RIMS-1938

# On the Bloch groups of finite fields and their quotients by the relation corresponding to a tetrahedral symmetry

By

Tomotada OHTSUKI

February 2021

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

# On the Bloch groups of finite fields and their quotients by the relation corresponding to a tetrahedral symmetry

Tomotada Ohtsuki

**Abstract**

In this note, we show that the natural homomorphisms between the Bloch groups of finite fields and their extensions of odd degree are injective. Further, we give concrete orders of the quotients of the Bloch groups of finite fields by the relation corresponding to a tetrahedral symmetry. In fact, the results are elementary consequences of known results, but these are practically useful and fundamental in studies of the Dijkgraaf–Witten invariant in Bloch groups of finite fields.

## 1 Introduction

Let $p$ be an odd prime, and let $\mathbb{F}_q$ be the field of order $q = p^f$. The Bloch group $\mathcal{B}(\mathbb{F}_q)$ of $\mathbb{F}_q$ is an abelian group generated by $\mathbb{F}_q - \{0, 1\}$ subject to a certain relation (2), which is related to the scissors congruence; see [2, 6] and [3] for details. From the topological viewpoint, the relation (2) corresponds to the 2–3 Pachner move (the pentagon relation) among tetrahedral decompositions of a 3-manifold; see *e.g.* [5]. Further, we consider a quotient group $\check{\mathcal{B}}(\mathbb{F}_q)$ of $\mathcal{B}(\mathbb{F}_q)$ by another relation (5), which corresponds to a tetrahedral symmetry of a tetrahedron in a tetrahedral decomposition of a 3-manifold. By Lemma 4.1, we note that, if a field $F$ satisfies certain conditions (see [2, Lemma 5.11]), the relation (5) can be derived from the relation (2). In this sense, the relation (5) is natural, though our field $\mathbb{F}_q$ do not satisfy the conditions of [2, Lemma 5.11]. Further, it is known ([3, Lemma 7.4] and [8, Remark VI.5.1.1]) that $\mathcal{B}(\mathbb{F}_q) \cong \mathbb{Z}/(\frac{q+1}{2})\mathbb{Z}$, and hence, $\check{\mathcal{B}}(\mathbb{F}_q)$ is also a finite cyclic group.

In this note, we show that, for odd $n > 0$, the natural homomorphisms $\mathcal{B}(\mathbb{F}_q) \to \mathcal{B}(\mathbb{F}_{q^n})$ and $\check{\mathcal{B}}(\mathbb{F}_q) \to \check{\mathcal{B}}(\mathbb{F}_{q^n})$ are injective in Theorems 3.1 and 4.3. Further, we give the concrete order of $\check{\mathcal{B}}(\mathbb{F}_q)$ in Theorem 4.4. We note that Karuo [4] studies invariants of (cusped) 3-manifolds in $\check{\mathcal{B}}(\mathbb{F}_q)$, which are related to the Dijkgraaf–Witten invariants [1] for $\mathrm{SL}_2(\mathbb{F}_q)$ by the Bloch–Wigner map (see [2]). In fact, our theorems can be obtained by elementary calculations from known facts, but our theorems are fundamental and useful in such studies of the Dijkgraaf–Witten invariants in $\check{\mathcal{B}}(\mathbb{F}_q)$; see Section 2.

The author would like to thank Kevin Hutchinson and Hiroaki Karuo for helpful comments. The author is partially supported by JSPS KAKENHI Grant Numbers JP16H02145 and JP16K13754.

## 2 The Dijkgraaf–Witten invariants in $\check{\mathcal{B}}(\mathbb{F}_q)$

To explain a motivation of our theorems, we briefly review the Dijkgraaf–Witten invariant in $\check{\mathcal{B}}(\mathbb{F}_q)$ studied by Karuo [4], in this section. For definitions of $\mathcal{B}(\mathbb{F}_q)$ and $\check{\mathcal{B}}(\mathbb{F}_q)$, see Sections 3 and 4.

Let $M$ be a closed oriented 3-manifold, and let $G$ be a finite group. We consider a representation $\rho : \pi_1(M) \to G$. For a 3-cocycle $\alpha$ of $G$, the *Dijkgraaf–Witten invariant* of $(M, \rho)$ is defined to be $(\rho^*\alpha)[M]$, where $[M]$ denotes the fundamental class of $M$. We note that this is rewritten as $\alpha(\rho_*[M])$.

We consider the case where $G = \mathrm{SL}_2(\mathbb{F}_q)$, and we consider a representation $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{F}_q)$. We consider the composition

$$H_3\big(\mathrm{SL}_2(\mathbb{F}_q); \mathbb{Z}\big) \longrightarrow \mathcal{B}(\mathbb{F}_q) \longrightarrow \check{\mathcal{B}}(\mathbb{F}_q) \tag{1}$$

of the Bloch-Wigner map and the projection. By the universal coefficient theorem, the above map can be given by a $\check{\mathcal{B}}(\mathbb{F}_q)$-valued 3-cocycle $\alpha$ of $\mathrm{SL}_2(\mathbb{F}_q)$. As in [4], we define the *reduced Dijkgraaf–Witten invariant* of $(M, \rho)$ to be $(\rho^*\alpha)[M] \in \check{\mathcal{B}}(\mathbb{F}_q)$. We note that this is rewritten as the image of $\rho_*[M]$ by the map (1).

We briefly review a construction of the reduced Dijkgraaf–Witten invariant. We consider a tetrahedral decomposition of $M$, and consider its lift to $\tilde{M}$ as a tetrahedral decomposition of $\tilde{M}$, where $\tilde{M}$ denotes the universal cover of $M$. We label the vertices of the tetrahedral decomposition of $\tilde{M}$ by elements of $P^1(\mathbb{F}_q)$, in such a way that this labeling is equivariant under the action of $\pi_1(M)$, which acts on $P^1(\mathbb{F}_q)$ by $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{F}_q)$ and the natural action of $\mathrm{SL}_2(\mathbb{F}_q)$ on $P^1(\mathbb{F}_q)$. We assume that labels of four vertices of each tetrahedron are distinct. For labels $a, b, c, d \in P^1(\mathbb{F}_q) = \mathbb{F}_q \cup \{\infty\}$ of four vertices of a tetrahedron, we consider their cross-ratio $\frac{(a-d)(b-c)}{(a-c)(b-d)} \in \mathbb{F}_q - \{0, 1\}$. Karuo [4] showed that

$$\begin{pmatrix} \text{the reduced Dijkgraaf–Witten} \\ \text{invariant of } (M,\rho) \end{pmatrix} = \sum_{\Delta} \begin{bmatrix} \text{cross-ratio of labels of} \\ \text{four vertices of a tetrahedron } \Delta \end{bmatrix} \in \check{\mathcal{B}}(\mathbb{F}_q),$$

where the sum runs over tetrahedra in the fundamental domain of the universal cover $\tilde{M} \to M$. For details, see [4]. See also [5] for the idea of this construction.

When we study the reduced Dijkgraaf–Witten invariant, our theorems are fundamental and useful in the following sense. In Theorem 4.4, we give the concrete order of the cyclic group $\check{\mathcal{B}}(\mathbb{F}_q)$, in which the reduced Dijkgraaf–Witten invariant is defined. Further, Theorem 4.3 shows that the natural homomorphism $\check{\mathcal{B}}(\mathbb{F}_q) \to \check{\mathcal{B}}(\mathbb{F}_{q^n})$ is injective for odd $n$. This theorem is useful, when we label the vertices of a complicated tetrahedral decomposition satisfying the above mentioned assumption, since we might need many labels for such decomposition and we can increase the number of labels by replacing $\mathbb{F}_q$ with $\mathbb{F}_{q^n}$ noting that $\check{\mathcal{B}}(\mathbb{F}_q)$ can be embedded in $\check{\mathcal{B}}(\mathbb{F}_{q^n})$ by Theorem 4.3.

## 3 The Bloch group of a finite field of odd characteristic

The aim of this section is to show Theorem 3.1, which give an injective homomorphism from the Bloch group of $\mathbb{F}_q$ to the Bloch group of $\mathbb{F}_{q^n}$ for odd $n$. For the Bloch groups of finite fields, see [3] and [8, Section VI.5].

The *pre-Bloch group* $\mathcal{P}(\mathbb{F}_q)$ of $\mathbb{F}_q$ is the abelian group generated by $[x]$ for $x \in \mathbb{F}_q^\times - \{1\}$ subject to the relations

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] \;=\; 0 \qquad (2)$$

for $x \neq y$. The *Bloch group* $\mathcal{B}(\mathbb{F}_q)$ of $\mathbb{F}_q$ is the kernel of the homomorphism

$$\lambda : \mathcal{P}(\mathbb{F}_q) \longrightarrow \mathbb{F}_q^\times \underset{\mathbb{Z}}{\wedge} \mathbb{F}_q^\times \;\cong\; \mathbb{Z}/2\mathbb{Z}$$

defined by $\lambda\big([z]\big) = z \wedge (1-z)$.

It is known [3] that

$$\mathcal{B}(\mathbb{F}_q) \;\cong\; \mathbb{Z}/(\tfrac{q+1}{2})\mathbb{Z}\,.$$

For a positive integer $n$, the natural inclusion $\mathbb{F}_q \subset \mathbb{F}_{q^n}$ induces a natural homomorphism $\mathcal{B}(\mathbb{F}_q) \to \mathcal{B}(\mathbb{F}_{q^n})$.

**Theorem 3.1.** *If $n$ is odd $> 0$, the natural homomorphism $\mathcal{B}(\mathbb{F}_q) \to \mathcal{B}(\mathbb{F}_{q^n})$ is injective.*

*Proof.* It is known [3, 7] that

$$H_3\big(\mathrm{SL}_2(\mathbb{F}_q), \mathbb{Z}[\tfrac{1}{p}]\big) \;\cong\; K_3(\mathbb{F}_q) \;\cong\; \mathbb{Z}/(q^2-1)\mathbb{Z}\,,$$

and it is known [7, Theorem 8] that the natural homomorphism $K_3(\mathbb{F}_q) \to K_3(\mathbb{F}_{q^n})$ is injective. Further, it is known [3] that there is the following natural surjective homomorphism,

$$\begin{array}{ccc} H_3\big(\mathrm{SL}_2(\mathbb{F}_q), \mathbb{Z}[\tfrac{1}{p}]\big) & \longrightarrow & \mathcal{B}(\mathbb{F}_q) \\ \wr\Vert & & \wr\Vert \\ \mathbb{Z}/(q^2-1)\mathbb{Z} & & \mathbb{Z}/(\tfrac{q+1}{2})\mathbb{Z}\,. \end{array}$$

Hence, since $n$ is odd, we have by Lemma 3.2 below that the natural homomorphism

$$\begin{array}{ccc} \mathcal{B}(\mathbb{F}_q) & \longrightarrow & \mathcal{B}(\mathbb{F}_{q^n}) \\ \wr\Vert & & \wr\Vert \\ \mathbb{Z}/(\tfrac{q+1}{2})\mathbb{Z} & & \mathbb{Z}/(\tfrac{q^n+1}{2})\mathbb{Z} \end{array}$$

is injective, as required. $\square$

**Lemma 3.2.** *Let $n$ be odd $> 0$. We consider the following commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}/(q^2-1)\mathbb{Z} & \xrightarrow{g_1} & \mathbb{Z}/(\tfrac{q+1}{2})\mathbb{Z} \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_2} \\ \mathbb{Z}/(q^{2n}-1)\mathbb{Z} & \xrightarrow{g_2} & \mathbb{Z}/(\tfrac{q^n+1}{2})\mathbb{Z} \end{array}$$

*where $g_1$ and $g_2$ are natural surjective homomorphisms, and $f_1$ is a natural injective homomorphism. Then, $f_2$ is injective.*

*Proof.* The kernels of $g_1$ and $g_2$ are $\{$multiples of $\frac{q+1}{2}\}$ and $\{$multiples of $\frac{q^n+1}{2}\}$. The image of $f_1$ is $\{$multiples of $\frac{q^{2n}-1}{q^2-1} = \frac{q^n+1}{q+1} \cdot \frac{q^n-1}{q-1}$ $\}$. Further, $f_1(\mathrm{kernel}\, g_1)$ is $\{$multiples of $\frac{q+2}{2} \cdot \frac{q^{2n}-1}{q^2-1} = \frac{q^n+1}{2} \cdot \frac{q^n-1}{q-1}$ $\}$. Hence,

$$\mathrm{image}\, f_1 \supset f_1(\mathrm{kernel}\, g_1) \quad \text{and} \quad \mathrm{kernel}\, g_2 \supset f_1(\mathrm{kernel}\, g_1).$$

In order to show that $f_2$ is injective, it is sufficient to show that

$$\mathrm{image}\, f_1 \cap \mathrm{kernel}\, g_2 = f_1(\mathrm{kernel}\, g_1).$$

Hence, it is sufficient to show that

$$\mathrm{lcm}\big(\tfrac{q^n+1}{q+1} \cdot \tfrac{q^n-1}{q-1}, \ \tfrac{q^n+1}{2}\big) = \tfrac{q^n+1}{2} \cdot \tfrac{q^n-1}{q-1}.$$

Therefore, it is sufficient to show that

$$\gcd\big(\tfrac{q^n+1}{q+1} \cdot \tfrac{q^n-1}{q-1}, \ \tfrac{q^n+1}{2}\big) = \tfrac{q^n+1}{q+1}.$$

Hence, it is sufficient to show that

$$\gcd\big(\tfrac{q^n-1}{q-1}, \ \tfrac{q+1}{2}\big) = 1. \tag{3}$$

Since

$$\tfrac{q^n-1}{q-1} = q^{n-1} + q^{n-2} + \cdots + 1 = \tfrac{q+1}{2} \cdot 2(q^{n-2} + q^{n-4} + \cdots + q) + 1,$$

we obtain (3). Hence, we obtain the lemma. □

By Theorem 3.1, we have a commutative diagram,

$$\begin{array}{ccc} H_3\big(\mathrm{SL}_2(\mathbb{F}_q), \mathbb{Z}\big) & \longrightarrow & \mathcal{B}(\mathbb{F}_q) \\ \big\downarrow & & \big\downarrow \\ H_3\big(\mathrm{SL}_2(\mathbb{F}_{q^n}), \mathbb{Z}\big) & \longrightarrow & \mathcal{B}(\mathbb{F}_{q^n}) \end{array} \tag{4}$$

where the right vertical homomorphism is injective.

# 4 A quotient of the Bloch group of a finite field of odd characteristic

The aim of this section is to show Theorem 4.3, which gives an injective homomorphism from a quotient of the Bloch group of $\mathbb{F}_q$ to a quotient of the Bloch group of $\mathbb{F}_{q^n}$ for odd $n$ by the relation corresponding to a tetrahedral symmetry.

As in [5], we consider the relation

$$[x] = \Big[1 - \frac{1}{x}\Big] = \Big[\frac{1}{1-x}\Big] = -\Big[\frac{1}{x}\Big] = -\Big[\frac{x-1}{x}\Big] = -[1-x] \tag{5}$$

for $x \in \mathbb{F}_q^\times - \{1\}$; this relation corresponds to a tetrahedral symmetry when we consider the Dijkgraaf–Witten invariant of 3-manifolds for $\mathrm{SL}_2(\mathbb{F}_q)$. Let $\check{\mathcal{P}}(\mathbb{F}_q)$ be the quotient abelian group of $\mathcal{P}(\mathbb{F}_q)$ by this relation, and let $\check{\mathcal{B}}(\mathbb{F}_q)$ be the image of $\mathcal{B}(\mathbb{F}_q)$ by the projection homomorphism $\mathcal{P}(\mathbb{F}_q) \to \check{\mathcal{P}}(\mathbb{F}_q)$.

In order to calculate a concrete form of $\check{\mathcal{B}}(\mathbb{F}_q)$, we review some properties of the Bloch group. The following two lemmas are due to Suslin [6]; see also [2], [3], [8, Section VI.5] for related useful formulas.

**Lemma 4.1** ([6])**.** *The following equations hold in* $\mathcal{P}(\mathbb{F}_q)$*,*

$$2\big([z] + \big[\,\frac{1}{z}\,\big]\big) \;=\; 0, \tag{6}$$

$$[z^2] + \big[\,\frac{1}{z^2}\,\big] \;=\; 0, \tag{7}$$

$$[x] + [1-x] \;=\; [y] + [1-y], \tag{8}$$

*for* $x, y, z \in \mathbb{F}_q^\times - \{1\}$*.*

Since it might be difficult for readers to see [6], we review a proof of the lemma. In fact, (6) and (7) are written in [2, Lemma 5.4], but we review the proof because we use (9) in the proof of Lemma 4.2 later. Further, in fact, (8) is written in [3, Lemma 7.3 (2)], but we review the proof because the proof is not given in [3, Lemma 7.3 (2)].

*Proof.* We review proofs of [6].

We can show (6), as follows. By replacing $x$ and $y$ in (2) with $\frac{1}{x}$ and $\frac{1}{y}$, and adding the resulting relation and (2), we obtain that

$$\big[\,\frac{x}{y}\,\big] + \big[\,\frac{y}{x}\,\big] \;=\; [y] + \big[\,\frac{1}{y}\,\big] - [x] - \big[\,\frac{1}{x}\,\big]. \tag{9}$$

Further, by replacing $x$ and $y$ in (9), and adding the resulting relation and (9), we obtain (6), where we put $z = \frac{x}{y}$.

We can obtain (7) from (9) by putting $x = z^2$ and $y = z$.

We can show (8), as follows. By replacing $x$ and $y$ in (2) with $1-y$ and $1-x$, we obtain that

$$[1-y] - [1-x] + \big[\,\frac{1-x}{1-y}\,\big] - \big[\,\frac{1-x^{-1}}{1-y^{-1}}\,\big] + \big[\,\frac{y}{x}\,\big] \;=\; 0.$$

Further, by subtracting this relation from (2), we obtain (8), as required. □

For $x \in \mathbb{F}_q^\times - \{1\}$, we put $C_{\mathbb{F}_q}(x) = [x] + [1-x] \in \mathcal{P}(\mathbb{F}_q)$. Then, by (8), $C_{\mathbb{F}_q}(x) \in \mathcal{P}(\mathbb{F}_q)$ does not depend on the choice of $x \in \mathbb{F}_q^\times - \{1\}$. Hence, we put $C_{\mathbb{F}_q} = C_{\mathbb{F}_q}(x) \in \mathcal{P}(\mathbb{F}_q)$. Further, we note that $C_{\mathbb{F}_q} \in \mathcal{B}(\mathbb{F}_q)$, since

$$\lambda(C_{\mathbb{F}_q}) \;=\; \lambda\big([x] + [1-x]\big) \;=\; x \wedge (1-x) + (1-x) \wedge x \;=\; 0.$$

For $x \in \mathbb{F}_q^\times - \{1\}$, we put $\langle\!\langle x \rangle\!\rangle = [x] + [\frac{1}{x}] \in \mathcal{P}(\mathbb{F}_q)$. We set $\langle\!\langle 1 \rangle\!\rangle = 0$.

**Lemma 4.2** ([6]). *There is a homomorphism*

$$\begin{array}{cccc} \Phi_{\mathbb{F}_q}: & \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 & \longrightarrow & \mathcal{P}(\mathbb{F}_q) \\ & \wr\| & & \\ & \mathbb{Z}/2\mathbb{Z} & & \end{array}$$

*defined by* $\Phi_{\mathbb{F}_q}(x) = \langle\langle x\rangle\rangle$.

In fact, the lemma is written in [3, Lemma 7.3 (1)], but we review the proof because the proof is not given in [3, Lemma 7.3 (1)].

*Proof.* We review a proof of [6].

By (6), we have that $2\langle\langle z\rangle\rangle = 0$ for $z \in \mathbb{F}_q^\times - \{1\}$. We note that this relation also holds for any $z \in \mathbb{F}_q^\times$, since $\langle\langle 1\rangle\rangle = 0$ by definition. Further, by (9), we have that $\langle\langle \frac{x}{y}\rangle\rangle = \langle\langle y\rangle\rangle - \langle\langle x\rangle\rangle$ for $x \neq y \in \mathbb{F}_q^\times - \{1\}$. We note that this relation also holds for any $x, y \in \mathbb{F}_q^\times$, since $\langle\langle 1\rangle\rangle = 0$ and $\langle\langle \frac{1}{y}\rangle\rangle = \langle\langle y\rangle\rangle$ by definition. By replacing $x$ in this relation with $xy$, we have that $\langle\langle xy\rangle\rangle = \langle\langle y\rangle\rangle - \langle\langle x\rangle\rangle = \langle\langle x\rangle\rangle + \langle\langle y\rangle\rangle$ for $x, y \in \mathbb{F}_q^\times$, since $2\langle\langle x\rangle\rangle = 0$. Hence, we obtain a homomorphism $\mathbb{F}_q^\times \to \mathcal{P}(\mathbb{F}_q)$ which takes $x$ to $\langle\langle x\rangle\rangle$. Further, since $\langle\langle z^2\rangle\rangle = 0$ by (7), we obtain the required homomorphism. □

Since the equalities of (5) are generated by the equalities that $[x] + [1-x] = 0$ and $[x] + [\frac{1}{x}] = 0$, we have that

$$\check{\mathcal{P}}(\mathbb{F}_q) \;=\; \mathcal{P}(\mathbb{F}_q) \,/\, \langle C_{\mathbb{F}_q},\, \mathrm{image}\,\Phi_{\mathbb{F}_q}\rangle.$$

Hence, noting that $C_{\mathbb{F}_q} \in \mathcal{B}(\mathbb{F}_q)$, we have that

$$\check{\mathcal{B}}(\mathbb{F}_q) \;=\; \mathcal{B}(\mathbb{F}_q) \,/\, \langle C_{\mathbb{F}_q},\, \mathcal{B}(\mathbb{F}_q) \cap \mathrm{image}\,\Phi_{\mathbb{F}_q}\rangle.$$

By Lemma 4.2,

$$\mathrm{image}\,\Phi_{\mathbb{F}_q} \;=\; \big\{0,\, \Phi_{\mathbb{F}_q}(a)\big\},$$

where $a$ is a quadratic nonresidue in $\mathbb{F}_q$; we note that $\Phi_{\mathbb{F}_q}(a)$ is 0 or the element of order 2 in $\mathcal{P}(\mathbb{F}_q)$. Hence, since $\mathcal{B}(\mathbb{F}_q) \cong \mathbb{Z}/(\frac{q+1}{2})\mathbb{Z}$,

$$\mathcal{B}(\mathbb{F}_q) \cap \mathrm{image}\,\Phi_{\mathbb{F}_q} \;=\; \begin{cases} \big\{0,\, \Phi_{\mathbb{F}_q}(a)\big\} & \text{if } \frac{q+1}{2} \text{ is even,} \\ \{0\} & \text{if } \frac{q+1}{2} \text{ is odd.} \end{cases}$$

Therefore,

$$\check{\mathcal{B}}(\mathbb{F}_q) \;=\; \begin{cases} \mathcal{B}(\mathbb{F}_q) \,/\, \langle C_{\mathbb{F}_q},\, \Phi_{\mathbb{F}_q}(a)\rangle & \text{if } \frac{q+1}{2} \text{ is even,} \\ \mathcal{B}(\mathbb{F}_q) \,/\, \langle C_{\mathbb{F}_q}\rangle & \text{if } \frac{q+1}{2} \text{ is odd.} \end{cases}$$

Further, it is known ([3, Lemma 7.4] and [8, Remark VI.5.1.1]) that the order of $C_{\mathbb{F}_q} \in \mathcal{B}(\mathbb{F}_q)$ is $\gcd(6, \frac{q+1}{2})$, $\Phi_{\mathbb{F}_q}(a)$ is a multiple of $C_{\mathbb{F}_q}$ when $\frac{q+1}{2}$ is even. Hence,

$$\check{\mathcal{B}}(\mathbb{F}_q) \;=\; \mathcal{B}(\mathbb{F}_q) \,/\, \langle C_{\mathbb{F}_q}\rangle. \tag{10}$$

For a positive odd integer $n$, the homomorphism of Theorem 3.1 induces a natural homomorphism $\check{\mathcal{B}}(\mathbb{F}_q) \to \check{\mathcal{B}}(\mathbb{F}_{q^n})$.

**Theorem 4.3.** *If $n$ is odd $> 0$, the natural homomorphism $\check{\mathcal{B}}(\mathbb{F}_q) \to \check{\mathcal{B}}(\mathbb{F}_{q^n})$ is injective.*

*Proof.* We denote the homomorphism of Theorem 3.1 by $\iota : \mathcal{B}(\mathbb{F}_q) \to \mathcal{B}(\mathbb{F}_{q^n})$. It is sufficient to show that

$$\iota\big(\langle C_{\mathbb{F}_q}\rangle\big) \;=\; \iota\big(\mathcal{B}(\mathbb{F}_q)\big) \cap \langle C_{\mathbb{F}_{q^n}}\rangle.$$

Hence, it is sufficient to show that the order of $C_{\mathbb{F}_q}$ in $\mathcal{B}(\mathbb{F}_q)$ is equal to the order of $C_{\mathbb{F}_{q^n}}$ in $\mathcal{B}(\mathbb{F}_{q^n})$.

We show this, as follows. It is known [3, Lemma 7.11] that the order of $C_{\mathbb{F}_q} \in \mathcal{B}(\mathbb{F}_q)$ is $\gcd(6, \frac{q+1}{2})$. Since

$$\gcd\big(6, \frac{q+1}{2}\big) \;=\; \begin{cases} 1 & \text{if} \;\; q \equiv 1, -3 \mod 12, \\ 2 & \text{if} \;\; q \equiv 3, -5 \mod 12, \\ 3 & \text{if} \;\; q \equiv 5 \mod 12, \\ 6 & \text{if} \;\; q \equiv -1 \mod 12, \end{cases} \tag{11}$$

we can verify by concrete calculation that

$$\gcd\big(6, \frac{q+1}{2}\big) \;=\; \gcd\big(6, \frac{q^n+1}{2}\big).$$

Hence, since $\iota$ is injective, the order of $C_{\mathbb{F}_q}$ is equal to the order of $C_{\mathbb{F}_{q^n}}$, as required. □

The commutative diagram (4) induces a commutative diagram,

$$\begin{array}{ccc} H_3\big(\mathrm{SL}_2(\mathbb{F}_q), \mathbb{Z}\big) & \longrightarrow & \check{\mathcal{B}}(\mathbb{F}_q) \\ \downarrow & & \downarrow \\ H_3\big(\mathrm{SL}_2(\mathbb{F}_{q^n}), \mathbb{Z}\big) & \longrightarrow & \check{\mathcal{B}}(\mathbb{F}_{q^n}) \end{array} \tag{12}$$

where the right vertical homomorphism is injective by Theorem 4.3.

Since the order of $C_{\mathbb{F}_q} \in \mathcal{B}(\mathbb{F}_q)$ is $\gcd(6, \frac{q+1}{2})$, we obtain the following theorem by (10) and (11).

**Theorem 4.4.**

$$\check{\mathcal{B}}(\mathbb{F}_q) \;\cong\; \begin{cases} \mathbb{Z}/(\frac{q+1}{2})\mathbb{Z} & \textit{if} \;\; q \equiv 1, -3 \mod 12, \\ \mathbb{Z}/(\frac{q+1}{4})\mathbb{Z} & \textit{if} \;\; q \equiv 3, -5 \mod 12, \\ \mathbb{Z}/(\frac{q+1}{6})\mathbb{Z} & \textit{if} \;\; q \equiv 5 \mod 12, \\ \mathbb{Z}/(\frac{q+1}{12})\mathbb{Z} & \textit{if} \;\; q \equiv -1 \mod 12. \end{cases}$$

## References


[1] Dijkgraaf, R., Witten, E., *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990) 393–429.

[2] Dupont, J. L., Sah, C. H., *Scissors congruences. II*, J. Pure Appl. Algebra **25** (1982) 159–195.

[3] Hutchinson, K., *A Bloch–Wigner complex for* $\mathrm{SL}_2$, J. K-Theory **12** (2013) 15–68.

[4] Karuo, H., *The reduced Dijkgraaf–Witten invariant of twist knots in the Bloch group of a finite field*, Master Thesis, RIMS, Kyoto University, January 2019.

[5] Neumann, W. D., *Extended Bloch group and the Cheeger-Chern-Simons class*, Geom. Topol. **8** (2004) 413–474.

[6] Suslin, A. A., $K_3$ *of a field, and the Bloch group* (Russian), Translated in Proc. Steklov Inst. Math. **1991**, no. 4, 217–239. Galois theory, rings, algebraic groups and their applications (Russian). Trudy Mat. Inst. Steklov. **183** (1990) 180–199, 229.

[7] Quillen, D., *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (2) **96** (1972) 552–586.

[8] Weibel, C. A., *The K-book. An introduction to algebraic K-theory*, Graduate Studies in Mathematics **145**. American Mathematical Society, Providence, RI, 2013.

Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto, 606-8502, Japan
E-mail address: `tomotada@kurims.kyoto-u.ac.jp`