\Pi_{U_2} \rtimes N & \longrightarrow & \Pi_{X_2} \rtimes N \\
\downarrow & & \downarrow \\
\Pi_U \rtimes N & \longrightarrow & \Pi_X \rtimes N
\end{array}
$$

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as in Remark 3.3.4. We regard $U$ as an open subscheme of $X$ via the natural open immersion $U \hookrightarrow X$. For each sequence

$$U \subseteq V \subseteq W \subseteq X$$

of open subschemes of $X$, write $V_2, W_2$ for the second configuration spaces associated to the hyperbolic curves $V, W$, respectively;

$$h_{V,W} : \text{Out}^F(\Pi_{V_2})^{\text{cusp}} \to \text{Out}^F(\Pi_{W_2})^{\text{cusp}}$$

for the homomorphism induced by the upper horizontal arrow of the above commutative diagram [cf. Theorem 2.1; [HMM], Corollary B; the well-known elementary structure of $\mathfrak{S}_5$]; $N_{V_2} \subseteq \text{Out}^F(\Pi_{V_2})^{\text{cusp}}$ for the image via the composite

$$N \hookrightarrow \text{Out}^F(\Pi_{V_2})^{\text{cusp}} \overset{h_{V,W}}{\to} \text{Out}^F(\Pi_{W_2})^{\text{cusp}}$$

[cf. Remark 3.3.4]. Suppose that $\mathbb{B}^\times$ satisfies the CS-property [cf. Definition 3.3, (iv)]. Then, for any $V, W$ as above, the composite

$$Z_{\text{Out}^F(\Pi_{V_2})^{\text{cusp}}}(N_{V_2}) \subseteq \text{Out}^F(\Pi_{V_2})^{\text{cusp}} \overset{h_{V,W}}{\to} \text{Out}^F(\Pi_{W_2})^{\text{cusp}}$$

is injective.

Proof. Write $h \overset{\text{def}}{=} h_{V,W} : \text{Cusp}(V), \text{Cusp}(W)$ for the set of cusps of $V, W$, respectively. First, let us note that we may assume without loss of generality [i.e., by forming the composite of the $h_{V,W}$ for suitable $V, W$] that the cardinality of the set $\text{Cusp}(V) \setminus \text{Cusp}(W)$ is 1. Let

$$\beta \in Z_{\text{Out}^F(\Pi_{V_2})^{\text{cusp}}}(N_{V_2}) \subseteq \text{Out}^F(\Pi_{V_2})^{\text{cusp}}$$

be such that $h(\beta) = 1$. Then it suffices to verify that

$$\beta = 1.$$

Note that the natural composites

$$N \overset{\sim}{\twoheadrightarrow} N_{V_2} \subseteq \text{Out}^F(\Pi_{V_2})^{\text{cusp}}, \quad N \overset{\sim}{\twoheadrightarrow} N_{W_2} \subseteq \text{Out}^F(\Pi_{W_2})^{\text{cusp}}$$

determine natural outer actions of $N$ on $\Pi_{V_2}, \Pi_{W_2}$, hence also on $\Pi_V, \Pi_W$ [by applying the natural outer surjections $\Pi_{V_2} \to \Pi_V, \Pi_{W_2} \to \Pi_W$ determined by the respective first projections].

Next, let us write

- $y$ for the unique element $\in \text{Cusp}(V) \setminus \text{Cusp}(W)$;
- $\eta_j : \text{Out}^F(\Pi_{V_2}) \to \text{Out}^F(\Pi_V)$ for the natural homomorphism induced by the $j$-th projection, where $j \in \{1, 2\}$ [where we note that in fact, $\eta_1 = \eta_2$ — cf. Corollary 2.2; [CmbCsp], Proposition 1.2, (iii)];
\begin{itemize}
    
    \item $Y^\log$ for the [uniquely determined, up to unique isomorphism] smooth log curve over \text{Spec} $\mathbb{Q}$ such that $U_V = V$;
    
    \item $Y_2^\log$ for the second log configuration space associated to $Y^\log$;
    
    \item $y^\log \overset{\text{def}}{=} y \times_Y Y^\log$ [where the fiber product is determined by the natural map $Y^\log \to Y$ obtained by forgetting the log structure];
    
    \item $Y^\log_y \overset{\text{def}}{=} Y^\log_{Y_y}$ [where the fiber product is determined by the first projection $Y^\log_y \to Y^\log$ and the natural map $y^\log \to Y^\log$];
    
    \item $G_y$ for the semi-graph of anabelioids of pro-$\mathfrak{P}$-Primes PSC-type determined by the stable log curve $Y^\log_y$ [cf. \cite{CmbGC}, Definition 1.1, (i)];
    
    \item $c_y, c_\Delta$ for the cusps of $G_y$ that arise from $y$, the diagonal divisor of $Y^\log_2$, respectively;
    
    \item $v_y$ for the vertex of $G_y$ associated to the irreducible component that does not contain $c_\Delta$;
    
    \item $\Pi_{G_y}$ for the pro-$\mathfrak{P}$-Primes fundamental group of $G_y$ [cf. \cite{CmbGC}, Definition 1.1, (ii)].
    
\end{itemize}

Then we have a commutative diagram of profinite groups

\[
\begin{array}{cccccc}
1 & \longrightarrow & \Pi_{G_y} & \longrightarrow & \Pi_{V_2}^\text{out} \times N & \longrightarrow & \Pi_V^\text{out} \times N & \longrightarrow & 1 \\
& & \downarrow q_y & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \Pi_V & \longrightarrow & \Pi_{W_2}^\text{out} \times N & \longrightarrow & \Pi_W^\text{out} \times N & \longrightarrow & 1,
\end{array}
\]

where the middle and right-hand vertical arrows denote surjections that represent the outer surjection induced by the natural open immersion $V \hookrightarrow W$; $\Pi_{V_2}^\text{out} \times N \to \Pi_V^\text{out} \times N, \Pi_{W_2}^\text{out} \times N \to \Pi_W^\text{out} \times N$ denote surjections that represent the outer surjections induced by the respective first projections; $q_y$ denotes the induced surjection. [Note that $\text{Ker}(q_y)$ coincides with the normal closed subgroup topologically generated by the cuspidal inertia subgroups of $\Pi_{G_y}$ associated to $c_y$.]

Since $\beta \in Z_{\text{Out}^\text{er}(\Pi_{V_2})}(N_{V_2}) \subseteq \text{Out}^\text{erF}(\Pi_{V_2})$, and $\Pi_{V_2}$ is center-free [cf. \cite{MT}, Proposition 2.2, (ii)], $\beta$ determines a $\Pi_{V_2}$-outer automorphism $\gamma_V$ of $\Pi_{V_2}^\text{out} \times N$ that lies over $N$. Let $I_y$ be a cuspidal inertia subgroup of $\Pi_V$ associated to $y$; $\tilde{\gamma}_V \in \text{Aut}(\Pi_{V_2}^\text{out} \times N)$ a lifting of $\gamma_V$. Write $(\tilde{\gamma}_V)_1$ for the automorphism of $\Pi_V^\text{out} \times N$ induced by $\tilde{\gamma}_V$ via the surjection $\Pi_{V_2}^\text{out} \times N \to \Pi_V^\text{out} \times N$ in the above commutative diagram. Then since $\beta \in \text{Out}^\text{erF}(\Pi_{V_2})^\text{cusp}$, by replacing $\tilde{\gamma}_V$ by a suitable composite with an inner automorphism of $\Pi_{V_2}^\text{out} \times N$ [determined by an element of $\Pi_{V_2}$], we may assume without loss of generality that

\[
(\tilde{\gamma}_V)_1(I_y) = I_y.
\]
Let $\Pi_v \subseteq \Pi_{\bar{g}_v}$ be a vertical subgroup associated to $v_y$. Note that since $V \subseteq W$, $v_y$ is not of type $(0, 3)$. Thus, it follows immediately from [ChTpII], Theorem 1.9, (ii), that the restriction $\overline{\gamma}_V|_{\Pi_v}$ of $\overline{\gamma}_V$ to $\Pi_{\bar{g}_v}$ preserves and fixes the conjugacy class of $\Pi_v$. Moreover, by replacing $\overline{\gamma}_V$ by a suitable composite with an inner automorphism of $\Pi_{\bar{g}_v}$ determined by an element of $\Pi_{\bar{g}_v}$ if necessary, we may assume without loss of generality that $\overline{\gamma}_V|_{\Pi_v}(\Pi_v) = \Pi_v$.

Write $\overline{\gamma}_V \in \text{Aut}(\Pi_{W_2} \rtimes N)$ for the automorphism [that lies over $N$] induced by $\overline{\gamma}_V$ [cf. Theorem 2.1] via the surjection $\Pi_{W_2} \rtimes N \twoheadrightarrow \Pi_{W_2} \rtimes N$ in the above commutative diagram.

Next, we verify the following assertion:

Claim 3.6.A: The outer automorphism $\gamma \in \text{Out}(\Pi V)$ determined by the restriction $\overline{\gamma}_V|_{\Pi V}$ of $\overline{\gamma}_V$ to $\Pi V$ ($\to \Pi_{W_2} \rtimes N$) coincides with $\eta_2(\beta) \in \text{Out}(\Pi V)$.

Recall that $\overline{\gamma}_V|_{\Pi_{v_y}}$ preserves the cuspidal inertia subgroups of $\Pi_{\bar{g}_v}$ [cf. Corollary 2.2]. Write $q_\Delta : \Pi_{\bar{g}_v} \twoheadrightarrow \Pi_{v_y}$ for the natural outer surjection induced by the second projection $V_2 \twoheadrightarrow V$. Note that $\text{Ker}(q_\Delta)$ coincides with the normal closed subgroup topologically generated by the cuspidal inertia subgroups of $\Pi_{\bar{g}_v}$ associated to $c_\Delta$. On the other hand, it follows immediately from the various definitions involved that

- $\gamma$ (respectively, $\eta_2(\beta)$) coincides with the outer automorphism induced by $\overline{\gamma}_V|_{\Pi_v}$ via the surjection $q_y$ (respectively, $q_\Delta$);
- $q_y$ and $q_\Delta$ determine the same outer isomorphism $(\Pi_{\bar{g}_v} \supseteq) \Pi_{v_y} \cong \Pi_{v_y}$.

Thus, since $\overline{\gamma}_V|_{\Pi_{v_y}}(\Pi_v) = \Pi_v$, we obtain the desired conclusion. This completes the proof of Claim 3.6.A.

Next, since $h(\beta) = 1$, we have

$$\overline{\gamma}_W \in \text{Inn}(\Pi_{W_2} \rtimes N) \subseteq \text{Aut}(\Pi_{W_2} \rtimes N),$$

where the inner automorphism $\overline{\gamma}_W$ is determined by an element $\in \Pi_{W_2}$. Write

- $(\overline{\gamma}_W)$, for the inner automorphism of $\Pi_{W_2} \rtimes N$ determined by an element $\in \Pi_{W_2}$ induced by $\overline{\gamma}_W$ via the surjection $\Pi_{W_2} \rtimes N \twoheadrightarrow \Pi_{W_2} \rtimes N$ in the above commutative diagram;
- $D_y (\to N)$ for the image of $N_{\Pi_{v_y}} \rtimes N(I_y)$ via the surjection $\Pi_{W_2} \rtimes N \twoheadrightarrow \Pi_{W_2} \rtimes N$ in the above commutative diagram.
Then it follows from our assumption that \((\gamma_W)_1(I_y) = I_y\) that \((\gamma_W)_1(D_y) = D_y\). Recall that since \(\mathbb{B}^\times\) satisfies the CS-property, \(D_y\) is normally terminal in \(\Pi_W \rtimes N\) [cf. the final sentence of Definition 3.3, (iv)]. Thus, we conclude that the inner automorphism \((\gamma_W)_1 \in \text{Inn}(\Pi_W \rtimes N)\) is determined by an \((n)\) element \(\in D_y \cap \Pi_W = \{1\}\), hence, in particular, that the inner automorphism \(\gamma_W\) is determined by an element \(\in \Pi_V \subseteq \Pi_{W_z}\), i.e., that \(\gamma = 1\). Finally, it follows immediately from the injectivity of \(\eta_{\beta}\) [cf. Corollary 2.2; [CmbCsp], Theorem A, (i)], together with Claim 3.6.A, that \(\beta = 1\). This completes the proof of Theorem 3.6.

**Corollary 3.7 (The CS-property implies the RGC-property).** Let \(J \subseteq \mathcal{G}T\) be a closed subgroup satisfying the CS-property [cf. Definition 3.3, (iv)]. Then \(J\) satisfies the RGC-property [cf. Definition 3.3, (iii)].

**Proof.** In the notation of Definition 3.3, (i), let \(\phi, \phi'\) be arithmetic dominations of \(\mathbb{B}^\times\) by \(\mathbb{B}^\times\), defined over a normal open subgroup \(M \subseteq J\). Then it suffices to prove that \(\phi = \phi'\). Since \(\text{Ker}(\phi)\) and \(\text{Ker}(\phi')\) are topologically generated by [certain of the] cuspidal inertia subgroups of \(\Pi_{U^2}\) [cf. Definition 3.3, (i), (a)], it follows immediately from the CS-property [where we take the \(\mathcal{T}\) of Definition 3.3, (iv), to be \(\text{Cusp}(U^2)\)], together with Definition 3.3, (i), (b), that

\[
\text{Ker}(\phi) = \text{Ker}(\phi').
\]

Fix \(\Pi_{U^2}\)-outer surjections

\[
\phi_2 : \Pi_{U^2} \times M \twoheadrightarrow \Pi_{U^2} \rtimes M, \quad \phi'_2 : \Pi_{U^2} \times M \twoheadrightarrow \Pi_{U^2} \rtimes M
\]

[that lie over \(\phi, \phi'\)] as in Definition 3.3, (i), (a), respectively.

Next, we observe the following assertion:

Claim 3.7.A: \(\phi_2\) and \(\phi'_2\) map the inertia subgroups of \(\Pi_{U^2}\) associated to the diagonal divisor of \(U^2\) to the inertia subgroups of \(\Pi_{U^2}\) associated to the diagonal divisor of \(U_2\).

Indeed, Claim 3.7.A follows immediately from Definition 3.3, (i), (a).

Next, we verify the following assertion:

Claim 3.7.B: \(\text{Ker}(\phi_2) = \text{Ker}(\phi'_2)\).

Write

\[
\phi : \Pi_U \times M \twoheadrightarrow \Pi_U \rtimes M, \quad \phi'_o : \Pi_{U^1} \rtimes M \twoheadrightarrow \Pi_U \rtimes M
\]

for the \(\Pi_U\)-outer surjections determined by \(\phi_2, \phi'_o\), respectively, via the outer surjections \(\Pi_{U^2} \twoheadrightarrow \Pi_U, \Pi_{U^2} \twoheadrightarrow \Pi_{U^1}\) induced by the respective second projections. Then it follows immediately from Claim 3.7.A, together with a similar argument to the argument applied in the proof of [CmbCsp], Proposition 1.2, (iii), that the following assertion holds:
Claim 3.7.C: $\phi = \phi_\ast$, $\phi' = \phi'_\ast$. In particular, $\text{Ker}(\phi_\ast) = \text{Ker}(\phi) = \text{Ker}(\phi') = \text{Ker}(\phi'_\ast)$.

Thus, since $\text{Ker}(\phi_2)$ and $\text{Ker}(\phi'_2)$ are topologically generated by [certain of the] cuspidal inertia subgroups of fiber subgroups of $\Pi_{U_2}$ of length 1 [cf. Definition 3.3, (i), (a)], we conclude, again from Claim 3.7.A [cf. also [CbTpII], Lemma 3.6, (i), (ii)], that $\text{Ker}(\phi_2) = \text{Ker}(\phi'_2)$. This completes the proof of Claim 3.7.B.

It follows immediately from Claim 3.7.B that there exists a unique $U_2$-outer automorphism $\varphi : \Pi_{U_2} \rtimes M \rightarrow \Pi_{U_2} \rtimes M$ such that $\phi_2 = \alpha \circ \phi'_2$. On the other hand, it follows from the $CS$-property, together with Definition 3.3, (i), (b), that we may apply Theorem 3.6 to conclude that $\alpha$ is the identity, hence that $\phi_2 = \phi'_2$, $\phi = \phi'$. This completes the proof of Corollary 3.7.

\section{Combinatorial construction of the field $\overline{\mathbb{Q}}_{BGT}$}

In §3, we defined a certain class of closed subgroups BGT of GT [cf. Definition 3.3, (v)]. In this section, for each such closed subgroup BGT, we give a purely combinatorial/group-theoretic construction of a field $\overline{\mathbb{Q}}_{BGT}$ associated to BGT equipped with a natural action by $C_{\text{GT}}(\text{BGT})$. Finally, by observing that this associated field is isomorphic to the field $\overline{\mathbb{Q}}$, we construct a natural outer homomorphism $C_{\text{GT}}(\text{BGT}) \rightarrow G_{\overline{\mathbb{Q}}} \overset{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ [cf. Theorem 4.4].

Write $X \overset{\text{def}}{=} \mathbb{P}^1 \backslash \{0, 1, \infty\}$.

\textbf{Definition 4.1.} Let $\text{BGT} \subseteq \text{GT}$ be a closed subgroup satisfying the $BC$-property [cf. Definition 3.3, (v)]. For any arithmetic Belyi diagram $\mathbb{B}^\ast$

$$
\begin{array}{ccc}
\Pi_U & \overset{\rtimes N}{\longrightarrow} & \Pi_X \\
\downarrow & & \\
\Pi_X & \overset{\rtimes N}{\longrightarrow} & N
\end{array}
$$

[where $N$ is a normal open subgroup of $\text{BGT}$], write $\Pi_B \overset{\text{def}}{=} \Pi_U$;

$$\text{Cusp}(\mathbb{B}^\ast)$$

for the set of conjugacy classes of cuspidal inertia subgroups [cf. [Tsjm], Theorem 1.3, (i)] of $\Pi_B$. Write

$$I_{\text{BGT}}$$

for the set of the arithmetic Belyi diagrams over normal open subgroups of $\text{BGT}$. We shall regard $I_{\text{BGT}}$ as a preordered set [i.e., a set equipped with a reflexive and transitive binary relation] by means of the relation determined by domination, i.e., the existence of an arithmetic domination [cf. Definition

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3.3, (i); Proposition 3.4: Claim 3.7.C in the proof of Corollary 3.7. It follows immediately from Remarks 3.3.2, 3.3.3; Proposition 3.5; [Tsjm], Definition 1.4, that there is a natural action of $C_{GT}(BGT)$ on the preordered set $I_{BGT}$. Since BGT satisfies the COF-property [cf. Definition 3.3, (ii)], it follows formally that the preordered set $I_{BGT}$ is directed, i.e., any pair of elements of the set admits a(n) [not necessarily minimal!] upper bound. Since BGT also satisfies the RGC-property [cf. Definition 3.3, (iii)], if $z \in B \rtimes I_{BGT}$ dominates $y \in B \rtimes I_{BGT}$, then the unique geometric domination $z \rtarr y$ of $y$ by $z$ determines [cf. Proposition 3.4] a natural injection $\kappa_{1,1} : \text{Cusp}(B^\infty) \hookrightarrow \text{Cusp}(B^\infty)$ [which we shall often use to regard $\text{Cusp}(B^\infty)$ as a subset of $\text{Cusp}(B^\infty)$]. Thus, we obtain a direct system $(\text{Cusp}(z) \rtarr y, z)$. We shall write $Q_{BGT}^0 = \lim B \rtimes I_{BGT} \text{Cusp}(B \rtimes)_{\text{nf}1g}$, $Q_{BGT} = Q_{BGT}^0 \amalg Q_{BGT}^1$, $Q_{BGT}^{<1} = Q_{BGT}^0 \amalg Q_{BGT}^1$, where $0, 1, 1 \in \text{Cusp}(B^\infty)$ denote the elements determined by the $X$-outer surjection $\Pi_X \rtarr \text{Out}(B^\infty)$ and the conjugacy classes of cuspidal inertia subgroups of $\Pi_X$ associated to $0, 1, 1$, respectively. We shall refer to $Q_{BGT}$ as the BGT-realization of $\bar{\Pi}$.

Remark 4.1.1. In the notation of Definition 4.1, it follows immediately from the various definitions involved that the kernel of the unique geometric domination $\Pi_{B^\infty} : \Pi_{B^\infty}$ of $B^\infty$ by $B^\infty$ is the normal closed subgroup of $\Pi_{B^\infty}$ topologically generated by the cuspidal inertia subgroups associated to $\text{Cusp}(B^\infty)$.

Proposition 4.2 (Countability of $I_{BGT}$). In the notation of Definition 4.1, $I_{BGT}$ is countable.

Proof. Let us observe that since $\Pi_X$ is topologically finitely generated,

- the set of open subgroups of $\Pi_X$ is countable;
- there exists a countable open basis of $BGT \subseteq \text{Out}(\Pi_X)$.

Thus, since $\text{Cusp}(B^\infty)$ is finite, it follows from the various definitions involved that $I_{BGT}$ is countable. This completes the proof of Proposition 4.2. \qed
Proposition 4.3 (Natural action of $C_{GT}(BGT)$ on the set $\mathfrak{U}_{BGT}$). There is a natural continuous action of $C_{GT}(BGT)$ on the discrete set $\mathfrak{U}_{BGT}$ [cf. Definition 4.1].

Proof. In the notation of Definition 4.1, let $\sigma \in C_{GT}(BGT); x \in \mathfrak{U}_{BGT}; B^\times \in I_{BGT}$ an arithmetic Belyi diagram

$$\begin{array}{ccc}
\Pi_U \times N & \longrightarrow & \Pi_X \times N \\
\sigma & & \sigma \\
\Pi_X \times N & \longrightarrow & \Pi_X \times N^\sigma,
\end{array}$$

[where $N$ is a normal open subgroup of $BGT$] such that $N^\sigma \overset{\text{def}}{=} \sigma N \sigma^{-1} \subseteq BGT$ and $x \in \text{Cusp}(B^\times)$. Recall that $x$ is the conjugacy class of some cuspidal inertia subgroup $I_x$ of $U$.

Next, let us recall the right-hand square in the diagram of the final display of the proof of [Tsjm], Corollary 1.6, (i), in the case where we take “$J$” to be GT [cf. also Remark 3.3.2]. In the notation of the present discussion, this right-hand square determines a commutative diagram of profinite groups

$$\begin{array}{ccc}
\Pi_U \times N & \longrightarrow & \Pi_X \times N \\
\sigma & & \sigma \\
\Pi_X \times N & \longrightarrow & \Pi_X \times N^\sigma,
\end{array}$$

where the horizontal arrows are the $X$-outer surjections induced by the natural open immersions $U \hookrightarrow X, U^\sigma \hookrightarrow X$ of hyperbolic curves; the left- (respectively, the right-) hand vertical arrow is a $\Pi_U$-outer (respectively, $\Pi_X$-outer) isomorphism of profinite groups. Write $x^\sigma \in \mathfrak{U}_{BGT}$ for the element determined by $\sigma(I_x)$. Thus, to obtain a well-defined action of $C_{GT}(BGT)$ on $\mathfrak{U}_{BGT}$, it suffices to show that $x^\sigma$ does not depend on the choice of $B^\times$. But this follows formally from the $COF$-property of $BGT$, together with Proposition 3.5 and the construction of $x^\sigma$. To verify that the resulting action is continuous, it suffices to observe that there exists an open subgroup $H \subseteq C_{GT}(BGT)$ [which may be obtained, for instance, by forming the intersection of $C_{GT}(BGT)$ with the open subgroup “$N \subseteq GT$” of [Tsjm], Definition 1.4] such that $x^\sigma = x$ for $\sigma \in H$.

This completes the proof of Proposition 4.3.

\[\square\]

Theorem 4.4 (Natural field structure on $\mathfrak{U}_{BGT}$). There exists a natural field structure on the set $\mathfrak{U}_{BGT}$ that is preserved by the natural action of $C_{GT}(BGT)$ [cf. Proposition 4.3]. Moreover, there exists a field isomorphism $\mathfrak{Q} \to \mathfrak{U}_{BGT}$. In particular, there exists a natural outer homomorphism $C_{GT}(BGT) \to G_{\mathfrak{Q}}$. 

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Proof. First, we construct a natural field structure on the set $\mathbb{Q}_{BGT}$. Write $0, 1 \in \mathbb{Q}_{BGT}$ for the elements determined, respectively, by the conjugacy classes of cuspidal inertia subgroups of $\Pi_X$ associated to the cusps “0”, “1” of $X$. Let $y \in \mathbb{Q}^\times_{BGT}$

(respectively, $y \in \mathbb{Q}^\times_{BGT}$; $x \in \mathbb{Q}^\times_{BGT}$; $y \in \mathbb{Q}_{BGT}$)

$\mathbb{B}^\times$ an arithmetic Belyi diagram

$$
P_{U, out} \times N \xrightarrow{\pi} \Pi_X \times N \xrightarrow{f} \Pi_{U, out} \times N$$

[where $N$ is a normal open subgroup of BGT] such that $x, y \in \text{Cusp}(\mathbb{B}^\times)$. Write $\iota : U \hookrightarrow X$ for the open immersion that gives rise to the horizontal arrow of $\mathbb{B}^\times$ [cf. [Tsjm], Definition 1.1, (i); [Tsjm], Definition 1.4]; $t$ for the standard coordinate on $X \defeq \mathbb{P}_\mathbb{Q}^1 \setminus \{0,1,\infty\}$;

$$
\iota^{-1} : U \hookrightarrow X
$$

(respectively, $\iota^{-1} : U \hookrightarrow X$; $\iota^{t/x} : U \hookrightarrow X$)

for the open immersion obtained from $\iota : U \hookrightarrow X$ by composing with the automorphism $t \mapsto t^{-1}$ of $X$ [i.e., the automorphism of $X$ that switches the cusps “0” and “$\infty$”] (respectively, composing with the automorphism $t \mapsto 1-t$ of $X$ [i.e., the automorphism of $X$ that switches the cusps “0” and “1”]; compactifying at the cusp “1” but not at the cusp “$\infty$”). Then it follows immediately from [Tsjm], Theorem 1.3, (ii) [cf. Remark 3.3.2], that the open immersion $\iota^{-1} : U \hookrightarrow X$ (respectively, $\iota^{-1} : U \hookrightarrow X$; $\iota^{t/x} : U \hookrightarrow X$) determines a $\Pi_X$-outer surjection

$$
f^{-1} : \Pi_{U, out} \times N \twoheadrightarrow \Pi_X \times N$$

(respectively, $f^{-1} : \Pi_{U, out} \times N \twoheadrightarrow \Pi_X \times N$; $f^{t/x} : \Pi_{U, out} \times N \twoheadrightarrow \Pi_X \times N$).

Thus, by considering $y$ via $f^{-1}$ (respectively, $f^{1-t}$; $f^{t/x}$) [cf. Definition 4.1], we obtain a new element $y^{\iota^{-1}} \in \mathbb{Q}_{BGT}$ (respectively, $y^{1-t} \in \mathbb{Q}_{BGT}$; $y^{t/x} \in \mathbb{Q}_{BGT}$).

In particular, by the COF-property of BGT, we obtain natural bijections
\( \{ -1 \} : \mathbb{Q}^{\times}_{\text{BGT}} \sim \mathbb{Q}^{\times}_{\text{BGT}}; \)

\( (1 - t) : \mathbb{Q}_{\text{BGT}} \sim \mathbb{Q}_{\text{BGT}}; \)

\( \{ t/x \} : \mathbb{Q}_{\text{BGT}} \sim \mathbb{Q}_{\text{BGT}} \)

such that \( \{ -1 \}(y) = y^{-1}, \) \( (1 - t)(y) = y^{1-t}, \) and \( \{ t/x \}(y) = y^{t/x}. \) Here, we observe that \( \{ -1 \} \) and \( \{ 1 - t \} \) are involutions, while \( \{ t/x \} \) and \( \{ t/x^{-1} \}, \) where we write \( x^{-1} \overset{\text{def}}{=} \{ t/x \}(1) \in \mathbb{Q}_{\text{BGT}}, \) are inverse to one another.

For each \( (x, y) \in \mathbb{Q}^{\times}_{\text{BGT}} \times \mathbb{Q}_{\text{BGT}}, \) write

\[ \Box_{\text{BGT}}(x, y) \overset{\text{def}}{=} \{ t/t^{-1} \}(x)(y), \]

\[ \Box_{\text{BGT}}(0, \mathbb{Q}_{\text{BGT}}) \overset{\text{def}}{=} 0, \quad \Box_{\text{BGT}}(1, y) \overset{\text{def}}{=} y. \]

Thus, we obtain a multiplication map

\[ \Box_{\text{BGT}} : \mathbb{Q}^{\times}_{\text{BGT}} \times \mathbb{Q}_{\text{BGT}} \to \mathbb{Q}_{\text{BGT}}. \]

Write

\[ \mathbb{B}^{1}_{\Sigma} \]

for the arithmetic Belyi diagram [over a suitable normal open subgroup of BGT — cf. the subgroup "M" of [Tsjm], Definition 1.4] determined by the unique [up to isomorphism] connected finite étale covering of \( X \) of degree 2 ramified over 0 and \( \infty: \)

\[ -1_{\text{BGT}} \in \mathbb{Q}_{\text{BGT}} \]

for the element of \( \mathbb{Q}_{\text{BGT}} \) determined by the unique element of \( \text{Cusp}(\mathbb{B}^{1}_{\Sigma}) \setminus \{ 0, 1, \infty \}. \) Then we obtain an addition map

\[ \boxplus_{\text{BGT}} : \mathbb{Q}^{\times}_{\text{BGT}} \times \mathbb{Q}_{\text{BGT}} \to \mathbb{Q}_{\text{BGT}} \]

by taking

\[ \boxplus_{\text{BGT}}(x, y) \overset{\text{def}}{=} \Box_{\text{BGT}}(x, (1 - t)(\Box_{\text{BGT}}(-1_{\text{BGT}}, \Box_{\text{BGT}}(\{ t^{-1} \}(x), y))), \]

\[ \boxplus_{\text{BGT}}(0, y) \overset{\text{def}}{=} y, \]

where \( (x, y) \in \mathbb{Q}^{\times}_{\text{BGT}} \times \mathbb{Q}_{\text{BGT}}. \)

Next, we verify the following assertion:

Claim 4.4.A: \( \Box_{\text{BGT}} \) and \( \boxplus_{\text{BGT}} \) determine a field structure on \( \mathbb{Q}_{\text{BGT}} \)

such that there exists a field isomorphism \( \mathbb{Q} \sim \mathbb{Q}_{\text{BGT}}. \)

In the following discussion, we shall identify \( X(\mathbb{Q}) \) with \( \mathbb{Q}^{\times}. \) We begin by observing that, for any pair consisting of
• an arithmetic Belyi diagram $\mathbb{B}^\times$

\[
\begin{array}{c}
\Pi_U \rtimes N \longrightarrow \Pi_X \rtimes N \\
\downarrow \\
\Pi_X \rtimes N
\end{array}
\]

[where $N$ is a normal open subgroup of BGT] and

• a finite subset $F \subseteq \overline{\mathbb{Q}}^h$,

there exist

• an open immersion $U^\dagger \hookrightarrow U \hookrightarrow X$ over $\mathbb{Q}$ such that

\[
F \subseteq X(\mathbb{Q}) \setminus U^\dagger(\mathbb{Q}) \subseteq X(\mathbb{Q}) = \overline{\mathbb{Q}}^h
\]

[where we regard $U^\dagger(\mathbb{Q})$ as a subset of $X(\mathbb{Q})$ by means of the composite of the open immersion $U^\dagger \hookrightarrow U$ with the open immersion $U \hookrightarrow X$ that gives rise to the horizontal arrow of the given arithmetic Belyi diagram],

• a normal open subgroup $M^\dagger \subseteq N$ of BGT, and

• an arithmetic Belyi diagram $\mathbb{B}^\times$

\[
\begin{array}{c}
\Pi_U^\dagger \rtimes M^\dagger \longrightarrow \Pi_X \rtimes M^\dagger \\
\downarrow \\
\Pi_X \rtimes M^\dagger
\end{array}
\]

[where the restriction $\Pi_U^\dagger \rightarrow \Pi_X$ of the horizontal arrow to $\Pi_U^\dagger$ is the $\Pi_X$-outer surjection that arises from the above open immersion $U^\dagger \hookrightarrow U \hookrightarrow X$ over $\mathbb{Q}$]

such that the outer action of $M^\dagger$ on $\Pi_U^\dagger$ is compatible, relative to the outer surjection $\Pi_U^\dagger \rightarrow \Pi_U$ [induced by the open immersion $U^\dagger \hookrightarrow U$], with the restriction to $M^\dagger \subseteq N$ of the outer action of $N$ on $\Pi_U$. Indeed, write $g : U \rightarrow X$ for the connected finite étale covering that gives rise to the vertical arrow of the given arithmetic Belyi diagram. Let $^*\mathbb{B}^\times$ be an arithmetic Belyi diagram

\[
\begin{array}{c}
\Pi_U^* \rtimes M^* \longrightarrow \Pi_X \rtimes M^* \\
\downarrow \\
\Pi_X \rtimes M^*
\end{array}
\]

[where $M^*$ is a normal open subgroup of BGT] such that

\[
U^* (\mathbb{Q}) \subseteq X(\mathbb{Q}) \setminus g(U(\mathbb{Q}) \cap F) \subseteq X(\mathbb{Q}) = \overline{\mathbb{Q}}^h
\]

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[where we regard \( U^* (\mathbb{Q}) \) as a subset of \( X (\mathbb{Q}) \) by means of the open immersion \( U^* \hookrightarrow X \) that gives rise to the horizontal arrow of \(" B^* \). Write \( U^\dagger \overset{\text{def}}{=} g^{-1} (U^*) \). Thus, we conclude that there exist a normal open subgroup \( M^\dagger \subseteq M^* \subseteq N \) of BGT and a diagram

\[
\begin{array}{cccc}
\Pi_{U^\dagger} \times M^\dagger & \longrightarrow & \Pi_U \times N|_{M^\dagger} & \longrightarrow & \Pi_X \times N|_{M^\dagger} \\
\downarrow & & \downarrow & & \downarrow \\
\Pi_{U^\dagger} \times M^*|_{M^\dagger} & \longrightarrow & \Pi_X \times N|_{M^\dagger} & \longrightarrow & \Pi_X \times M^*|_{M^\dagger}
\end{array}
\]

— where the upper right-hand portion of the diagram is the diagram determined by \( B^* \); the lower left-hand portion of the diagram is the diagram determined by \(" B^* \); the upper left-hand square of the diagram is cartesian— such that the composite of the upper horizontal arrows and the composite of the left-hand vertical arrows determine an arithmetic Belyi diagram \( \uparrow B^* \)

\[
\begin{array}{cccc}
\Pi_{U^\dagger} \times M^\dagger & \longrightarrow & \Pi_X \times M^\dagger \\
\downarrow & & \downarrow \\
\Pi_X \times M^\dagger
\end{array}
\]

satisfying the desired property. This completes the proof of the above observation.

Next, let us fix an element \( B^* \in I_{BGT} \). Then by applying the above observation in a recursive fashion [i.e., by applying the observation to \( B^* \) and some finite subset \( F \) to obtain \( \uparrow B^* \), then applying the observation to \( \uparrow B^* \) and some other finite subset \( F' \) to obtain \( \uparrow B^* \), etc.], we conclude [cf. the definition of \( \mathfrak{Q}_{BGT} \)] that one may construct a family of injections

\[
\left\{ \phi_{B^*, F} : F \cup \{0, 1\} \hookrightarrow \mathfrak{Q}_{BGT} \right\}_{\{F \subseteq \mathfrak{Q}^\dagger\}}
\]

[indexed by the finite subsets \( F \subseteq \mathfrak{Q}^\dagger \)] such that the following conditions are satisfied:

- \( \operatorname{Cusp}(B^*) \subseteq \bigcup_{F \subseteq \mathfrak{Q}^\dagger} \operatorname{Im}(\phi_{B^*, F}). \)

- If \( F_1 \subseteq F_2 \subseteq \mathfrak{Q}^\dagger \), then \( \left( \phi_{B^*, F_2} \right)|_{F_1} = \phi_{B^*, F_1}. \)

Thus, the various injections \( \phi_{B^*, F}, \) indexed by the finite subsets \( F \subseteq \mathfrak{Q}^\dagger \), determine an injection

\[
\phi_{B^*} : \mathfrak{Q} \hookrightarrow \mathfrak{Q}_{BGT}
\]

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associated to $\mathbb{B}^\times \in I_{BGT}$ such that $\text{Cusp}(\mathbb{B}^\times) \setminus \{\infty\} \subseteq \text{Im}(\phi_{\mathbb{B}^\times})$. In particular, if we allow $\mathbb{B}^\times \in I_{BGT}$ to vary, then we obtain an equality of sets as follows:

$$\bigcup_{\mathbb{B}^\times \in I_{BGT}} \text{Im}(\phi_{\mathbb{B}^\times}) = \mathbb{Q}_{BGT}. $$

In fact, the following stronger property holds:

Claim 4.4.B: For each finite subset $F_{BGT} \subseteq \mathbb{Q}_{BGT}$, there exists an arithmetic Belyi diagram $\mathbb{B}^\times$ such that $F_{BGT} \subseteq \text{Im}(\phi_{\mathbb{B}^\times})$.

Indeed, Claim 4.4.B follows formally from the $\text{COF}$-property of $BGT$, together with the inclusion $\text{Cusp}(\mathbb{B}^\times) \setminus \{\infty\} \subseteq \text{Im}(\phi_{\mathbb{B}^\times})$, for $\mathbb{B}^\times \in I_{BGT}$.

Next, we verify the following assertion:

Claim 4.4.C: Fix an arithmetic Belyi diagram $\mathbb{B}^\times$. Then $\mathbb{B}_{BGT}$ preserves $\text{Im}(\phi_{\mathbb{B}^\times})$ and induces the usual operation of multiplication on $\mathbb{Q}$.

Indeed, recall that $\{t^{-1}\}$ and $\{t/x\}$ ($x \in \mathbb{Q}_{BGT}$) are defined by using the scheme-theoretic morphisms $t^{-1}$, $t/x$. Thus, Claim 4.4.C follows immediately from the definition of the multiplication map $\mathbb{B}_{BGT}$.

Next, we verify the following assertion:

Claim 4.4.D:

(i) For any $\mathbb{B}^\times \in I_{BGT}$, the maps $\{t^{-1}\}$ and $\mathbb{B}_{BGT}$ determine an abelian group structure on $\mathbb{Q}_{BGT}$ with respect to which $\phi_{\mathbb{B}^\times}$ induces a group homomorphism

$$\mathbb{Q}^\times \to \mathbb{Q}_{BGT}^\times. $$

(ii) $\phi_{\mathbb{B}^\times}(-1) = -1_{BGT}$.

(iii) For any $\mathbb{B}^\times \in I_{BGT}$,

$$\phi_{\mathbb{B}^\times}(-1) \in \mu_2(\mathbb{Q}_{BGT}) \overset{\text{def}}{=} \{x \in \mathbb{Q}_{BGT} \mid \mathbb{B}_{BGT}(x, x) = 1\}. $$

(iv) $\mu_2(\mathbb{Q}_{BGT}) = \{1, -1_{BGT}\}$.

Since each axiom for an abelian group may be written as a condition concerning finitely many elements of the set under consideration, we conclude, by applying Claim 4.4.C for suitable $\mathbb{B}^\times \in I_{BGT}$ [i.e., $\mathbb{B}^\times$ such that the subset $\text{Im}(\phi_{\mathbb{B}^\times}) \subseteq \mathbb{Q}_{BGT}$ is sufficiently large — cf. Claim 4.4.B], that $\{t^{-1}\}$ and $\mathbb{B}_{BGT}$ determine an abelian group structure on $\mathbb{Q}_{BGT}$. Thus, Claim 4.4.D, (i), follows from Claims 4.4.B, 4.4.C. Claim 4.4.D, (ii), follows immediately from the definitions of $\mathbb{B}^\times_{-1}$ and $-1_{BGT}$. Claim 4.4.D, (iii), follows immediately from Claim 4.4.D, (i), together with the equality $(-1)^2 = 1 \in \mathbb{Q}$. Next, we verify Claim 4.4.D, (iv). The inclusion $\mu_2(\mathbb{Q}_{BGT}) \supseteq \{1, -1_{BGT}\}$ follows immediately from Claim
4.4.D, (ii), (iii). Let $x \in \mu_2(\overline{\mathbb{Q}}_{B^\times}) \setminus \{1\}$. Then Claim 4.4.B implies that there exists an arithmetic Belyi diagram $\mathbb{B}^\times$ such that $-1_{B^\times}, x \in \text{Im}(\phi_{B^\times})$. Thus, by applying Claim 4.4.D, (i), (iii), we conclude that $-1_{B^\times} = x$. This completes the proof of Claim 4.4.D.

Next, we verify the following assertion:

Claim 4.4.E: Fix an arithmetic Belyi diagram $B^\times$. Then $B^\times$ preserves $\text{Im}(\phi_{B^\times})$ and induces the usual operation of addition on $\overline{\mathbb{Q}}$.

Indeed, recall that $\{t^{-1}\}, \{1 - t\}$ are defined by using the scheme-theoretic morphisms $\iota_{t^{-1}}, \iota_{1 - t}$. Moreover, we observe that $B^\times$ is completely determined by $\{t^{-1}\}, \{1 - t\}, B^\times$, and $-1_{B^\times}$. Thus, Claim 4.4.E follows immediately from Claims 4.4.C, 4.4.D, (iv).

Since each axiom for a field may be written as a condition concerning finitely many elements of the set under consideration, we conclude, by applying Claims 4.4.C, 4.4.E, for suitable $B^\times$ such that the subset $\text{Im}(\phi_{B^\times}) \subseteq \overline{\mathbb{Q}}_{B^\times}$ is sufficiently large — cf. Claim 4.4.B, that $\mathbb{B}^\times$ and $\mathbb{B}^\times$ determine a field structure on $\overline{\mathbb{Q}}_{B^\times}$ such that $\phi_{B^\times}$ is a field homomorphism for each $B^\times \in I_{B^\times}$. Moreover, since $\bigcup_{B^\times \in I_{B^\times}} \text{Im}(\phi_{B^\times}) = \overline{\mathbb{Q}}_{B^\times}$, every element of $\overline{\mathbb{Q}}_{B^\times}$ is algebraic over $\mathbb{Q}$. Thus, we also conclude that

Claim 4.4.F: $\phi_{B^\times}$ is, in fact, a field isomorphism for every $B^\times \in I_{B^\times}$.

This completes the proof of Claim 4.4.A.

Next, we prove that the natural action of $C_{\text{GT}}(B^\times)$ on the set $\overline{\mathbb{Q}}_{B^\times}$ [cf. Proposition 4.3] is compatible with the field structure constructed above. Let $\sigma \in C_{\text{GT}}(B^\times)$. Recall that the maps $\mathbb{B}^\times$ and $\mathbb{B}^\times$ are completely determined by $\{t^{-1}\}, \{1 - t\}, \{t/x\} (x \in \overline{\mathbb{Q}}_{B^\times}),$ and $-1_{B^\times}$. Thus, since $0^\sigma = 0$ and $1^\sigma = 1$, it suffices to prove the following assertion:

Claim 4.4.G: Let $x \in \overline{\mathbb{Q}}_{B^\times}, y \in \overline{\mathbb{Q}}_{B^\times}$. Then

- $\{t^{-1}\}(y^\sigma) = \{t^{-1}\}(y)^\sigma$,
- $\{1 - t\}(y^\sigma) = (\{1 - t\}(y))^\sigma$,
- $\{t/x\}(y^\sigma) = (\{t/x\}(y))^\sigma$.

In particular, $(-1_{B^\times})^\sigma = -1_{B^\times}$ [cf. Claim 4.4.D, (iv)].

Let $\mathbb{B}^\times$ be an arithmetic Belyi diagram

$$
\begin{array}{ccc}
\Pi_U^\text{out} \times N & \xrightarrow{f} & \Pi_X^\text{out} \times N \\
\downarrow & & \downarrow \\
\Pi_X^\text{out} \times N
\end{array}
$$

[where $N$ is a normal open subgroup of $B^\times$] such that $N^\sigma \overset{\text{def}}{=} \sigma N \sigma^{-1} \subseteq B^\times$, and $x, y \in \text{Cusp}(\mathbb{B}^\times)$. Then, by recalling the [right-hand square in the final
display of the] proof of [Tsjm], Corollary 1.6, (i) [cf. also Remark 3.3.2; the functorial algorithm of Remark 3.3.3], in the case where \( J = GT \), we obtain a commutative diagram

\[
\begin{array}{ccc}
\Pi_U \otimes N & \xrightarrow{f} & \Pi_X \otimes N \\
\sigma \downarrow t & & \sigma \downarrow t \\
\Pi_U \otimes N^\sigma & \xrightarrow{f^\sigma} & \Pi_X \otimes N^\sigma,
\end{array}
\]

where the horizontal arrows are the \( \Pi_X \)-outer surjections induced by the natural open immersions \( U \hookrightarrow X, U^\sigma \hookrightarrow X \) of hyperbolic curves; the left- (respectively, the right-) hand vertical arrow is a \( \Pi_U \)-outer (respectively, \( \Pi_X \)-outer) isomorphism of profinite groups.

Note that \( \{t^{-1}\}(y^\sigma) \) (respectively, \( \{1-t\}(y^\sigma); \{t/x^\sigma\}(y^\sigma) \)) is completely determined by \( y^\sigma \) and the \( \Pi_X \)-outer surjection

\[
(f^\sigma)^{-1} : \Pi_U \otimes N^\sigma \to \Pi_X \otimes N^\sigma
\]

(respectively,

\[
(f^\sigma)^{1-t} : \Pi_U \otimes N^\sigma \to \Pi_X \otimes N^\sigma;
\]

\[
(f^\sigma)^{t/x^\sigma} : \Pi_U \otimes N^\sigma \to \Pi_X \otimes N^\sigma)
\]

which sends \((\infty, 1, 0)\) (respectively, \((1, 0, \infty); (0, x^\sigma, \infty))\) to \((0, 1, \infty)\).

On the other hand, \( \{(t^{-1})(y)\}^\sigma \) (respectively, \( \{(1-t)(y)\}^\sigma; \{(t/x)(y)\}^\sigma \)) is completely determined by \( y^\sigma \) and the \( \Pi_X \)-outer surjection

\[
\sigma \circ f^{t^{-1}} \circ \sigma^{-1} : \Pi_U \otimes N^\sigma \to \Pi_X \otimes N^\sigma
\]

(respectively,

\[
\sigma \circ f^{1-t} \circ \sigma^{-1} : \Pi_U \otimes N^\sigma \to \Pi_X \otimes N^\sigma;
\]

\[
\sigma \circ f^{t/x} \circ \sigma^{-1} : \Pi_U \otimes N^\sigma \to \Pi_X \otimes N^\sigma)
\]

which sends \((\infty, 1, 0)\) (respectively, \((1, 0, \infty); (0, x^\sigma, \infty))\) to \((0, 1, \infty)\).

Let us note that the \( \Pi_X \)-outer surjections of the last two displays exhibit analogous behavior on the cusps [i.e., more precisely, on the conjugacy classes of cuspidal inertia subgroups]. Thus, we conclude from the above commutative diagram that

- \((f^\sigma)^{-1} = \sigma \circ f^{t^{-1}} \circ \sigma^{-1},\)
- \((f^\sigma)^{1-t} = \sigma \circ f^{1-t} \circ \sigma^{-1},\)
- \((f^\sigma)^{t/x} = \sigma \circ f^{t/x} \circ \sigma^{-1}.

This completes the proof of Claim 4.4.G, hence of Theorem 4.4. \(\square\)
Remark 4.4.1. Let $p$ be a prime number, $F$ a field which is a finite extension of the field of rational numbers $\mathbb{Q}$ or the field of $p$-adic numbers $\mathbb{Q}_p$. Thus, we have a natural inclusion $\mathbb{Q} \subseteq F$. Let $\overline{F}$ be an algebraic closure of $F$. By abuse of notation, we shall identify $\overline{\mathbb{Q}}$ with the algebraic closure of $\mathbb{Q}$ in $\overline{F}$. Write $G_F \overset{\text{def}}{=} \text{Gal}(\overline{F}/F)$. Thus, we obtain natural injections

$$G_F \hookrightarrow G_\mathbb{Q} \hookrightarrow G_\mathbb{Q} \hookrightarrow \text{Out}(\Pi_X)$$

[cf. the discussion of the beginning of [Tsjm], Introduction], which we use to identify $G_F$ with its image in $G_\mathbb{Q}$. Then it follows immediately from the fact that $F$ is Kummer-faithful [cf. [AbsTopIII], Definition 1.5; [AbsTopIII], Remark 1.5.4, (i)], together with a similar argument to the argument applied in the proof of [Tsjm], Corollary 3.2, that $G_F$ satisfies the CS-property. Thus, we conclude from Corollary 3.7 that $G_F$ satisfies the RGC-property. Since, in this situation, the COF-property is immediate, we thus conclude that $G_F$ satisfies the BC-property, i.e., that we may take “BGT” to be $G_F$. Moreover, the scheme-theoretic interpretation of the various arithmetic Belyi diagrams that appear determines a natural isomorphism of fields $\overline{\mathbb{Q}}_{G_F} \cong \overline{\mathbb{Q}}$ that is compatible, relative to the natural injection $G_F \hookrightarrow G_\mathbb{Q}$, with the respective natural actions, i.e., we obtain a diagram as follows:

$$G_F \hookrightarrow G_\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_{G_F} \cong \overline{\mathbb{Q}}.$$

Remark 4.4.2. It is not clear to the authors at the time of writing whether or not $G_T$ satisfies the BC-property, i.e., whether or not “$G_T = BGT$”.

Corollary 4.5 (Group-theoretic nature of BGT). Let $n$ be an integer such that $n \geq 2$. Write $X_n$ for the $n$-th configuration space of $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$; $\text{GT}_n \overset{\text{def}}{=} \text{Out}^{\text{top}}(\Pi_{X_n}) \subseteq \text{Out}(\Pi_{X_n})$. Recall that we have a natural isomorphism $\text{GT}_n \cong \text{GT}$ [cf. the first display of [HMM], Corollary C]. Then one may reconstruct from $\Pi_{X_n}$ in a purely combinatorial/group-theoretic way, i.e., in a way that only involves the structure of $\Pi_{X_n}$ as a topological group [cf. also Remark 4.5.1 below],

- the subgroups $\text{GT}_n \subseteq \text{Out}(\Pi_{X_n})$, $\text{GT} \subseteq \text{Out}(\Pi_X)$, where we regard $\Pi_X$ as the quotient of $\Pi_{X_n}$ by a generalized fiber subgroup;
- the natural isomorphism $\text{GT}_n \cong \text{GT}$;
- the collection of closed subgroups $J \subseteq \text{GT}_n$ such that $J$ satisfies [i.e., the image of $J$, via the natural isomorphism, in $\text{GT}$ satisfies] the BC-property [cf. Definition 3.3, (v)].
If, moreover, a closed subgroup \( J = \text{BGT} \subseteq \text{GT} \subseteq \text{Out}(\Pi_X) \) satisfies the BC-property, then the construction from \( \Pi_{X_n} \) [cf. also Remark 4.5.1 below] of

- the preordered set of arithmetic Belyi diagrams \( I_{\text{BGT}} \) [cf. Definition 4.1],
- the natural action of \( C_{\text{GT}(\text{BGT})} \) on the preordered set \( I_{\text{BGT}} \) [cf. Definition 4.1],
- the set \( \text{Cusp}(-) \) associated to any element of \( I_{\text{BGT}} \) [cf. Definition 4.1],
- the direct limit \( \mathcal{U}_{\text{BGT}} \) [cf. Definition 4.1],
- the natural continuous action of \( C_{\text{GT}(\text{BGT})} \) on \( \mathcal{U}_{\text{BGT}} \) [cf. Proposition 4.3], and
- the field structure on \( \mathcal{U}_{\text{BGT}} \) [cf. Theorem 4.4]

may be phrased in purely combinatorial/group-theoretic terms, i.e., in terms that only involve the structure of \( \Pi_{X_n} \) as a topological group.

**Proof.** The various assertions of Corollary 4.5 follow immediately from Definitions 3.3, 4.1; Remarks 3.3.2, 3.3.3 [cf. also Remark 4.5.1 below]; Proposition 4.3 [and its proof]; Theorem 4.4 [and its proof]; [HMM], Theorem A, (ii); the first display of [HMM], Corollary C; [Tsjm], Theorem 1.3, (i); [Tsjm], Definition 1.4. \( \Box \)

**Remark 4.5.1.**

(i) Here, in the context of Remark 3.3.3, (i), we observe that the natural isomorphism \( \text{GT}_n \xrightarrow{\sim} \text{GT} \) [cf. the first display of [HMM], Corollary C], together with the algorithm of Corollary 3.1, (ii), implies that there is in fact no substantive difference between

- constructions starting from \( \Pi_{X_n} \) [where we recall that \( n \geq 2 \)] and
- constructions starting from \( \Pi_{X_3} \).

(ii) In the situation discussed in (i) [cf. also Remark 3.3.3, (i)], suppose that we apply the constructions discussed in Corollary 4.5 to \( \Pi_X \), regarded as an abstract topological group. Then the algorithm of Corollary 3.1, (ii), determines a subgroup

\[ \mathfrak{S}_3 \subseteq \text{Out}(\Pi_X), \]

[i.e., where, by a slight abuse of notation, we use the notation “\( \mathfrak{S}_3 \)” to denote this subgroup which is isomorphic to the symmetric group on 3 letters] of the group of outer automorphisms \( \text{Out}(\Pi_X) \) of the quotient \( \Pi_X \) of the given abstract topological group \( \Pi_{X_3} \) discussed in Remark 3.3.3, (i), (d).
We maintain the notation of (ii). Then observe that since the quotient $X$ of the given abstract topological group $\Pi_X$, is not equipped with a natural bijection between its set of cusps and the set of symbols $\{0, 1, \infty\}$, it follows that this quotient $X$ is only related to any of the “$\Pi$’s” that appear in the arithmetic Belyi diagrams discussed in the statement of Corollary 4.5 not by a single outer isomorphism, but rather by an $S_3$-torsor of outer isomorphisms.

5 Combinatorial construction of the conjugacy class of subgroups of $G_T$ determined by $G_\mathbb{Q}$

Write $X \overset{\text{def}}{=} \mathbb{P}^1 \setminus \{0, 1, \infty\}$; $X_n$ for the $n$-th configuration space associated to $X$, where $n \geq 2$. In this section, we reconstruct from the topological group $\Pi_{X_n}$, in a purely combinatorial/group-theoretic way, the conjugacy class of subgroups of the Grothendieck-Teichmüller group $G_T \subseteq \text{Out}(\Pi_X)$ determined by the absolute Galois group of $\mathbb{Q}$ as the set of maximal closed subgroups $B_{G_T}$ of $G_T$ satisfying a certain purely combinatorial/group-theoretic condition that we refer to as the AA-property [cf. Definition 5.12; Theorem 5.17, (ii)].

Write $\Pi_{X_{\text{nuc}}}$ for the quotient of $\Pi_X$ by the normal closed subgroup topologically generated by the cuspidal inertia subgroups associated to the cusp “1” [so $\Pi_{X_{\text{nuc}}}$ is isomorphic to $\mathbb{Z}$ as an abstract topological group]. Let $J$ be a closed subgroup of $G_T \subseteq \text{Out}(\Pi_X)$. Then we shall write [by a slight abuse of notation]

$$\Pi_X \overset{\text{out}}{\times} J \twoheadrightarrow \Pi_{X_{\text{nuc}}} \times J$$

for the quotient by the normal closed subgroup topologically generated by the cuspidal inertia subgroups associated to the cusp “1”.

**Definition 5.1.** In the notation of Definition 4.1:

(i) Write

$$\Pi \overset{\text{def}}{=} \varprojlim_{B^* \in I_{G_T}} \Pi_{B^*},$$

where the transition morphisms are the unique geometric dominations

$$\Pi_{B^*} \twoheadrightarrow \Pi_{B^*}.$$
a profinite group, up to inner automorphisms. It follows immediately from Proposition 3.5, together with the various definitions involved, that the natural action of $C_{GT}(BGT)$ on $I_{BGT}$ [cf. Definition 4.1] induces a natural outer action of $C_{GT}(BGT)$ on the group $I_{BGT}$.

(ii) In the context of the inverse limit of Definition 5.1, (i), we shall refer to an inverse limit of cuspidal inertia subgroups of some cofinal collection of $\Pi_\alpha$’s [where the induced transition morphisms are necessarily isomorphisms] as a cuspidal inertia subgroup of $\Pi$. For each open subgroup $\Pi^*$ of $\Pi$, we shall refer to the intersection of $\Pi^*$ with a cuspidal inertia subgroup of $\Pi$ as a cuspidal inertia subgroup of $\Pi^*$ and write

$$\text{Cusp}(\Pi^*)$$

for the set of $\Pi^*$-conjugacy classes of cuspidal inertia subgroups of $\Pi^*$. Thus, it follows immediately from the definitions that we obtain a natural surjection

$$\text{Cusp}(\Pi^*) \twoheadrightarrow \text{Cusp}(\Pi)$$

with finite fibers. For each finite subset $E^* \subseteq \text{Cusp}(\Pi^*)$, write

$$\Pi^* \twoheadrightarrow \Pi_{E}^*.$$

for the topologically finitely generated [cf. Remark 5.1.1 below] quotient profinite group of $\Pi^*$ obtained by forming the quotient of $\Pi^*$ by the normal closed subgroup topologically generated by the cuspidal inertia subgroups associated to $\text{Cusp}(\Pi^*) \setminus E^*$. Observe that the natural outer action of $C_{GT}(BGT)$ on $\Pi$ [cf. Definition 5.1, (i)] induces a natural action of $C_{GT}(BGT)$ on $\text{Cusp}(\Pi)$. Finally, we observe that it follows immediately from the various definitions involved [cf., especially, Definition 4.1] that we have a natural $C_{GT}(BGT)$-equivariant bijection

$$\text{Cusp}(\Pi) \cong \mathcal{C}_{BGT} \cup \{\infty\}.$$

(iii) Write

$$C_{BGT}$$

for the set of finite subsets of $\text{Cusp}(\Pi)$ that contain $\{0, 1, \infty\}$. Observe that the natural action of $C_{GT}(BGT)$ on $\text{Cusp}(\Pi)$ [cf. Definition 5.1, (ii)] induces a natural action of $C_{GT}(BGT)$ on $C_{BGT}$. We shall write

$$C_{BGT}^G \subseteq C_{BGT}$$

for the subset of $C_{GT}(BGT)$-stable elements, i.e., elements fixed by the action of $C_{GT}(BGT)$. Finally, we observe that the assignment $I_{BGT} \ni B^* \mapsto \text{Cusp}(B^*) \in C_{BGT}$ induces a natural $C_{GT}(BGT)$-equivariant map

$$I_{BGT} \to C_{BGT}.$$
Remark 5.1.1. In the notation of Definition 5.1, it follows immediately from Remark 4.1.1 that the kernel of the natural outer surjection

$$\Pi \twoheadrightarrow \Pi_{\mathbb{B}^\times}$$

is the normal closed subgroup of $\Pi$ topologically generated by the cuspidal inertia subgroups associated to $\text{Cusp}(\Pi) \setminus \text{Cusp}(\mathbb{B}^\times)$. In particular, the quotient $\Pi \twoheadrightarrow \Pi_{\mathbb{B}^\times}$ may be naturally identified with the quotient

$$\Pi \twoheadrightarrow \Pi_{\text{Cusp}(\mathbb{B}^\times)}$$

of the third display of Definition 5.1, (ii) [i.e., where we take “$\Pi^*$” to be $\Pi$ and “$E^*$” to be $\text{Cusp}(\mathbb{B}^\times)$].

Remark 5.1.2. Let $E \in C^\text{st}_{\text{BGT}}$ [cf. Definition 5.1, (iii)]. Then it follows immediately from the various definitions involved that the natural outer action of $C_{\text{GT}}(\text{BGT})$ on $\Pi$ [cf. Definition 5.1, (i)] induces, via the natural outer surjection $\Pi \twoheadrightarrow \Pi_E$, a natural continuous outer action of $C_{\text{GT}}(\text{BGT})$ on the topologically finitely generated profinite group $\Pi_E$ [cf. the discussion entitled “Topological groups” in Notations and Conventions: Definition 5.1, (ii); [Tsjm], Lemma 1.2, (b); [Tsjm], Theorem 1.3, (ii); [Tsjm], Definition 1.4].

Remark 5.1.3. Observe that it follows immediately from the continuity [cf. Proposition 4.3] of the natural action of $C_{\text{GT}}(\text{BGT})$ on $\overline{\mathbb{Q}}_{\text{BGT}} \cup \{\overline{\infty}\}$ (\sim Cusp(\Pi)) [cf. Definition 5.1, (ii)], together with the COF-property of BGT, that for any $E \in C_{\text{BGT}}$, there exists an element $E^{\text{st}} \in C^\text{st}_{\text{BGT}}$ (respectively, $\mathbb{B}^\times \in I_{\text{BGT}}$) such that $E \subseteq E^{\text{st}}$ (respectively, $E \subseteq \text{Cusp}(\mathbb{B}^\times)$).

In particular, we conclude [cf. Remarks 5.1.1, 5.1.2; Proposition 5.2, (ii) below] that we may write

$$\Pi = \lim_{E \in C_{\text{BGT}}} \Pi_E = \lim_{E^{\text{st}} \in C_{\text{BGT}}} \Pi_{E^{\text{st}}}$$

$$\Pi \times \text{BGT} = \lim_{E^{\text{st}} \in C_{\text{BGT}}} \Pi_{E^{\text{st}}} \times \text{BGT}$$

— where, in the inverse limits, we regard $C_{\text{BGT}}$ and $C^\text{st}_{\text{BGT}}$ as directed preordered sets by means of the relation of inclusion of subsets of $\text{Cusp}(\Pi)$.

Proposition 5.2 (Basic properties of $\Pi$). In the notation of Definition 5.1, the following hold:

(i) There exists an isomorphism of profinite groups between $\Pi$ and the absolute Galois group of the function field of $X$ that induces a bijection between the respective sets of cuspidal inertia subgroups.
(ii) For each $E \in C_{\text{BGT}}$, $\Pi_E$ is slim. In particular, $\Pi$ is slim.

(iii) The group $\Pi^* \rtimes \text{BGT}$ admits a natural structure of profinite group.

(iv) Let $\Pi^*$ be a normal open subgroup of $\Pi$. Then, for any sufficiently small normal open subgroup $M \subseteq \text{BGT}$, there exist an outer action of $M$ on $\Pi^*$ and an open injection $\Pi^* \rtimes M \hookrightarrow \Pi^* \rtimes \text{BGT}$ such that

(a) the outer action of $M$ on $\Pi^*$ preserves the set of cuspidal inertia subgroups of $\Pi^*$;

(b) the outer action of $M$ on $\Pi^*$ extends uniquely [cf. the slimness of $\Pi$] to a $\Pi^*$-outer action on $\Pi$ that is compatible with the outer action of $\text{BGT}(\geq M)$ on $\Pi$; the injection $\Pi^* \rtimes M \hookrightarrow \Pi^* \rtimes \text{BGT}$ is the injection determined by the inclusions $\Pi^* \subseteq \Pi$ and $M \subseteq \text{BGT}$, together with the $\Pi^*$-outer actions of $M$ on $\Pi^*$ and $\Pi$.

(v) In the notation of (iv), the homomorphism $\Pi^* \rtimes M \rightarrow \text{Aut}(\Pi^*)$ determined by conjugation is injective.

(vi) Let $\Pi^*$ be an open subgroup of $\Pi$. Then any surjective homomorphism $\Pi^* \twoheadrightarrow \Pi^*$ of profinite groups that induces a bijection $\text{Cusp}(\Pi^*) \cong \text{Cusp}(\Pi^*)$ is an isomorphism.

Proof. Assertion (i) follows immediately from Claim 4.4.F in the proof of Theorem 4.4. Assertion (ii) follows immediately from [MT], Proposition 1.4. Assertion (iii) follows immediately, in light of the second line of the final display of Remark 5.1.3, from Remark 5.1.2. Next, since, in the notation of Definition 5.1, (i), $\Pi^*$ arises as the inverse image in $\Pi$ of some normal open subgroup of some $\Pi_{\delta^*}$, assertion (iv) follows immediately from a similar argument to the argument applied in the proof of [Tsjm], Lemma 1.2.

Next, we verify assertion (v). First, we note that since $\Pi$, hence also $\Pi^*$, is slim [cf. (iii)], the restriction of the homomorphism $\Pi^* \rtimes M \rightarrow \text{Aut}(\Pi^*)$ to $\Pi^*$ is injective. Note also that since the natural surjection $\Pi \twoheadrightarrow \Pi_X$ is compatible with the respective outer actions of $M$, and $M \subseteq \text{GT} \subseteq \text{Out}(\Pi_X)$, the natural homomorphism $M \rightarrow \text{Out}(\Pi)$ is injective. Since $\Pi$ is slim, it follows immediately from condition (b) of assertion (iv) that the natural homomorphism $M \rightarrow \text{Out}(\Pi^*)$ is injective. Thus, we conclude that the homomorphism $\Pi^* \rtimes M \rightarrow \text{Aut}(\Pi^*)$ is injective. This completes the proof of assertion (v).

Finally, we verify assertion (vi). Let $\phi : \Pi^* \rightarrow \Pi^*$ be a surjective homomorphism of profinite groups that induces a bijection $\delta^* : \text{Cusp}(\Pi^*) \cong \text{Cusp}(\Pi^*)$. Then, for each finite subset $E^* \subseteq \text{Cusp}(\Pi^*)$, the surjective homomorphism $\phi$ induces a surjective homomorphism $\phi_{E^*} : \Pi_{E^*} \rightarrow \Pi^*_{\delta^*(E^*)}$ of profinite groups. On the other hand, since the cardinality of $E^*$ and the cardinality of $\delta^*(E^*)$ are equal, there exists an isomorphism $\Pi^*_{\delta^*(E^*)} \cong \Pi^*_{E^*}$ of profinite groups. Then since $\Pi^*_{E^*}$ is topologically finitely generated [hence satisfies the “Hopfian
property\)\], the surjective homomorphism \(\phi_{E^*}\) is an isomorphism. Thus, by allowing \(E^* \subseteq \text{Cusp}(\Pi^*)\) to vary, we conclude that \(\phi\) is an isomorphism. This completes the proof of assertion (vi), hence of Proposition 5.2.

\[\]

**Definition 5.3.** In the following, we consider the analogues of [Tsjm], Definition 1.5, (i), (ii); [Tsjm], Corollary 1.6, (ii), obtained by replacing \("\Pi_X\) by \(\Pi_{X_{0\infty}}\).

Let \(J\) be a closed subgroup of \(\text{GT} \subseteq \text{Out}(\Pi_X)\).

(i) Fix an arithmetic Belyi diagram \(\mathcal{B}^\times\)

\[
\begin{array}{ccc}
\Pi_U & \cong & M \\
\downarrow & & \downarrow \\
\Pi_X & \cong & M
\end{array}
\]

[cf. [Tsjm], Definition 1.4]. Write

\[
\mathcal{D}_{0\infty}^{\text{ext}}(\mathcal{B}^\times, M, J)
\]

for the set consisting of the images via the natural composite \(\Pi_{X_{0\infty}}\)-outer homomorphism \(\Pi_U \cong M \rightarrow \Pi_X \cong M \leftarrow \Pi_X \cong J \rightarrow \Pi_{X_{0\infty}}\times J\) of the normalizers in \(\Pi_U \cong M\) of the cuspidal inertia subgroups of \(\Pi_U\) that are not associated to 0 and \(\infty\);

\[
\mathcal{D}_{0\infty}(\mathcal{B}^\times, J)
\]

for the quotient set \(\bigcup_{M \subseteq J} \mathcal{D}_{0\infty}(\mathcal{B}^\times, M, J) / \sim\), where \(M\) ranges over all sufficiently small normal open subgroups of \(J\), and we write \(\mathcal{D}_{0\infty}(\mathcal{B}^\times, M, J) \ni G_M \sim G_M^1 \in \mathcal{D}_{0\infty}(\mathcal{B}^\times, M^1, J)\) if \(G_M \cap G_M^1\) is open in both \(G_M\) and \(G_M^1\). Observe that \(\Pi_{X_{0\infty}}\) acts naturally on \(\mathcal{D}_{0\infty}(\mathcal{B}^\times, M, J)\) and \(\mathcal{D}_{0\infty}(\mathcal{B}^\times, J)\).

(ii) Write

\[
\mathcal{D}_{0\infty}^{\text{ext}}(J)
\]

for the quotient set \(\bigcup_{\mathcal{B}^\times \in \mathcal{D}_{0\infty}^{\text{ext}}(\mathcal{B}^\times, J)} / \sim\), where \(\mathcal{B}^\times\) ranges over all arithmetic Belyi diagrams, and we write \(\mathcal{D}_{0\infty}^{\text{ext}}(\mathcal{B}^\times, J) \ni G_{\mathcal{B}^\times} \sim G_{\mathcal{B}^\times}^1 \in \mathcal{D}_{0\infty}(\mathcal{B}^\times, J)\) if \(G_{\mathcal{B}^\times} \cap G_{\mathcal{B}^\times}^1\) is open in both \(G_{\mathcal{B}^\times}\) and \(G_{\mathcal{B}^\times}^1\) for some representative \(G_{\mathcal{B}^\times}\) (respectively, \(G_{\mathcal{B}^\times}^1\)) of \(G_{1\mathcal{B}^\times}\) (respectively, \(G_{1\mathcal{B}^\times}^1\)). Observe that \(\Pi_{X_{0\infty}}\) acts naturally on \(\mathcal{D}_{0\infty}(J)\).

(iii) Write

\[
D_{0\infty}(J)
\]
for the quotient set $D_{0\infty}(J)/\Pi_{X_{0\infty}}$.

Remark 5.3.1. In the following, we consider certain slightly generalized analogues of [Tsjm], Corollary 1.6, (ii), (iii), obtained by replacing “$\Pi_X$” by $\Pi_{X_{0\infty}}$. Let $J$ be a closed subgroup of $\Gamma \subseteq \text{Out}(\Pi_X)$. Then it follows immediately from a similar argument [cf. also Remarks 3.3.2, 3.3.3] to the proof of [Tsjm], Corollary 1.6, together with the various definitions involved, that:

- $D_{0\infty}(J)$ admits a natural action by $C_{\Gamma}(J)$, hence, in particular, by $J$.

- Let $J_1$ and $J_2$ be closed subgroups of $\Gamma$. If $J_1 \subseteq J_2 \subseteq \Gamma$, then the inclusion $J_1 \subseteq J_2$ induces, by considering the intersection of subgroups of $\Pi_{X_{0\infty}} \rtimes J_2$ with $\Pi_{X_{0\infty}} \rtimes J_1$, a natural surjection

$$D_{0\infty}(J_2) \twoheadrightarrow D_{0\infty}(J_1)$$

that is equivariant with respect to the natural actions of $J_1 (\subseteq J_2)$ on the domain and codomain.

Lemma 5.4 (Kummer classes of group-theoretic constant functions).
We maintain the notation of Definitions 4.1, 5.3. Then the following hold:

(i) There exists a natural injection

$$\iota_{\Gamma} : D_{0\infty}(\Gamma) \hookrightarrow \lim_{\substack{\longrightarrow \\\{M \subset \Gamma\}}} H^1(M, \Pi_{X_{0\infty}}),$$

where $M$ ranges over the normal open subgroups of $\Gamma$.

(ii) There exists a natural surjection

$$\psi_{\Gamma} : \mathbb{Q}_{\Gamma}^\times \twoheadrightarrow D_{0\infty}(\Gamma).$$

(iii) The above maps $\iota_{\Gamma}$ and $\psi_{\Gamma}$ are compatible with the respective natural actions of $C_{\Gamma}(\Gamma)$ [cf. Theorem 4.4, Remark 5.4.1].

(iv) The composite

$$\iota_{\Gamma} \circ \psi_{\Gamma} : \mathbb{Q}_{\Gamma}^\times \twoheadrightarrow \lim_{\substack{\longrightarrow \\\{M \subset \Gamma\}}} H^1(M, \Pi_{X_{0\infty}})$$

is a group homomorphism [cf. Theorem 4.4].

Proof. First, we verify assertion (i). Let $I_1$ be a cuspidal inertia subgroup of $\Pi_X$ associated to the cusp “1”. Then the image of the normalizer

$$N \rtimes \Gamma (I_1) \subseteq \Pi_X \rtimes \Gamma$$


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via the natural surjection $\Pi_X \times \text{BGT} \to \Pi_{X_{0\infty}} \times \text{BGT}$ determines a section $s_1$ [cf. [CmbGC], Proposition 1.2, (ii)] of the second to last arrow of the natural exact sequence

$$1 \rightarrow \Pi_{X_{0\infty}} \rightarrow \Pi_{X_{0\infty}} \times \text{BGT} \rightarrow \text{BGT} \rightarrow 1.$$ 

On the other hand, note that an element $x \in D_{0\infty}(\mathbb{B}^\infty, M, \text{BGT})$, where $\mathbb{B}^\infty$ denotes an arithmetic Belyi diagram as in Definition 5.3, (i) [i.e., where we take “$J$” to be $\text{BGT}$], determines a section $s_x$ [cf. [CmbGC], Proposition 1.2, (ii)] of the restriction to $M$ of the second to last arrow of the above exact sequence. Thus, by forming the difference $\kappa_x$ between $s_x$ and the restriction to $M$ of $s_1$, one verifies immediately that the assignment $s_x \mapsto \kappa_x$ determines, by allowing $\mathbb{B}^\infty \in \text{I}_{\text{BGT}}$ [hence also “$M$” to vary], a natural map

$$\iota_{\text{BGT}} : D_{0\infty}(\text{BGT}) \to \lim_{M \leq \text{BGT}} H^1(M, \Pi_{X_{0\infty}}),$$

where $M$ ranges over the normal open subgroups of $\text{BGT}$. Finally, the injectivity of $\iota_{\text{BGT}}$ follows immediately from the definitions of $D_{0\infty}(\cdot)$ and $H^1(\cdot, \cdot, \cdot)$. This completes the proof of assertion (i). Assertion (ii) follows immediately from the definitions of $\overline{D}^\infty_{\text{BGT}}$ and $D_{0\infty}(\text{BGT})$. Assertion (iii) follows immediately from the definitions of the natural actions of $C_{\text{GT}}(\text{BGT})$ [cf., especially, the proof of Claim 4.4.G in the proof of Theorem 4.4]. Assertion (iv) follows immediately from the construction of the multiplication operation on the field $\overline{D}^\infty_{\text{BGT}}$ [i.e., the construction of $\mathbb{E}_{\text{BGT}}$ in the proof of Theorem 4.4] by means of the well-known natural group structure on $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, \infty\}$, i.e., “$(\mathbb{G}_m)_{\mathbb{Q}}^{-1}$”. This completes the proof of Lemma 5.4.

In the remainder of the present paper, we shall identify $D_{0\infty}(\text{BGT})$ with $\text{Im}(\iota_{\text{BGT}})$ via the natural injection $\iota_{\text{BGT}}$.

**Proposition 5.5 (Synchronizations of cuspidal inertia subgroups).** We maintain the notation of Definition 5.1. Then the following hold:

(i) For each cuspidal inertia subgroup $I_x$ of $\Pi$ associated to $x \in \text{Cusp}(\Pi)$, the natural scheme-theoretic isomorphism

$$I_x \xrightarrow{\sim} \Pi_{X_{0\infty}}$$

may be reconstructed, in a purely combinatorial/group-theoretic way, from the collection of data

$$(\Pi; \text{Cusp}(\Pi); \{0, \infty\} \subseteq \text{Cusp}(\Pi))$$

consisting of

- a profinite group $\Pi$;


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• a set $\text{Cusp}(\Pi)$ of the conjugacy classes of subgroups of $\Pi$;
• a subset $\{0, \infty\} \subseteq \text{Cusp}(\Pi)$ of cardinality 2 [equipped with labels “0”, “\infty”] of the set $\text{Cusp}(\Pi)$.

(ii) Let $\Pi^* \subseteq \Pi$ be an open subgroup; $x \in \text{Cusp}(\Pi^*)$; $I_x^*$ a cuspidal inertia subgroup of $\Pi^*$ associated to $x$. Then one may construct a natural isomorphism

$$I_x^* \xrightarrow{\sim} \Pi_{X0\infty}$$

as follows: Write $I_x \overset{\text{def}}{=} N_\Pi(I_x^*)$. Note that $I_x = N_\Pi(I_x) = C_\Pi(I_x) = C_\Pi(I_x^*)$ is the unique cuspidal inertia subgroup of $\Pi$ containing $I_x^*$ [cf. Proposition 5.2, (i); [CmbGC], Proposition 1.2, (ii)], and the subgroup $I_x^* \subseteq I_x$ is of finite index $m$. Then since cuspidal inertia subgroups are abstractly isomorphic to $\hat{\mathbb{Z}}$ [cf. [CmbGC], Remark 1.1.3], division by $m$ determines an isomorphism $I_x^* \xrightarrow{\sim} I_x$. Thus, by composing with the isomorphism of (i), we obtain a natural isomorphism $I_x^* \xrightarrow{\sim} \Pi_{X0\infty}$.

Proof. First, we verify assertion (i). Let $I_0$ be a cuspidal inertia subgroup of $\Pi$ associated to the cusp “0 $\in \text{Cusp}(\Pi^*)$”. Write

$$\Pi \twoheadrightarrow \Pi_{(0,x)}$$

for the quotient profinite group of $\Pi$ obtained by forming the quotient of $\Pi$ by the normal closed subgroup topologically generated by the cuspidal inertia subgroups associated to $\text{Cusp}(\Pi) \setminus \{0, x\}$. Then the surjection $\Pi \twoheadrightarrow \Pi_{(0,x)}$ induces isomorphisms

$$\alpha_0 : I_0 \xrightarrow{\sim} \Pi_{(0,x)}; \quad \alpha_x : I_x \xrightarrow{\sim} \Pi_{(0,x)}.$$ 

Write $\alpha : I_x \xrightarrow{\sim} I_0$ for the composite of $\alpha_0^{-1} \circ \alpha_x$ with the inversion map $I_0 \xrightarrow{\sim} I_0$. Thus, $\alpha$ and the natural surjection $\Pi \twoheadrightarrow \Pi_{X0\infty}$ determine an isomorphism $I_x \xrightarrow{\sim} \Pi_{X0\infty}$. The desired functoriality follows immediately from the construction. This completes the proof of assertion (i).

Assertion (ii) follows immediately from the various definitions involved. This completes the proof of Proposition 5.5. \qed

**Definition 5.6.** In the notation of Definition 5.1, let $\Pi^* \subseteq \Pi$ be an open subgroup. Fix

• a normal open subgroup $M \subseteq \text{BGT}$,
• an outer action of $M$ on $\Pi^*$, and
• an open injection $f_M : \Pi^* \xrightarrow{\text{out}} M \leftrightarrow \Pi^* \xrightarrow{\text{out}} \text{BGT}$

such that
(a) the outer action of $M$ on $\Pi^*$ preserves the set of cuspidal inertia subgroups of $\Pi^*$;

(b) the outer action of $M$ on $\Pi^*$ extends uniquely [cf. the slinness of $\Pi$] to a $\Pi^*$-outer action on $\Pi$ that is compatible with the outer action of $\text{BGT} (\supseteq M)$ on $\Pi$; the injection $\Pi^* \ltimes M \hookrightarrow \Pi^* \ltimes \text{BGT}$ is the injection determined by the inclusions $\Pi^* \subseteq \Pi$ and $M \subseteq \text{BGT}$, together with the $\Pi^*$-outer actions of $M$ on $\Pi^*$ and $\Pi$

[cf. Proposition 5.2, (iv)]. Write $I(\Pi^*, \Pi)$ for the set of open injections $f_{\Pi^*} : \Pi^* \hookrightarrow \Pi$ satisfying the following properties:

1. For each cuspidal inertia subgroup $I^*$ of $\Pi^*$, the commensurator $C_{\Pi}(f_{\Pi^*}(I^*))$ of $f_{\Pi^*}(I^*)$ in $\Pi$ is a cuspidal inertia subgroup of $\Pi$ [which implies, by Proposition 5.5, (ii), that $C_{\Pi}(f_{\Pi^*}(I^*)) = N_{\Pi}(f_{\Pi^*}(I^*))$].

2. For each cuspidal inertia subgroup $I$ of $\Pi$, the inverse image $f^{-1}_{\Pi^*}(I) \subseteq \Pi^*$ is a cuspidal inertia subgroup of $\Pi^*$.

3. Let $I^*$ be a cuspidal inertia subgroup of $\Pi^*$; $I$ a cuspidal inertia subgroup of $\Pi$ such that $I^* = f_{\Pi^*}^{-1}(I)$. Then the composite

$$\Pi_{\Pi_{\infty}} \leftarrow I^* \hookrightarrow I \rightarrow \Pi_{\Pi_{\infty}}$$

— where the first and final arrows are the isomorphisms of Proposition 5.5, (i), (ii) — coincides with the homomorphism determined by multiplication by some positive integer.

4. For any sufficiently small normal open subgroup $N^* \subseteq M$ of $\text{BGT}$, there exists a (necessarily unique — cf. Remark 5.6.1 below) open injection $\Pi^* \ltimes N^* \hookrightarrow \Pi \ltimes N^*$ that is compatible with the open injections between respective subgroups $f_{\Pi^*} : \Pi^* \hookrightarrow \Pi$ and the surjections to $N^*$ ($\subseteq \text{BGT}$).

Remark 5.6.1. Note that any open injection $\Pi^* \ltimes N^* \hookrightarrow \Pi \ltimes N^*$ as in Definition 5.6, (4), is unique. Indeed, let $f : \Pi^* \ltimes N^* \hookrightarrow \Pi \ltimes N^*$ be an open injection as in Definition 5.6, (4); $\Pi^{**} \subseteq \Pi$ an open subgroup such that $\Pi^{**} \subseteq f_{\Pi^*}(\Pi^*)$, and $\Pi^{**} \subseteq \Pi \ltimes N^*$ is a normal closed subgroup. Write $\text{Aut}_{\Pi^*}(\Pi) \subseteq \text{Aut}(\Pi)$ for the subgroup of automorphisms that preserve the normal open subgroup $\Pi^{**} \subseteq \Pi$. Then we have a commutative diagram

$$\begin{array}{ccc}
\Pi^* \ltimes N^* & \overset{f}{\longrightarrow} & \Pi \ltimes N^* \\
\downarrow & \downarrow & \\
\text{Aut}(\Pi^{**}) & \overset{\longrightarrow}{\text{Aut}_{\Pi^*}(\Pi)} & \\
\end{array}$$

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where the vertical arrows denote the injections determined by the respective actions by conjugation; the lower horizontal arrow denotes the natural injection [cf. Proposition 5.2, (ii)]. Thus, we conclude that the open injection \( f : \Pi^* \times N^* \mapsto \Pi \times N^* \) is uniquely determined by the open injection \( f_{\Pi^*} \) and the respective outer actions of \( N^* \) on \( \Pi^* \) and \( \Pi \), hence that any open injection as in Definition 5.6, (4), is unique.

Remark 5.6.2. In the notation of Definition 5.6, let \( \Pi^{**} \subseteq \Pi \) be an open subgroup contained in \( \Pi^* \). Then the inclusion \( \Pi^{**} \subseteq \Pi^* \) determines a natural map \( I(\Pi^*, \Pi) \to I(\Pi^{**}, \Pi) \) [cf. Propositions 5.2, (iv); 5.5, (ii)].

Proposition 5.7 (Kummer classes of group-theoretic nonconstant functions). In the notation of Definition 5.6, let \( f_{\Pi^*} \in I(\Pi^*, \Pi) \). Then \( f_{\Pi^*} \) naturally determines an element of

\[
\lim_{N^* \in \text{BGT}} H^1(\Pi^* \times N^*, \Pi_{X_0,\infty}),
\]

where \( N^* \) ranges over the normal open subgroups of BGT. In particular, we obtain a natural map

\[
\kappa_{\Pi^*} : I(\Pi^*, \Pi) \to \lim_{N^* \in \text{BGT}} H^1(\Pi^* \times N^*, \Pi_{X_0,\infty}).
\]

Proof. Let \( \Pi^* \times N^* \hookrightarrow \Pi \times N^* \) be an open injection as in Definition 5.6, (4). Write

\[
s_{f_{\Pi^*}} : \Pi^* \times N^* \hookrightarrow \Pi \times N^* \twoheadrightarrow \Pi_{X_0,\infty} \times N^*
\]

for the composite of this open injection \( \Pi^* \times N^* \hookrightarrow \Pi \times N^* \) with the natural \( \Pi_{X_0,\infty} \)-outer surjection \( \Pi \times N^* \twoheadrightarrow \Pi_{X_0,\infty} \times N^* \). Let \( I_1 \) be a cuspidal inertia subgroup of \( \Pi_X \) associated to the cusp \( \infty \). Then \( I_1 \) determines a section \( s_1|_{N^*} \) of the surjection \( \Pi_{X_0,\infty} \times N^* \twoheadrightarrow N^* \) [cf. the proof of Lemma 5.4, (i)]. In particular, by composing \( s_1|_{N^*} \) with the natural surjection \( \Pi^* \times N^* \twoheadrightarrow N^* \), we obtain a homomorphism

\[
s_1|_{\Pi^* \times N^*} : \Pi^* \times N^* \to \Pi_{X_0,\infty} \times N^*.
\]

Thus, by forming the difference between \( s_{f_{\Pi^*}} \) and \( s_1|_{\Pi^* \times N^*} \), we obtain an element \( \in H^1(\Pi^* \times N^*, \Pi_{X_0,\infty}) \), hence an element

\[
f_{\Pi^*} \in \lim_{N^* \in \text{BGT}} H^1(\Pi^* \times N^*, \Pi_{X_0,\infty}).
\]
Finally, it follows immediately from the various definitions involved that $f_{\Pi^*}$ is independent of the choice of $I_x$ [within its $\Pi_X$-conjugacy class]. This completes the proof of Proposition 5.7. 

**Definition 5.8.** We maintain the notation of Definition 5.6. Let $f_{\Pi^*} \in I(\Pi^*, \Pi); x \in \text{Cusp}(\Pi^*); I_x$ a cuspidal inertia subgroup of $\Pi^*$ associated to $x$. Then we define the value 

$$f_{\Pi^*}(x) \in \mathbb{Q}_{\text{BGT}} \cup \{\infty\}$$

of $f_{\Pi^*}$ at $x$ to be the image of the element $\in \text{Cusp}(\Pi)$ determined by the cuspidal inertia subgroup $N_{\Pi}(f_{\Pi^*}(I_x)) \subseteq \Pi$ via the natural $C_{\text{GT}}(\text{BGT})$-equivariant bijection $\text{Cusp}(\Pi) \rightarrow \mathbb{Q}_{\text{BGT}} \cup \{\infty\}$ [cf. Definition 5.1, (ii)]. It follows immediately from the various definitions involved that $f_{\Pi^*}(x) \in \mathbb{Q}_{\text{BGT}} \cup \{\infty\}$ does not depend on the choice of $I_x$ within its $\Pi^*$-conjugacy class.

**Definition 5.9.** We maintain the notation of Definition 5.8.

(i) Write

$$F_{\Pi^*} : I(\Pi^*, \Pi) \rightarrow \text{Fn}(\text{Cusp}(\Pi^*), \mathbb{Q}_{\text{BGT}} \cup \{\infty\})$$

(respectively, 

$$B_{\Pi^*} : \mathbb{Q}_{\text{BGT}} \rightarrow \text{Fn}(\text{Cusp}(\Pi^*), \mathbb{Q}_{\text{BGT}} \cup \{\infty\})$$)

for the natural map determined by considering the value (respectively, the constant value) at each of the elements $\in \text{Cusp}(\Pi^*)$. Then we shall write

$$L_{\Pi^*} \overset{\text{def}}{=} \text{Im } F_{\Pi^*} \cup \text{Im } B_{\Pi^*} \subseteq \text{Fn}(\text{Cusp}(\Pi^*), \mathbb{Q}_{\text{BGT}} \cup \{\infty\}).$$

(ii) For each finite subset $S \subseteq \text{Cusp}(\Pi^*)$, we shall write

$$\Pi_S^*$$

for the quotient of $\Pi^*$ by the normal closed subgroup topologically generated by the cuspidal inertia subgroups associated to $\text{Cusp}(\Pi^*) \setminus S$. Suppose that

$N \subseteq \text{BGT}$ induces the identity automorphism on $S$

[cf. Definition 5.1, (ii)]. Then we shall write

$$\Pi_{S,N}^* \overset{\text{def}}{=} \Pi^* \mathbin{\times} N, \quad \Pi_{S,N}^* \overset{\text{def}}{=} \Pi_S^* \mathbin{\times} N.$$

Write

$$I_S(\Pi^*, \Pi)$$

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for the inverse image of
\[ \text{Fn}(\text{Cusp}(\Pi^*) \setminus S, \overline{\mathbb{T}_{\text{BGT}}}) \subseteq \text{Fn}(\text{Cusp}(\Pi^*) \setminus S, \overline{\mathbb{T}_{\text{BGT}}} \cup \{\infty\}) \]
by the composite of \( F_{\Pi^*} \) with the restriction map
\[ \text{Fn}(\text{Cusp}(\Pi^*), \overline{\mathbb{T}_{\text{BGT}}} \cup \{\infty\}) \to \text{Fn}(\text{Cusp}(\Pi^*) \setminus S, \overline{\mathbb{T}_{\text{BGT}}} \cup \{\infty\}) ; \]
\[ F_{\Pi^*,S} : I_S(\Pi^*, \Pi) \to \text{Fn}(\text{Cusp}(\Pi^*) \setminus S, \overline{\mathbb{T}_{\text{BGT}}}) \]
for the natural map induced by \( F_{\Pi^*} ; S \) : \( I_S(\Pi^*, \Pi) \to \text{lim}_{N^* \subseteq N} H^1(\Pi^*_{S \times N^*}, \Pi_{X_{0\infty}}) \)
— where \( N^* \) ranges over the normal open subgroups of BGT contained in \( N \) — for the restriction of \( \kappa_{\Pi^*} \) to \( I_S(\Pi^*, \Pi) \) [cf. Proposition 5.7]. Here, we note that it follows immediately from the various definitions involved [cf. the proof of Proposition 5.7] that \( \kappa_{\Pi^*,S} \) factors as the composite of a natural map
\[ \kappa_{\Pi^*} : I_S(\Pi^*, \Pi) \to \lim_{N^* \subseteq N} H^1(\Pi^*_{S \times N^*}, \Pi_{X_{0\infty}}) \]
with the injection given by the inflation map
\[ \lim_{N^* \subseteq N} H^1(\Pi^*_{S \times N^*}, \Pi_{X_{0\infty}}) \hookrightarrow \lim_{N^* \subseteq N} H^1(\Pi^*_{S \times N^*}, \Pi_{X_{0\infty}}) . \]

(iii) In the notation of (ii), let \( x \in \text{Cusp}(\Pi^*) \setminus S ; N_x \) a normal open subgroup of BGT contained in \( N \) that stabilizes \( x ; I_x \subseteq \Pi^* \) a cuspidal inertia subgroup associated to \( x \). Then the image of \( N_{I_x} \rightarrow (I_x) \) via the natural surjection \( \Pi^*_{S \times N_x} \twoheadrightarrow \Pi^*_{S \times N^*} \) determines a section \( \overline{N_x} \to \Pi^*_{S \times N^*} \) of the natural surjection \( \Pi^*_{S \times N^*} \twoheadrightarrow N_x \) [cf. the proof of Lemma 5.4, (i)]. Thus, in particular, by allowing \( “N_x“ \) to vary, we obtain a natural map
\[ D_{\Pi^*} : \lim_{N^* \subseteq N} H^1(\Pi^*_{S \times N^*}, \Pi_{X_{0\infty}}) \to \text{Fn}(\text{Cusp}(\Pi^*) \setminus S, H^1(N, \Pi_{X_{0\infty}})) , \]
where \( H^1(N, \Pi_{X_{0\infty}}) \overset{\text{def}}{=} \lim_{N^* \subseteq N} H^1(N^*, \Pi_{X_{0\infty}}) . \)

Remark 5.9.1. In the remainder of the present paper, we shall use the injection given by the inflation map in the final display of Definition 5.9, (ii) to regard the group \( \lim_{N^* \subseteq N} H^1(\Pi^*_{S \times N^*}, \Pi_{X_{0\infty}}) \) as a subgroup of \( \lim_{N^* \subseteq N} H^1(\Pi^*_{S \times N^*}, \Pi_{X_{0\infty}}) . \)
Remark 5.9.2. We maintain the notation of Definition 5.9. Note that, for each element $f_{II^*} \in I(\Pi^*, \Pi)$, the set of $\Pi^*$-conjugacy classes of cuspidal inertia subgroups $I^*$ of $\Pi^*$ such that $f_{II^*}(I^*)$ is contained in a fixed $\Pi$-conjugacy class of a cuspidal inertia subgroup of $\Pi$ is finite. Indeed, this follows immediately from the fact that $f_{II^*}$ is an open injection that induces a bijection between the cuspidal inertia subgroups of $\Pi^*$ and $\Pi$ — cf. Definition 5.6, (1), (2). Thus, it follows immediately from the various definitions involved that

$$I(\Pi^*, \Pi) = \bigcup_{S \subseteq \text{Cusp}(\Pi^*)} I_S(\Pi^*, \Pi),$$

where $S$ ranges over the finite subsets of Cusp($\Pi^*$).

Definition 5.10. We maintain the notation of Definition 5.9, (ii). Suppose that $S \neq \emptyset$, and that, for each normal open subgroup $N^*$ of BGT contained in $N$,

$$H^1(\Pi^*_{S^0}, \Pi_{X_{0,\infty}})^{N^*} = \{0\}.$$ 

Then we shall construct a subgroup

$$K^N_{\Pi^*} \subseteq \lim_{\overrightarrow{N^* \subseteq N}} H^1(\Pi^*_{S \times N^*}, \Pi_{X_{0,\infty}})$$

as follows: First, we observe that the natural exact sequence

$$1 \longrightarrow \Pi^*_S \longrightarrow \Pi^*_{S \times N^*} \longrightarrow N^* \longrightarrow 1$$

determines an exact sequence

$$0 \longrightarrow H^1(N^*, \Pi_{X_{0,\infty}}) \longrightarrow H^1(\Pi^*_{S \times N^*}, \Pi_{X_{0,\infty}}) \overset{r}{\longrightarrow} H^1(\Pi^*_{S^0}, \Pi_{X_{0,\infty}})^{N^*}.$$ 

Thus, by allowing the normal open subgroup $N^*$ to vary, we obtain an exact sequence

$$0 \longrightarrow \lim_{\overrightarrow{N^* \subseteq N}} H^1(N^*, \Pi_{X_{0,\infty}}) \longrightarrow \lim_{\overrightarrow{N^* \subseteq N}} H^1(\Pi^*_{S \times N^*}, \Pi_{X_{0,\infty}}) \longrightarrow \lim_{\overrightarrow{N^* \subseteq N}} H^1(\Pi^*_{S^0}, \Pi_{X_{0,\infty}})^{N^*}.$$ 

Here, we observe that

$$H^1(\Pi^*_S, \Pi_{X_{0,\infty}})^{N^*} = H^1((\Pi^*_S)^{ab}, \Pi_{X_{0,\infty}})^{N^*}.$$ 

Next, for each $x \in S$, let $I_x$ be a cuspidal inertia subgroup of $\Pi^*$ associated to $x$. Then we have an exact sequence of $N^*$-modules

$$\bigoplus_{x \in S} I_x \longrightarrow (\Pi^*_S)^{ab} \longrightarrow (\Pi^*_S)^{ab} \longrightarrow 0,$$

which determines an exact sequence of modules

$$0 \longrightarrow H^1((\Pi^*_S)^{ab}, \Pi_{X_{0,\infty}})^{N^*} \longrightarrow H^1((\Pi^*_S)^{ab}, \Pi_{X_{0,\infty}})^{N^*} \longrightarrow \bigoplus_{x \in S} H^1(I_x, \Pi_{X_{0,\infty}}).$$
Thus, by applying our assumption that $H^1((\Pi_0^*)^{ab}, \Pi_{X_0})^N = \{0\}$, we obtain a natural injection

$$i : H^1((\Pi_0^*)^{ab}, \Pi_{X_0})^N \hookrightarrow \bigoplus_{x \in S} H^1(I_x, \Pi_{X_0}).$$

Write

$$1_x \in H^1(I_x, \Pi_{X_0}) = \text{Hom}(I_x, \Pi_{X_0})$$

for the isomorphism $I_x \xrightarrow{\sim} \Pi_{X_0}$ of Proposition 5.5, (ii);

$$Z_x \subseteq H^1(I_x, \Pi_{X_0})$$

for the subgroup generated by $1_x$;

$$i_x : N^* \hookrightarrow \Pi_{X_0}^*$$

for the section of the natural surjection $\Pi_{X_0}^* \twoheadrightarrow N^*$ determined by the image of $N \Pi_{X_0}^* (I_x)$ via the natural surjection $\Pi_{X_0}^* \twoheadrightarrow \Pi_{X_0}^*$. [cf. the proof of Lemma 5.4, (i)]. Next, we fix $x_0 \in S$. Write

$$D_x \in H^1(N^*, (\Pi_0)^{ab})$$

for the element obtained by forming the difference between $i_{x_0}$ and $i_x$;

$$\mathcal{P}_S \subseteq \bigoplus_{x \in S} Z_x \subseteq \bigoplus_{x \in S} H^1(I_x, \Pi_{X_0})$$

for the subgroup consisting of $(n_x)_{x \in S} \in \bigoplus_{x \in S} Z_x$ such that

$$\sum_{x \in S} n_x = 0, \quad \sum_{x \in S} n_x \cdot D_x = 0 \ (\in H^1(N^*, (\Pi_0)^{ab}))$$

[where we note that one verifies immediately that these conditions on $(n_x)_{x \in S}$ are independent of the choice of $x_0 \in S$];

$$\mathcal{P}_S^x$$

for the image of $(i \circ r)^{-1}(\mathcal{P}_S)$ via the natural homomorphism $H^1(\Pi_{X_0}^*, \Pi_{X_0}) \rightarrow \lim_{M^* \subseteq N^*} H^1(\Pi_{X_0}^*, \Pi_{X_0})$, where $M^*$ ranges over the normal open subgroups of $\mathfrak{H}^G$ contained in $N$. Then we define

$$K_{\mathcal{P}_S}^x \overset{\text{def}}{=} \mathcal{P}_S \cap D_x^{-1}(\mathfrak{Cusp}(\Pi^*)\setminus S, D_{0\infty}(\mathfrak{B}G)) \subseteq \lim_{M^* \subseteq N^*} H^1(\Pi_{X_0}^*, \Pi_{X_0}),$$

[cf. Lemma 5.4, (i); Definition 5.9, (iii)] and

$$K_{\mathcal{P}_S}^x \overset{\text{def}}{=} \bigcup_{T} K_{\mathcal{P}_S}^T \subseteq \lim_{M^* \subseteq N^*} H^1(\Pi_{X_0}^*, \Pi_{X_0}),$$

where $T$ ranges over the finite subsets of $\mathfrak{Cusp}(\Pi^*)$. 65
Remark 5.10.1. In the notation of Definition 5.10, suppose that $BGT = G_Q$.
Then the above construction of $K^\kappa_{H^*_S}$ coincides with the reconstruction of the Kummer classes of rational functions associated to $\Pi^*$ [cf. [AbsTopIII], Proposition 1.8].

Lemma 5.11 (Kummer classes of abstract functions). We maintain the notation of Definitions 5.9, 5.10. Suppose that the restriction $D_{H^*_S}|K^\kappa_{H^*_S}$ of $D_{H^*_S}$ to $K^\kappa_{H^*_S}$ [cf. Definition 5.9, (iii); Definition 5.10] is injective for arbitrary choices of “$S” and “$N” as in Definition 5.9, (ii). Then there exists a unique map

$$\operatorname{Im}(F_{\Pi^*,S}) \to \operatorname{Im}(\kappa_{H^*_S})$$

[cf. Definition 5.9, (ii)] whose composite with the natural surjection $I_S(\Pi^*,\Pi) \to \operatorname{Im}(F_{\Pi^*,S})$ determined by $F_{\Pi^*,S}$ coincides with the natural surjection $I_S(\Pi^*,\Pi) \to \operatorname{Im}(\kappa_{H^*_S})$ determined by $\kappa_{H^*_S}$, and whose image lies in $K^\kappa_{H^*_S}$. Moreover, by allowing $S$ to vary, one obtains a natural map

$$L_{\Pi^*} \setminus \{0\} \to \lim_{\mathbf{N} \subseteq \mathbf{N}} H^1(\Pi^* \times N^*,\Pi_{X_{\infty}}),$$

whose image lies in $K^\kappa_{H^*_S}$.

Proof. First, we observe that it follows from the various definitions involved that there exists a commutative diagram

$$\begin{array}{ccc}
I_S(\Pi^*,\Pi) & \xrightarrow{F_{\Pi^*,S}} & \operatorname{Fn}(\operatorname{Cusp}(\Pi^* \setminus \{0\}, \mathbb{Q}^{\infty}_{BGT}) \\
\kappa_{H^*_S} & & \\
\lim_{N^* \subseteq \mathbf{N}} H^1(\Pi^* S \times N^*,\Pi_{X_{\infty}}) & \xrightarrow{D_{H^*_S}} & \operatorname{Fn}(\operatorname{Cusp}(\Pi^* \setminus \{0\}, \mathbb{Q}^{\infty}_{BGT}) \\
\end{array}$$

where the right-hand vertical arrow is the natural map induced by the homomorphism

$$\psi_{BGT} : \mathbb{Q}^{\infty}_{BGT} \to \lim_{N^* \subseteq \mathbf{N}} H^1(N^*,\Pi_{X_{\infty}})$$

[cf. Lemma 5.4, (iv)].

Next, we observe that $\kappa_{H^*_S}$ factors as the composite of a map

$$I_S(\Pi^*,\Pi) \to K^\kappa_{H^*_S}$$

with the inclusion $K^\kappa_{H^*_S} \subseteq \lim_{N^* \subseteq \mathbf{N}} H^1(\Pi^* S \times N^*,\Pi_{X_{\infty}})$ [cf. Definition 5.10]. Indeed, since $(\Pi_0)^{ab} = \{0\}$ [hence, in particular, $H^1(N^*,(\Pi_0)^{ab}) = \{0\}$], it follows immediately from the various definitions involved that $\kappa_{\Pi_0(\infty)}$ maps $\operatorname{id} \in I_0(\Pi_0,\Pi)$ [cf. Proposition 5.5, (ii)] to an element of $K^\kappa_{H^*_S(\infty)}$. Thus, since any element $f_{\Pi^*} \in I_S(\Pi^*,\Pi)$ may be thought of as the pull-back \"via
of id $\in I_{(0, \infty)}(\Pi, \Pi)$, by applying the functoriality of the constructions involved [cf. also Definition 5.6, (3)], we obtain the desired conclusion.

Next, we apply our assumption that $D_{\Pi^0} | \kappa_{\Pi_1}$ is injective. Thus, since the above diagram is commutative, there exists a unique map $\text{Im}(F_{\Pi^0, S}) \to \text{Im}(\kappa_{\Pi_1})$ compatible with the maps $F_{\Pi^0, S}$ and $\kappa_{\Pi_1}$ in the desired sense. In particular, since all of the constructions involved are functorially compatible with enlargement of the finite subset $S \subseteq \text{Cusp}(\Pi^0)$, by allowing $S \subseteq \text{Cusp}(\Pi^0)$ to vary, we obtain a natural map

$$\text{Im}(F_{\Pi^0}) \to \lim_{N^* \subseteq N} H^1(\Pi^*_X \cap N^*, \Pi_{X_\infty})$$

[cf. Remarks 5.9.1, 5.9.2]. On the other hand, by considering the composite of $\psi_{\Pi^0} \circ \psi_{\Pi^0}$ with the restriction map

$$\lim_{N^* \subseteq N} H^1(N^*, \Pi_{X_\infty}) \to \lim_{N^* \subseteq N} H^1(\Pi^*_X \cap N^*, \Pi_{X_\infty}),$$

we obtain a natural map

$$\text{Im}(B_{\Pi^0}) \setminus \{0\} \to \lim_{N^* \subseteq N} H^1(\Pi^*_X \cap N^*, \Pi_{X_\infty}).$$

Thus, since $L_{\Pi^0} = \text{Im} F_{\Pi^0} \cup \text{Im} B_{\Pi^0}$ [where we note that $\text{Im} F_{\Pi^0} \cap \text{Im} B_{\Pi^0} = \emptyset$ — cf. Remark 5.9.2], we obtain the desired conclusion. This completes the proof of Lemma 5.11. \hfill $\square$

**Definition 5.12.** Let $\Pi \subseteq GT$ be a closed subgroup satisfying the $\mathcal{B}$-property [cf. Definition 3.3, (v)]. We apply the notation of Definitions 4.1, 5.1, 5.6, 5.8, 5.9. Write $t \in L_{\Pi^0}$ for the element determined by id $\in I(\Pi^0, \Pi)$ [cf. Proposition 5.5, (ii)]. Then, if $\Pi$ satisfies the following conditions (i), (ii), (iii) (respectively, (i), (ii), (iii), (iv)), then we shall say that $\Pi$ satisfies the $\mathcal{QAA}$-property [i.e., “quasi-algebraically ample property”] (respectively, $\mathcal{AA}$-property [i.e., “algebraically ample property”]):

(i) Write $(\mathcal{Q}_{\Pi^0})_{\text{div}} \subseteq \mathcal{Q}_{\Pi^0}$ for the subfield generated over $\mathbb{Q}$ by $\text{Ker}(\psi_{\Pi^0} \circ \psi_{\Pi^0})$ [cf. Lemma 5.4, (iv)]. Then $(\mathcal{Q}_{\Pi^0})_{\text{div}} \subseteq \mathcal{Q}_{\Pi^0}$ is an infinite extension of fields.

(ii) For

- each normal open subgroup $\Pi^0 \subseteq \Pi$,
- each nonempty finite subset $S \subseteq \text{Cusp}(\Pi^0)$,
- any sufficiently small normal open subgroup $N^0$ of $\Pi$,

it holds that $H^1(\Pi^0 \cap N^0, \Pi_{X_\infty}) = \{0\}$ [cf. Definition 5.10], and the restriction $D_{\Pi^0} | \kappa_{\Pi_1}$ of $D_{\Pi^0}$ to $\kappa_{\Pi_1}$ [cf. Definition 5.9, (iii); Definition 5.10] is injective [cf. Lemma 5.11].

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(iii) Assume that condition (ii) holds. There exists a family of subsets

\[ \{ K^t \subseteq L^t \}_{t \in T} \]

where \( t \) ranges over the normal open subgroups of \( \Pi \) satisfying the following conditions:

(a) Let \( \Pi^t \subseteq \Pi \) be normal open subgroups of \( \Pi \). Then the natural injection \( L^t \hookrightarrow L^t \) [determined by the natural surjection \( \text{Cusp}^t \rightarrow \text{Cusp}^{t'} \) — cf. Proposition 5.5, (ii); Remarks 5.6.2, 5.9.2] induces an injection

\[ K^t \hookrightarrow K^t. \]

In the remainder of the present paper, we regard \( K^t \) as a subset of \( K^t \) via this injection.

(b) For each normal open subgroup \( \Pi^t \subseteq \Pi \), and each finite subset \( R \subseteq \text{Cusp}^t \), the restriction to \( K^t \) of the natural restriction map

\[ \text{Fn}(\text{Cusp}^t, \mathbb{Q}_{\text{BGT}} \{ \mathbb{R} \}) \rightarrow \text{Fn}(\text{Cusp}^t \setminus R, \mathbb{Q}_{\text{BGT}} \cup \{ \mathbb{R} \}) \]

is injective.

(c) For each normal open subgroup \( \Pi^t \subseteq \Pi \), \( K^t \) admits a [necessarily unique — cf. (b)] field structure compatible with the ring structure of \( \text{Fn}(\text{Cusp}^t, \mathbb{Q}_{\text{BGT}}) \) in the following sense: Let \( f, g \in K^t \), \( T \subseteq \text{Cusp}^t \) a finite subset such that \( f(x), g(x) \in \mathbb{Q}_{\text{BGT}} \) for any \( x \in \text{Cusp}^t \setminus T \). [For given elements \( f, g \in K^t \), the existence of such a finite set \( T \) follows immediately from Remark 5.9.2.] Then the images of \( f + g \) and \( fg \) via the restriction map

\[ \text{Fn}(\text{Cusp}^t, \mathbb{Q}_{\text{BGT}} \cup \{ \mathbb{R} \}) \rightarrow \text{Fn}(\text{Cusp}^t \setminus T, \mathbb{Q}_{\text{BGT}} \cup \{ \mathbb{R} \}) \]

coincide, respectively, with the functions

\[ \text{Cusp}^t \setminus T \ni x \mapsto f(x) + g(x) \in \mathbb{Q}_{\text{BGT}}, \]

\[ \text{Cusp}^t \setminus T \ni x \mapsto f(x)g(x) \in \mathbb{Q}_{\text{BGT}}. \]

Moreover, relative to these unique field structures, \( K^t \subseteq K^t \) is a finite Galois extension.

(d) \( \mathbb{Q}_{\text{BGT}} = \text{Im} B_{\Pi} \subseteq K_{\Pi} \), and \( t \in K_{\Pi} \). Moreover, if we write \( \mathbb{Q}_{\text{BGT}}(t) \subseteq K_{\Pi} \) for the subfield generated by \( \mathbb{Q}_{\text{BGT}}(t) \) and \( t \), then \( K^t \cong \mathbb{Q}_{\text{BGT}}(t) \).

(e) For each normal open subgroup \( \Pi^t \subseteq \Pi \), the natural action of \( \Pi \) on \( L_{\Pi} \) [cf. Proposition 5.2, (iv)] preserves \( K^t \). Moreover, the natural homomorphism

\[ \Pi/\Pi^t \rightarrow \text{Gal}(K_{\Pi}/K^t) \]

is an isomorphism.
(f) For each normal open subgroup $\Pi^1 \subseteq \Pi$, the image of $K^\infty_\Pi$ ($\subseteq L^1_\Pi$) via the natural map

$$L^1_\Pi \setminus \{0\} \to K^\infty_\Pi \left( \subseteq \lim_{N^1 \leq BGT} H^1(\Pi^1_{\infty N^1}, \Pi_{X_{\infty 0}}) \right)$$

[cf. condition (ii); Definition 5.10; Lemma 5.11] is surjective.

(iv) Assume that conditions (ii), (iii) hold. In the notation of condition (iii), write $Q_{BGT}[t; 1_t; 1_t; 1_t]$ for the $Q_{BGT}$-subalgebra generated by $t$, $1_t$, and $1_t$. $X_{BGT}$ is also the natural outer surjection $\Pi \twoheadrightarrow \Pi_X$ determined by a natural outer isomorphism $\Pi_{X_{BGT}} \twoheadrightarrow \Pi_X$. Then the natural outer isomorphism $\Pi_{X_{BGT}} \twoheadrightarrow \Pi_X$ is induced by an outer isomorphism $\Pi \twoheadrightarrow \Pi_X$ determined by the respective first projections [cf. Remark 5.12.2 below].

**Remark 5.12.1.** In the notation of Remark 4.4.1, it follows immediately from Remark 4.4.1, together with the various definitions involved and the fact that $F$ is Kummer-faithful [cf. [AbsTopIII], Definition 1.5; [AbsTopIII], Remark 1.5.4, (i)], that $G_F$ satisfies the AA-property [cf. the proof of Theorem 6.8, (i), below, for more details].

**Remark 5.12.2.** In condition (iv), we regard $\Pi_{X_{\infty}}$ as an abstract topological group and $\Pi_X$ as a quotient of $\Pi_{X_{\infty}}$, i.e., as in Corollary 4.5 [cf. also Remark 4.5.1].

**Lemma 5.13 (Geometric interpretation of the set of cuspidal inertia subgroups of $\Pi$).** Suppose that BGT satisfies conditions (ii), (iii) of Definition 5.12. Let

$$\{K^\infty_\Pi \subseteq L^1_\Pi\}_{\Pi^1 \subseteq \Pi}$$

be a family of subsets as in Definition 5.12, (iii). Write

$$\tilde{K}^\infty_\Pi \overset{\text{def}}{=} \lim_{\Pi^1 \subseteq \Pi} K^\infty_\Pi,$$

where $\Pi^1$ ranges over the normal open subgroups of $\Pi$. Then the following hold:
(i) Let $\Pi^1 \subseteq \Pi$ be a normal open subgroup. Write $Y_{\Pi^1} \to \mathbb{P}^1_{\mathbb{Q}_{BGT}}$ for the finite ramified Galois covering of smooth, proper, connected curves over $\overline{\mathbb{Q}}_{BGT}$ corresponding to the extension of function fields $\mathbb{Q}_{BGT}(t) = K_\Pi \subseteq K_{\Pi^1}$ [cf. Definition 5.12, (iii), (a), (c), (d), (e)]; $Y_{\Pi^1}(\overline{\mathbb{Q}}_{BGT})$ for the set of $\overline{\mathbb{Q}}_{BGT}$-valued points of $Y_{\Pi^1}$. Then the natural map

$$\text{ev}_{\Pi^1} : \text{Cusp}(\Pi^1) \to Y_{\Pi^1}(\overline{\mathbb{Q}}_{BGT})$$

induced by evaluating elements of $K_{\Pi^1}$ at elements of Cusp$(\Pi^1)$ is bijective.

(ii) $\tilde{K}_\Pi$ is an algebraic closure of $\mathbb{Q}_{BGT}(t) = K_\Pi$. Moreover, the natural action of $\Pi$ on $\tilde{K}_\Pi$ determines an isomorphism

$$\Pi \cong G_{\mathbb{Q}_{BGT}(t)} \overset{\text{def}}{=} \text{Gal}(\tilde{K}_\Pi/\mathbb{Q}_{BGT}(t))$$

that induces a bijection between the respective sets of cuspidal inertia subgroups of $\Pi$ and $G_{\mathbb{Q}_{BGT}(t)}$.

Proof. Let $\tilde{K}_{\Pi}^{\text{alg}}$ be an algebraic closure of $\tilde{K}_\Pi$. First, we verify assertion (i). Note that it follows immediately from the various definitions involved [cf. especially, Definition 5.12, (iii), (d)] that $\text{ev}_{\Pi^1}$ is bijective. Note, moreover, that the natural map $\text{ev}_{\Pi^1} : \text{Cusp}(\Pi^1) \to Y_{\Pi^1}(\overline{\mathbb{Q}}_{BGT})$ is compatible with the isomorphism $\Pi/\Pi^1 \cong \text{Gal}(K_{\Pi^1}/K_{\Pi^1})$ [cf. Definition 5.12, (iii), (e)] and the respective natural actions of $\Pi/\Pi^1$ and $\text{Gal}(K_{\Pi^1}/K_{\Pi^1})$. Thus, it follows immediately from the transitivity of the natural action of $\text{Gal}(K_{\Pi^1}/K_{\Pi^1})$ on the fibers of the finite ramified Galois covering $Y_{\Pi^1} \to \mathbb{P}^1_{\mathbb{Q}_{BGT}}$ that $\text{ev}_{\Pi^1}$ is surjective.

Write

$$\overline{\text{Cusp}}(\Pi) \overset{\text{def}}{=} \lim_{\Pi^1 \subseteq \Pi} \text{Cusp}(\Pi^1), \quad \overline{Y}(\overline{\mathbb{Q}}_{BGT}) \overset{\text{def}}{=} \lim_{\Pi^1 \subseteq \Pi} Y_{\Pi^1}(\overline{\mathbb{Q}}_{BGT}),$$

where $\Pi^1$ ranges over the normal open subgroups of $\Pi$. Observe that the natural maps $(\text{ev}_{\Pi^1})_{\Pi^1 \subseteq \Pi}$ induce a natural map $\overline{\text{ev}} : \overline{\text{Cusp}}(\Pi) \to \overline{Y}(\overline{\mathbb{Q}}_{BGT})$ that, for each normal open subgroup $\Pi^1$ of $\Pi$, fits into a commutative diagram

$$\begin{array}{ccc}
\overline{\text{Cusp}}(\Pi) & \xrightarrow{\overline{\text{ev}}} & \overline{Y}(\overline{\mathbb{Q}}_{BGT}) \\
\downarrow & & \downarrow \\
\text{Cusp}(\Pi^1) & \xrightarrow{\text{ev}^1} & Y_{\Pi^1}(\overline{\mathbb{Q}}_{BGT}).
\end{array}$$

One verifies easily that this commutative diagram is compatible with the natural isomorphism $\Pi \cong \text{Gal}(\tilde{K}_\Pi/\mathbb{Q}_{BGT}(t))$ [cf. Definition 5.12, (iii), (e)] and the respective natural actions of $\Pi$ and $\text{Gal}(\tilde{K}_\Pi/\mathbb{Q}_{BGT}(t))$.

Suppose that $\text{ev}_{\Pi^1}(c_1) = \text{ev}_{\Pi^1}(c_2)$, where $c_1, c_2 \in \text{Cusp}(\Pi^1)$. Let $I_1 \subseteq \Pi^1$, $I_2 \subseteq \Pi^1$, $J \subseteq \text{Gal}(\tilde{K}_\Pi/K_{\Pi^1})$ be cuspidal inertia subgroups associated to $c_1$, $c_2$. 

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the QAA-property \cite{De} and the respective natural actions, one verifies immediately that by choosing suitable conjugates of $I_1$, $I_2$, and $J$, we may assume without loss of generality that the natural isomorphism $\Pi^1 \xrightarrow{\sim} \operatorname{Gal}(\tilde{K}_\Pi/K_{\Pi^1})$ induces inclusions $\iota_1 : I_1 \hookrightarrow J$, $\iota_2 : I_2 \hookrightarrow J$. Next, observe that any cuspidal inertia subgroup of $\operatorname{Gal}(K_{\Pi^1}/K_{\Pi^1})$ is a quotient of some cuspidal inertia subgroup of $\operatorname{Gal}(\tilde{K}_\Pi^\text{alg}/\tilde{K}_\Pi)$ via the natural surjection $\operatorname{Gal}(K_{\Pi^1}^\text{alg}/K_{\Pi^1}) \twoheadrightarrow \operatorname{Gal}(\tilde{K}_\Pi/K_{\Pi^1})$, and that every cuspidal inertia subgroup of $\operatorname{Gal}(K_{\Pi^1}^\text{alg}/K_{\Pi^1})$ is isomorphic to $\tilde{Z}$. Thus, we conclude that $J$ is abelian, and hence, by applying the inclusions $\iota_1$, $\iota_2$, that $I_1 \subseteq N_{\Pi^1}(I_2)$, $I_2 \subseteq N_{\Pi^1}(I_1)$, which \cite{Proposition 5.5, (ii)} implies that $I_1 = I_2$, as desired. This completes the proof of the injectivity of $\operatorname{ev}_\Pi^1$ and hence of assertion (i).

Next, we verify assertion (ii). Recall from Proposition 5.2, (i), that there exists an isomorphism $\xi : \Pi \xrightarrow{\sim} \operatorname{Gal}(K_{\Pi^1}^\text{alg}/\overline{BGT}(t))$ of profinite groups that induces a bijection between the respective sets of cuspidal inertia subgroups. In particular, since the natural isomorphism $\Pi \xrightarrow{\sim} \operatorname{Gal}(\tilde{K}_\Pi^\text{alg}/\overline{BGT}(t))$ \cite{Definition 5.12, (iii), (a)} induces a bijection between the respective sets of cuspidal inertia subgroups of $\Pi$ and $\operatorname{Gal}(\tilde{K}_\Pi/\overline{BGT}(t))$ \cite{assertion (i)}, the composite morphism

$$\operatorname{Gal}(K_{\Pi^1}^\text{alg}/\overline{BGT}(t)) \twoheadrightarrow \operatorname{Gal}(\tilde{K}_\Pi^\text{alg}/\overline{BGT}(t)) \xrightarrow{\sim} \Pi \xrightarrow{\sim} \operatorname{Gal}(K_{\Pi^1}^\text{alg}/\overline{BGT}(t))$$

is a surjection that induces a bijection between the respective sets of conjugacy classes of cuspidal inertia subgroups of the domain and codomain \cite{i.e., both of which are equal to $\operatorname{Gal}(K_{\Pi^1}^\text{alg}/\overline{BGT}(t))$. Thus, we conclude from Proposition 5.2, (vi), that this composite morphism is an isomorphism, hence that $K^\text{alg}_{\Pi^1} = \tilde{K}_\Pi$. This completes the proof of assertion (ii), hence of Lemma 5.13. \hfill $\square$

**Theorem 5.14 (Uniqueness of function fields).** Suppose that $\overline{BGT}$ satisfies the QAA-property \cite{cf. Definition 5.12}. Then any family

$$\{K_{\Pi^1} \subseteq L_{\Pi^1}\}_{\Pi^1 \subseteq \Pi}$$

of subsets as in Definition 5.12, (iii), is unique.

**Proof.** Let $\{K_{\Pi^1} \subseteq L_{\Pi^1}\}_{\Pi^1 \subseteq \Pi}$, $\{\star K_{\Pi^1} \subseteq L_{\Pi^1}\}_{\Pi^1 \subseteq \Pi}$ be families of subsets as in Definition 5.12, (iii). Recall that, if $\Pi^1 \subseteq \Pi^1$ are normal open subgroups of $\Pi$, then $K_{\Pi^1} \subseteq K_{\Pi^1}$ and $\star K_{\Pi^1} \subseteq \star K_{\Pi^1}$ \cite{cf. Definition 5.12, (iii), (a)}. Write

$$\tilde{K}_\Pi \overset{\text{def}}{=} \operatorname{lim}_{\Pi^1 \subseteq \Pi} K_{\Pi^1}, \quad \star \tilde{K}_\Pi \overset{\text{def}}{=} \operatorname{lim}_{\Pi^1 \subseteq \Pi} \star K_{\Pi^1},$$

where $\Pi^1$ ranges over the normal open subgroups of $\Pi$. Then since $\tilde{K}_\Pi$ and $\star \tilde{K}_\Pi$ are algebraic closures of $K_\Pi$ \cite{cf. Lemma 5.13, (ii)}, there exists an abstract
field isomorphism $\beta : \tilde{K}_\Pi \xrightarrow{\sim} \cdot \tilde{K}_\Pi$ over $K_\Pi$, which determines an isomorphism of profinite groups $\alpha : \text{Gal}(\cdot \tilde{K}_\Pi/K_\Pi) \xrightarrow{\sim} \text{Gal}(\tilde{K}_\Pi/K_\Pi)$. Fix a normal open subgroup $\Pi^\dagger \subseteq \Pi$.

Write

- $\circ K_\Pi^\dagger \overset{\text{def}}{=} \beta^{-1}(\cdot K_\Pi^\dagger) \subseteq \tilde{K}_\Pi$;
- $Y \rightarrow \mathbb{P}^1_{\BGT}$ (respectively, $\cdot Y \rightarrow \mathbb{P}^1_{\BGT}$, $\circ Y \rightarrow \mathbb{P}^1_{\BGT}$) for the finite ramified Galois covering of smooth, proper, connected curves over $\mathbb{Q}_{BGT}$ corresponding to the extension of function fields $\mathbb{Q}_{BGT}(t) = K_\Pi \subseteq \cdot K_\Pi^\dagger$, $\mathbb{Q}_{BGT}(t) = K_\Pi \subseteq \circ K_\Pi$ [cf. Definition 5.12, (iii), (a), (c), (d), (e)];
- $\mathbb{P}^1_{\BGT}(\mathbb{Q}_{BGT})$, $\cdot Y(\mathbb{Q}_{BGT})$, $\circ Y(\mathbb{Q}_{BGT})$ for the respective sets of $\mathbb{Q}_{BGT}$-valued points of $\mathbb{P}^1_{\BGT}$, $Y$, $\cdot Y$, $\circ Y$.

Observe that there exist natural bijections

$$\text{Cusp}(\Pi) \xrightarrow{\text{evn}_{\text{evn}}} \mathbb{P}^1_{\BGT}(\mathbb{Q}_{BGT}), \quad \text{Cusp}(\Pi^\dagger) \xrightarrow{\text{evn}_{\text{evn}}} Y(\mathbb{Q}_{BGT}), \quad \text{Cusp}(\Pi^\dagger) \xrightarrow{\text{evn}_{\text{evn}}} \cdot Y(\mathbb{Q}_{BGT})$$

[cf. Lemma 5.13, (i)] that fit into a commutative diagram

$$\begin{array}{ccccccc}
\text{Gal}(\tilde{K}_\Pi/K_\Pi) & \xrightarrow{\sim} & \Pi & \xrightarrow{\sim} & \text{Gal}(\cdot \tilde{K}_\Pi/K_\Pi) & \xrightarrow{\alpha} & \text{Gal}(\tilde{K}_\Pi/K_\Pi) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Gal}(K_\Pi^\dagger/K_\Pi) & \xrightarrow{\sim} & \Pi/\Pi^\dagger & \xrightarrow{\sim} & \text{Gal}(\cdot K_\Pi^\dagger/K_\Pi) & \xrightarrow{\alpha} & \text{Gal}(\circ K_\Pi^\dagger/K_\Pi) \\
\ldots & \xrightarrow{\text{evn}} & \text{Cusp}(\Pi^\dagger) & \xrightarrow{\text{evn}} & \cdot Y(\mathbb{Q}_{BGT}) & \xrightarrow{\beta} & \circ Y(\mathbb{Q}_{BGT}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{P}^1_{\BGT}(\mathbb{Q}_{BGT}) & \xrightarrow{\text{evn}} & \text{Cusp}(\Pi) & \xrightarrow{\text{evn}} & \mathbb{P}^1_{\BGT}(\mathbb{Q}_{BGT}) & = & \mathbb{P}^1_{\BGT}(\mathbb{Q}_{BGT}),
\end{array}$$

where the vertical arrows denote the natural surjections; the horizontal arrows $\text{Gal}(\cdot K_\Pi^\dagger/K_\Pi) \xrightarrow{\alpha} \text{Gal}(\cdot K_\Pi^\dagger/K_\Pi)$ and $\cdot Y(\mathbb{Q}_{BGT}) \xrightarrow{\beta} \circ Y(\mathbb{Q}_{BGT})$ denote the bijections induced, respectively, by $\alpha$ and $\beta$.

Note that it follows immediately from the above commutative diagram that the sets $\subseteq \mathbb{P}^1_{\BGT}(\mathbb{Q}_{BGT})$ of branch points of the finite ramified Galois coverings $Y \rightarrow \mathbb{P}^1_{\BGT}$ and $\circ Y \rightarrow \mathbb{P}^1_{\BGT}$ coincide. Write $T \subseteq \text{Cusp}(\Pi)$ for the image of the set of branch points of the finite ramified Galois covering $Y \rightarrow \mathbb{P}^1_{\BGT}$ via the bijection $\text{evn}_{\text{evn}}^{-1}$. Then, by replacing the normal open subgroup $\Pi^\dagger \subseteq \Pi$ by the pull-back of a suitable characteristic open subgroup of $\Pi_T$ [cf. Definition 5.1, (ii)] via the natural surjection $\Pi \rightarrow \Pi_T$, we may assume without loss of generality that $K_\Pi = \circ K_\Pi^\dagger$, $Y = \circ Y$.

Write

$$\sigma : Y(\mathbb{Q}_{BGT}) \xrightarrow{\sim} \circ Y(\mathbb{Q}_{BGT}) = Y(\mathbb{Q}_{BGT})$$
for the composite of the horizontal arrows in the third row of the above commutative diagram. Recall that the images of $K_{\Pi}^\times$, $\bullet K_{\Pi}^\times$ ($\subseteq L_{\Pi}$) via the natural map

$$L_{\Pi} \setminus \{0\} \rightarrow \lim_{N \supseteq \text{BGT}}^\Pi H^1(\Pi_{x,N}, \Pi_{x,\infty})$$

coincide with $K_{\Pi}^\times$ [cf. Definition 5.12, (iii), (f)]. In particular, for each $f \in K_{\Pi}^\times$, there exist

$$\phi_f \in \text{Fn}((Y(Q_{\text{BGT}}), (\varpi_{\Pi}^{\text{BGT}})) \subseteq \text{Fn}((Y(Q_{\text{BGT}}), \varpi_{\text{BGT}} \cup \{\infty\})),$$

such that $f^\sigma \overset{\text{def}}{=} f \circ \sigma = \phi_f \cdot g_f$ [cf. Definitions 5.9; 5.10; 5.12, (i)]. Note that it follows immediately from the above commutative diagram that $\sigma$ lies over the identity automorphism of $\mathbb{P}_1^0 \varpi_{\text{BGT}}(\varpi_{\text{BGT}})$. Thus, we conclude from Corollary 1.3 [cf. also Definition 5.12, (i)] that, relative to the notational conventions of loc. cit., $\sigma \in \text{Gal}(K_{\Pi} / K_{\Pi})$ and hence that $K_{\Pi} = \bullet K_{\Pi}$. This completes the proof of Theorem 5.14.

\textbf{Theorem 5.15 (Injectivity of $C_{\text{GT}}(\text{BGT}) \rightarrow \text{Aut}(\varpi_{\text{BGT}})$).} Suppose that BGT satisfies the QAA-property [cf. Definitions 3.3, (v); 5.12]. Write

$$\{K_{\Pi} \subseteq L_{\Pi} \mid \Pi \subseteq \Pi\}$$

for the unique family of subsets as in Definition 5.12, (iii) [cf. Theorem 5.14];

$$\tilde{K}_{\Pi} \overset{\text{def}}{=} \lim_{\Pi \subseteq \Pi} K_{\Pi},$$

where $\Pi$ ranges over the normal open subgroups of $\Pi$;

$$G_{\varpi_{\text{BGT}}(t)} \overset{\text{def}}{=} \text{Gal}(\tilde{K}_{\Pi} / K_{\Pi}) \overset{=} {=} \text{Gal}(\tilde{K}_{\Pi} / \varpi_{\text{BGT}}(t))$$

[cf. Definition 5.12, (iii), (d)];

$$\rho : C_{\text{GT}}(\text{BGT}) \rightarrow G_{\varpi_{\text{BGT}}} \overset{\text{def}}{=} \text{Aut}(\varpi_{\text{BGT}})$$

for the homomorphism induced by the natural action of $C_{\text{GT}}(\text{BGT})$ on the field $\varpi_{\text{BGT}}$ [cf. Theorem 4.4]. Then the following hold:

(i) $\Pi \overset{\text{out}}{\times} C_{\text{GT}}(\text{BGT})$ acts naturally on the algebraically closed field $\tilde{K}_{\Pi}$ [cf. Lemma 5.13, (ii)]. Moreover, this action induces a commutative diagram

$$\begin{array}{ccc}
C_{\text{GT}}(\text{BGT}) & \overset{\rho}{\longrightarrow} & G_{\varpi_{\text{BGT}}} \\
\downarrow & & \downarrow \\
\text{Out}(\Pi) & \overset{\sim}{\longrightarrow} & \text{Out}(G_{\varpi_{\text{BGT}}(t)}),
\end{array}$$

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where the left-hand vertical arrow denotes the homomorphism induced by the natural outer action of $C_{\Gamma}(B\Gamma)$ on $\Pi$; the right-hand vertical arrow denotes the natural outer representation; the lower horizontal arrow denotes the isomorphism induced by the isomorphism $\Pi \rightarrow G_{\mathbb{Q}B\Gamma(t)}$ [cf. Lemma 5.13, (ii)].

(ii) The commutative diagram of (i) induces a commutative diagram

$$\begin{array}{ccc}
C_{\Gamma}(B\Gamma) & \longrightarrow & G_{\mathbb{Q}B\Gamma} \\
\downarrow & & \downarrow \\
\text{Out}(\Pi_X) & \xrightarrow{\sim} & \text{Out}(\Pi_{X_{\mathbb{Q}B\Gamma}}),
\end{array}$$

where the left-hand vertical arrow denotes the homomorphism induced by the natural faithful outer action of $C_{\Gamma}(B\Gamma) \subseteq \Gamma$ on $\Pi_X$; the right-hand vertical arrow denotes the natural outer representation; the lower horizontal arrow denotes the isomorphism $\Pi_X \rightarrow \Pi_{X_{\mathbb{Q}B\Gamma}}$ [cf. Lemma 5.13, (ii)].

(iii) The homomorphism $\rho$ is injective. In particular, the restriction $\rho|_{B\Gamma}$ of $\rho$ to $B\Gamma$ is injective.

(iv) Suppose, moreover, that $B\Gamma$ satisfies the AA-property. Write $G_{\text{B\Gamma}} \subseteq \text{Out}(\Pi_{(X_{\mathbb{Q}B\Gamma})^2})$ for the Grothendieck-Teichmüller group associated [cf. Corollary 4.5] to $\Pi_{(X_{\mathbb{Q}B\Gamma})^2}$. Then the commutative diagram of (ii) induces a commutative diagram

$$\begin{array}{ccc}
C_{\Gamma}(B\Gamma) & \longrightarrow & G_{\mathbb{Q}B\Gamma} \\
\downarrow & & \downarrow \\
\Gamma & \xrightarrow{\sim} & G_{\text{B\Gamma}},
\end{array}$$

where the vertical arrows denote the natural injections; the lower horizontal arrow denotes the isomorphism induced by an outer isomorphism $\Pi_{X_{\mathbb{Q}B\Gamma}} \rightarrow \Pi_{(X_{\mathbb{Q}B\Gamma})^2}$ as in Definition 5.12, (iv).

Proof. First, we verify assertion (i). Note that it follows immediately from the various definitions involved that $\Pi^\text{out} \times C_{\Gamma}(B\Gamma)$ acts naturally on the family of sets $\{L_{\Pi}\}_{\Pi \subseteq \Pi}$, where $\Pi^\text{out}$ ranges over the normal open subgroups of $\Pi$ [cf. Definition 5.8]. Thus, we conclude from the uniqueness of the family of subsets $\{K_{\Pi} \subseteq L_{\Pi}\}_{\Pi \subseteq \Pi}$ [cf. Theorem 5.14] that $\Pi^\text{out} \times C_{\Gamma}(B\Gamma)$ acts naturally on the algebraically closed field $K_{\Pi}$. Moreover, it follows immediately from the various definitions involved that this natural action induces the desired commutative diagram. This completes the proof of assertion (i). Next, since the natural surjection $\Pi \rightarrow \Pi_X$ is compatible with the respective outer actions of $C_{\Gamma}(B\Gamma)$ [cf. Definition 5.1, (i)], assertion (ii) follows immediately from
assertion (i). Assertion (iii) follows immediately from assertion (ii). Assertion (iv) follows immediately from the various definitions involved. This completes the proof of Theorem 5.15.

Lemma 5.16 (Elementary property of profinite groups). Let $G$ be a profinite group, $H \subseteq G$ a closed subgroup, $g \in G$ an element such that $H \subseteq H^g := g \cdot H \cdot g^{-1}$. Then $H = H^g$.

Proof. By considering quotients of $G$ by normal open subgroups, one reduces immediately to the case where $G$ is finite. Then the equality $H = H^g$ follows immediately from the fact that $H$ and $H^g$ have the same cardinality. This completes the proof of Lemma 5.16.

Theorem 5.17 (Combinatorial construction of $G_\mathbb{Q}$).

(i) Write $\text{Out}^C(\Pi_X) \subseteq \text{Out}(\Pi_X)$ for the closed subgroup of outer automorphisms that induce the identity automorphisms on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_X$. Then the conjugacy class of subgroups of $\text{Out}^C(\Pi_X)$ determined by the absolute Galois group of $\mathbb{Q}$ may be constructed from the abstract topological group $\Pi_X$, [cf. Corollary 4.5, Remark 4.5.1], in a purely combinatorial/group-theoretic way, as the set of maximal elements [relative to the relation of inclusion] in the set of closed subgroups of $\text{Out}^C(\Pi_X)$ that arise as $\text{Out}^C(\Pi_X)$-conjugates of closed subgroups of $G_\mathbb{Q}$ that satisfy the QAA-property [cf. Definitions 3.3, (v); 5.12].

(ii) The conjugacy class of subgroups of $G_\mathbb{Q}$ determined by the absolute Galois group of $\mathbb{Q}$ may be constructed from the abstract topological group $\Pi_X$, [cf. Corollary 4.5, Remark 4.5.1], in a purely combinatorial/group-theoretic way, as the set of maximal elements [relative to the relation of inclusion] in the set of closed subgroups of $G_\mathbb{Q}$ that arise as closed subgroups of $G_\mathbb{Q}$ that satisfy the AA-property [cf. Definitions 3.3, (v); 5.12].

Proof. Recall from Remark 5.12.1 that $G_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ may be regarded as a closed subgroup of $G_\mathbb{Q}$ that satisfies the AA-property, hence may itself be taken to be “BGT”. Thus, it follows formally from Theorem 5.15, (ii) [cf. also Lemma 5.13, (ii)] (respectively, Theorem 5.15, (iv)), that any $\text{Out}^C(\Pi_X)$-conjugate (respectively, $G_\mathbb{Q}$-conjugate) of a closed subgroup of $G_\mathbb{Q}$ that satisfies the QAA-property (respectively, AA-property) is contained in — hence equal to, whenever it is maximal with respect to the relation of inclusion among such conjugates of closed subgroups — some $\text{Out}^C(\Pi_X)$-conjugate (respectively, GT-conjugate) of $G_\mathbb{Q}$. In particular, the maximality of any $\text{Out}^C(\Pi_X)$-conjugate (respectively, GT-conjugate) of $G_\mathbb{Q}$ follows formally from Lemma 5.16. This completes the proof of Theorem 5.17.
6 Application to semi-absolute anabelian geometry over TKND-AVKF-fields

In this section, we introduce the notion of a TKND-AVKF-field [cf. Definition 6.6, (iii)] and show that the absolute Galois group of a TKND-AVKF subfield of $\mathbb{Q}$ satisfies the AA-property [cf. Theorem 6.8, (i)]. We then apply the theory developed in the present paper to prove a semi-absolute version of the Grothendieck Conjecture for higher dimensional configuration spaces [of dimension $\geq 2$] associated to hyperbolic curves of genus 0 over TKND-AVKF-fields [cf. Theorem 6.10, (ii)].

Write $\mathbb{Q}^{ab} (\subseteq \overline{\mathbb{Q}})$ for the maximal abelian extension of $\mathbb{Q}$.

**Definition 6.1.** Let $p \in \text{Primes}$; $\Sigma \subseteq \text{Primes}$ a nonempty subset.

(i) Let $M$ be an abelian group. Then we shall say that $M$ is $p^\infty$-tor-finite if the subgroup of $p$-power torsion elements of $M$ is finite. We shall say that $M$ is $\Sigma^\infty$-tor-finite if, for each $l \in \Sigma$, $M$ is $l^\infty$-tor-finite.

(ii) Let $G$ be a profinite group. Then we shall say that $G$ is $p$-subfree if there exists a closed subgroup of $G$ isomorphic to $\mathbb{Z}_p$. We shall say that $G$ is $\Sigma$-subfree if, for each $l \in \Sigma$, $G$ is $l$-subfree. We shall say that $G$ is $p$-sparse if the maximal pro-$p$ quotient of every open subgroup of $G$ is finite. We shall say that $G$ is $\Sigma$-sparse if, for each $l \in \Sigma$, $G$ is $l$-sparse.

(iii) Let $K$ be a field. If $K$ satisfies the following condition, then we shall say that $K$ is an AVKF-field [i.e., “abelian variety Kummer–faithful field”]:

Let $A$ be an abelian variety over a finite extension $L$ of $K$. Write $A(L)^\infty$ for the group of divisible elements $\in A(L)$. Then $A(L)^\infty = \{1\}$.

If $K$ is an AVKF-field, then we shall also say that $K$ is AVK.

(iv) Let $K$ be a field. If $K$ satisfies the following condition, then we shall say that $K$ is $p$-AV-tor-indivisible (respectively, $p^\infty$-AV-tor-finite):

Let $A$ be an abelian variety over a finite extension $L$ of $K$. Write

- $A(L)^{p^\infty}$ for the group of $p$-divisible elements $\in A(L)$;
- $A(L)_\infty$ for the group of torsion elements $\in A(L)$;
- $A(L)_{p^\infty}$ for the group of $p$-power torsion elements $\in A(L)$.

Then $A(L)^{p^\infty} \subseteq A(L)^\infty$ (respectively, $A(L)_{p^\infty}$ is finite).

We shall say that $K$ is $\Sigma$-AV-tor-indivisible (respectively, $\Sigma^\infty$-AV-tor-finite) if, for each $l \in \Sigma$, $K$ is $l^\infty$-AV-tor-finite.

(v) Let $K$ be a field. Then we shall say that $K$ is stably $\Sigma$-$\mu$-indivisible (respectively, stably $\mu^\infty$-finite) if, for each $l \in \Sigma$, $K$ is stably $l$-$\mu$-indivisible (respectively, stably $\mu^\infty$-finite) [cf. [Tsjm], Definition 3.3, (v), (vii)].

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Remark 6.1.1. If a profinite group $G$ is $\Sigma$-subfree (respectively, $\Sigma$-sparse), then so is any open subgroup of $G$.

Remark 6.1.2. Let $\square$ be one of the following properties:

- AVKF,
- $\Sigma$-AV-tor-indivisible,
- $\Sigma^\infty$-AV-tor-finite,
- stably $\Sigma\times\mu$-indivisible,
- stably $\mu\Sigma^\infty$-finite.

Then one verifies immediately that if $L$ is an extension field of a field $K$, then the following implication holds:

$$L \text{ is } \square \Rightarrow K \text{ is } \square.$$  

Remark 6.1.3. In the notation of Definition 6.1, (iii), suppose further that $K$ is of characteristic $0$. Then it follows immediately from [AbsTopIII], Definition 1.5, that the following implication concerning $K$ holds:

$$\left(\text{torally Kummer-faithful and AVKF} \right) \iff \text{Kummer-faithful}.$$  

Lemma 6.2 (Stably $p\times\mu$-indivisible and $p$-AV-tor-indivisible fields). Let $p \in \mathbb{P}$, $K$ a field of characteristic $\neq p$. Then:

(i) Let $L$ be a \textit{not necessarily finite!} Galois extension of $K$ such that $\text{Gal}(L/K)$ is $p$-sparse. Let $\square$ be one of the following properties:

- stably $p\times\mu$-indivisible,
- stably $\mu_p^\infty$-finite,
- $p$-AV-tor-indivisible,
- $p^\infty$-AV-tor-finite.

Then if $K$ is $\square$, then so is $L$.

(ii) Let $L$ be a \textit{not necessarily finite!} Galois extension of $K$.

(ii$^*$) Suppose that $L$ is stably $\mu_p^\infty$-finite. Then if $K$ is stably $p\times\mu$-indivisible, then so is $L$.

(ii$^{AV}$) Suppose that $L$ is $p^\infty$-AV-tor-finite. Then if $K$ is $p$-AV-tor-indivisible, then so is $L$.

(iii) The following properties hold:
Suppose that $K$ is stably $p$-$\times$-$\mu$-indivisible, stably $\mu_{\varphi_{\text{Primes}}\infty}$-finite, and of characteristic 0. Then $K$ is torally Kummer-faithful. If, moreover, $K$ is AVKF, then $K$ is Kummer-faithful [cf. Remark 6.1.3].

(iii\textsuperscript{AV}) Suppose that $K$ is $p$-AV-tor-indivisible and $\varphi_{\text{Primes}}\infty$-AV-tor-finite. Then $K$ is AVKF.

(iv) The following properties hold:

(iii\textsuperscript{x}) If $K$ is torally Kummer-faithful, then $K$ is stably $\mu_{\varphi_{\text{Primes}}\infty}$-finite.

(iii\textsuperscript{AV}x) If $K$ is AVKF, then $K$ is $\varphi_{\text{Primes}}\infty$-AV-tor-finite.

(v) Suppose that $K$ is a $p$-adic field [cf. [LocAn], Definition 15.4, (i)]. Then $K$ is

- stably $p$-$\times$-$\mu$-indivisible,
- stably $\mu_{\varphi_{\text{Primes}}\infty}$-finite,
- $p$-AV-tor-indivisible,
- $\varphi_{\text{Primes}}\infty$-AV-tor-finite.

Proof. First, we consider assertion (i). We begin by observing that any finite extension field $L^\dagger$ of $L$ arises as a Galois extension of some finite extension field $K^\dagger$ of $K$ such that $\text{Gal}(L^\dagger/K^\dagger)$ is $p$-sparse. Next, we observe that the Galois group $\text{Gal}(M/K)$ of any [not necessarily finite] Galois extension $M$ of $K$ that arises by

- adjoining compatible systems of $p$-power roots of elements of $K$ or by
- adjoining infinitely many $p$-power roots of unity, 

admits an open subgroup which is a pro-$p$ group. Assertion (i) in the case where $\Box$ is taken to be one of the first two properties then follows immediately from the above observations, together with our assumption that $\text{Gal}(L/K)$ is $p$-sparse. Assertion (i) in the case where $\Box$ is taken to be one of the latter two properties follows by a similar argument. This completes the proof of assertion (i).

Assertion (ii\textsuperscript{x}) follows immediately from [Tsjm], Lemma D, (v). Next, we verify assertion (ii\textsuperscript{AV}). Let $L^\dagger$ be a finite extension field of $L$; $A^\dagger$ an abelian variety over $L^\dagger$. To verify assertion (ii\textsuperscript{AV}), it suffices to prove that $A^\dagger(L^\dagger)^{p\infty} \subseteq A^\dagger(L^\dagger)_\infty$. Let $x \in A^\dagger(L^\dagger)^{p\infty}$. By replacing $K$ by a finite extension field of $K$, we may assume without loss of generality that

- $L^\dagger = L$;
- $A^\dagger = A \times_K L$, where $A$ is an abelian variety over $K$;
- $x \in A(K)$.

Thus, since $K$ is $p$-AV-tor-indivisible, it suffices to verify the following assertion:

Claim 6.2.A: $x \in A(K)^{p\infty}$.
Indeed, let \( n \) be a positive integer. Since \( L \) is \( p^\infty \)-AV-tor-finite, \( A(L)_{p^\infty} \) is finite. Write \( p^n \) for the cardinality of \( A(L)_{p^\infty} \). Then since \( x \in A(L)_{p^\infty} \), there exists an element \( x_{m+n} \in A(L) \) such that \( p^{m+n} \cdot x_{m+n} = x \). Write \( x_n \overset{\text{def}}{=} p^m \cdot x_{m+n} \). Thus, since \( p^n \cdot x_n = x \), it suffices to prove that \( x_n \in A(K) \). Let \( \sigma \in \Gal(L/K) \). Observe that
\[
p^{m+n} \cdot ((x_{m+n})^\sigma - x_{m+n}) = x^\sigma - x = 0,
\]
hence, in particular, that \((x_{m+n})^\sigma - x_{m+n} \in A(L)_{p^\infty} \). Thus, we conclude that
\[
x_n^\sigma - x_n = p^m \cdot ((x_{m+n})^\sigma - x_{m+n}) = 0,
\]
hence that \( x_n \in A(K) \). This completes the proof of Claim 6.2.A, hence of assertion (ii AV).

Assertions (iii AV), (iii AV) follow immediately from the fact that, for any \( l \in \Primes \), the divisible group \( \Q_l / \Z_l \) has no nontrivial finite quotient.

Next, we verify assertion (iv). Recall that, for any \( l \in \Primes \), the group of \( l \)-torsion points of an abelian variety over an algebraically closed field is finite [cf. e.g., [Mumf], p. 64]. Thus, assertion (iv) follows immediately from the fact that, for any \( l \in \Primes \), every infinite subgroup of \( \Q_l / \Z_l \) is divisible.

Assertion (v) follows immediately from a similar argument to the argument applied in [AbsTopIII], Remark 1.5.4, (i). This completes the proof of Lemma 6.2.

Remark 6.2.1. The argument applied in the proof of Claim 6.2.A [in the proof of Lemma 6.2, (ii AV)] is similar to the argument applied in the proof of [Moon], Proposition 7.

Proposition 6.3 (Examples of AVKF-fields). Let \( F \subseteq \Qbar \) be a number field.

(i) Let \( L \) be a [not necessarily finite!] Galois extension of \( F \cdot \Qab \subseteq \Qbar \) such that \( \Gal(L/F) \cdot \Qab \) is \( \Primes \)-sparse. Then \( L \) is

- stably \( \Primes \times \mu \)-indivisible,
- \( \Primes \)-AV-tor-indivisible,
- \( \Primes^\infty \)-AV-tor-finite.

In particular, \( L \) is a stably \( \times \mu \)-indivisible AVKF-field [cf. Lemma 6.2, (iii AV); [Tsjm], Lemma D, (i)].

(ii) Let \( \{v_1, v_2, \ldots \} \) be an infinite set of non-archimedean primes of \( F \). [Here, we assume, for simplicity, that the indices of the “\( v_j \)” are chosen in such a way that \( v_j \neq v_j' \) for \( j \neq j' \).] Let \( \{\Sigma_j \subseteq \Primes\}_{j \geq 1} \) be a family of subsets such that, for any positive integer \( j \),

\[
\bigcup_{i \geq j} \Sigma_i = \Primes.
\]
where $i$ ranges over the positive integers $\geq j$; $M \subseteq \overline{\mathbb{Q}}$ a [not necessarily finite!] Galois extension of $F$; $L$ a [not necessarily finite!] Galois extension of $M \subseteq \overline{\mathbb{Q}}$ such that $\text{Gal}(L/M)$ is $\text{Primes}$-sparse. Suppose that for each positive integer $j$, the absolute Galois group of the residue field of the ring of integers of $M$ at [every prime that divides] $v_j$ is $\Sigma_j$-subfree. Then $L$ is

- stably $\text{Primes} \times \mu$-indivisible,
- stably $\mu_{\text{Primes}}$-$\infty$-finite,
- $\text{Primes}$-$\text{AV}$-tor-indivisible,
- $\text{Primes}^\infty$-$\text{AV}$-tor-finite.

In particular, $L$ is a Kummer-faithful field [cf. Lemma 6.2, (iii$^\ast$), (iii$^\ast\text{AV}$)].

**Proof.** First, we verify assertion (i). Note that it follows immediately from Lemma 6.2, (i), that we may assume without loss of generality that $L = F \cdot \mathbb{Q}^{ab}$. Then since $L$ is an abelian extension of a number field, it follows immediately from [Tshinj], Lemma D, (iii), (iv), that $L$ is stably $\text{Primes} \times \mu$-indivisible. On the other hand, it follows immediately from [KLR], Appendix, Theorem 1, that $L$ is $\text{Primes}^\infty$-$\text{AV}$-tor-finite. Next, observe that $F$ is $\text{Primes}$-$\text{AV}$-tor-indivisible [cf. Lemma 6.2, (v)]. Thus, since $L$ is a $\text{Primes}^\infty$-$\text{AV}$-tor-finite Galois extension of $F$, we conclude from Lemma 6.2, (ii$^\ast\text{AV}$), that $L$ is $\text{Primes}$-$\text{AV}$-tor-indivisible. This completes the proof of assertion (i).

Next, we verify assertion (ii). Note that it follows immediately from Lemma 6.2, (i), that we may assume without loss of generality that $L = M$. For each positive integer $j$, write $p_j$ for the residue characteristic of $v_j$. Then it follows immediately from our assumption on various unions of the subsets $\Sigma_i \subseteq \text{Primes}$ that, for any positive integer $j$, the morphism $A^\dagger |_{\mathbb{L}_y}$ given by multiplication by a power of $p_j$ is finite étale, it follows immediately that there exists a natural injection

$$A^\dagger(L^\dagger)_{p^\infty} \hookrightarrow A^\dagger_{\mathbb{L}_y}(k_{\mathbb{L}_y}^\dagger).$$

Thus, it follows immediately from
our assumption [cf. Remark 6.1.1] that the absolute Galois group of $k^1_{v_j}$ is
$\Sigma_j$-subfree,
the well-known fact that the absolute Galois group of a finite field is iso-
morphic to $\mathbb{Z}$, and
the well-known fact that, for any positive integer $n$, $GL_n(\mathbb{Z}_p)$ contains an
open subgroup which is a pro-$p$-group

that $A^1(L^1)p^\infty$ is finite. Thus, by allowing $p$ to vary, we conclude that $L$
is $\mathbb{Primes}^\infty$-tor-finite. A similar argument applied to the multiplicative
group $\mathbb{G}_m$ implies that $L$ is stably $\mu_{\mathbb{Primes}^\infty}$-finite. Next, observe that $L$
is a $\mathbb{Primes}^\infty$-AV-tor-indivisible Galois extension of the $\mathbb{Primes}$-AV-tor-
indivisible field $F$ [cf. Lemma 6.2, (v)]. Thus, we conclude from Lemma 6.2, (ii$^\mathbb{AV}$), that $L$
is $\mathbb{Primes}$-AV-tor-indivisible. A similar argument implies that $L$ is stably $\mathbb{Primes}$-
$\times\mu$-indivisible. This completes the proof of assertion (ii), hence of Proposition
6.3.

Remark 6.3.1. The following example was suggested to the authors of the
present paper by A. Tamagawa. Let $\{G_i\}_{i \in I}$ be a family of nonabelian finite
simple groups [i.e., such as the alternating group on $n$ letters $\mathfrak{A}_n$, where $n \geq 5$].
Then the direct product group

$$G \overset{\text{def}}{=} \prod_{i \in I} G_i$$

endowed with the product topology is $\mathbb{Primes}$-sparse. Indeed, this follows imme-
diately from the definition of the product topology, together with the elementary
fact that, for each $p \in \mathbb{Primes}$, $i \in I$, the maximal pro-$p$ quotient of $G_i$ is trivial.
If $I$ is countable, and we assume that $G_i$ and $G_j$ are non-isomorphic whenever
$I \ni i \neq j \in I$, then it follows immediately from the well-known fact that number
fields are Hilbertian [cf. [FJ], §6.2; [FJ], Theorem 13.4.2] that $G$ may be realized
as the Galois group of a Galois extension $E$ of a number field $F$. Here, we note
that such a Galois extension $E$ of $F$ is necessarily linearly disjoint from any
abelian field extension of $F$.

Remark 6.3.2. Later [cf. Remark 6.6.3 below], we shall see that the fields “$L$”
of Proposition 6.3, (i), (ii), are in fact “TKND-AVKF-fields”.

Remark 6.3.3. Let $F \subseteq \overline{\mathbb{Q}}$ be a number field such that $\sqrt{-1} \in F$; \{v_1, v_2, \ldots\}
an infinite set of non-archimedean primes of $F$. [Here, we assume, for simplicity,
that the indices of the “$v_j$” are chosen in such a way that $v_i \neq v_j$, for $j \neq j$.]
Let $\{\Sigma_i \subseteq \mathbb{Primes}\}_{i \geq 1}$ be a family of finite subsets such that, for any positive
integer $j$,

$$\bigcup_{i \geq j} \Sigma_i = \mathbb{Primes},$$

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where \( i \) ranges over the positive integers \( \geq j \). For each positive integer \( j \), write \( \mathbb{Q} \setminus \Sigma_j = \{p_j, m\}_{m \geq 1} \): \( F_{v_j} \) for the completion of \( F \) at \( v_j \). For each pair of positive integers \( i, j \) such that \( j \leq i \), write \( F_{v_j}^i[i] \) for the \textit{finite unramified [abelian] extension} of \( F_{v_j} \) of degree \( \prod_{1 \leq m \leq i} p_j^m \).

For each positive integer \( j \), let \( F_{v_j}^i \) be an \textit{abelian totally wildly ramified infinite extension} of \( F_{v_j} \). For each pair of positive integers \( i, j \) such that \( j \leq i \), let \( F_{v_j}^i[i] \subseteq F_{v_j}^i \) be a \textit{finite subextension} of \( F_{v_j} \) such that

\[
F_{v_j}^i[i] \subseteq F_{v_j}^i[i + 1], \quad \bigcup_{j \leq m} F_{v_j}^i[m] = F_{v_j}^i,
\]

where \( m \) ranges over the positive integers \( \geq j \). [Here, we observe that the \textit{existence} of such extensions of \( F_{v_j} \) follows immediately from [Neu], Chapter II, Proposition 10.2; [Neu], Chapter V, Theorems 1.3, 6.2.] Next, let \( i \) be a positive integer; \( M_i \) an \textit{abelian extension} of \( F \) such that, for each pair of positive integers \( i, j \) such that \( j \leq i \), the local extensions of \( M_i/F \) at \( v_j \) coincide with the extension \( F_{v_j}^i[i] \cdot F_{v_j}^i[i]/F_{v_j} \). [Here, we observe that, in light of our assumption that \( \sqrt{-1} \in F \), the \textit{existence} of such an abelian extension \( M_i \) of \( F \) follows immediately from [NSW], Definitions 9.1.5, 9.1.7; [NSW], Theorem 9.2.8.] Write

\[
M \subseteq \mathbb{Q}
\]

for the field generated by \( \{M_i\}_{i \geq 1} \) over \( F \). Then we make the following \textit{observations}, each of which follows immediately from the construction of \( M \):

(a) \( M \) is an \textit{abelian extension} of \( F \);

(b) for each positive integer \( j \), the \textit{absolute Galois group} of the residue field of the ring of integers of \( M \) at [every prime that divides] \( v_j \) is \( \Sigma_j \)-subfree;

(c) for each positive integer \( j \), the \textit{ramification index} of the extension \( M/F \) at \( v_j \) is \textit{infinite} [so if \( \{v_1, v_2, \ldots\} \) coincides with the set of all non-archimedean primes of \( F \), then \( M \) is \textit{not} a \textit{generalized sub-p-adic field} for any prime number \( p \) — cf. [AnabTop], Definition 4.11];

(d) for each positive integer \( j \), the \textit{residue field} of the ring of integers of \( M \) at [every prime that divides] \( v_j \) is \textit{infinite}.

Thus, in particular, any Galois extension \( L \) of \( M \) whose Galois group is \( \mathbb{Q} \)-sparse — such as, for instance, a composite field \( L \overset{\text{def}}{=} M \cdot E \), where \( E \) is as in Remark 6.3.1 — satisfies the assumptions of Proposition 6.3, (ii), as well as the properties discussed in (c), (d).
Remark 6.3.4. Note that it follows immediately from the various definitions involved that the field “$L$” of Proposition 6.3, (i), satisfies properties analogous to the properties (c), (d) of Remark 6.3.3. That is to say, in the notation of Proposition 6.3, (i),

- the ramification index of the extension $L/F$ at every non-archimedean prime of $L$ is infinite [so L is not a generalized sub-$p$-adic field for any prime number $p$ — cf. [AnabTop], Definition 4.11];
- the residue field of the ring of integers of $L$ at every non-archimedean prime of $L$ is algebraically closed, hence infinite.

Remark 6.3.5. The properties (c), (d) of Remark 6.3.3 [cf. also Remark 6.3.4] are of interest in that they show that

anabelian geometry over fields such as the fields $L$ of Proposition 6.3, (i), (ii) [cf. Theorem 6.10 below] cannot be treated by means of well-known techniques of anabelian geometry that require the use of $p$-adic Hodge theory or Frobenius elements of absolute Galois groups of finite fields [cf. [Tama], Theorem 0.4; [LocAn], Theorem A; [AnabTop], Theorem 4.12].

Corollary 6.4 (AVKF-fields satisfy the CS-property). Let $K \subseteq \overline{\mathbb{Q}}$ be an AVKF-field [cf. Definition 6.1, (iii)]. Write $G_K \overset{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/K) \subseteq G_\mathbb{Q} \overset{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus, we obtain natural injections

$$G_K \subseteq G_\mathbb{Q} \hookrightarrow \text{GT} \subseteq \text{Out}(\Pi_X)$$

[cf. the discussion of the beginning of [Tsjm], Introduction], which we use to identify $G_K$ with its image in $\text{GT}$. Then the closed subgroup $G_K \subseteq \text{GT}$ satisfies the CS-property.

Proof. Indeed, it follows immediately from a similar argument to the argument applied in the proof of [Tsjm], Theorem 3.1, and [Tsjm], Corollary 3.2, that

(ISC) the injectivity portion of the Section Conjecture for arbitrary hyperbolic curves over AVKF-fields holds.

The CS-property for the closed subgroup $G_K \subseteq \text{GT}$ then follows formally from this property (ISC). This completes the proof of Corollary 6.4.

Corollary 6.5 (AVKF-fields satisfy the BC-property). In the notation of Corollary 6.4, the closed subgroup $G_K \subseteq \text{GT}$ satisfies the BC-property. Moreover, if one takes “BGT” to be $G_K$ [cf. Definition 3.3, (v)], then the following hold:
(i) In the notation of Theorem 4.4, there exists a natural isomorphism of fields

$$\mathbb{K}_{G_K} \xrightarrow{\sim} \mathbb{Q}$$

that is compatible with the inclusion $G_K \subseteq G_\mathbb{Q}$. In the remainder of the present §6, we shall use this natural isomorphism to identify $\mathbb{K}_{G_K}$ with $\mathbb{Q}$.

(ii) In the notation of Definition 5.1, there exists a natural outer isomorphism

$$\Pi \xrightarrow{\sim} G_{K_X}$$

between the profinite group $\Pi$ and the absolute Galois group $G_{K_X}$ of the function field $K_X$ of $X \overset{\text{def}}{=} \mathbb{P}^1 \setminus \{0, 1, \infty\}$. This natural outer isomorphism is compatible with the respective natural outer actions of $\text{BGT} = G_K$ on $\Pi$ and $G_{K_X}$.

(iii) There exists a natural homomorphism

$$C_{\text{GT}}(G_K) \rightarrow G_\mathbb{Q}$$

whose restriction to $G_K$ is the natural inclusion $G_K \subseteq G_\mathbb{Q}$.

Proof. First, we observe that it follows immediately — from the evident scheme-theoretic interpretation of the various arithmetic Belyi diagrams that arise — that the closed subgroup $G_K \subseteq \text{GT}$ satisfies the COF-property. Thus, it follows immediately from Corollaries 3.7, 6.4, together with the various definitions involved, that the closed subgroup $G_K \subseteq \text{GT}$ satisfies the BC-property. Next, we observe that it follows immediately — from the evident scheme-theoretic interpretation of the various arithmetic Belyi diagrams that arise — that these arithmetic Belyi diagrams determine

- a natural isomorphism of fields $\mathbb{K}_{G_K} \xrightarrow{\sim} \mathbb{Q}$ that is compatible with the inclusion $G_K \subseteq G_\mathbb{Q}$, and
- a natural outer isomorphism $\Pi \xrightarrow{\sim} G_{K_X}$ that is compatible with the respective natural outer actions of $\text{BGT} = G_K$ on $\Pi$ and $G_{K_X}$

[cf. Claim 4.4.F in the proof of Theorem 4.4]. Thus, we conclude [cf. the proof of Theorem 4.4] that there exists a natural homomorphism

$$C_{\text{GT}}(G_K) \rightarrow G_\mathbb{Q}$$

whose restriction to $G_K$ is the natural inclusion $G_K \subseteq G_\mathbb{Q}$. This completes the proof of Corollary 6.5. \qed

Definition 6.6. Let $K$ be a field, $\overline{K}$ an algebraic closure of $K$. Write $K_{\text{prim}} \subseteq K$ for the prime field of $K$. 

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(i) Write

\[ K_{\text{div}} \overset{\text{def}}{=} \bigcup_{L/K} L_{x,\infty} \subseteq \overline{K}, \]

where \( L (\subseteq \overline{K}) \) ranges over the finite extensions of \( K \), and we write

\[ L_{x,\infty} \overset{\text{def}}{=} K_{\text{prim}}(L^{\times,\infty}) \subseteq L. \]

(ii) If \( K_{\text{div}} \subseteq \overline{K} \) is an infinite field extension, then we shall say that \( K \) is a TKND-field [i.e., “torally Kummer-nondegenerate field”]. If \( K \) is a TKND-field, then we shall say that \( K \) is TKND.

(iii) If \( K \subseteq \overline{K} \) is both TKND and AVKF, then we shall say that \( K \) is a TKND-AVKF-field. If \( K \) is a TKND-AVKF-field, then we shall say that \( K \) is TKND-AVKF.

**Remark 6.6.1.** One verifies immediately that if \( L \) is an algebraic extension of a field \( K \), then the following implication holds:

\[ L \text{ is TKND} \Rightarrow K \text{ is TKND}. \]

**Remark 6.6.2.** In the notation of Definition 6.6, suppose further that \( K \) is of characteristic 0. Then the following implications concerning \( K \) hold [cf. Definition 6.1, (iii); [AbsTopIII], Definition 1.5; [Tsjm], Definition 3.3, (v); the well-known fact that \( \mathbb{Q}^{ab} \subseteq \overline{\mathbb{Q}} \) is an infinite field extension]:

- torally Kummer-faithful \( \Rightarrow \) stably \( \times \mu \)-indivisible \( \Rightarrow \) TKND;
- Kummer-faithful \( \Rightarrow \) stably \( \times \mu \)-indivisible and AVKF \( \Rightarrow \) TKND-AVKF.

**Remark 6.6.3.** It follows immediately from Remark 6.6.2 that the fields “\( L \)” of Proposition 6.3, (i), (ii), are TKND-AVKF-fields.

**Remark 6.6.4.** Recall that

- the TKND-field “\( L \)” of Proposition 6.3, (i) [cf. Remark 6.6.3], contains the entire subset \( \mu(\overline{\mathbb{Q}}) \), while
- the TKND-field “\( L \)” of Proposition 6.3, (ii) [cf. Remark 6.6.3], is stably \( \mu_{\text{prim}}^{\infty \text{-finite}} \).

That is to say, the TKND-fields of Proposition 6.3, (i), (ii), may be thought of as two “extremal cases”, i.e., with regard to the property of containing roots of unity. On the other hand, a detailed analysis of the various “intermediate cases” that, in some sense, lie in between these two “extremal cases” is beyond the scope of the present paper.
Lemma 6.7 (Generalities on rational functions). Let $K$ be a field of characteristic 0; $\overline{K}$ an algebraic closure of $K$; $Y$ a smooth curve over $K$. For each algebraic extension $M (\subseteq K)$ of $K$, write $\text{Gal}(K/M)$ for the group of $M$-rational points of $Y$; $O^\times_{Y_M}$ for the group of invertible regular functions on $Y_M$; $\Pi^1_{Y_K}$ for the set of $K$-rational points of $Y$; $O^\times_{Y_K}$ for the group of invertible regular functions on $Y_K$; $\text{Kummer}(K) = \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_2(K))$. Let $K$ be a field of characteristic 0; $\overline{K}$ an algebraic closure of $K$; $Y$ a smooth curve over $K$. For each algebraic extension $M (\subseteq K)$ of $K$, write $\text{Gal}(K/M)$ for the group of $M$-rational points of $Y$; $O^\times_{Y_M}$ for the group of invertible regular functions on $Y_M$.

$$\kappa_Y : O^\times_{Y_K} = \lim_{K \subseteq K^1} O^\times_{Y_{K^1}} \rightarrow \lim_{K \subseteq K^1} H^1(\Pi^1_{Y_{K^1}}, \mu_2(\overline{K}))$$

for the Kummer map, where $\mu_2(\overline{K}) = \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mu(\overline{K}))$; $K^1 (\subseteq K)$ ranges over the finite extensions of $K$. Let $y \in Y(K^1)$, where $K^1 (\subseteq K)$ is a finite extension of $K$. Thus, $y \in Y(K^1)$ determines a section $G_{K^1} \rightarrow \Pi^1_{Y_{K^1}}$ [i.e., strictly speaking, an outer homomorphism] of the natural surjection $\Pi^1_{Y_{K^1}} \rightarrow G_{K^1}$. In particular, by allowing $K^1$ and $y \in Y(K^1)$ to vary, we obtain a natural homomorphism

$$D_Y : \lim_{K \subseteq K^1} H^1(\Pi^1_{Y_{K^1}}, \mu_2(\overline{K})) \rightarrow \text{Fn}(Y(\overline{K})), \lim_{K \subseteq K^1} H^1(G_{K^1}, \mu_2(\overline{K}))).$$

Then the following hold:

(i) Suppose that $K$ is AVKF, and $Y$ is proper over $K$. Then

$$H^1(\Pi^1_{Y_{K^1}}, \mu_2(\overline{K}))^G_K = \{0\}.$$
On the other hand, Poincaré duality yields a $G_K$-equivariant isomorphism of topological modules

$$H^1(\Pi_{Y,K}, \mu_\mathbb{Z}(\mathbb{K})) = \text{Hom}(\Pi_{Y,K}^b, \mu_\mathbb{Z}(\mathbb{K})) \xrightarrow{\sim} \Pi_{Y,K}^b.$$  

Thus, we conclude that $H^1(\Pi_{Y,K}, \mu_\mathbb{Z}(\mathbb{K}))^{G_K} = \{0\}$. This completes the proof of assertion (i). Assertion (ii) follows immediately from the various definitions involved [cf. the argument applied in the proof of [Tsjm], Theorem 3.1].

Finally, we verify assertion (iii). First, we observe that it follows from the various definitions involved that there exists a commutative diagram

$$\begin{array}{ccc}
O_{Y,K}^\times 
\downarrow \kappa_Y & \xrightarrow{ev_Y} & \text{Fn}(Y(\mathbb{K}), K^\times) \\
\lim_{K \subseteq K^1} H^1(\Pi_{Y,K}, \mu_\mathbb{Z}(\mathbb{K})) 
\downarrow & & \downarrow \\
\lim_{K \subseteq K^1} H^1(G_{K^1}, \mu_\mathbb{Z}(\mathbb{K})), 
\end{array}$$

where $ev_Y$ denotes the homomorphism induced by evaluating elements of $O_{Y,K}^\times$ at elements of $Y(\mathbb{K})$; the right-hand vertical arrow denotes the natural homomorphism induced by the Kummer map

$$\mathbb{K}^\times = \lim_{K \subseteq K^1} (K^1)^\times \longrightarrow \lim_{K \subseteq K^1} H^1(G_{K^1}, \mu_\mathbb{Z}(\mathbb{K})).$$

Let $f \in \text{Ker}(D_Y \circ \kappa_Y)$. Then the commutativity of the above diagram implies that $\text{Im}(ev_Y(f)) \subseteq K^\times_{\text{div}} \subseteq \mathbb{K}^\times$. On the other hand, we note that, for any nonconstant rational function $g \in O_{Y,K}^\times$, the complement $\mathbb{K}^\times \setminus \text{Im}(ev_Y(g))$ is a finite set. In particular, it follows immediately from our assumption that $K$ is TKND [i.e., the fact that $K_{\text{div}} \subseteq \mathbb{K}$ is an infinite field extension] that $f$ is a constant function such that $\kappa_Y(f) = 0$. Thus, we conclude that $D_Y\mid_{\text{Im}(\kappa_Y)}$ is injective. This completes the proof of assertion (iii), hence of Lemma 6.7. □

**Theorem 6.8 (TKND-AVKF-fields satisfy the AA-property).** Let $K \subseteq \mathbb{Q}$ be a TKND-AVKF-field. Then the following hold:

(i) The closed subgroup $G_K \subseteq G_T$ satisfies the **AA-property** [cf. Definition 5.12].

(ii) The natural homomorphism

$$C_{G_T}(G_K) \rightarrow G_{\mathbb{Q}}$$

[cf. Corollary 6.5, (iii)] is injective.

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Proof. First, we verify assertion (i). Since $K$ is AVKF, it follows from Corollary 6.5 that the closed subgroup $G_K \subseteq \text{GT}$ satisfies the $BC$-property. Next, since $K$ is TKND, it follows immediately from the various definitions involved that the closed subgroup $G_K \subseteq \text{GT}$ satisfies condition (i) of Definition 5.12. Moreover, since $K$ is TKND-AVKF, it follows immediately from Lemma 6.7, (i), (iii), together with the various definitions involved, that the closed subgroup $G_K \subseteq \text{GT}$ satisfies condition (ii) of Definition 5.12. On the other hand, since $K$ is AVKF, it follows immediately from Lemma 6.7, (ii), together with the various definitions involved, that the function fields of finite ramified Galois coverings of $\mathbb{P}^1_\mathbb{Q}$ [i.e., the projective line over $\mathbb{Q}$] determine a family

\[ \{K_n \subseteq L_n\}_{n \in \mathbb{N}} \]

of subsets as in Definition 5.12, (iii). Finally, it follows immediately from the various definitions involved that condition (iv) of Definition 5.12 holds. Thus, we conclude that the closed subgroup $G_K \subseteq \text{GT}$ satisfies the AA-property. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i), together with Theorem 5.15, (iii). This completes the proof of Theorem 6.8.

Remark 6.8.1. Theorem 6.8, (i), may be regarded as a generalization of Remark 5.12.1 [cf. Remark 6.6.2]. In this context, we observe that the proof of Theorem 6.8, (i), (ii), can be simplified considerably in the case where $K$ is assumed to be Kummer-faithful, in which case one may combine the techniques of [AbsTopIII], Theorem 1.11, or [Hsh1], Theorem A, with the combinatorial approach to Belyi cuspidalizations developed in §3 of the present paper.

Corollary 6.9 (Semi-absolute Grothendieck Conjecture-type result over TKND-AVKF-fields for tripods). Let $n$ be an integer $\geq 2$; $K, L \subseteq \overline{\mathbb{Q}}$ TKND-AVKF-fields. Write $X_K \overset{\text{def}}{=} \mathbb{P}^1_K \setminus \{0, 1, \infty\}$; $X_L \overset{\text{def}}{=} \mathbb{P}^1_L \setminus \{0, 1, \infty\}$; $(X_K)_n$ (respectively, $(X_L)_n$) for the $n$-th configuration space associated to $X_K$ (respectively, $X_L$); $G_K \overset{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/K)$ (respectively, $G_L \overset{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/L)$);

\[ \text{Out}(\Pi((X_K)_n)/G_K, \Pi((X_L)_n)/G_L) \]

for the set of outer isomorphisms $\Pi((X_K)_n) \overset{\sim}{\rightarrow} \Pi((X_L)_n)$ that induce outer isomorphisms $G_K \overset{\sim}{\rightarrow} G_L$. Then the natural map

\[ \text{Isom}((X_K)_n, (X_L)_n) \rightarrow \text{Out}(\Pi((X_K)_n)/G_K, \Pi((X_L)_n)/G_L) \]

is bijective.

Proof. Write $X \overset{\text{def}}{=} \mathbb{P}^1 \setminus \{0, 1, \infty\}$; $X_n$ for the $n$-th configuration space associated to $X$. Let $\sigma \in \text{Out}(\Pi((X_K)_n)/G_K, \Pi((X_L)_n)/G_L)$;

\[ \tilde{\sigma} : \Pi((X_K)_n) \overset{\sim}{\rightarrow} \Pi((X_L)_n) \]

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an isomorphism that lifts $\sigma$. Write $\sigma_{\text{Gal}} \in \text{Out}(\Pi_{X_n})$ for the outer automorphism determined by the restriction of $\bar{\sigma}$ to $\Pi_{X_n}$; $\bar{\sigma}_{\text{Gal}} : G_K \overset{\sim}{\to} G_L$ for the isomorphism induced by the isomorphism $\bar{\sigma}$. Thus, it follows immediately from the various definitions involved that there exists a commutative diagram

$$
\begin{array}{c}
G_K \\ \downarrow \sigma_{\text{Gal}} \downarrow i \\
G_L \\ \downarrow i \\
\text{Out}(\Pi_{X_n}),
\end{array}
$$

where the horizontal arrows denote the natural outer representations; the right-hand vertical arrow denotes the automorphism $i_{\sigma_{\text{Gal}}}$ obtained by conjugating by $\sigma_{\text{Gal}}$. Next, we verify the following assertion:

Claim 6.9.A: The isomorphism $\bar{\sigma}_{\text{Gal}}$ arises from an isomorphism $\overline{Q} \overset{\sim}{\to} \overline{Q}$ that maps $K \subseteq \overline{Q}$ onto $L \subseteq \overline{Q}$.

Indeed, [cf. the above commutative diagram] since the closed subgroups $G_K \subseteq GT$ and $G_L \subseteq GT$ satisfy the BC-property [cf. Corollary 6.5], the functorial constructions of Corollary 4.5, together with the isomorphism of Corollary 6.5, (i) [applied to $G_K$ and $G_L$], determine a commutative diagram

$$
\begin{array}{c}
G_K \\ \sim \\
\overline{Q} \\ \sim \\
\text{Out}(\Pi_{X_n}) = G_K
\end{array}
\begin{array}{c}
\sim \\
\sim \\
\sim \\
\sim \\
\text{Out}(\Pi_{X_n}) \sim G_L \\
\text{Out}(\Pi_{X_n}) = G_L
\end{array}
$$

where the lower horizontal arrows are isomorphisms of fields. Thus, we obtain the desired conclusion. This completes the proof of Claim 6.9.A.

Now it follows from Claim 6.9.A that we may assume without loss of generality that $K \simeq L \subseteq \overline{Q}$. Next, it follows from Theorem 6.8, (ii), together with the various definitions involved, that

$$
N_{\text{GT}}(G_K) \subseteq C_{\text{GT}}(G_K) \subseteq \overline{Q}.
$$

In particular, we conclude that $N_{\text{GT}}(G_K)/G_K = N_{G_Q}(G_K)/G_K$. Note that since $\Pi_{X_n}$ is center-free [cf. [MT], Proposition 2.2, (ii)], there exists [cf. the above commutative diagram] a natural isomorphism

$$
\text{Out}(\Pi_{X_n})/G_K \overset{\sim}{\to} N_{\text{Out}(\Pi_{X_n})}(G_K)/G_K,
$$

where $\text{Out}(\Pi_{X_n})/G_K$ denotes the set of outer automorphisms of $\Pi_{X_n}$ that induce outer automorphisms of $G_K$. In particular, $\sigma \in \text{Out}(\Pi_{X_n})/G_K$ determines an element of

$$
N_{\text{Out}(\Pi_{X_n})}(G_K)/G_K = N_{\text{GT} \times S_{n+3}}(G_K)/G_K
= (N_{\text{GT}}(G_K)/G_K) \times S_{n+3}
$$

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[cf. the first display of [HMM], Corollary C]. Thus, in light of the natural isomorphism

$$\text{Aut}(K) \cong N_{G_K}(G_K)/G_K = N_{GT}(G_K)/G_K,$$

we conclude that the natural group homomorphism

$$\text{Aut}((X_K)_n) \to \text{Out}(\Pi((X_K)_n)/G_K)$$

is surjective, and [by considering the various fiber subgroups of $\Pi_X$ and cuspidal inertia subgroups of $\Pi_X$] that any element $\alpha \in \text{Aut}((X_K)_n)$ in the kernel of this group homomorphism is $K$-linear and compatible with the identity automorphism of $X_K$ relative to any of the $n + 3$ generalized projection morphisms $(X_K)_n \to X_K$ [cf. [HMM], Definition 2.1, (i)]. But this implies that any such $\alpha$ is equal to the identity automorphism of $(X_K)_n$. This completes the proof of Corollary 6.9.

Theorem 6.10 (Semi-absolute Grothendieck Conjecture-type result over TKND-AVKF-fields for arbitrary hyperbolic curves). Let $(m, n)$ be a pair of positive integers; $K, L \subseteq \mathbb{Q}$ TKND-AVKF-fields; $X_K$ (respectively, $Y_L$) a hyperbolic curve over $K$ (respectively, $L$). Write $g_X$ (respectively, $g_Y$) for the genus of $X_K$ (respectively, $Y_L$); $(X_K)_m$ (respectively, $(Y_L)_n$) for the $m$-th (respectively, $n$-th) configuration space associated to $X_K$ (respectively, $Y_L$); $G_K \overset{\text{def}}{=} \text{Gal}(\mathbb{Q}/K)$ (respectively, $G_L \overset{\text{def}}{=} \text{Gal}(\mathbb{Q}/L)$); $\text{Out}(\Pi((X_K)_m)/G_K, \Pi((Y_L)_n)/G_L)$

for the set of outer isomorphisms $\Pi((X_K)_m) \overset{\sim}{\to} \Pi((Y_L)_n)$ that induce outer isomorphisms between $G_K$ and $G_L$. Then the following hold:

(i) Suppose that

- $m \geq 4$ or $n \geq 4$ if $X$ or $Y$ is proper;
- $m \geq 3$ or $n \geq 3$ if $X$ or $Y$ is affine.

Then the outer isomorphism

$$G_K \overset{\sim}{\to} G_L$$

induced by any outer isomorphism $\in \text{Out}(\Pi((X_K)_m)/G_K, \Pi((Y_L)_n)/G_L)$ arises from a field isomorphism $K \overset{\sim}{\to} L$.

(ii) Suppose that

- $m \geq 2$ or $n \geq 2$;
- $g_X = 0$ or $g_Y = 0$. 

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Then the natural map

\[ \text{Isom}( (X_K)_m, (Y_L)_n ) \to \text{Out}( \Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L ) \]

is bijective.

Proof. Write

- \( Z_K \stackrel{\text{def}}{=} \mathbb{P}_K \setminus \{0, 1, \infty\}; \) \( Z_L \stackrel{\text{def}}{=} \mathbb{P}_L \setminus \{0, 1, \infty\}; \)
- \( X \stackrel{\text{def}}{=} X_K \times_K \mathbb{Q}_K; \) \( Y \stackrel{\text{def}}{=} Y_L \times_L \mathbb{Q}_L; \) \( Z \stackrel{\text{def}}{=} Z_K \times_K \mathbb{Q}_K = Z_L \times_L \mathbb{Q}_L; \)
- \( r_X \) (respectively, \( r_Y \)) for the cardinality of the set of cusps of \( X \) (respectively, \( Y \)).

For each positive integer \( i \), write

- \( X_i \) (respectively, \( Y_i, Z_i \)) for the \( i \)-th configuration space associated to \( X \) (respectively, \( Y, Z \)).

Note that, to verify assertions (i), (ii), it follows immediately from the various definitions involved that we may assume without loss of generality that

\[ \text{Out}( \Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L ) \neq \emptyset. \]

Thus, we conclude from [HMM], Theorem A, (i), that

\[ m = n \geq 2, \quad g_X = g_Y, \quad r_X = r_Y. \]

Let \( \sigma \in \text{Out}( \Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L ); \)

\[ \tilde{\sigma} : \Pi_{(X_K)_m} \xrightarrow{\sim} \Pi_{(Y_L)_n} \]

an isomorphism that lifts \( \sigma \). Write \( \sigma_{\mathbb{Q}} : \Pi_{X_n} \xrightarrow{\sim} \Pi_{Y_n} \) for the outer isomorphism determined by the restriction of \( \tilde{\sigma} \) to \( \Pi_{X_n}; \) \( \tilde{\sigma}_{\text{Gal}} : G_K \xrightarrow{\sim} G_L \) for the isomorphism induced by the isomorphism \( \tilde{\sigma} \).

Next, we verify assertion (i). Note that \( m = n \geq 3 \). Let \( \Pi_{X_3}^{\text{ctpd}} \subseteq \Pi_{X_3} \) (respectively, \( \Pi_{Y_3}^{\text{ctpd}} \subseteq \Pi_{Y_3} \)) be a 3-central \( \{1, 2, 3\} \)-tripod of \( \Pi_{X_3} \) (respectively, \( \Pi_{Y_3} \)) [cf. [CbTpII], Definition 3.7, (ii)]. Then since \( m = n, \) \( g_X = g_Y, \) and \( r_X = r_Y \), it follows immediately from [HMM], Theorem B; [CbTpII], Theorem A, (ii); [CbTpII], Theorem C, (ii); the discussion of [CbTpII], Remark 4.14.1, that we may assume without loss of generality that

- \( \sigma_{\mathbb{Q}} \) induces bijections between the respective sets of fiber subgroups and inertia subgroups;
- the outer isomorphism \( \Pi_{X_3} \xrightarrow{\sim} \Pi_{Y_3} \) induced by \( \sigma_{\mathbb{Q}} \) determines an outer isomorphism \( \sigma_{\text{ctpd}} : \Pi_{X_3}^{\text{ctpd}} \xrightarrow{\sim} \Pi_{Y_3}^{\text{ctpd}}. \)
there exists a commutative diagram of profinite groups

\[
\begin{array}{c}
\text{Out}^{\text{FC}}(\Pi_{X_n}) \xrightarrow{T_X} \text{Out}(\Pi_{X}^{\text{ctpd}}) \\
\downarrow i \quad \downarrow i \\
\text{Out}^{\text{FC}}(\Pi_{Y_n}) \xrightarrow{T_Y} \text{Out}(\Pi_{Y}^{\text{ctpd}}),
\end{array}
\]

where the vertical arrows denote the isomorphisms induced by the outer isomorphisms \(\sigma_Z\) and \(\sigma_{\text{ctpd}}\), and \(T_X\) and \(T_Y\) denote the respective tripod homomorphisms.

Here, we identify \(\Pi_{Z}\) with \(\Pi_{X}^{\text{ctpd}}, \Pi_{Y}^{\text{ctpd}}\), via outer isomorphisms \(\Pi_{Z} \xrightarrow{\sim} \Pi_{X}^{\text{ctpd}}, \Pi_{Z} \xrightarrow{\sim} \Pi_{Y}^{\text{ctpd}}\) that arise from the respective \(\mathfrak{S}_3\)-torsors of scheme-theoretic isomorphisms of tripods over \(\mathbb{Q}\) in such a way that

- the outer automorphism \(\sigma_Z : \Pi_{Z} \xrightarrow{\sim} \Pi_{X}^{\text{ctpd}} \xrightarrow{\sim} \Pi_{Y}^{\text{ctpd}} \xrightarrow{\sim} \Pi_{Z}\) obtained by conjugating \(\sigma_{\text{ctpd}}\) by these identifying outer isomorphisms determines an element \(\in \text{GT} \subseteq \text{Out}(\Pi_{Z})\)

[cf. [CbTpII], Theorem C, (iv), together with our assumptions on \(m = n\)]. Moreover, it follows immediately [again from [CbTpII], Theorem C, (iv), together with our assumptions on \(m = n\)] that

- the images of \(T_X\) and \(T_Y\) are contained in \(\text{GT} \subseteq \text{Out}(\Pi_{Z})\).

In particular, the above commutative diagram, together with the natural outer representations \(G_K \rightarrow \text{Out}^{\text{FC}}(\Pi_{X_n}), G_L \rightarrow \text{Out}^{\text{FC}}(\Pi_{Y_n})\), determines a commutative diagram of profinite groups

\[
\begin{array}{c}
G_K \longrightarrow \text{GT} \\
\downarrow \sigma_{\text{Gal}} \quad \downarrow \sigma_Z \\
G_L \longrightarrow \text{GT},
\end{array}
\]

where the right-hand vertical arrow denotes the inner automorphism obtained by conjugating by \(\sigma_Z\); the horizontal arrows denote the natural injections. Observe that since \(\Pi_{Z}\) is center-free [cf. [MT], Proposition 2.2, (ii)], this last commutative diagram determines an outer isomorphism \(\Pi_{(Z_{K})_2} \xrightarrow{\sim} \Pi_{(Z_{L})_2}\) that lies over \(\sigma_{\text{Gal}}\) between the second configuration spaces \((Z_K)_2, (Z_L)_2\) associated to \(Z_K, Z_L\), respectively. Thus, we conclude from Corollary 6.9 that the outer isomorphism determined by \(\sigma_{\text{Gal}} : G_K \xrightarrow{\sim} G_L\) arises from a field isomorphism \(K \xrightarrow{\sim} L\). This completes the proof of assertion (i).

Next, we verify assertion (ii). First, it follows from a similar argument to the argument applied in the final portion of the proof of Corollary 6.9 that the natural map

\[
\text{Isom}((X_K)_n, (Y_L)_n) \longrightarrow \text{Out}(\Pi_{(X_K)_n}/G_K, \Pi_{(Y_L)_n}/G_L)
\]

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is injective. Thus, it suffices to prove that this map is surjective. Observe that since \( \Pi_{\{Y\}} \) is slim [cf. [MT], Proposition 2.2, (ii)], it follows immediately from [NodNon], Theorem C, (ii), that \( \Pi_{\{Y\}} \) is slim. In particular, it follows immediately, by applying Galois descent, that we may assume without loss of generality that every cusp of \( X \) (respectively, \( Y \)) is \( K \)-rational (respectively, \( L \)-rational). On the other hand, since \( g_X = g_Y = 0 \), it suffices to consider the case where \( r_X = r_Y \geq 4 \) [cf. Corollary 6.9].

Next, we verify the following assertion:

Claim 6.10.A: There exists an isomorphism of schemes \( X_K \cong Y_L \).

Indeed, observe that it follows from Theorem 2.1 [cf. our assumption that \( r_X = r_Y \geq 4 \)] that there exist open immersions \( X_K \hookrightarrow Z_K \), \( Y_L \hookrightarrow Z_L \) over \( K, L \), respectively, which, together with \( \sigma \), determine a \( \Pi_{\mathbb{Z}} \)-outer isomorphism \( \sigma_{Z_K} : \Pi_{(Z_K)} \cong \Pi_{(Z_L)} \) that lies over the isomorphism \( \tilde{\sigma}_{\text{Gal}} \). Thus, by applying Corollary 6.9, we may assume without loss of generality that

- \( K = L \);
- \( \tilde{\sigma}_{\text{Gal}} \) is the identity automorphism;
- \( \sigma_{Z_K} \) is the identity \( \Pi_{\mathbb{Z}} \)-outer automorphism.

In particular, since \( \sigma_{\mathbb{Z}} \) induces a bijection between the respective sets of fiber subgroups and inertia subgroups [cf. Corollary 2.2; the discussion of [CbTpII], Remark 4.14.1], \( \tilde{\sigma} \) determines a \( \Pi_{\mathbb{Y}} \)-outer isomorphism \( \sigma_1 : \Pi_{X_K} \cong \Pi_{Y_K} \) [cf. [CbTpI], Theorem A, (i)] such that

- \( \sigma_1 \) lies over \( G_K \);
- \( \sigma_1 \) induces a bijection between the respective sets of cuspidal inertia subgroups.

Thus, we conclude from the property (ISC) [cf. the proof of Corollary 6.4], applied to \( Z_K \), that there exists an isomorphism \( X_K \cong Y_K \) over \( K \). This completes the proof of Claim 6.10.A.

In summary, it follows formally from Claim 6.10.A, together with the above discussion, that we may assume without loss of generality that

- \( K = L \), \( X_K = Y_K \);
- \( \tilde{\sigma} \) is an automorphism of \( \Pi_{(X_K)} \) that lies over the identity automorphism of \( G_K \);
- the \( \Pi_{\mathbb{Z}} \)-outer automorphism \( \sigma_{Z_K} : \Pi_{(Z_K)} \cong \Pi_{(Z_L)} \) induced by \( \tilde{\sigma} \) and the open immersion \( X_K \hookrightarrow Z_K \) over \( K \) is the identity \( \Pi_{\mathbb{Z}} \)-outer automorphism;
- the outer automorphism \( \sigma_{\mathbb{Z}} : \Pi_{X_K} \cong \Pi_{X_K} \) induced by \( \tilde{\sigma} \) induces the identity automorphism on the set of fiber subgroups;
• the $\Pi_X$-outer automorphism $\sigma_1 : \Pi_X \to \Pi_X$ (determined by $\sigma$) induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_X$ [cf. the discussion above of the property (ISC) applied to $Z_K$].

Thus, if we regard $G_K$ as a subgroup of $\text{Out}^G F(\Pi_{X_n})^{\text{cusp}}$ via the natural injection $G_K \hookrightarrow \text{Out}^G F(\Pi_{X_n})^{\text{cusp}}$ [cf. [NodNon], Theorem C, (ii)], then $\sigma_{\overline{\chi}} \in Z_{\text{Out}^G F(\Pi_{X_n})^{\text{cusp}}}(G_K)$. Write

$$\beta \in Z_{\text{Out}^G F(\Pi_{X_2})^{\text{cusp}}}(G_K)$$

for the element determined by $\sigma_{\overline{\chi}}$ via the natural injection $\text{Out}^G F(\Pi_{X_n})^{\text{cusp}} \hookrightarrow \text{Out}^G F(\Pi_{X_2})^{\text{cusp}}$ [cf. [NodNon], Theorem B];

$$h : \text{Out}^G F(\Pi_{X_2})^{\text{cusp}} \to \text{Out}^G F(\Pi_{Z_2})^{\text{cusp}}$$

for the natural homomorphism induced by the natural open immersion $X_2 \hookrightarrow Z_2$ [cf. Theorem 2.1]. Then it follows immediately from our assumption that $\sigma_{\overline{\chi}} : \Pi_{Z_2} \to \Pi_{Z_2}$ is the identity $\Pi_{Z_2}$-outer automorphism that $h(\beta) = 1$.

Thus, we conclude from Theorem 3.6 [where we take $V \subseteq W$ to be the open immersion $X \hookrightarrow Z$ in the above discussion], together with Corollary 6.4, that $\beta = 1$, hence that $\sigma_{\overline{\chi}} = 1$. Finally, since $\Pi_{X_n}$ is center-free [cf. [MT], Proposition 2.2, (ii)], it holds that $\sigma$ is an inner automorphism, hence that $\sigma = 1$. Thus, we obtain the desired surjectivity. This completes the proof of assertion (ii), hence of Theorem 6.10.

Remark 6.10.1. In the notation of Theorem 6.10, write

$$\text{Out}(\Pi_{(X_K)_m}, \Pi_{(Y_L)_n})$$

for the set of outer isomorphisms $\Pi_{(X_K)_m} \to \Pi_{(Y_L)_n}$. Suppose that $G_K$ and $G_L$ are very elastic [cf. [AbsTopII], Definition 1.1, (ii)]. Then since $\Pi_{(X_K)_m}$ and $\Pi_{(Y_L)_n}$ are topologically finitely generated [cf. [MT], Proposition 2.2, (ii)], it follows formally that

$$\text{Out}(\Pi_{(X_K)_m}, \Pi_{(Y_L)_n}) = \text{Out}(\Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L),$$

i.e., that the “absolute version” of Theorem 6.10 holds.

Remark 6.10.2. In the notation of Theorem 6.10, suppose that $K$ and $L$ arise as fields “$L$” of the sort discussed in Proposition 6.3, (i), (ii) [cf. Remark 6.6.3]. Suppose, further, that $K$ and $L$ are abelian extensions of number fields. Then $K$ and $L$ are very elastic [cf. [FJ], Theorem 13.4.2; [FJ], Theorem 16.11.3; [Mi], Theorem 2.1]. In particular, it follows immediately from Remark 6.10.1 that the absolute version of Theorem 6.10 holds.
References


[AbsTopI] S. Mochizuki, Topics in absolute anabelian geometry I: Generalities, 


Updated versions of [HMM], [CbTpII] may be found at the following URL: 
http://www.kurims.kyoto-u.ac.jp/~motizuki/

[Tsji] may be found at the following URL: 
http://www.kurims.kyoto-u.ac.jp/~stsuji/

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