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Outstanding problems on normal projective surfaces admitting non-isomorphic surjective endomorphisms

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OUTSTANDING PROBLEMS ON NORMAL PROJECTIVE SURFACES ADMITTING NON-ISOMORPHIC SURJECTIVE ENDOMORPHISMS

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ABSTRACT. The structure of a normal projective surface X admitting a non-isomorphic surjective endomorphism f is determined except for log del Pezzo surfaces of Picard number 1. The structure of X is also determined in the case where the first dynamical degree λ_f is not equal to the positive square root of $\deg f$.

1. INTRODUCTION

By the study of normal Moishezon surfaces X admitting non-isomorphic surjective endomorphisms f in [20] and [21], we have determined the structure of such surfaces except the case where X is a rational surface with only quotient singularities and $K_X + S_f$ is not pseudo-effective. Here, K_X stands for the canonical divisor and S_f for the *characteristic completely invariant divisor* of f (cf. [20, Def. 2.16]). By [21, §4], the following three subcases remain as unsolved cases:

(R1) X is a *log del Pezzo surface* of Picard number $\boldsymbol{\rho}(X) = 1$;

(R2) $\boldsymbol{\rho}(X) = 2$ and $-K_X$ is big;

(R3) (X, S_f) is log-canonical, $-(K_X + S_f)$ is nef and big, and the number $\boldsymbol{n}(S_f)$ of prime components of S_f equals $\boldsymbol{\rho}(X) \geq 3$.

Determining the structure of (X, f) belonging to (R2) or (R3) is an outstanding problem bothering the author many years, but at last, this is settled in this article.

Remark. The existence of non-isomorphic surjective endomorphism implies that X is projective, by [20, Cor. B]. Suppose that X has only quotient singularities and $K_X + S_f$ is not pseudo-effective. If $\boldsymbol{\rho}(X) = 1$, then X is a log del Pezzo surface, since $-K_X$ is ample. The following is an additional information for (X, f) with $\boldsymbol{\rho}(X) \geq 2$ not belonging to (R2) nor (R3):

Assume that $\boldsymbol{\rho}(X) = 2$. If X is irrational or if $-K_X$ is not big, then we have known the structure of X by [20, Thm. 4.16] and [21, Thm. 4.7]. We shall prove a stronger result for this X as Theorem 5.1 in Section 5.1 below.

Assume next that $\boldsymbol{\rho}(X) \geq 3$. Then (X, S_f) is an $\mathcal{L}$-*surface* in the sense of [21, Def. 4.2] (cf. [21, Prop. 4.3]). In particular, X is rational. Moreover, we have the following by the structure theorem [21, Thm. 4.5] on $\mathcal{L}$-surfaces:

- Divisors $-K_X$ and S_f are big, and $-(K_X + S_f)$ is semi-ample;

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- If $\boldsymbol{n}(S_f) \neq \boldsymbol{\rho}(X)$, then $(X, S_f + B)$ is a *toric surface* for a prime divisor B.

Furthermore, by [21, Cor. 4.6], if $-(K_X + S_f)$ is not big, then $(X, S_f + B)$ is a toric surface or a *half-toric surface* for a prime divisor B.

Remark. For a toric surface X and its *boundary divisor* D, the complement of the open torus, the pair (X, D) is called also a *toric surface* by abuse of notation (cf. [18, Conv. 1.4]). A *half-toric* surface is defined in [18, Def. 7.1], and it is a key notion in this article: This is a pair (X, S) of a normal projective surface X and a reduced divisor S such that $K_X + S \not\sim 0$, $2(K_X + S) \sim 0$, and $(V, \tau^{-1}S)$ is a toric surface for a double cover $\tau\colon V \to X$ étale in codimension 1: We call τ the *characteristic double cover* of (X, S) (cf. [18, §7.1]).

As a solution to the outstanding problem, we shall show in Theorems 4.23 and 5.6 below that one of the two conditions below is satisfied for any (X, f) belonging to (R2) and that the latter condition is satisfied for any (X, f) belonging to (R3):

- There is a finite Galois cover $X' \to X$ étale in codimension 1 such that X' is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ or over an elliptic curve, the Galois group preserves the $\mathbb{P}^1$-bundle structure, and some power $f^k = f \circ f \circ \cdots \circ f\colon X \to X$ lifts to an endomorphism of X'.
- There is a reduced divisor B such that $(X, S_f + B)$ is a toric surface or a half-toric surface.

Combining results in [21], we have:

Theorem 1.1. *Let X be a normal projective surface. If X admits a non-isomorphic surjective endomorphism f such that $K_X + S_f$ is not pseudo-effective, then one of the following holds*:

1. *The surface X is log del Pezzo of Picard number* 1.
2. *There is a finite Galois cover $V \to X$ étale in codimension* 1 *from the product $V = \mathbb{P}^1 \times T$ for a non-singular projective curve T of genus ≥ 2.*
3. *There is a finite Galois cover $V \to X$ étale in codimension* 1 *from one of the following $\mathbb{P}^1$-bundles V over an elliptic curve T*:
   - $V = \mathbb{P}^1 \times T$;
   - $V = \mathbb{P}_T(\mathcal{E})$ *for an indecomposable locally free sheaf $\mathcal{E}$ of rank* 2 *and degree* 0;
   - $V = \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{L})$ *for an invertible sheaf $\mathcal{L}$ of degree $\neq 0$.*
4. *There is a finite Galois cover $V \to X$ étale in codimension* 1 *from a* projective cone *V over an* elliptic curve (*cf.* [20, Def. 1.16]).
5. *There is a finite Galois cover $V \to X$ étale in codimension* 1 *from a $\mathbb{P}^1$-bundle V over $\mathbb{P}^1$ in which the Galois group* $\mathrm{Gal}(V/X)$ *preserves the $\mathbb{P}^1$-bundle structure.*
6. *The pair $(X, S_f + B)$ is a toric surface for a non-zero reduced divisor B having at most two prime components.*
7. *The pair $(X, S_f + B)$ is a half-toric surface for a prime divisor B, and B is an end component of $S_f + B$.*

Moreover, in cases (2)*,* (3)*,* (4)*, and* (5)*, some power* $f^k\colon X \to X$ *lifts to an endomorphism of* V*.*

Remark. The divisor B in (6) and (7) of Theorem 1.1 has no common prime component with S_f, since $S_f + B$ is reduced. Moreover, every prime component of B is nef, since S_f contains all the negative curves (cf. [20, Prop. 2.20(3)]).

Theorems 1.1, 4.23, and 5.6 are proved by arguments in [21, Sect. 4] and by certain properties of $\mathbb{P}^1$-fibrations, *pseudo-toric surfaces* (cf. [18, Def. 6.1]), and $\mathcal{V}$*-surfaces*, which are discussed in Sections 2, 3, and 4 below, respectively. By endomorphisms of toric and half-toric surfaces constructed in [22] and by adding some results on the converse of Theorem 1.1, we have:

Theorem 1.2. *Let* X *be a normal projective surface which is not a log del Pezzo surface of Picard number* 1*. Then* X *admits a non-isomorphic surjective endomorphism* f *such that* $K_X + S_f$ *is not pseudo-effective if and only if one of the following conditions is satisfied*:

(1) *There is a finite Galois cover* $V \to X$ *étale in codimension* 1 *which satisfies one of conditions* (2)–(5) *of Theorem* 1.1*.*
(2) *There is a reduced divisor* D *such that* (X, D) *is a toric surface and that some prime component of* D *is not a negative curve.*
(3) *There is a reduced divisor* D *with a prime component* Γ *such that*
    - (X, D) *is a half-toric surface and* Γ *is an end component of the linear chain* D *of rational curves,*
    - *each prime component of* $\tau^*\Gamma$ *is not a negative curve for the characteristic double cover* $\tau\colon V \to X$ *of the half-toric surface* (X, D)*.*

We have the following update of [21, Thm. 1.1] by Theorem 1.1:

Theorem 1.3. *Let* X *be a normal projective surface which is not a log del Pezzo surface of Picard number* 1*. Then* X *admits a non-isomorphic surjective endomorphism if and only if there is a finite Galois cover* $V \to X$ *étale in codimension* 1 *from one of the following normal projective surfaces* V:

(1) $V = C \times T$ *for an elliptic curve* C *and a non-singular projective curve* T *of genus* ≥ 2;
(2) V *is an abelian surface*;
(3) $V = \mathbb{P}^1 \times T$ *for a non-singular projective curve* T *of genus* ≥ 2;
(4) V *is a* $\mathbb{P}^1$*-bundle over an elliptic curve*;
(5) V *is a projective cone over an elliptic curve*;
(6) V *is a* $\mathbb{P}^1$*-bundle over* $\mathbb{P}^1$ *and the Galois group* $\mathrm{Gal}(V/X)$ *preserves the* $\mathbb{P}^1$*-bundle structure*;
(7) V *is a toric surface and the Galois group* $\mathrm{Gal}(V/X)$ *preserves the open torus.*

Remark. By results in this article, only the case of log del Pezzo surface of Picard number 1 remains unsolved in the problem of classifying normal Moishezon surfaces admitting non-isomorphic surjective endomorphisms. Even in the case, our arguments in this article, e.g. those in Section 3.3 below, seem to be effective for the problem when the characteristic completely invariant divisor S_f is not zero. However, now the author has no good idea if $S_f = 0$.

On the *first dynamical degree* λ_f (cf. [20, Def. 3.1]) of a non-isomorphic surjective endomorphism $f\colon X \to X$, we have a fundamental theorem as [20, Thm. D]. In particular, if X has Picard number 1 or if X contains a negative curve, then λ_f equals the positive square root $\delta_f := (\deg f)^{1/2}$. By results in Section 5 below, we can improve [20, Thm. D] to the following in the case where $\lambda_f > \delta_f$:

Theorem 1.4. *Let X be a normal projective surface admitting a non-isomorphic surjective endomorphism f such that $\lambda_f > \delta_f$. Then $\boldsymbol{\rho}(X) \geq 2$ and there exists a finite Galois cover $V \to X$ étale in codimension 1 from one of the normal projective surfaces V listed below, in which some power f^k lifts to an endomorphism of V:*

(1) *V is an abelian surface;*
(2) *$V = C \times T$ for an elliptic curve C and a non-singular projective curve T of genus ≥ 2;*
(3) *$V = \mathbb{P}^1 \times T$ for a non-singular projective curve T of genus ≥ 2;*
(4) *V is a $\mathbb{P}^1$-bundle $\mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{L})$ over an elliptic curve T for an invertible sheaf $\mathcal{L}$ of degree 0;*
(5) *$V = \mathbb{P}^1 \times \mathbb{P}^1$.*

Moreover, $\lambda_f = \deg f$ in cases (2) and (3), and λ_f is an integer dividing into $\deg f$ in cases (4) and (5).

Remark. The inequality $\lambda_f \geq \delta_f$ holds for any f (cf. [20, Prop. 3.3(2)]). If $\lambda_f = \delta_f$, then the pullback homomorphism $(f^k)^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ is a scalar map for some $k > 0$ by [20, Thm. D], where $\mathsf{N}(X)$ denotes the real vector space of numerical classes of $\mathbb{R}$-divisors on X (cf. [18, Def. 2.7], [20, §1.1]).

Crucial ideas. As is mentioned above, three cases (R1), (R2), and (R3) remain unsolved in the problem determining the structure of a normal Moishezon surface X admitting a non-isomorphic surjective endomorphism f.

Before explaining our crucial ideas, we shall note a relation between studies in (R2) and (R3). For (X, f) in (R3), we know the following by the structure theorem [21, Thm. 4.5] on $\mathcal{L}$-surfaces:

- $-(K_X + S_f)$ is semi-ample;
- S_f is a *linear chain of rational curves* (cf. [18, Def. 4.1]);
- $(K_X + S_f)C < 0$ for one end component C of S_f;
- the union $S_f^{\natural}$ of non-end components is negative definite.

The target $\overline{X}$ of the contraction morphism $\phi\colon X \to \overline{X}$ of $S_f^{\natural}$ is a normal projective surface with $\boldsymbol{\rho}(\overline{X}) = 2$ and f descends to an endomorphism $\bar{f}$ of $\overline{X}$ such that $K_X + S_f = \phi^*(K_{\overline{X}} + S_{\bar{f}})$. Hence, the study in the case (R3) is reduced to that in the case (R2). Conversely, for (X, f) in (R2), if S_f is connected and reducible and if $\lambda_f = \delta_f$, then a toroidal blowing up $\widetilde{X} \to X$ at a singular point of S_f produces a non-isomorphic surjective endomorphism $\tilde{f}$ of a normal projective surface $\widetilde{X}$ belonging to (R3) (cf. [19, Cor. 5.7], [20, Thm. D]).

We have three crucial ideas on pseudo-toric surfaces, half-toric surfaces, and a comparison of ramification divisors. We shall explain how these idea are applied to the study of (X, f) but not explain the ideas directly. The first crucial idea is analyzing negative curves on some pseudo-toric surface which is a compactification of the universal cover of a certain open subset of another pseudo-toric surface. Here, we assume the following for (X, f):

- $-(K_X + S_f)$ is nef,
- S_f is connected, and
- $(K_X + S_f)C_1 < 0$ and $(K_X + S_f)C_2 < 0$ for two prime components C_1 and C_2 of S_f.

Then, by Lemma 3.12 below, S_f is a linear chain of rational curves with C_1 and C_2 as end components, and there is an effective divisor $B \sim -(K_X + S_f)$ such that

- $(X, B + S_f)$ is log-canonical along S_f, and
- $B \cap \operatorname{Supp} S_f$ consists of two points which are in C_1 and C_2.

By an argument generating a pencil on X, we have a situation of Proposition 3.11 below, by which we can find another non-zero reduced divisor B' such that $(X, B' + S_f)$ is a toric surface. The proof of Proposition 3.11 uses results on the universal cover of a certain open subset of a pseudo-toric surface, which are prepared in Section 3.2. If B is reducible, then we can prove easily that $(X, S_f + B)$ is a toric surface. When B is irreducible, we can show that $(X, S_f + B)$ is a *pseudo-toric surface of defect* 1 (cf. [18, Defs. 2.23 and 6.1]), and we shall consider the universal cover above for $(X, S_f + B)$, which extends to a finite Galois cover $\widetilde{X} \to X$ from a normal projective surface $\widetilde{X}$ by the Grauert–Remmert extension theorem (cf. [6], [7, XII, Thm. 5.4]). Here, $\widetilde{X}$ is also a pseudo-toric surface with the inverse image of $S_f + B$ as the boundary divisor (cf. Lemma 3.4), and we have an endomorphism $\tilde{f}$ of $\widetilde{X}$ as a lift of f (cf. Lemma 3.10). Our first crucial idea is analyzing negative curves on $\widetilde{X}$, which are all contained in $S_{\tilde{f}}$. Necessary results on negative curves on $\widetilde{X}$ are prepared in Proposition 3.9, by which we can find the expected divisor B' such that $(X, S_f + B')$ is a toric surface.

The second crucial idea concerns Theorem 4.10 below on half-toric surfaces and $\mathcal{V}_{\mathrm{A}}$-*surfaces* (cf. Definitions 4.1 and 4.5(3)). Before the idea, we shall explain how $\mathcal{V}$-*surfaces* (cf. Definition 4.1) are related to the study of (X, f) in (R2): Suppose that X contains two negative curves C_1 and C_2. Then $\lambda_f = \delta_f$, S_f equals the linear chain $C_1 + C_2$, and by the discussion above on the first crucial idea (cf. Theorem 3.14), we may assume that $-(K_X + S_f)$ is not ample; for example, $(K_X + S_f)C_1 < 0$ and $(K_X + S_f)C_2 = 0$. Then (X, C_1, C_2) is a $\mathcal{V}$-surface. Even if X contains a unique negative curve, in many cases, we can find another prime divisor C' such that (X, C, C') or (X, C', C) is a $\mathcal{V}$-surface, by arguments in Section 5.2. Conversely, $-K_V$ is big for any $\mathcal{V}$-surface $(V, \Lambda_1, \Lambda_2)$ by Lemma 4.4(2). The $\mathcal{V}$-surfaces are divided into three subclasses: $\mathcal{V}_{\mathrm{A}}$-*surfaces*, *ordinary* $\mathcal{V}_{\mathrm{B}}$-*surfaces*, and *extraordinary* $\mathcal{V}_{\mathrm{B}}$-*surfaces* (cf. Definitions 4.5(3) and 4.18). The $\mathcal{V}$-surfaces (X, C_1, C_2), (X, C, C'), and (X, C', C) above are not extraordinary $\mathcal{V}_{\mathrm{B}}$-surfaces (cf. Lemma 4.22). In Theorem 4.10 (resp. 4.21), we can prove that if $(V, \Lambda_1, \Lambda_2)$ is a $\mathcal{V}_{\mathrm{A}}$-surface (resp. an ordinary $\mathcal{V}_{\mathrm{B}}$-surface), then $(V, \Lambda_1 + \Lambda_2 + B)$ is a half-toric surface for a prime divisor B: Theorem 4.21 is proved by Theorem 4.10. These theorems prove Theorem 5.6 for many cases of (X, f) belonging to (R2) in which X contains a negative curve and $-(K_X + S_f)$ is not ample. Moreover, these theorems and Theorem 3.13 prove Theorem 4.23 on (R3) by the above-mentioned reduction of the study in (R3) to that in (R2).

Our second crucial idea appears in the proof of Theorem 4.10 in the case where $\Lambda_1^2 < 0$: A $\mathcal{V}_\mathrm{A}$-surface $(V, \Lambda_1, \Lambda_2)$ with $\Lambda_1^2 < 0$ satisfies the following (cf. Lemma 4.4):

- V is rational with $\boldsymbol{\rho}(V) = 2$;
- $\Lambda_1 + \Lambda_2$ is connected and $(V, \Lambda_1 + \Lambda_2)$ is log-canonical with $\Lambda_1^2 < 0$, $\Lambda_2^2 < 0$, $(K_V + \Lambda_1 + \Lambda_2)\Lambda_1 < 0$, and $(K_V + \Lambda_1 + \Lambda_2)\Lambda_2 = 0$;
- $(\Lambda_1 \setminus \Lambda_2) \cap \operatorname{Sing} V$ is empty or consists of one point at which (V, Λ_1) is 1-log-terminal;
- $(\Lambda_2 \setminus \Lambda_1) \cap \operatorname{Sing} V$ consists of two A_1-singular points.

Let $Y \to V$ be the minimal resolution of singularities lying on $\Lambda_1 + \Lambda_2$. Then the dual graph of the total transform of $\Lambda_1 + \Lambda_2$ is of type D (cf. Lemma 4.13). Applying Lemma 2.3 and Corollary 2.31 to the dual graph, we can find a $\mathbb{P}^1$-fibration on $Y \to \mathbb{P}^1$ with a double section which is exceptional for $Y \to V$ (cf. Lemma 4.14). The second crucial idea is analyzing singular fibers of the $\mathbb{P}^1$-fibration. In Section 2.3, we study singular fibers of a $\mathbb{P}^1$-fibration with a double section satisfying certain conditions by introducing the notion of *PDS configurations* (cf. Definition 2.17). Using properties of *H-surfaces* in [18, §7.2], we have some global properties of PDS configurations in Proposition 2.28 below. These properties and the equality in [18, Prop. 2.33(7)] on reducible fibers are applied to removing bad cases for the $\mathbb{P}^1$-fibration $Y \to \mathbb{P}^1$ in proofs of Lemmas 4.14 and 4.15 below. As a consequence, we can find the expected prime divisor B such that $(V, \Lambda_1 + \Lambda_2 + B)$ is a half-toric surface.

The third crucial idea is in the proof of Lemma 5.18 below on comparison of ramification divisors for a certain finite morphism of normal projective surfaces with compatible non-isomorphic surjective endomorphisms. Lemma 5.18 is applied to proving Theorem 5.17, which is a structure theorem for (X, f) belonging to (R2) containing no negative curves: Theorem 5.6 mentioned above for this (X, f) is a consequence of Theorem 5.17. In this case, X has two $\mathbb{P}^1$-fibrations over $\mathbb{P}^1$ as contraction morphisms of extremal rays, and f descends to an endomorphism of each base curve $\mathbb{P}^1$. Thus, we have an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ and the induced finite morphism $\phi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is compatible with endomorphisms of X and $\mathbb{P}^1 \times \mathbb{P}^1$. We can determine the ramification divisor R_ϕ by Lemma 5.18 under a certain condition (cf. Proposition 5.24). The description of R_ϕ is a key to prove Theorem 5.17. We apply the Perron–Frobenius theorem [3] to the proof of Lemma 5.18.

Organization of this article. In Section 2, we discuss some properties of linear chains of rational curves and $\mathbb{P}^1$-fibrations over a non-singular curve. We recall elementary properties of linear chains of rational curves and exceptional divisors for the minimal resolutions of cyclic quotient singularities, in Section 2.1. Here, we shall prove some results on simple normal crossing divisors of rational curves having the dual graph of type D, as an application (cf. Lemma 2.3). We investigate singularities arising on $\mathbb{P}^1$-fibrations with respect to sections or double sections in Sections 2.2 and 2.3. Especially, in Section 2.3, we introduce *PDS configurations* for double sections satisfying certain conditions, and study their properties. In Section 2.4, we shall give sufficient conditions for a reduced divisor of rational curves on a normal Moishezon surface to be a set-theoretic fiber of a $\mathbb{P}^1$-fibration.

Section 3 is devoted to the study of pseudo-toric surfaces. In Section 3.1, we discuss in detail the structure of a pseudo-toric surface having a fibration to $\mathbb{P}^1$. In Section 3.2, we analyze the universal cover of the complement of a part of the boundary divisor of a pseudo-toric surface. As applications, in Section 3.3, we shall prove Theorems 3.13 and 3.14 on endomorphisms.

In Section 4, we introduce $\mathcal{V}$-surfaces and study their structures. After giving some remarks on half-toric surfaces in Section 4.1, we shall prove basic properties of $\mathcal{V}$-surfaces in Section 4.2, where two subclasses $\mathcal{V}_{\mathrm{A}}$-surfaces and $\mathcal{V}_{\mathrm{B}}$-surfaces are introduced. Section 4.3 is devoted to proving Theorem 4.10 as a structure theorem for $\mathcal{V}_{\mathrm{A}}$-surfaces. The ordinary $\mathcal{V}_{\mathrm{B}}$-surfaces and extraordinary $\mathcal{V}_{\mathrm{B}}$-surfaces are introduced in Section 4.4, where we shall prove Theorem 4.21 as an analogy of Theorem 4.10 for ordinary $\mathcal{V}_{\mathrm{B}}$-surfaces, and prove Theorem 4.23 on the structure of a normal projective surface admitting a non-isomorphic surjective endomorphism belonging to the case (R3). Extraordinary $\mathcal{V}_{\mathrm{B}}$-surfaces are studied in detail in Section 4.5, where we obtain Theorem 4.29 as a structure theorem.

Section 5 is devoted to determining structures of surfaces X with non-isomorphic surjective endomorphisms f such that $\boldsymbol{\rho}(X) = 2$ and $K_X + S_f$ is not pseudo-effective. This covers (X, f) belonging to (R2). In Section 5.1, we treat the case where X is irrational or $-K_X$ is not big, and we shall prove Theorem 5.1 as a structure theorem for such X. The structure theorem for (R2) is Theorem 5.6 mentioned above, and this is proved in Sections 5.2 and 5.3: Section 5.2 (resp. 5.3) treats the case where X contains (resp. does not contain) a negative curve. Theorem 5.17 related to our third crucial idea is proved in Section 5.3, which implies Theorem 5.6 in the case of no negative curves.

Section 6 is the final section, where theorems in the introduction are all proved.

Notation and conventions. We use the same notation and conventions as in [20] and [21]. In particular, we treat complex analytic spaces rather than schemes over $\mathbb{C}$, and a complex analytic variety is called a variety for short. A variety of dimension 1 (resp. 2) is called a curve (resp. surface). As in [19, Rem. 2.3], we avoid the use of "log terminal" in the sense of [25] and [14], and the notion of "purely log terminal" is called "1-log-terminal" in this article. The important notions of "pseudo-toric surfaces" and "half-toric surfaces" are defined in [18]. We list our specific notation in Table 1. Here, we note that the Weil–Picard number $\hat{\boldsymbol{\rho}}(X) = \dim \mathsf{N}(X)$ equals the Picard number $\boldsymbol{\rho}(X)$ if X is $\mathbb{Q}$-factorial.

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TABLE 1. List of notations

$\mathbb{C}^\star$	1-dimensional algebraic torus ($= \mathbb{C} \setminus \{0\}$)
$\operatorname{Sing} X$	singular locus of a reduced analytic space X
X_{reg}	non-singular locus $X \setminus \operatorname{Sing} X$
$\boldsymbol{\rho}(X)$	Picard number of a normal projective variety X
$\mathsf{N}(X)$	vector space of numerical classes of $\mathbb{R}$-divisors on a normal projective surface X (cf. [20, §1.1])
$\overline{\mathrm{NE}}(X)$	pseudo-effective cone in $\mathsf{N}(X)$ (cf. [20, §1.1])
$\mathrm{Nef}(X)$	nef cone in $\mathsf{N}(X)$ (cf. [20, §1.1])
$\mathrm{cl}(D)$	numerical class of an $\mathbb{R}$-divisor D
$\operatorname{Supp} D$	support of an $\mathbb{R}$-divisor D (= the union of prime components of D)
D_{red}	reduced divisor identified with $\operatorname{Supp} D$ for an effective $\mathbb{R}$-divisor D
$\boldsymbol{n}(D)$	number of prime components of a reduced divisor D (cf. [18], [21, Def. 4.1])
$\boldsymbol{\pi}_1(U)$	fundamental group of a topological space U
$\mathbb{T}_{\mathsf{N}}(\triangle)$	toric variety defined by a fan $\triangle$ of a free abelian group N of finite rank
$\boldsymbol{\delta}(X, S)$	defect of (X, S) $(= \hat{\boldsymbol{\rho}}(X) + 2 - \boldsymbol{n}(S))$ (cf. [18, Def. 2.23])
	For an endomorphism f:
R_f	ramification divisor (cf. [19, §1.5])
S_f	characteristic completely invariant divisor (cf. [20, Def. 2.16])
Δ_f	refined ramification divisor (cf. [20, Def. 2.16])
λ_f	the first dynamical degree (cf. [20, Def. 3.1])
$\deg f$	(mapping) degree
δ_f	$:= (\deg f)^{1/2} > 0$ (cf. [20, Def. 3.2])

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2. LINEAR CHAINS OF RATIONAL CURVES AND $\mathbb{P}^1$-FIBRATIONS

In Section 2.1, we recall and prove some elementary properties of linear chains of rational curves and exceptional divisors for the minimal resolutions of cyclic quotient singularities. Singularities arising on $\mathbb{P}^1$-fibrations are studied in Sections 2.2 and 2.3 with respect to sections or double sections. In Section 2.4, we shall give sufficient conditions for a reduced divisor of rational curves to be a set-theoretic fiber of a $\mathbb{P}^1$-fibration.

2.1. **Remarks on linear chains of rational curves.**

Definition. Let D be a non-zero compact reduced divisor on a normal surface X and let $D = \sum_{i=1}^{k} D_i$ be the prime decomposition. We say that D is *negative definite* (resp. *negative semi-definite*) if the intersection matrix $(D_i D_j)_{1 \le i,j \le k}$ is so.

Remark. The divisor D is negative definite if and only if $E^2 < 0$ for any non-zero divisor E on X such that $\operatorname{Supp} E \subset D$. The divisor D is not negative semi-definite if and only if there is a divisor B on X such that $\operatorname{Supp} B \subset D$ and $B^2 > 0$. These are proved by considering eigenvalues of the matrix $(D_i D_j)_{1 \le i,j \le k}$. Here, B is taken as an effective divisor. In fact, if $B = B_1 - B_2$ for two effective divisors B_1 and B_2 without common prime components, then $B^2 > 0$ implies that $B_1^2 > 0$ or $B_2^2 > 0$.

Definition. Let D be a negative definite compact reduced divisor on a normal surface X. Then there is a bimeromorphic morphism $\tau\colon X \to \overline{X}$ to another normal surface $\overline{X}$ such that $\tau(D)$ is a finite set, $D = \tau^{-1}(\tau(D))$, and τ induces an isomorphism $X \setminus D \simeq Y \setminus \tau(D)$. This is known as the Grauert contraction theorem in case X is non-singular (cf. [5, (e), pp. 366–367], [16], [2, Cor. 6.12(b)]), and this is proved in the singular case by [24, Thm. (1.2)] (cf. [18, Thm. 2.6]). We call τ the *contraction morphism* of D.

Remark. Let $\sigma\colon X \to X'$ be a bimeromorphic morphism of normal surfaces and let D be a non-zero compact reduced divisor on X. If $\dim \sigma(D) = 0$, then D is negative definite. When $D = \sigma^{-1}D'$ for a reduced divisor D' on X', D is negative definite (resp. negative semi-definite) if and only if D' is so.

Lemma 2.1. *Let D be a compact and connected reduced divisor on a normal surface X such that D is negative semi-definite but not negative definite. Then there is an effective divisor F on X such that* $\operatorname{Supp} F = D$ *and that $FD_i = 0$ for any prime component D_i of D. If C is a compact reduced divisor on X such that $C \cap D$ is a non-empty finite set, then $C + D$ is not negative semi-definite.*

Proof. By considering eigenvalues of the intersection matrix $(D_i D_j)_{i,j}$, we can find a non-zero divisor F on X such that $\operatorname{Supp} F \subset D$ and $FD_i = 0$ for any $1 \le i \le k$. We write $F = G - H$ for two effective divisors G and H having no common prime components. Then $G^2 = GH = H^2$ by $FG = FH = 0$. Since $GH \ge 0$ and D is negative semi-definite, we have $G^2 = GH = H^2 = 0$. In particular, $\operatorname{Supp} G \cap \operatorname{Supp} H = \emptyset$. We may assume that $G \ne 0$ by replacing F with $-F$ if necessary. If $\Gamma \cap G$ is a non-empty finite set for a prime component Γ of D, then $(\Gamma + mG)^2 = \Gamma^2 + 2m\Gamma G > 0$ for $m \gg 0$, violating the negative semi-definite property of D. Since D is connected, we have $\operatorname{Supp} G = D$ and $F = G$. If C is a compact reduced divisor such that $C \cap D$ is a non-empty finite set, then $(C + mF)^2 = C^2 + 2mCF > 0$ for $m \gg 0$, which shows the last assertion. □

Lemma 2.2. *Let D be a compact simple normal crossing divisor on a non-singular surface M forming a linear chain of rational curves. Then D is negative semi-definite if and only if there exist a bimeromorphic morphism $\phi\colon M \to N$ to a non-singular surface N and a compact simple normal crossing divisor E on N forming a linear chain of rational curves satisfying one of conditions* (1)–(6) *below, where $E = E_1 + E_2 + \cdots + E_n$ is a prime decomposition with a dual graph*

$$\overset{E_1}{\bullet} \text{———} \overset{E_2}{\bullet} \text{——} \quad \cdots \quad \text{——} \overset{E_n}{\bullet}$$

and where ϕ is either an isomorphism or a succession of blowings up at nodes of inverse images of E:

(1) $n = 1$ *and* $E_1^2 \leq 0$;
(2) $n = 2$ *and* $E_1^2 = E_2^2 = -1$;
(3) $n \geq 2$ *and* $E_i^2 \leq -2$ *for any* $1 \leq i \leq n$;
(4) $n \geq 2$, $E_1^2 = -1$, *and* $E_i^2 \leq -2$ *for any* $1 < i \leq n$;
(5) $n \geq 3$, $E_1^2 = E_n^2 = -1$, $E_i^2 \leq -2$ *for any* $1 < i < n$, *and* $E_k^2 \leq -3$ *for some* $1 < k < n$;
(6) $n \geq 3$, $E_1^2 = E_n^2 = -1$, *and* $E_i^2 = -2$ *for any* $1 < i < n$.

Here, D *is not negative definite if and only if* E *satisfies* (1) *with* $E_1^2 = 0$, (2), *or* (6).

Proof. The "if" part and the last assertion are shown as follows. Since $D = \phi^{-1}E$, E is negative semi-definite (resp. negative definite) if and only of D is so. It is well known or easily shown that E is negative semi-definite by conditions (1)–(6). Moreover, E is not negative definite if and only if (1) with $E_1^2 = 0$, (2), or (6) is satisfied. This shows the "if" part and the last assertion.

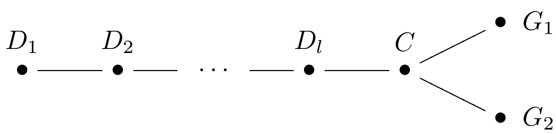
The "only if" part is shown as follows. Now, D is negative semi-definite. If D is irreducible, then it is enough to take ϕ as the identity morphism and (1) holds. If D consists of two prime components, then it is enough to take ϕ as the identity morphism and one of (2), (3), and (4) holds. Thus, we may assume that the number $\boldsymbol{n}(D)$ of prime components of D is greater than 2. Then the union $D^\natural$ of non-end components of D is not zero, and it is negative definite by Lemma 2.1. Let $M \to V$ be the contraction morphism of $D^\natural$ and let $N \to V$ be the minimal resolution of singularities. Then we have a morphism $\phi\colon M \to N$ over V, which is a succession of contractions of (-1)-curves contained in images of $D^\natural$. Then $E = \phi_*(D)$ is a simple normal crossing divisor on N forming a linear chain of rational curves such that

- $D = \phi^{-1}E$, and E is negative semi-definite,
- end components of E are images of end components of D under ϕ,
- ϕ is a succession of blowings up at nodes of inverse images of E,
- any non-end component of E is not a (-1)-curve.

Therefore, E satisfies one of conditions (2)–(6). Thus, we are done. $\square$

Lemma 2.3. *Let* Z *be a compact simple normal crossing divisor on a non-singular surface* M *with a dual graph*

(II-1)



for a prime decomposition $Z = D_1 + D_2 + \cdots + D_l + C + G_1 + G_2$, *where* $l > 0$. *Assume that every prime component of* Z *is a non-singular rational curve with negative self-intersection number and that* $G_1^2 = G_2^2 = -2$, *and* $C^2 \leq -2$. *Then the following hold*:

(1) *Assume that $l = 1$ or $D_i^2 \leq -2$ for any $2 \leq i \leq l$. Then Z is negative semi-definite. If Z is not negative definite, then $D_1^2 = -1$ and $D_i^2 = C^2 = -2$ for any $2 \leq i \leq l$.*

(2) *Assume that Z is not negative semi-definite and that $\sum_{i=1}^{l} D_i$ is negative definite. Then there is an integer $1 \leq a < l$ such that $\sum_{i=a+1}^{l} D_i + C + G_1 + G_2$ is not negative definite but negative semi-definite.*

Proof. (1): We set $c := -C^2$ and $d_i := -D_i^2$ for $1 \leq i \leq l$. Then $c \geq 2$, $d_1 \geq 1$, and $d_i \geq 2$ for any $2 \leq i \leq l$. We consider a $\mathbb{Q}$-divisor

$$A = \sum_{i=1}^{l} x_i D_i + yC + z_1 G_1 + z_2 G_2$$

for rational numbers x_i, y, z_1, and z_2. Then

$$\begin{aligned} A^2 &= -\Big(\sum_{i=1}^{l} d_i x_i^2\Big) - cy^2 - 2(z_1^2 + z_2^2) + 2\Big(x_l y + y(z_1 + z_2) + \sum_{i=1}^{l-1} x_i x_{i+1}\Big) \\ &= (1 - d_1)x_1^2 + \Big(\sum_{i=2}^{l} (2 - d_i)x_i^2\Big) + (2 - c)y^2 - (z_1 - z_2)^2 \\ &\qquad - \Big(\sum_{i=1}^{l-1} (x_i - x_{i+1})^2\Big) - (x_l - y)^2 - (y - (z_1 + z_2))^2. \end{aligned}$$

Thus, $A^2 \leq 0$, and Z is negative semi-definite. The equality $A^2 = 0$ holds if and only if there is a rational number u such that

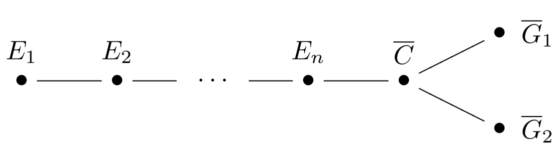
$$u = z_1 = z_2, \quad x_1 = x_i = y = 2u, \quad (1 - d_1)u = (2 - d_i)u = (2 - c)u = 0$$

for any $2 \leq i \leq l$. Thus, if $A^2 = 0$ and $A \neq 0$, then $u \neq 0$, $d_1 = 1$, $d_i = 2$ for any $2 \leq i \leq l$, and $c = 2$. Thus, (1) has been shown.

(2): By Lemma 2.2, there exists a morphism $\phi\colon M \to N$ to a non-singular surface N with a compact simple normal crossing divisor $E = E_1 + E_2 + \cdots + E_n$ forming a linear chain of rational curves such that

- ϕ is an isomorphism or a succession of blowings up at nodes of inverse images of E,
- $D_1 + \cdots + D_l = \phi^{-1}(E)$,
- the linear chain E of rational curves with this order $E_1, E_2, \ldots, E_n$ or the reverse order $E_n, E_{n-1}, \ldots, E_1$ satisfies one of conditions (1), (3), (4), and (5) of Lemma 2.2.

We set $\overline{C} := \phi(C)$ and $\overline{G}_j = \phi(G_j)$ for $j = 1, 2$. We may assume that $E_1 = \phi(D_1)$ and $E_n = \phi(D_l)$. Then $\overline{Z} := \phi_*(Z) = \sum_{i=1}^{n} E_i + \overline{C} + \overline{G}_1 + \overline{G}_2$ has a dual graph

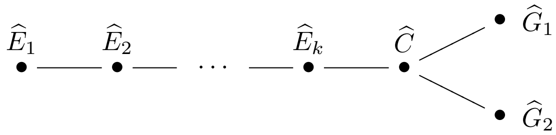


and we have $Z = \phi^{-1}(\overline{Z})$. In particular, $\overline{Z}$ is not negative semi-definite, $E = \sum_{i=1}^{n} E_i$ is negative definite, $\overline{C}^2 = C^2 \leq -2$, and $\overline{G}_j^2 = G_j^2 = -2$ for $j = 1, 2$. By (1) applied to $\overline{Z}$, one of the following conditions is satisfied:

(i) $n \geq 2$, $E_n^2 = -1$, and $E_i^2 \leq -2$ for any $i < n$ (cf. Lemma 2.2(4));

(ii) $n \geq 3$, $E_1^2 = E_n^2 = -1$, $E_i^2 \leq -2$ for any $1 < i < n$, and $E_k^2 \leq -3$ for some $1 < k < n$ (cf. Lemma 2.2(5)).

By considering successive contractions of (-1)-curves in images of $\sum_{i=2}^{n} E_i$, we have a morphism $\psi \colon N \to \widehat{N}$ to a non-singular surface $\widehat{N}$ such that $\widehat{C} := \psi(\overline{C})$ is a negative curve. The pushforward $\widehat{Z} = \psi_*\overline{Z}$ is a compact simple normal crossing divisor $\sum_{i=1}^{k} \widehat{E}_i + \widehat{C} + \widehat{G}_1 + \widehat{G}_2$ consisting of rational curves with a dual graph



in which $k \geq 1$ and the following conditions are satisfied for $\widehat{E} := \sum_{i=1}^{k} \widehat{E}_i$ and for some integer $1 < b \leq l$:

- ψ is a succession of blowings up at nodes of inverse images of $\widehat{E} + \widehat{C}$;
- $E + \overline{C} = \psi^{-1}(\widehat{E} + \widehat{C})$, $E = \psi^{-1}(\widehat{E})$, and $\sum_{i=b}^{l} E_i + \overline{C} = \psi^{-1}(\widehat{C})$;
- $\widehat{E}_1 = \psi(E_1)$, and $\widehat{G}_j = \psi(\overline{G}_j)$ for $j = 1, 2$.

Obviously, $\widehat{G}_j^2 = G_j^2 = -2$ for $j = 1, 2$. We may assume that $\widehat{C}^2 = -1$. In fact, if $\widehat{C}^2 \leq -2$ and if $\widehat{E}_k^2 = -1$, then we can contract $\widehat{E}_k$ to get a similar situation. If $\widehat{C}^2 \leq -2$ and if $\widehat{E}_k^2 \leq -2$, then we can apply (1) to $\widehat{Z}$, which is not negative semi-definite, and as a consequence, $k \geq 2$ and $\widehat{E}_p^2 = -1$ for some $2 \leq p \leq k$; however, in this situation, $E = \psi^{-1}\widehat{E}$ does not satisfy (i) nor (ii): This is a contradiction.

The linear chain $\widehat{C} + \widehat{G}_1 + \widehat{G}_2$ is not negative definite but negative semi-definite. Now, $G_j = \varphi^{-1}(\psi^{-1}(\widehat{G}_j))$ for $j = 1, 2$, and $\sum_{i=a+1}^{l} D_i + C = \varphi^{-1}(\psi^{-1}(C))$ for some $1 \leq a < l$. Thus, $\sum_{i=a+1}^{l} D_i + C + G_1 + G_2$ is not negative definite but negative semi-definite. □

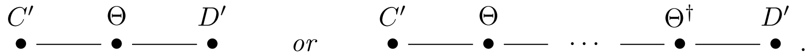
Lemma 2.4. *Let $\mu \colon Y \to X$ be a bimeromorphic morphism of non-singular surfaces whose exceptional locus is $\mu^{-1}(P)$ for a point $P \in X$. Assume that $\mu^{-1}(P)$ contains a unique (-1)-curve Θ and that Θ intersects the proper transform C' in Y of a non-singular prime divisor $C \subset X$ containing P. Then $\mu^{-1}(P)$ is a simple normal crossing divisor forming a linear chain of rational curves such that*

(1) *Θ is an end component of $\mu^{-1}(P)$ with $C'\Theta = 1$, and*
(2) *the other prime component is a (-2)-curve not intersecting C'.*

If D is a non-singular prime divisor on X such that $\{P\} = C \cap D$ and D intersects C transversely, then the following hold for the proper transform D' in Y of D:

(3) *If $\mu^{-1}(P) = \Theta$, then $D'\Theta = 1$ and $C' \cap D' = \emptyset$.*
(4) *If $\mu^{-1}(P) \neq \Theta$, then $D'\Theta^{\dagger} = 1$ for the other end component $\Theta^{\dagger}$ of $\mu^{-1}(P)$, and D' does not intersect C' nor $\mu^{-1}(P) - \Theta^{\dagger}$.*

In other words, $\mu^{-1}(P) \cup C' \cup D'$ has a dual graph



Proof. We set $B := \mu^{-1}(P)$ as a reduced divisor on Y. Then $(K_Y + C')\Theta = -1 + C'\Theta \geq 0$ and $(K_Y + C')\Gamma \geq K_Y\Gamma \geq 0$ for any prime component Γ of $B - \Theta$.

Thus, $K_Y + C'$ is μ-nef and $K_Y + C' = \mu^*(K_X + C) - E$ for a μ-exceptional effective divisor E. On the other hand, $-E$ is effective by the logarithmic ramification formula for the "birational pair" $(C\&X)$ (cf. [27, Prop. 2.1], [11, Part 2, Prop. 1], [19, Prop. 1.40(2)]). This is also deduced from a property that (X, C) is 1-*log-terminal* (cf. [19, Def. 2.1, Rem. 2.3]), since E is Cartier. Thus,

$$K_Y + C' = \mu^*(K_X + C).$$

Consequently, $C'\Theta = 1$, and Γ is a (-2)-curve with $C' \cap \Gamma = \emptyset$ for any prime component Γ of $B - \Theta$. In particular, (2) and the latter part of (1) hold. By shrinking X, we may assume the existence of the prime divisor D above. Then

$$K_Y + C' + D' + B = \mu^*(K_X + C + D) + G$$

for a μ-exceptional effective divisor G by the logarithmic ramification formula (cf. [9, §4, (R)], [10, Thm. 11.5]) or by a property that $(X, C + D)$ is log-canonical. Hence, $\mu^*D = D' + B - G$, and we have $G = 0$. As a consequence, $(Y, C' + D' + B)$ is also log-canonical. Then μ is a *toroidal blowing up* (cf. [18, Def. 4.19, Cor. 4.22]). Hence, B is a linear chain of rational curves with Θ as an end component and it satisfies (3) and (4). □

Here, we recall some well-known properties on 2-dimensional cyclic quotient singularities.

Fact 2.5 (cf. [18, Exam. 3.2]). Let X be a normal surface with a unique singular point P such that (X, P) is a cyclic quotient singularity of order $n \geq 2$. Then (U, D) is *toroidal* at P for an open neighborhood U of P and for a reduced divisor $D \ni P$, i.e., $U \setminus D \hookrightarrow U$ is a *toroidal embedding* at P (cf. [13, II, §1], [18, Def. 3.12]). Let $\mu\colon M \to X$ be the minimal resolution of singularity. This is described as a toric morphism. In particular, $\mu^{-1}(P)$ is a simple normal crossing divisor $\sum_{i=1}^k \Gamma_i$ forming a linear chain of rational curves, and moreover,

is a dual graph of $\mu^{-1}(D)$, where D'_1 and D'_2 are proper transforms of prime components D_1 and D_2 of D locally at P. By Hirzebruch–Jung's method (cf. [8, §3.4]), there exist two series of integers $1 = p_1 < p_2 < \cdots < p_k < n$ and $n > q_1 > q_2 > \cdots > q_k = 1$ such that

$$\text{(II-2)} \qquad \mu^*D_1 = D'_1 + \sum\nolimits_{i=1}^k (q_i/n)\Gamma_i \quad \text{and} \quad \mu^*D_2 = D'_2 + \sum\nolimits_{i=1}^k (p_i/n)\Gamma_i$$

(cf. [15, Lem. 2.7, Rem. 2.9]). In particular, $D_1D_2 = (\mu^*D_1)D'_2 = D'_1(\mu^*D_2) = 1/n$. By the dual graph of $\mu^{-1}(P)$ and by (II-2), we have

$$p_{i-1} + p_1\Gamma_i^2 + p_{i+1} = q_{i-1} + q_i\Gamma_i^2 + q_{i+1} = 0$$

for any $1 \leq i \leq k$, where we set $p_{-1} = q_{k+1} = 0$ and $p_{k+1} = q_{-1} = n$. Since $K_M + D'_1 + D'_2 + \sum_{i=1}^{k} \Gamma_i = \mu^*(K_X + D)$, we have

$$K_M = \mu^* K_X - \sum\nolimits_{i=1}^{k} (1 - (p_i + q_i)/n)\Gamma_i \qquad \text{and}$$
$$K_M + D'_1 = \mu^*(K_X + D_1) - \sum\nolimits_{i=1}^{k} (1 - p_i/n)\Gamma_i$$

by (II-2). Since $p_1 = 1$, the second equality induces an equality

$$(K_X + D_1)|_{D_1} = K_{D_1} + (1 - 1/n)P$$

of $\mathbb{Q}$-divisors on D_1 by adjunction for (M, D'_1) and by the isomorphism $D'_1 \simeq D_1$ induced by μ. This means that $\mathrm{Diff}_{D_1}(0) = (1 - 1/n)P$, where $\mathrm{Diff}_{D_1}(0)$ is the *different* of (X, D_1) in the sense of [14, Prop.-Def. 16.5] (cf. [14, Prop. (16.6.3)]).

Remark 2.6. For integers p_i and q_j above, p_k and q_1 are coprime to n, and $p_k q_1 \equiv 1 \bmod n$ (cf. [15, Lem. 2.7]). Moreover, (X, P) is a cyclic quotient singularity of type (n, p_k) or (n, q_1) in the sense of [18, Exam. 3.2], i.e., it is isomorphic to the germ of the quotient surface of $\mathbb{C}^2$ by the action of an automorphism

$$(\mathtt{x}, \mathtt{y}) \mapsto (\zeta^{p_k}\mathtt{x}, \zeta\mathtt{y}) \qquad (\text{resp.} \quad (\mathtt{x}, \mathtt{y}) \mapsto (\zeta\mathtt{x}, \zeta^{q_1}\mathtt{y}))$$

for a primitive n-th root ζ of unity, where $(\mathtt{x}, \mathtt{y})$ is a coordinate of $\mathbb{C}^2$.

Remark 2.7. In the situation of Fact 2.5, n equals the *numerical factorial index* of X at P (cf. [19, Def. 1.26]). This is shown as follows: We define $\mathbb{Q}$-divisors $E_{(i)}$ for $1 \leq i \leq k$ on M inductively by

$$E_{(1)} = -\sum\nolimits_{i=1}^{k} (q_i/n)\Gamma_i, \quad E_{(2)} = \Gamma_1 + b_1 E_{(1)}, \quad E_{(i+1)} = \Gamma_i + b_i E_{(i)} - E_{(i-1)}$$

for $2 \leq i \leq k-1$ (cf. (II-2)), where $b_i = -(\Gamma_i)^2$. Then $E_{(i)}\Gamma_j = \delta_{i,j}$ for any $1 \leq i, j \leq n$. Hence, n equals the numerical factorial index by [19, Lem. 1.27]. Here, we have $E_{(k)} = -\sum_{i=1}^{k} (p_i/n)\Gamma_i$ by (II-2).

Lemma 2.8. *For X and P in Fact* 2.5*, let C be a prime divisor on X containing P and let C' be the proper transform of C in M. Then (X, C) is* 1-log-terminal *at P in the sense of* [19, Def. 2.1] *if and only if*

$$\text{(II-3)} \qquad \textit{either} \quad C'\Gamma_i = \begin{cases} 1, & \textit{if} \quad i = 1, \\ 0, & \textit{if} \quad i > 0, \end{cases} \quad \textit{or} \quad C'\Gamma_i = \begin{cases} 0, & \textit{if} \quad i < k, \\ 1, & \textit{if} \quad i = k. \end{cases}$$

Assume that (X, C) is 1*-log-terminal with $C'\Gamma_k = 1$. Then the following hold for any prime divisor B on X such that $B \cap C = \{P\}$, where B' is the proper transform of B in M:*

(1) *If $BC = 1/n$, then $(X, B + C)$ is log-canonical at P, $B' \cap C' = \emptyset$, and*

$$\overset{B'}{\bullet} \text{———} \overset{\Gamma_1}{\bullet} \text{———} \overset{\Gamma_2}{\bullet} \text{——} \quad \cdots \quad \text{——} \overset{\Gamma_k}{\bullet} \text{———} \overset{C'}{\bullet}$$

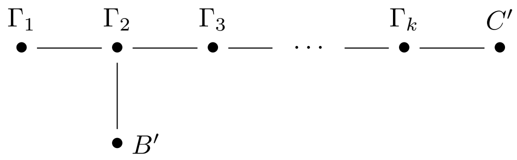
is a dual graph of $\mu^{-1}(B + C)$.

(2) *If $BC = 2/n$, then $(X, (1/2)B + C)$ is log-canonical at P, $B' \cap C' = \emptyset$, and one of the following two cases occurs:*

(a) $B'\Gamma_1 = 2$ *and* $B'\Gamma_i = 0$ *for any* $i > 1$; *In particular,*

is a dual graph of $\mu^{-1}(B + C)$;

(b) $k \geq 2$, $\Gamma_1^2 = -2$, $B'\Gamma_2 = 1$, *and* $B'\Gamma_i = 0$ *for any* $i \neq 2$, *and* $p_2 = 2$; *In particular,*



is a dual graph of $\mu^{-1}(B + C)$.

(3) *If* $k \geq 2$, $\Gamma_1^2 = -2$, $B'\Gamma_2 = 1$, *and* $B'\Gamma_i = 0$ *for any* $i \neq 2$, *then* $BC = 2/n$.

Proof. If (X, C) is 1-log-terminal at P, then, by [19, Fact 2.5], we may assume that $C = D_1$ or $C = D_2$ for the divisor D in Fact 2.5, and hence, (II-3) holds. Assume that the second case of (II-3) holds. Then

$$\mu^* C = C' + \sum\nolimits_{i=1}^{k} (p_i/n)\Gamma_i \tag{II-4}$$

by (II-2) in Fact 2.5. Furthermore, by an argument in Fact 2.5, we have

$$K_M + C' = \mu^*(K_X + C) - \sum\nolimits_{i=1}^{k} (1 - q_i/n)\Gamma_i,$$

and it implies that (X, C) is 1-log-terminal at P, since $1 - q_i/n < 1$ for any i. This proves the first assertion.

In the second assertion, if $BC = 1/n$ (resp. $BC = 2/n$), then $(X, B + C)$ (resp. $(X, (1/2)B + C)$) is log-canonical at P by "inversion of adjunction" (cf. [14, Thm. 17.7]), since

$$\mathrm{Diff}_C(B) = B|_C + (1 - 1/n)P = P$$
$$\bigl(\text{resp.}\quad \mathrm{Diff}_C((1/2)B) = (1/2)B|_C + (1 - 1/n)P = P\bigr)$$

by Fact 2.5. In case $BC = 1/n$, $(X, B+C)$ is toroidal at P by [19, Fact 2.5], and we have (1) by Fact 2.5, since we may assume that $D_1 = B$ and $D_2 = C$. If $BC = 2/n$, then one of two conditions (i) and (ii) below holds by $n \geq 2$, $B' \cap \mu^{-1}(P) \neq \emptyset$, and the equality

$$BC = B'(\mu^* C) = B'C' + \sum\nolimits_{i=1}^{k} (p_i/n) B'\Gamma_i$$

induced by (II-4):

(i) $B'C' = 0$, $(B'\Gamma_1, B'\Gamma_2) = (2, 0)$, and $B'\Gamma_i = 0$ for any $i > 2$;
(ii) $B'C' = 0$, $(B'\Gamma_1, B'\Gamma_2) = (0, 1)$, $p_2 = 2$, and $B'\Gamma_i = 0$ for any $i > 2$.

Consequently, $B' \cap C' = \emptyset$, and (2a) holds in case (i). Moreover, (2b) holds in case (ii), since we have $\Gamma_1^2 = -2$ by the equality $p_0 + p_1\Gamma_1^2 + p_2 = 0$ in Fact 2.5. Thus, (2) has been proved. In the situation of (3), we have $BC = B'\mu^* C = p_2/n$ and $0 = \Gamma_1 \mu^* C = -2/n + p_2/n$ by (II-4). In particular, $p_2 = 2$ and $BC = 2/n$. Thus, we are done. □

2.2. **Singularities on $\mathbb{P}^1$-fibrations.** For a $\mathbb{P}^1$-fibration $\pi\colon X \to T$ from a normal surface X to a non-singular curve T, we shall study the structure of X under certain conditions on singularity of pairs $(X, C + F_{\mathrm{red}})$ for a fiber F of π and for a prime divisor C on X which is either a section or a *double section* of π. Here, C is called a double section if $\pi|_C\colon C \to T$ is a finite morphism of degree 2. We fix the $\mathbb{P}^1$-fibration $\pi\colon X \to T$ throughout Section 2.2. We begin with:

Remark 2.9. The vanishing $R^1\pi_*\mathcal{O}_X = 0$ holds: This is well known and is proved by the same method as in the proof of [18, Prop. 2.33(2)]. As a consequence, $H^1(E, \mathcal{O}_E) = 0$ for any effective divisor E on X contained in fibers of π, since $R^1\pi_*\mathcal{O}_X \to R^1\pi_*\mathcal{O}_E$ is surjective. In particular, every prime component of any fiber is a non-singular rational curve, and if a fiber F is reduced and irreducible, then $F \simeq \mathbb{P}^1$ and π is smooth along F (cf. [18, Prop. 2.33(3), (4)]).

Remark 2.10. Suppose that X is non-singular. Then every reducible fiber of π contains a (-1)-curve. For, if not, $K_X\Gamma \geq 0$ for any prime component Γ on the reducible fiber, since $\Gamma^2 < 0$, but it implies that $K_XF \geq 0$ for a general fiber F, contradicting that π is a $\mathbb{P}^1$-fibration.

Lemma 2.11. *For any $t \in T$, $\pi^{-1}(t) \cap X_{\mathrm{reg}}$ is a simple normal crossing divisor. Moreover, if $\pi^{-1}(t)$ is reducible, then the following hold for any prime component Γ of $\pi^{-1}(t)$:*

(1) *If a connected and reduced divisor D on X is contained in $\pi^{-1}(t) - \Gamma$, then $\#D \cap \Gamma \leq 1$. If $D \cap \Gamma \subset X_{\mathrm{reg}}$ in addition, then $D\Gamma \leq 1$.*
(2) *If $\Gamma \subset X_{\mathrm{reg}}$ and if $\Gamma^2 = -1$, then $\#(\pi^{-1}(t) - \Gamma) \cap \Gamma = (\pi^{-1}(t) - \Gamma)\Gamma \leq 2$.*

Proof. If $\pi^{-1}(t)$ is irreducible, then it is isomorphic to $\mathbb{P}^1$ by Remark 2.9. Thus, we may assume that $\pi^{-1}(t)$ is reducible. Assume that $D \cap \Gamma \subset X_{\mathrm{reg}}$ for a prime component Γ of $\pi^{-1}(t)$ and the divisor D in (1). Then D is Cartier along $D \cap \Gamma$, and we have an the exact sequence

$$0 \to \mathcal{O}_\Gamma(-D) = \mathcal{O}_X(-D) \otimes \mathcal{O}_\Gamma \to \mathcal{O}_{\Gamma+D} \to \mathcal{O}_D \to 0.$$

Then $H^1(\Gamma, \mathcal{O}_\Gamma(-D)) = 0$ by $H^0(D, \mathcal{O}_D) \simeq \mathbb{C}$ and $H^1(X, \mathcal{O}_{D+\Gamma}) = 0$ (cf. Remark 2.9). In particular, $-1 \leq \deg \mathcal{O}_\Gamma(-D) = -\Gamma D$. This shows the latter half of (1). The inequality $D\Gamma \leq 1$ for arbitrary such Γ and D implies that $\pi^{-1}(t) \cap X_{\mathrm{reg}}$ is a simple normal crossing divisor (cf. [18, Rem. 2.34]). For the first half of (1), let us consider the minimal resolution $\mu\colon M \to X$ of singularities and the proper transform Γ' of Γ in M. Then $D' = \mu^{-1}(D)$ is connected, and $\Gamma' + D'$ is contained in the fiber over t of the $\mathbb{P}^1$-fibration $\pi \circ \mu\colon M \to T$. Thus, $\Gamma'D' \leq 1$ by the latter half of (1) for $\pi \circ \mu$. Hence, $\Gamma \cap D = \mu(\Gamma' \cap D')$ consists of at most one point, and we have proved (1).

In the situation of (2), $\pi^{-1}(t)$ is normal crossing along Γ, and $\#(\pi^{-1}(t) - \Gamma) \cap \Gamma = (\pi^{-1}(t) - \Gamma)\Gamma$. Let $\phi\colon X \to \widehat{X}$ be the contraction morphism of the (-1)-curve Γ. Then there is a $\mathbb{P}^1$-fibration $\hat{\pi}\colon \widehat{X} \to T$ such that $\pi = \hat{\pi} \circ \phi$. Here, ϕ is the blowing up at a non-singular point $\widehat{P}$ of $\widehat{X}$, and the fiber $\hat{\pi}^{-1}(t)$ is normal crossing at $\widehat{P}$. This implies that $\#(\pi^{-1}(t) - \Gamma) \cap \Gamma \leq 2$, and we have proved (2). $\square$

Lemma 2.12. *For a point $t \in T$ and an integer $m > 1$, assume that*

- *there is a cyclic cover $\tau\colon T' \to T$ of degree m from a non-singular curve T' such that $\tau^*(t) = mt'$ for a point $t' \in T'$, and*
- *$F = mF_{\mathrm{red}}$ for the fiber $F = \pi^*(t)$.*

Let X' be the normalization of $X \times_T T'$ with morphisms $\nu\colon X' \to X$ and $\pi'\colon X' \to T'$ induced by projections, which make a commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\ \nu\ } & X \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ T' & \xrightarrow{\ \tau\ } & T. \end{array}$$

Then $F' = \pi'^*(t')$ is reduced and ν is étale in codimension 1 along F'. Here, (X, F_{red}) is 1-log-terminal along F_{red} if and only if π' is smooth along F'.

Proof. We may assume that τ is étale outside t' by shrinking T. Then the first assertion is a consequence of [20, Lem. 4.2]. Now, we have an equality $K_{X'} + F' = \nu^*(K_X + F_{\mathrm{red}})$. By [19, Lem. 2.10(2), Prop. 2.12(2)], (X, F_{red}) is 1-log-terminal along F_{red} if and only if (X', F') is 1-log-terminal along F'. If π' is smooth along F', then (X', F') is 1-log-terminal along F'. Conversely if (X', F') is 1-log-terminal along F', then F' is non-singular (cf. [19, Fact 2.5]), and π' is smooth along F'. Thus, we are done. □

Lemma 2.13. *Let C be a section of π and F a fiber of π such that $(X, C + F_{\mathrm{red}})$ is log-canonical at a point $P \in C \cap F_{\mathrm{red}}$. Then the prime component Γ of F intersecting C is unique, and moreover:*

(1) *If $P \in X_{\mathrm{reg}}$, then $\mathrm{mult}_\Gamma F = 1$, and if $P \in \operatorname{Sing} X$, then $\mathrm{mult}_\Gamma F$ equals the order of the cyclic quotient singularity (X, P).*

Assume that $P \in \operatorname{Sing} X$ and that F is irreducible, i.e., $F_{\mathrm{red}} = \Gamma$. Then:

(2) *The pair (X, F_{red}) is 1-log-terminal along F_{red}, and $F_{\mathrm{red}} \cap \operatorname{Sing} X \setminus \{P\}$ consists of one point at which X has a cyclic quotient singularity of the same order as (X, P).*

(3) *If $C^\dagger$ is another section of π such that $C \cap C^\dagger \cap F = \emptyset$, then $(X, C + C^\dagger + F_{\mathrm{red}})$ is log-canonical along F_{red}.*

Proof. Since $(X, C + F_{\mathrm{red}})$ is *toroidal* at P (cf. [19, Fact. 2.5(1)]), F_{red} is locally irreducible at P. Thus, Γ is unique. If $P \in X_{\mathrm{reg}}$, then $CF = C\Gamma = 1$, and hence, $\mathrm{mult}_\Gamma F = 1$. If $P \in \operatorname{Sing} X$, then (X, P) is a cyclic quotient singularity of order $m > 1$ such that $C\Gamma = 1/m$, since $(X, C + F_{\mathrm{red}})$ is toroidal at P; thus, $m = \mathrm{mult}_\Gamma F$ by $CF = 1$. This proves the first assertion and (1). Next, we shall prove (2) and (3), where $P \in \operatorname{Sing} X$ and $F = m\Gamma$ for $m > 1$.

(2): By replacing T with an open neighborhood of $t := \pi(P)$, we have a cyclic cover $\tau\colon T' \to T$ from a non-singular curve T' such that $\tau^*(t) = mt'$ for a point $t' \in T'$ and that τ is branched only at t. For morphisms $\nu\colon X' \to X$ and $\pi'\colon X' \to T'$ in Lemma 2.12 defined by τ, we know that ν is étale in codimension 1 and that the fiber $F' = \pi'^*(t')$ is reduced. Hence,

$$K_{X'} + C' + F' = \nu^*(K_X + C + F_{\mathrm{red}})$$

for the section $C' := \nu^* C$ of π', and $(X', C'+F')$ is log-canonical at the point $\{P'\} = \nu^{-1}(P) = C' \cap F'$ by [19, Lem. 2.10(1)]. This implies that F' is irreducible, since each prime component of F' contains P'. Hence, $F' \simeq \mathbb{P}^1$ and π' is smooth along F' (cf. Remark 2.9). Then (X, F_{red}) is 1-log-terminal along F_{red} by Lemma 2.12, and it implies that $(X, C + F_{\mathrm{red}})$ is log-canonical along F_{red}. Since

$$(K_X + C + F_{\mathrm{red}})F_{\mathrm{red}} = (1/\deg\nu)(K_{X'} + C' + F')F' = -1/m < 0,$$

the rest of (2) follows from [18, Prop. 3.29] concerning conditions (E)–(H) there.

(3): We have $C^\dagger F_{\mathrm{red}} = (1/m)C^\dagger F = 1/m$. Let $P^\dagger$ be the intersection point of $C^\dagger$ and F_{red}. Then $P^\dagger \in \operatorname{Sing} X$ and $(X, P^\dagger)$ is a cyclic quotient singularity of order m by (2). Moreover, the equality $C^\dagger F_{\mathrm{red}} = 1/m$ implies that $(X, C^\dagger + F_{\mathrm{red}})$ is log-canonical at $P^\dagger$ by Lemma 2.8(1). Thus, we have (3) by (2). □

Lemma 2.14. *Let C be a section of π such that*

(∗) *$(X, C + G_t)$ is log-canonical at Q_t for any $t \in \mathcal{S}(\pi) := \pi(C \cap \operatorname{Sing} X)$,*

where $G_t := \pi^{-1}(t)$ and $\{Q_t\} := C \cap G_t$. Then the following hold:

1. *For any $t \in \mathcal{S}(\pi)$, there is an integer $m_t > 1$ such that $CG_t = 1/m_t$ and that (X, Q_t) is a cyclic quotient singularity of order m_t.*
2. *The divisor $K_X + C + \sum_{t\in\mathcal{S}(\pi)} G_t$ is Cartier along C.*
3. *One has the following equalities of $\mathbb{Q}$-divisors on C*:

$$(K_X + C + \sum\nolimits_{t\in\mathcal{S}(\pi)} G_t)|_C = K_C + \sum\nolimits_{t\in\mathcal{S}(\pi)} Q_t, \tag{II-5}$$

$$(K_X + C)|_C = K_C + \sum\nolimits_{t\in\mathcal{S}(\pi)} (1 - m_t^{-1})Q_t. \tag{II-6}$$

Proof. We have (1) by (∗) and by Lemma 2.13. The divisor $K_X + C + \sum_{t\in\mathcal{S}(\pi)} G_t$ is Cartier at Q_t for any $t \in \mathcal{S}(\pi)$ by (∗), since Q_t is a node of $C + G_t$. On the other hand, π is smooth along $C \cap X_{\mathrm{reg}} = C \setminus \pi^{-1}\mathcal{S}(\pi)$, and $C \cap X_{\mathrm{reg}}$ is non-singular. Hence, $K_X + C + \sum_{t\in\mathcal{S}(\pi)} G_t$ is Cartier along C; thus (2) holds. For (3), we have (II-6) by (II-5), since $m_t G_t|_C = Q_t$ as a divisor on C. Thus, it is enough to prove (II-5).

Let $\mu\colon M \to X$ be the minimal resolution of singularities lying on C and let E_t be the exceptional divisor $\mu^{-1}(Q_t)$ for $t \in \mathcal{S}(\pi)$. Then

$$K_M + C' + \sum\nolimits_{t\in\mathcal{S}(\pi)} (G'_t + E_t) = \mu^*(K_X + C + \sum\nolimits_{t\in\mathcal{S}(\pi)} G_t) \tag{II-7}$$

for the proper transforms C' and G'_t of C and G_t in M, respectively, since μ is a toroidal blowing up (cf. [18, §4.3]). Here, $C' \cap G'_t = \emptyset$ and $E_t|_{C'}$ is identified with Q_t by the isomorphism $C' \simeq C$ induced by μ. Thus, we have (II-5) by (II-7) and by adjunction for (M, C'). □

Proposition 2.15. *Let C_1 and C_2 be two sections of π and let F be a fiber of π such that*

(i) *$C_1 \cap C_2 \cap F = \emptyset$,*
(ii) *$(X, C_1 + C_2 + F_{\mathrm{red}})$ is log-canonical at $C_1 \cap F_{\mathrm{red}}$, and*
(iii) *$(K_X + C_1 + C_2 + F_{\mathrm{red}})\Theta \le 0$ for any prime component Θ of F.*

Then F_{red} *is a linear chain of rational curves,* $(X, C_1 + C_2 + F_{\mathrm{red}})$ *is log-canonical along* F_{red}*, and the following hold*:

(1) $K_X + C_1 + C_2 + F_{\mathrm{red}}$ *is Cartier along* F_{red};
(2) $\mathcal{O}_X(K_X + C_1 + C_2 + F_{\mathrm{red}}) \otimes \mathcal{O}_{F_{\mathrm{red}}} \simeq \mathcal{O}_{F_{\mathrm{red}}}$.

Proof. For the intersection point P_1 of C_1 and F, let Γ be a prime component of F containing P_1. Then Γ is unique by (ii) and Lemma 2.13. Let $\varphi\colon X \to \overline{X}$ be the contraction morphism of $F_{\mathrm{red}} - \Gamma$, where φ is the identity morphism if $F_{\mathrm{red}} = \Gamma$. Let $\overline{\pi}\colon \overline{X} \to T$ be the induced $\mathbb{P}^1$-fibration such that $\pi = \overline{\pi} \circ \varphi$. We set $\overline{C}_i := \varphi(C_i)$ for $i = 1,\ 2$, $\overline{P}_1 = \varphi(P_1)$, $\overline{\Gamma} := \varphi(\Gamma)$, and $\overline{F} := \overline{\pi}^*(\pi(P_1))$. Then $\overline{F} = m\overline{\Gamma}$ for $m := \mathrm{mult}_\Gamma F$, and

$$K_X + C_1 + C_2 + F_{\mathrm{red}} = \varphi^*(K_{\overline{X}} + \overline{C}_1 + \overline{C}_2 + \overline{\Gamma}) + E \qquad \text{(II-8)}$$

for a φ-exceptional effective $\mathbb{Q}$-divisor E by (iii). Since φ is an isomorphism over an open neighborhood of $\overline{P}_1$, we see that $(\overline{X}, \overline{C}_1 + \overline{C}_2 + \overline{\Gamma})$ is log-canonical at $\overline{P}_1$ by (ii) and that $\overline{P}_1 \not\in \overline{C}_2$, i.e., $\overline{C}_1 \cap \overline{C}_2 \cap \overline{\Gamma} = \emptyset$. Thus, $(\overline{X}, \overline{C}_1 + \overline{C}_2 + \overline{\Gamma})$ is log-canonical along $\overline{\Gamma}$ by Lemma 2.13(3) applied to $\overline{\pi}$. Now $(K_{\overline{X}} + \overline{C}_1 + \overline{C}_2 + \overline{\Gamma})\overline{\Gamma} = 0$ by $\overline{F} = m\overline{\Gamma}$, and hence, $\Gamma \cap \operatorname{Supp} E = \emptyset$ and $(K_X + C_1 + C_2 + F_{\mathrm{red}})\Gamma = 0$ by (iii) and (II-8). If a prime component Θ of $F_{\mathrm{red}} - \Gamma$ is not contained in $\operatorname{Supp} E$, then $(K_X + C_1 + C_2 + F_{\mathrm{red}})\Theta = 0$ and $\Theta \cap \operatorname{Supp} E = \emptyset$ by (iii) and (II-8). Since F_{red} is connected, we have $E = 0$. Therefore,

$$K_X + C_1 + C_2 + F_{\mathrm{red}} = \varphi^*(K_{\overline{X}} + \overline{C}_1 + \overline{C}_2 + \overline{\Gamma}).$$

Consequently, $(X, C_1 + C_2 + F_{\mathrm{red}})$ is log-canonical along F_{red} and $(K_X + C_1 + C_2 + F_{\mathrm{red}})\Theta = 0$ for any prime component Θ of F. Then F_{red} is a linear chain of rational curves by [18, Lem. 4.5]. Moreover, we have (1) and (2) by [18, Prop. 3.29(C)] and by the canonical isomorphism

$$\mathrm{Pic}(F_{\mathrm{red}}) \simeq \prod\nolimits_{\Theta \subset F_{\mathrm{red}}} \mathrm{Pic}(\Theta)$$

of Picard groups (cf. [1, Thm. (1.7)], [18, Rem. 4.2]). □

Lemma 2.16. *Let* C *be a double section of* π*. For a fiber* F *of* π *and a point* $P \in F$*, assume that*

(i) $C \cap F_{\mathrm{red}} = \{P\}$,
(ii) $(X, C + F_{\mathrm{red}})$ *is log-canonical at* P*, and*
(iii) $(K_X + C + F_{\mathrm{red}})\Gamma \le 0$ *for any prime component* Γ *of* F_{red}.

Then $P \in C_{\mathrm{reg}}$, F_{red} *is a linear chain of rational curves,* $(X, C + F_{\mathrm{red}})$ *is log-canonical along* F_{red}*, and* $(K_X + C + F_{\mathrm{red}})\Gamma = 0$ *for any prime component* Γ *of* F_{red}*. Moreover,* $\Sigma := (F_{\mathrm{red}})_{\mathrm{reg}} \cap \operatorname{Sing} X \setminus \{P\}$ *is not contained in any non-end component of* $C + F_{\mathrm{red}}$*, and* Σ *consists of either*

(a) *two* A_1*-singular points of* X*, or*
(b) *one point at which* (X, F_{red}) *is a log-canonical singularity of* type $\mathcal{D}$ *in the sense of* [18, Def. 3.23]*, i.e.,* (X, F_{red}) *is log-canonical but not* 1*-log-terminal at the point.*

Consequently, $K_X + C + F_{\mathrm{red}}$ *is not Cartier at* Σ*, but* $2(K_X + C + F_{\mathrm{red}})$ *is Cartier along* F_{red} *with an isomorphism*

$$\mathcal{O}_X(2(K_X + C + F_{\mathrm{red}})) \otimes_{\mathcal{O}_X} \mathcal{O}_{F_{\mathrm{red}}} \simeq \mathcal{O}_{F_{\mathrm{red}}}.$$

Proof. By (ii), C is non-singular at P. By shrinking T, we may assume that C is non-singular, π is smooth over $T \setminus \{t\}$, where $t := \pi(P)$, and the double cover $\pi|_C : C \to T$ is branched only at t. We rewrite $\pi|_C : C \to T$ as a morphism $\tau : T' \to T$, and let X' be the normalization of $X \times_T T'$. Then we have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\nu} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ T' & \xrightarrow{\tau} & T \end{array}$$

for the induced $\mathbb{P}^1$-fibration $\pi' : X' \to T'$ and the induced double cover $\nu : X' \to X$, and we have two sections C'_1 and C'_2 of π' such that $\nu^* C = C'_1 + C'_2$. Now, $\tau^*(t) = 2t'$ for a point $t' \in T'$. We set $F' := \pi'^*(t')$. Then

$$K_{X'} + C'_1 + C'_2 + F'_{\mathrm{red}} = \nu^*(K_X + C + F_{\mathrm{red}}) \tag{II-9}$$

by [19, Lem. 1.39] as ν is étale over $X \setminus F_{\mathrm{red}}$. Hence, $(X, C'_1 + C'_2 + F'_{\mathrm{red}})$ is log-canonical along $\nu^{-1}(P)$ by (ii) and [19, Lem. 2.10(1)]. In particular, $C'_1 \cap C'_2 \cap F'_{\mathrm{red}} = \emptyset$ (cf. [19, Fact. 2.5]). Moreover, $C'_1 \cap C'_2 = \emptyset$, since $\pi|_C : C \to T$ is étale over $T \setminus \{t\}$. By (iii), $(K_{X'} + C'_1 + C'_2 + F'_{\mathrm{red}})\Gamma' \le 0$ for any prime component Γ' of F'. Hence, $(X', C'_1 + C'_2 + F'_{\mathrm{red}})$ is log-canonical along F'_{red}, and

$$(K_{X'} + C'_1 + C'_2 + F'_{\mathrm{red}})\Gamma' = 0$$

for any prime component Γ' of F' by Proposition 2.15 applied to $\pi' : X' \to T'$, C'_1, C'_2, and F'. Then $(X, C + F_{\mathrm{red}})$ is log-canonical along F_{red} by (II-9) and [19, Prop. 2.12(1)], and we have $(K_X + C + F_{\mathrm{red}})\Gamma = 0$ for any prime component Γ of F. As a consequence, F_{red} is a linear chain of rational curves by [18, Lem. 4.5].

Let Γ_0 be a prime component of F containing P, which is unique, since $(X, C + F_{\mathrm{red}})$ is log-canonical at P (cf. Lemma 2.13). Here, $(K_X + F_{\mathrm{red}})\Gamma_0 = -C\Gamma_0 < 0$, and $(K_X + F_{\mathrm{red}})\Gamma = -CF = 0$ for any prime component Γ of $F_{\mathrm{red}} - \Gamma_0$. In particular, if F_{red} is reducible, then Γ_0 is an end component of F_{red}, by [18, Lem. 4.5(3)].

If F is irreducible, i.e., $F_{\mathrm{red}} = \Gamma_0$, then $\Sigma \subset \Gamma_0$, and either (a) or (b) holds by [18, Prop. 3.29] concerning conditions (G), and (H) there, since $(K_X + C + F_{\mathrm{red}})\Gamma = 0$. If F is reducible, then Σ is contained in the other end component of the linear chain F_{red}, and either (a) or (b) holds by [18, Prop. 3.29] concerning (C), (G), and (H) there. In both cases, Σ is not contained in any non-end component of $C + F_{\mathrm{red}}$. The last assertion on $K_X + C + F_{\mathrm{red}}$ is a consequence of [18, Prop. 3.29, Rem. 4.2]. □

2.3. **PDS configurations.** We introduce the notion of *PDS configurations* for the study of double sections of a $\mathbb{P}^1$-fibration in Definition 2.17 below. We shall discuss relations among irreducible PDS configurations, basic PDS configurations, and standard PDS configurations (cf. Definition 2.20).

Definition 2.17. Let $\pi\colon X \to T$ be a $\mathbb{P}^1$-fibration from a normal surface X to a non-singular curve T. Let C be a double section of π and let D be a set-theoretic fiber of π. We say that $(\pi\colon X \to T, C, D)$ is a *practical double section configuration*, or a *PDS configuration* for short, if $C \cap D$ consists of one point P and the following conditions are satisfied:

(1) (X, C) is 1-log-terminal at P;
(2) $CD \in \{1/n, 2/n\}$ for the numerical factorial index n of X at P;
(3) a prime component B of D intersecting C is unique.

The point P and its image $\pi(P)$ are called the *intersection point* and the *base point* of $(X/T, C, D)$, respectively. The prime divisor B in (3) is called the *distinguished component* of D. The integer n in (2) is called the *index* of $(X/T, C, D)$. If $CD = 1/n$ (resp. $CD = 2/n$) in (2), then $(X/T, C, D)$ is called a PDS configuration of *type* I_n (resp. II_n), or *type* I (resp. II), for simplicity. The PDS configuration $(X/T, C, D)$ is said to be *irreducible* if D is irreducible.

Lemma 2.18. *Let $(\pi\colon X \to T, C, D)$ be a PDS configuration of index n with the intersection point $P \in X$, the base point $o = \pi(P) \in T$, and the distinguished component B of D. Then*:

(1) *Every prime component of D is isomorphic to $\mathbb{P}^1$, and $D \cap X_{\mathrm{reg}}$ is a simple normal crossing divisor. In particular, D is locally irreducible at P.*
(2) *The curve C is non-singular at P. If $P \in X_{\mathrm{reg}}$, then $n = 1$; if $P \in \operatorname{Sing} X$, then n equals the order of the cyclic quotient singularity (X, P).*
(3) *There is an open neighborhood $\mathcal{U}$ of o in T such that π is smooth over $\mathcal{U} \setminus \{o\}$ and that $\pi|_C\colon C \to T$ is étale over $\mathcal{U} \setminus \{o\}$. In particular, $C \cap \pi^{-1}\mathcal{U}$ is non-singular.*
(4) *If $(X/T, C, D)$ is of type I_n (resp. II_n), then $\operatorname{mult}_B \pi^*(o) = 2n$ (resp. $\operatorname{mult}_B \pi^*(o) = n$).*
(5) *Assume that D is reducible and let $\phi\colon X \to \overline{X}$ be the contraction morphism of $D - B$. Then $(\overline{X}/T, \overline{C}, \overline{D})$ is an irreducible PDS configuration of the same type as $(X/T, C, D)$ for $\overline{C} = \phi(C)$ and $\overline{D} = \phi_* D$.*

Proof. Assertion (1) follows from Remark 2.9 and Definition 2.17(3). We have (2) by (1), Definition 2.17(1), and Remark 2.7. For (3), it is enough to set $\mathcal{U}$ to be the complement of $\mathcal{S} \setminus \{o\}$ in T, where $\mathcal{S}$ is the set of points $t \in T$ such that $\pi^*(t)$ or $(\pi|_C)^*(t)$ is not smooth. We have (4) by $2 = C\pi^*(o) = (\operatorname{mult}_B \pi^*(o))CD$. Assertion (5) holds trivially by definition, since ϕ is an isomorphism along C. □

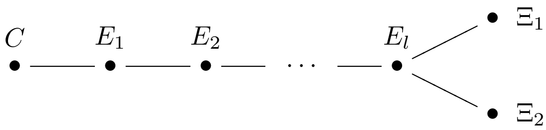
Lemma 2.19. *For a PDS configuration $(X/T, C, D)$, it is of type* I *if and only if $(X, C + D)$ is log-canonical at the intersection point.*

Proof. Let n be the index of $(X/T, C, D)$ and let P be the intersection point. If the type is I, i.e., $CD = 1/n$, then $(X, C + D)$ is log-canonical at P by Lemma 2.8(1). Assume that the type is II, i.e.. $CD = 2/n$. If $P \in X_{\mathrm{reg}}$, then $(X, C + D)$ is not log-canonical at P, since C intersects D tangentially at P. Thus, we may assume that $P \in \operatorname{Sing} X$. Then (X, P) is a cyclic quotient singularity of order $n > 1$ (cf. Lemma 2.18(2)), and we can apply Lemma 2.8(2) to (X, C, P) and D, since D

is locally irreducible at P (cf. Lemma 2.18(1)). Let $\mu\colon M \to X$ be the minimal resolution of singularity (X, P) and let B' be the proper transform in M of the distinguished component B of D. If $(X, C + D)$ is log-canonical at P, then $\mu^{-1}(P)$ is a linear chain of rational curves, and B' intersects $\mu^{-1}(P)$ transversely at just one point contained in an end component of the linear chain. However, this contradicts Lemma 2.8(2). Hence, if the type is II, then $(X, C + D)$ is not log-canonical at P. □

Definition 2.20. A PDS configuration $(X/T, C, D)$ is said to be *standard* if $D \subset X_{\mathrm{reg}}$ and if D is a simple normal crossing divisor expressed as $E + \Xi_1 + \Xi_2$ for a linear chain E of rational curves and for two (-2)-curves Ξ_1 and Ξ_2 such that

(II-10)



is a dual graph of $C + D$, where E_1, E_2, ..., E_l are prime components of E. If E is irreducible, then $(X/T, C, D)$ is said to be *basic*.

Remark. The standard PSD configuration is of type I_1, where E_1 is just the distinguished component of D. The linear chain E of rational curves contains a (-1)-curve by Remark 2.10. If $(X/T, C, D)$ is basic, then $\pi^*(o) = 2E + \Xi_1 + \Xi_2$ for the $\mathbb{P}^1$-fibration $\pi\colon X \to T$ and for the base point o.

Lemma 2.21. *For the standard PDS configuration* $(X/T, C, D)$ *in Definition* 2.20*, the following hold*:

1. *The* $\mathbb{Q}$*-divisor* $K_X + C + E + (1/2)(\Xi_1 + \Xi_2)$ *is* numerically trivial on D*, i.e.,*
$$\big(K_X + C + E + (1/2)(\Xi_1 + \Xi_2)\big)\Gamma = 0$$
for any prime component Γ *of* D.
2. *If* E_l *is a* (-1)*-curve, then* $(X/T, C, D)$ *is basic.*
3. *If* E *is reducible and if* E_1 *is a unique* (-1)*-curve contained in* E*, then* E_i *is a* (-2)*-curve for any* $i > 1$.
4. *If* E *is reducible, then there exists a basic PDS configuration* $(Y/T, C_Y, D_Y)$ *with a bimeromorphic morphism* $\phi\colon X \to Y$ *over* T *such that* $\phi(C) = C_Y$ *and* $\phi_*D = D_Y$ *and that* ϕ *is a succession of blowings up at nodes of inverse images of* $\phi(C + E)$.

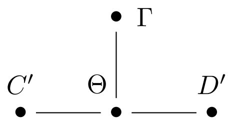
Proof. We have (1) from the dual graph (II-10) in Definition 2.20 by calculation of intersection numbers. If E_l is a (-1)-curve, then $E_l + \Xi_1 + \Xi_2$ is not negative definite, and hence, $D = E_l + \Xi_1 + \Xi_2$, i.e., E is irreducible. This proves (2). Assertion (3) follows from Lemma 2.3(1). We shall show (4), where E is reducible. Since E is negative definite, by Lemma 2.2, we have a bimeromorphic morphism $\varphi\colon X \to X^\dagger$ to a normal surface $X^\dagger$ such that

- $E^\dagger := \varphi_*(E)$ is a linear chain of rational curves contained in $(X^\dagger)_{\mathrm{reg}}$,
- E satisfies one of three conditions (3), (4), and (5) of Lemma 2.2,

- φ is a succession of blowings up at nodes of inverse images of $E^\dagger$.

In particular, $E^\dagger$ is reducible, $\varphi(E_1)$ and $\varphi(E_l)$ are the end components of $E^\dagger$. Moreover, a $\mathbb{P}^1$-fibration $X^\dagger \to T$ is induced and $(X^\dagger/T, C^\dagger, D^\dagger)$ is a standard PDS configuration for $C^\dagger = \varphi(C)$ and $D^\dagger = \varphi_* D$. Here, $D^\dagger = E^\dagger + \Xi_1^\dagger + \Xi_2^\dagger$ for (-2)-curves $\Xi_1^\dagger = \varphi(\Xi_1)$ and $\Xi_2^\dagger = \varphi(\Xi_2)$. There is a (-1)-curve contained in $E^\dagger$ by Remark 2.10. If $\varphi(E_l)$ is a (-1)-curve, then $E^\dagger = \varphi(E_l)$ by (2), contradicting the reducibility of $E^\dagger$. Hence, $\varphi(E_l)$ is not a (-1)-curve, $E^\dagger$ satisfies Lemma 2.2(4), and $\varphi(E_1)$ is a unique (-1)-curve in $E^\dagger$. By (3), every prime component of $E^\dagger - \varphi(E_1)$ is a (-2)-curve. Let $X^\dagger \to Y$ be the contraction morphism of $E^\dagger - \varphi(E_l)$ and let $\phi\colon X \to Y$ be the composite with φ. Then $(Y/T, \phi(C), \phi_*(D))$ is a basic PDS configuration for the induced $\mathbb{P}^1$-fibration $Y \to T$. Thus, we are done. □

Remark 2.22. Basic PDS configurations and irreducible PDS configurations of type II_1 are connected by bimeromorphic morphisms as follows: Let $(X/T, C, D)$ be an irreducible PDS configuration of type II_1. Then the $\mathbb{P}^1$-fibration $X \to T$ is smooth along D, and D intersects C tangentially at the intersection point P. In particular, $D \subset X_{\mathrm{reg}}$. Let $\mu\colon M \to X$ be the composite of two blowings up at points lying over P such that $\mu^{-1}(C + D)$ is normal crossing and let C' and D' be the proper transforms of C and D in M, respectively. Then $\mu^{-1}D = D' + \Gamma + \Theta$ for a (-2)-curve Γ and a (-1)-curve Θ, and

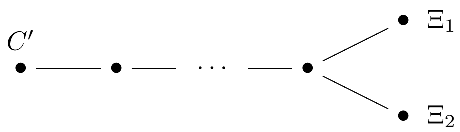


is a dual graph of $\mu^{-1}(C+D)$. Hence, $(M/T, C', \mu^{-1}D)$ is a basic PDS configuration with Θ as the distinguished component. Note that if C is compact, then

$$C^2 = (C')^2 + 2 \tag{II-11}$$

by construction. Conversely, every basic PDS configuration is obtained from an irreducible PDS configuration of type II_1 by this process. In fact, if $(M/T, \widetilde{C}, \widetilde{D})$ is a basic PDS configuration with two (-2)-curves Ξ_1 and Ξ_2 in $\widetilde{D}$, then the contraction morphism of $\widetilde{D} - \Xi_1$ (or $\widetilde{D} - \Xi_2$) produces an irreducible PDS configuration of type II_1.

Lemma 2.23. *Let $(\pi\colon Y \to T, C, D)$ be an irreducible PDS configuration of type* I *and let $\mu\colon M \to Y$ be the minimal resolution of singularities lying on D. Then $(M/T, C', D_M)$ is a standard PDS configuration for the proper transform C' of C in M and for $D_M = \mu^{-1}D$, in which the proper transform D in M is a unique (-1)-curve contained in D_M. In particular, $C' + D_M$ has a dual graph*

(II-12) 

for two (-2)-curves Ξ_1 and Ξ_2. Moreover, $(Y, C+D)$ is log-canonical along D, and the following hold for the index n of $(Y/T, C, D)$ and the set $\Sigma = (D \setminus C) \cap \operatorname{Sing} Y$:

(1) *If* $\#\Sigma = 2$*, then* $(M/T, C', D_M)$ *is basic and* Σ *consists of two* A_1*-singular points.*
(2) *If* $n > 1$*, then* Σ *consists of one point at which* (Y, D) *is log-canonical* of type $\mathcal{D}$ (*cf.* [18, Def. 3.23]).
(3) *If* $n = 1$ *and* $\#\Sigma \neq 2$*, then* Σ *consists of a rational double point of type* D_m *for some* $m \geq 3$*.*

Proof. Let P and $o = \pi(P)$, respectively, be the intersection point and the base point of $(Y/T, C, D)$. Then $CD = 1/n$, $(Y, C + D)$ is log-canonical at P and $\pi^*(o) = 2nD$ by Lemma 2.19. By applying Lemma 2.16 to $F_{\text{red}} = D$, we see that $(Y, C + D)$ is log-canonical along D, and $\Sigma = D \cap \operatorname{Sing} Y \setminus \{P\}$ consists of either

- two A_1-singular points, or
- one point at which (Y, D) is log-canonical of type $\mathcal{D}$.

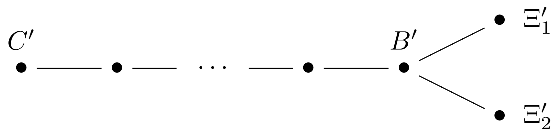
By well-known descriptions of minimal resolutions of such singularities (cf. [12, Thm. 9.6], [18, Thm. 3.22, Fig. 2]), we have a dual graph of $\mu^{-1}(C+D) = C' + D_M$ as (II-12). As a consequence, $(M/T, C', D_M)$ is a standard PDS configuration. For the proper transform D' of D in M, every prime component of $D_M - D'$ is not a (-1)-curve, since it is μ-exceptional and μ is minimal. Thus, D' is a (-1)-curve (cf. Remark 2.10). The remaining assertions are shown as follows.

(1): If $\#\Sigma = 2$, then Σ consists of two A_1-singular point; hence, D' equals the prime component of D intersecting $\Xi_1 + \Xi_2$, and $(M/T, C', D_M)$ is basic by Lemma 2.21(2).

(2): If $n > 1$, then $D' \neq B$, and we have $\#\Sigma \neq 2$ by (1). Thus, (2) follows from the possibility of Σ above.

(3): Assume that $n = 1$. Then $P \notin \operatorname{Sing} Y$, and D' equals the distinguished component B of D_M. If $\#\Sigma \neq 2$, then every prime component of $D_M - B$ is a (-2)-curve by Lemma 2.21(3), and hence, Σ consists of one D_m-singular point for the number m of prime components of $D_M - B$. Thus, we are done. ∎

Example 2.24. By Lemmas 2.21 and 2.23, every irreducible PDS configuration of type I is obtained from a basic PDS configuration by the following method: Let $(\pi \colon M \to T, C, D)$ be a basic PDS configuration with the distinguished component B of D and prime components Ξ_1 and Ξ_2 of $D - B$. Let $\beta \colon M' \to M$ be an isomorphism or a succession of blowings up whose center in each step is a node of the inverse image of $C + D$ contained in a (-1)-curve in the inverse image of D; in particular, the center is lying over the intersection point P of C and D. Then $\beta^{-1}(D)$ contains a unique (-1)-curve Γ, and $\beta^{-1}(C + D)$ has a dual graph



for the proper transforms C', B', Ξ'_1, and Ξ'_2 of C, B, Ξ_1, and Ξ_2 in M', respectively. Here, if β is not an isomorphism, then $\Gamma \neq B'$. By construction, $(M'/T, C', \beta^{-1}D)$ is a standard PDS configuration and β is the morphism in Lemma 2.21(4) for

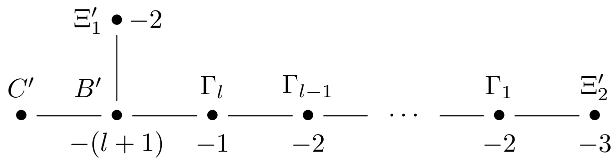
$(M'/T, C', \beta^{-1}D)$ when β is not an isomorphism. Let $\alpha\colon M' \to Y$ be the contraction morphism of $\beta^{-1}(D + \Xi_1 + \Xi_2) - \Gamma$, and set $C_Y := \alpha(C')$ and $D_Y = \alpha_*(\beta^{-1}D) = \alpha(\Gamma)$. Then α is the minimal resolution of singularities of Y lying on D_Y, and $(Y, C_Y + D_Y)$ is log-canonical along D_Y, since

$$K_{M'} + C' + E' + (1/2)(\Xi_1 + \Xi_2) = \alpha^*(K_Y + C_Y + D_Y)$$

for the linear chain $E' = \beta^{-1}D - \Xi'_1 - \Xi'_2$ of rational curves (cf. Lemma 2.21(1)). By Lemma 2.19, $(Y/T, C_Y, D_Y)$ is an irreducible PDS configuration of type I.

Example 2.25. We shall explain a method producing an irreducible PDS configuration of type II_n for an odd integer $n > 1$ from a basic PDS configuration by bimeromorphic morphisms. Let $(\pi\colon M \to T, C, D)$ be a basic PDS configuration with the distinguished component B and (-2)-curves Ξ_1 and Ξ_2 in D. Then B is a (-1)-curve and $\pi^*(o) = 2B + \Xi_1 + \Xi_2$ for the base point $o \in T$. Let $\beta\colon M' \to M$ be a succession of blowings up whose center in each step is a node of the inverse image of $B + \Xi_2$ contained in the proper transform of B. Assume that β is not an isomorphism. Then $\beta^{-1}(C + D)$ is a simple normal crossing divisor with a dual graph

(II-13)



for the proper transforms C', B', Ξ'_1, and Ξ'_2 in M of C, B, Ξ_1, and Ξ_2, respectively, where l is the number of point blowings up, $\Gamma_1, \ldots, \Gamma_l$ are the β-exceptional prime divisors, and -1, -2, -3, $-(l+1)$ indicate self-intersection numbers. From the dual graph (II-13), we have

$$2B' + \Xi'_1 + \Xi'_2 + \sum\nolimits_{i=1}^{l} (2i+1)\Gamma_i = \beta^*(2B + \Xi_1 + \Xi_2), \tag{II-14}$$

$$\begin{aligned} K_{M'} + C' + B' + (1/2)\bigl(\Xi'_1 + \Xi'_2 + \sum\nolimits_{i=1}^{l} \Gamma_i\bigr) \\ = \beta^*(K_M + C + B + (1/2)(\Xi_1 + \Xi_2)) \end{aligned} \tag{II-15}$$

by calculation. Let $\alpha\colon M' \to X$ be the contraction morphism of $\beta^{-1}(D) - \Gamma_l$ and let $\pi_X : X \to T$ be the induced $\mathbb{P}^1$-fibration such that $\pi_X \circ \alpha = \pi \circ \beta$. Note that α is the minimal resolution of singularities of X. We set $C_X := \alpha(C')$, $D_X := \alpha_*(\beta^{-1}D) = \alpha(\Gamma_l)$, and $\{P_X\} := C_X \cap D_X$. Then $\alpha^{-1}(P_X) = \Xi'_1 + B'$, and $(X, C_X + D_X)$ is not log-canonical at P_X by the dual graph (II-13). However, we have

$$K_{M'} + C' + B' + (1/2)\bigl(\Xi'_1 + \Xi'_2 + \sum\nolimits_{i=1}^{l} \Gamma_i\bigr) = \alpha^*(K_X + C_X + (1/2)D_X)$$

by (II-15), which implies that $(X, C_X + (1/2)D_X)$ is log-canonical at P_X and (X, C_X) is 1-log-terminal at P_X. On the other hand, we have

$$\alpha^* C_X = C' + \frac{1}{2l+1}(\Xi'_1 + 2B') \quad \text{and} \quad \pi_X^*(o) = (2l+1)D_X$$

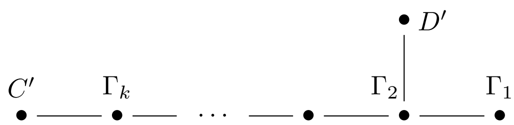
by (II-13) and (II-14). Hence, the order n of the cyclic quotient singularity (X, P_X) equals $2l+1$ and $C_X D_X = 2/n$. Therefore, $(X/T, C_X, B_X)$ is an irreducible PDS configuration of type II_n for $n = 2l+1$.

Theorem 2.26. *Every irreducible PDS configuration $(X/T, C, D)$ of type II_n for $n > 1$ is obtained from a basic PDS configuration $(M/T, C_M, D_M)$ by the method of Example* 2.25. *In particular, the following hold*:

(1) *The integer n is odd, (X, P) is a cyclic quotient singularity of type $(n, 2)$ (cf. Remark* 2.6) *for the intersection point P.*
(2) *There is a point $P^\dagger \in D$ such that $(D \setminus C) \cap \operatorname{Sing} X = \{P^\dagger\}$, (X, D) is* 1*-log-terminal at $P^\dagger$, and $(X, P^\dagger)$ is a cyclic quotient singularity of type $(n, n-2)$.*
(3) *If C is compact, then $C^2 = (C_M)^2 + 2/n$.*

Proof. Let $\pi\colon X \to T$ be the $\mathbb{P}^1$-fibration and let $o = \pi(P)$ be the base point. We shall prove Theorem 2.26 by the following seven steps.

Step 1 (On the minimal resolution of singularities of X along D). Let $\alpha\colon M' \to X$ be the minimal resolution of singularities lying on D and let C' and D' be the proper transforms of C and D in M', respectively. We also set $D_{M'} := \alpha^{-1}D$. Since (X, C) is 1-log-terminal at P and since $CD = 2/n$ for the numerical factorial index n of X at P, by applying Lemma 2.8, we see that $\alpha^{-1}(P)$ is a linear chain $\sum_{i=1}^{k} \Gamma_i$ of rational curves and that either (2a) or (2b) of Lemma 2.8 occurs for (D', C') instead of (B', C'). If Lemma 2.8(2a) occurs, then Γ_1 and D' are prime components of $\alpha^{-1}(\pi^{-1}(o))$ with $D'\Gamma_1 = 2$: This contradicts Lemma 2.11(1). Thus, Lemma 2.8(2b) occurs. In particular, $k = \boldsymbol{n}(\alpha^{-1}(P)) \geq 2$, $\Gamma_1^2 = -2$, $\Gamma_2 D' = 1$, $C' \cap D' = \emptyset$, and $\Gamma_i \cap D' = \emptyset$ for any $i \neq 2$; hence,



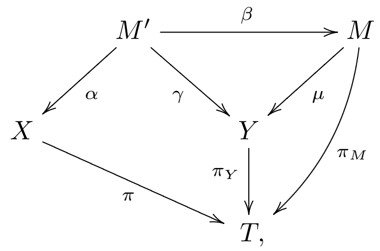
is a dual graph of $C' + D' + \alpha^{-1}(P)$. Since α is minimal, any prime component of $D_{M'} = \alpha^{-1}(\pi^{-1}(o))$ except D' has self-intersection number ≤ -2. Thus, D' is a (-1)-curve (cf. Remark 2.10).

Step 2 (Constructing some bimeromorphic morphisms). We set

$$E := \alpha^{-1}D - \alpha^{-1}(P) = D_{M'} - \sum\nolimits_{i=1}^{k} \Gamma_i$$

as a connected reduced divisor on M'. If $D' \neq E$, then $\alpha(E - D') \subset D \cap \operatorname{Sing} X \setminus \{P\}$. Let $\gamma\colon M' \to Y$ be the contraction morphism of E and set $\{Q\} := \gamma(E)$. Let $\mu\colon M \to Y$ be the minimal resolution of the singularity at Q. Then $\mu \circ \beta = \gamma$ for a bimeromorphic morphism $\beta\colon M' \to M$ by the minimality, and β and γ are isomorphisms along $C' + \sum_{i \neq 2} \Gamma_i$. We have $\mathbb{P}^1$-fibrations $\pi_Y : Y \to T$ and $\pi_M : M \to$

T with a commutative diagram



where α, β, γ, and μ are isomorphisms over $T \setminus \{o\}$. We set

$$C_Y := \gamma(C'), \qquad D_Y := \gamma_* D_{M'} = \pi_Y^{-1}(o), \qquad \Gamma_{Y,i} := \gamma(\Gamma_i).$$
$$C_M := \beta(C'), \qquad D_M := \beta_* D_{M'} = \pi_M^{-1}(o), \qquad \Gamma_{M,i} := \beta(\Gamma_i)$$

for $1 \le i \le k$. Then $D_Y = \sum_{i=1}^k \Gamma_{Y,i}$, $D_M = \sum_{i=1}^k \Gamma_{M,i} + \beta_* E$, C_Y (resp. C_M) is a double section of π_Y (resp. π_M), $Q \in \Gamma_{Y,2}$, $Q \notin \Gamma_{Y,i}$ for any $i \neq 2$, $E = \gamma^{-1}(Q)$, and $\beta_* E = \mu^{-1}(Q)$.

Step 3. We shall show that $D_Y \cap \operatorname{Sing} Y = \{Q\}$. Assume the contrary. Then Y is non-singular along D_Y, and $(\Gamma_{Y,i})^2 = \Gamma_i^2 \le -2$ for any $i \neq 2$. Thus, $\Gamma_{Y,2}$ is a (-1)-curve (cf. Remark 2.10). Since $(\Gamma_{Y,1})^2 = -2$ and since the linear chain $\sum_{i=1}^k \Gamma_{Y,i}$ of rational curves is negative semi-definite but not negative definite, the integer k is equal to 3, and $\Gamma_{Y,3}$ is a (-2)-curve by Lemma 2.2. However, in this case, $\pi_Y^*(o) = \Gamma_{Y,1} + 2\Gamma_{Y,2} + \Gamma_{Y,3}$, and it implies that $C_Y \pi_Y^*(o) = C_Y \Gamma_{Y,3} = 1$, contradicting that C_Y is a double section of π_Y. Therefore, $D_Y \cap \operatorname{Sing} Y = \{Q\}$.

Step 4 ($k = \boldsymbol{n}(\alpha^{-1}(P)) = 2$). Any prime component of D_M except $\Gamma_{M,2}$ has self-intersection number ≤ -2 by the minimality of μ and by $(\Gamma_{M,i})^2 = \Gamma_i^2 \le -2$ for any $i \neq 2$. Thus, $\Gamma_{M,2}$ is a (-1)-curve (cf. Remark 2.10). Since $D_Y \cap \operatorname{Sing} Y = \{Q\}$ by Step 3, there is a prime component Ξ of $\beta_* E \subset D_M$ such that $\mu(\Xi) = \{Q\}$ and $\Xi \cap \Gamma_{M,2} \neq \emptyset$. Here, $\Xi\Gamma_{M,2} = 1$ by Lemma 2.11(1), and $\Xi\Gamma_{M,1} = 0$ by $\beta_* E \cap \Gamma_{M,1} = \beta(E \cap \Gamma_1) = \emptyset$. If $k \ge 3$, then three prime divisors $\Gamma_{M,1}$, $\Gamma_{M,3}$, and Ξ intersect the (-1)-curve $\Gamma_{M,2}$: this contradicts Lemma 2.11(2). Therefore, $k = 2$. In particular, $D_M = \Gamma_{M,1} + \Gamma_{M,2} + \beta_*(E)$ and $D_Y = \Gamma_{Y,1} + \Gamma_{Y,2}$.

Step 5 (Proofs of (1) and (3)). The equality $k = 2$ implies that

$$\alpha^* C = C' + (1/n)\Gamma_1 + (2/n)\Gamma_2,$$

since Lemma 2.8(2b) occurs for (D', C') (cf. Step 1, Fact 2.5). Then n is odd and (X, P) is a cyclic quotient singularity of type $(n, 2)$ by Remark 2.6; hence (1) holds. Moreover, we have

$$\text{(II-16)} \qquad \Gamma_2^2 = -(n+1)/2$$

by $0 = \Gamma_2(\alpha^* C) = 1 + 1/n + (2/n)\Gamma_2^2$. Similarly, if C is compact, then

$$C^2 = C'(\alpha^* C) = (C')^2 + (2/n)C'\Gamma_2 = (C_M)^2 + 2/n.$$

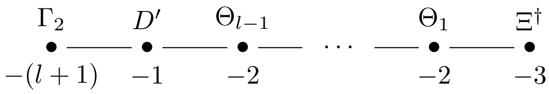
Thus, (3) has been proved.

Step 6. We shall show that Q is an A_1-singular point of Y. Let $\rho\colon Y \to \overline{Y}$ be the contraction morphism of the (-2)-curve $\Gamma_{Y,1}$ and let $\overline{\pi}\colon \overline{Y} \to T$ be the induced $\mathbb{P}^1$-fibration such that $\pi_Y = \overline{\pi} \circ \rho$. We set $\overline{C} := \rho(C_Y)$ and $\overline{D} := \rho_* D_Y = \rho(\Gamma_{Y,2}) = \overline{\pi}^{-1}(o)$, and define three points Q_0, Q_1, and Q_2 of $\overline{Y}$ by

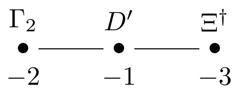
$$\{Q_0\} := \rho(C_Y \cap D_Y) = \overline{C} \cap \overline{D}, \quad \{Q_1\} := \rho(\Gamma_{Y,1}), \quad \text{and} \quad Q_2 := \rho(Q).$$

Then $\overline{C}$ is a double section of $\overline{\pi}$, $(\overline{Y}, \overline{C} + \overline{D})$ is log-canonical at Q_0, and $(K_{\overline{Y}} + \overline{C} + \overline{D})\overline{D} = 0$. Moreover, $\overline{D} \cap \operatorname{Sing} \overline{Y} \setminus \{Q_0\} = \{Q_1, Q_2\}$. By Lemma 2.16, we see that $(\overline{Y}, \overline{C} + \overline{D})$ is log-canonical along $\overline{D}$, and Q_2 is a A_1-singular point. Therefore, Q is an A_1-singular point of Y, since ρ is an isomorphism outside $\Gamma_{Y,1}$.

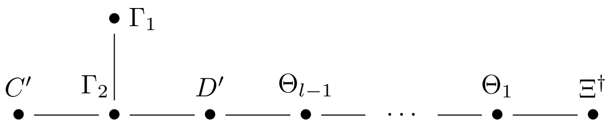
Step 7 (Final step). By Step 6, the μ-exceptional locus $\mu^{-1}(Q)$ equals the prime divisor Ξ introduced in Step 4, and it is a (-2)-curve. Then $\beta_* E = \Xi$ and $D_M = \Gamma_{M,1} + \Gamma_{M,2} + \Xi$. Therefore, $(M/T, C_M, D_M)$ is a basic PDS configuration. Let $\Xi^\dagger$ be the proper transform in Ξ in M'. It is a prime component of $E = \gamma^{-1}(Q) = \beta^{-1}\Xi$. Note that D' is a unique (-1)-curve contained in E, $D' \cap \Gamma_2 \neq \emptyset$, and $(E - D') \cap \Gamma_2 = \emptyset$ (cf. Step 1). By Lemma 2.4, E is a linear chain of rational curves with a prime decomposition $E = \sum_{i=0}^{l} \Theta_i$ for an integer $l > 0$ with $\Theta_0 = \Xi^\dagger$, $\Theta_l = D'$ such that



is a dual graph of $\Gamma_2 + E$, where -1, -2, -3, and $-(l+1)$ indicate self-intersection numbers, since $\Xi^2 = -2$ and $(\Gamma_{M,2})^2 = -1$. Note that the dual graph is



when $l = 1$. In particular, $n = 2l + 1$ by $-\Gamma_2^2 = (n+1)/2 = l + 1$ (cf. (II-16) in Step 5). Hence,



is a dual graph of $\alpha^{-1}(C + D)$, and $(X/T, C, D)$ is obtained from the basic PDS configuration $(M/T, C_M, D_M)$ by the method of Example 2.25. The dual graph of $\Gamma_2 + E$ above implies that $(D \setminus C) \cap \operatorname{Sing} X = \{P^\dagger\}$ for the image $P^\dagger$ of $E - D' = \Xi^\dagger + \sum_{i=1}^{l-1} \Theta_i$ under $\alpha\colon M' \to X$. By calculation, we have

$$\mu^* D = (1/n)\Gamma_1 + (2/n)\Gamma_2 + D' + (1/n)\Xi^\dagger + \sum\nolimits_{i=1}^{l-1} ((2i+1)/n)\Theta_i.$$

Hence, $(X, P^\dagger)$ is a cyclic quotient singularity of type $(2l+1, 2l-1) = (n, n-2)$ (cf. Fact 2.5) even in case $l = 1$, and moreover, (X, D) is 1-log-terminal at $P^\dagger$ by Lemma 2.8, since D' intersects $\alpha^{-1}(P^\dagger)$ transversely one point of an end component of the linear chain $\alpha^{-1}(P^\dagger) = \Xi^\dagger + \sum_{i=1}^{l-1} \Theta_i$ of rational curves. Thus, (2) has been proved, and the proof of Theorem 2.26 has been completed. □

Corollary 2.27. *For an irreducible PDS configuration* $(X/T, C, D)$*, if* $\#(D \setminus C) \cap \operatorname{Sing} X > 1$*, then it is of type* I_1*,* $D \cap \operatorname{Sing} X$ *consists of two* A_1*-singular points, and* $(M/T, C', \mu^{-1}D)$ *is a basic PDS configuration for the minimal resolution* $\mu\colon M \to X$ *of singularities and the proper transform* C' *of* C *in* M*.*

Proof. The type is not II by Theorem 2.26(2) (cf. Remark 2.22). Then the assertion follows from (1) and (2) of Lemma 2.23. □

Proposition 2.28. *Let* $\pi\colon X \to T \simeq \mathbb{P}^1$ *be a* $\mathbb{P}^1$*-fibration from a normal projective rational surface* X *and let* C *be a double section. For two points* t_1*,* $t_2 \in T$*, assume that* π *is smooth over* $T \setminus \{t_1, t_2\}$ *and that* $(X/T, C, D_i)$ *is a PDS configuration for any* $i = 1$*, 2, where* $D_i := \pi^{-1}(t_i)$*.*

1. *If* $(X/T, C, D_i)$ *is basic for any* $i = 1$*, 2, then* $(X, C + D_1 + D_2)$ *is an* H-surface (*cf.* [18, Def. 7.7]); *In particular,* $C^2 = 0$*.*
2. *If* $(X/T, C, D_i)$ *is irreducible of type* I *for any* $i = 1$*, 2, then* $(X, C+D_1+D_2)$ *is a half-toric surface.*
3. *If* $(X/T, C, D_1)$ *is basic and if* $(X/T, C, D_2)$ *is of type* II_n*, then* $C^2 = 2/n$*.*
4. *If* D_1 *is irreducible with* $\#(D_1 \setminus C) \cap \operatorname{Sing} Y > 1$ *and if* $(X/T, C, D_2)$ *is of type* II_n*, then* $C^2 = 2/n$*.*

Proof. (1): This follows from Definition 2.20 and [18, Lem. 7.8].

(2): Let $\mu\colon M \to X$ be the minimal resolution of singularities. For the proper transform C' of C in M, $(M/T, C', \mu^{-1}D_i)$ is a standard PDF configuration for $i = 1$, 2 by Lemma 2.23. Moreover, by Lemma 2.21(4), there is a birational morphism $\phi\colon M \to Y$ of non-singular surfaces over T such that

- $(Y/T, C_Y, D_{Y,i})$ is a basic PDS configuration for $i = 1$, 2, where $C_Y = \phi(C')$ and $D_{Y,i} = \phi_*(\mu^{-1}D_i)$,
- ϕ is a succession of blowings up at nodes of inverse images of $C_Y + D_{Y,1} + D_{Y,2}$ lying over $C_Y \cap (D_{Y,1} + D_{Y,2})$.

By (1), $(Y, C_Y + D_{Y,1} + D_{Y,2})$ is an H-surface. Let $\rho_Y : Y \to \overline{Y}$ be the contraction morphisms of four (-2)-curves in $D_{Y,1} + D_{Y,2}$ and let $\rho_M : M \to \overline{M}$ be the contraction morphism of the inverse images of these four (-2)-curves. Then $(\overline{Y}, \overline{E})$ is a half-toric surface of Picard number 2 for $\overline{E} := \rho_{Y*}(C_Y + D_{Y,1} + D_{Y,2})$ by [18, Prop. 7.15]. By construction, there exist birational morphisms $\bar{\phi}\colon \overline{M} \to \overline{Y}$ and $\bar{\mu}\colon \overline{M} \to X$ such that

- $\rho_Y \circ \phi = \bar{\phi} \circ \rho_M$ and $\bar{\mu} \circ \rho_M = \mu$,
- $\bar{\phi}$ is a toroidal blowing up with respect to the log-canonical pair $(\overline{Y}, \overline{E})$,
- $\bar{\phi}^{-1}\overline{E} = \rho_{M*}(C' + \mu^{-1}D_1 + \mu^{-1}D_2)$ and $\bar{\mu}_*(\bar{\phi}^{-1}\overline{E}) = C + D_1 + D_2$.

Hence, $(\overline{M}, \bar{\phi}^{-1}\overline{E})$ and $(X, C + D_1 + D_2)$ are half-toric surfaces by [18, Lem. 7.2(2), (3)].

(3): We may also assume that $D_2 = \pi^{-1}(t_2)$ is irreducible by replacing X with the surface obtained by contracting prime components of D_2 not intersecting C (cf.

Lemma 2.18(5)). By Theorem 2.26 and Remark 2.22, there exists a bimeromorphic map $\phi\colon X \cdots\to M$ to a non-singular surface M over T such that

- ϕ is an isomorphism over $T \setminus \{t_2\}$,
- $(M/T, C', \pi_M^{-1}(t_i))$ is a basic PDS configuration for $i = 1, 2$,
- $C^2 = (C')^2 + 2/n$,

where C' is the proper transform of C in M and $\pi_M\colon M \to T$ is the structure morphism. Here, $(C')^2 = 0$ by (1). Thus, $C^2 = 2/n$.

(4): By Corollary 2.27, $D_1 \cap \operatorname{Sing} X$ consisting of two A_1-singular points and $(Y/T, C', D_{Y,1})$ is a basic PDS configuration for the minimal resolution $\nu\colon Y \to X$ of the A_1-singularities, the proper transform C' of C in Y, and $D_{Y,1} := \nu^{-1}(D_1)$. Here, $(C')^2 = C^2$. Hence, by replacing Y with X, we may assume that $(X/T, C, D_1)$ is a basic PDS configuration. Then (4) follows from (3). Thus, we are done. □

2.4. **Existence of $\mathbb{P}^1$-fibrations.** We shall present some sufficient conditions for the existence of a $\mathbb{P}^1$-fibration on a normal surface.

Lemma 2.29. *Let X be a normal Moishezon surface with an effective divisor D such that*

(i) $\emptyset \neq \operatorname{Supp} D \subset X_{\mathrm{reg}}$,
(ii) $DD_i = 0$ *for any prime component* D_i *of* D,
(iii) D_{red} *is a simple normal crossing divisor forming a linear chain of rational curves.*

Then X is projective and there is a $\mathbb{P}^1$-fibration $\pi\colon X \to T$ to a non-singular projective curve T such that $D = m\pi^*(t)$ for a point $t \in T$ and a positive integer m and that $\pi^*(t)$ is reduced along the end components of D_{red}.

Proof. The reduced divisor D_{red} is not negative definite but negative semi-definite. In fact, if it is negative definite, then $D = 0$ by (ii), contradicting (i). If it is not negative semi-definite, then there is an effective divisor P supported on D_{red} such that $P^2 > 0$, where $DP = D^2 = 0$ by (ii), and we have $D = 0$ by the Hodge index theorem, contradicting (i). Let $X \to X'$ be the blowing down of a (-1)-curve which is a non-end component of D_{red}. Then the image of D_{red} is a simple normal crossing divisor in X'_{reg} forming a linear chain of rational curves, and D is the pullback of an effective divisor on X' by (ii). Thus, for the proof, we may replace X with X'. Hence, by Lemma 2.2, we may assume one of the following for the prime decomposition $D_{\mathrm{red}} = \sum_{i=1}^{k} D_i$:

(a) $k = 1$, i.e., D_{red} is irreducible;
(b) $k = 2$, and D_1 and D_2 are (-1)-curves;
(c) $k \geq 3$, D_1 and D_k are (-1)-curves being end components of D_{red}, and $D_i^2 = -2$ for any $2 \leq i \leq k-1$.

In cases (a) and (b), $D = mD_{\mathrm{red}}$ for some $m > 0$. In case (c), $D = m(D_1 + 2(D_2 + \cdots + D_{k-1}) + D_k)$ for some $m > 0$. In cases (b) and (c), we can consider the contraction morphism $\rho\colon X \to \overline{X}$ of the linear chain $D_{\mathrm{red}} - D_1$, where $\overline{X}$ is normal and $\overline{D} := \rho(D_{\mathrm{red}}) = \rho(D_1) \subset \overline{X}_{\mathrm{reg}}$. By construction, $\overline{D} \simeq \mathbb{P}^1$, $\overline{D}^2 = 0$, and

$D = m\rho^*(\overline{D})$. Therefore, for the proof, by replacing X with $\overline{X}$, we may assume that $D \simeq \mathbb{P}^1$ and $D^2 = 0$.

Let $\mu\colon M \to X$ be the minimal resolution of singularities. Then $\mu^*D \simeq \mathbb{P}^1$ and $(\mu^*D)^2 = 0$. Suppose that there is a surjective morphism $f\colon M \to T$ to a projective curve T which contracts μ^*D to a point. By Stein factorization, we may assume that f is a fibration and T is non-singular. Then the μ-exceptional locus is contained in a union of fibers of f, since it is away from μ^*D. Hence, there is a fibration $\pi\colon X \to T$ such that $f = \pi \circ \mu$. Since $D^2 = 0$, there is a point $t \in T$ such that $D = \pi^{-1}(t)$. Let c be a positive integer such that $\pi^*(t) = cD$. For a general fiber F of π, we have $K_X F = cK_X D = -2c$. Thus, $c = 1$ and $F \simeq \mathbb{P}^1$. Hence, π is a $\mathbb{P}^1$-fibration and $\pi^*(t) = D$. As a consequence, X is projective by [18, Prop. 2.33(1)].

Therefore, we may assume that X is a non-singular projective surface, $D \simeq \mathbb{P}^1$, and $D^2 = 0$, and it suffices to construct a surjection $\pi\colon X \to T$ to a projective curve T which contracts D to a point. If $H^1(X, \mathcal{O}_X) \neq 0$, then $D \simeq \mathbb{P}^1$ is contracted to a point by the non-trivial Albanese morphism of X; hence, we have such a morphism $X \to T$ for the image T of the Albanese morphism. Thus, we may assume that $H^1(X, \mathcal{O}_X) = 0$. Then the canonical exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_X(D) \otimes \mathcal{O}_D \simeq \mathcal{O}_D \to 0$$

induces a surjection $H^0(X, \mathcal{O}_X(D)) \to H^0(D, \mathcal{O}_D) \simeq \mathbb{C}$, and the linear system $|D|$ is a base point free pencil. Hence, D is contracted to a point by the morphism $X \to \mathbb{P}^1$ associated with the pencil. Thus, we are done. □

Lemma 2.30. *Let X be a normal Moishezon surface and D a non-zero reduced divisor on X such that $D \subset X_{\mathrm{reg}}$ and D is a simple normal crossing divisor forming a linear chain of rational curves. If each prime component of D is a negative curve and if D is not negative semi-definite, then there is a $\mathbb{P}^1$-fibration $\pi\colon X \to T \simeq \mathbb{P}^1$ such that $\pi^{-1}(t)$ is a linear chain of rational curves contained in D for some $t \in T$. If a prime component C of D satisfies $C \not\subset \pi^{-1}(t)$ and $C \cap \pi^{-1}(t) \neq \emptyset$, then C is a section of π.*

Proof. Let $D = D_1 + \cdots + D_n$ be a prime decomposition of D with a dual graph

$$\overset{D_1}{\bullet} \text{———} \overset{D_2}{\bullet} \text{——} \;\cdots\; \text{——} \overset{D_n}{\bullet}\;.$$

First, we shall prove the following assertion (∗) by induction on $n = \boldsymbol{n}(D)$:

(∗) There exist integers $1 \leq a < b \leq n$ such that $(a, b) \neq (1, n)$ and that $\sum_{i=a}^{b} D_i$ is not negative definite but negative semi-definite.

Since D is not negative semi-definite, we have $n \geq 3$ and D_p is a (-1)-curve for some $1 < p < n$ by Lemma 2.2. Let $\varphi\colon X \to \overline{X}$ be the contraction morphism of D_p. Then $\overline{D} := \varphi_*(D)$ is a simple normal crossing divisor on $\overline{X}$ expressed as a linear chain $\overline{D}_1 + \cdots + \overline{D}_{n-1}$ of rational curve, where

$$\overline{D}_i = \begin{cases} \varphi(D_i), & \text{if } i < p; \\ \varphi(D_{i+1}), & \text{if } i \geq p. \end{cases}$$

Assume that $n = 3$. Then $p = 2$, and either $\overline{D}_1^2 = 0$ or $\overline{D}_2^2 = 0$ by Lemma 2.2, since $\overline{D}$ is not negative semi-definite. Thus, either $D_1 + D_2$ or $D_2 + D_3$ is not negative definite but negative semi-definite. Hence $(*)$ holds when $n = 3$. Assume that $n > 3$. By induction, we have integers $1 \le a' < b' \le n - 1$ such that $(a', b') \ne (1, n-1)$ and that $\sum_{i=a'}^{b'} \overline{D}_i$ is not negative definite but negative semi-definite. Now, there exist integers $1 \le a < b \le n$ such that $(a, b) \ne (1, n)$ and that $\varphi^*(\sum_{i=a'}^{b'} \overline{D}_i) = \sum_{i=a}^{b} D_i$. Thus, $(*)$ holds for any n.

By $(*)$, we have a non-zero effective divisor G on X such that $G_{\mathrm{red}} = \sum_{i=a}^{b} D_i$ and that $GD_i = 0$ for any $a \le i \le b$. By applying Lemma 2.29 to G, we have a $\mathbb{P}^1$-fibration $\pi\colon X \to T$ such that $G = m\pi^*(t)$ for a point $t \in T$ and an integer $m > 0$. We may assume that $m = 1$ by replacing G. Then $\mathrm{mult}_{D_a} G = \mathrm{mult}_{D_b} G = 1$ by Lemma 2.29. Let C be a prime component of D such that $C \not\subset G$ and $C \cap G \ne \emptyset$. Then $C = D_{a-1}$ with $a \ne 1$ or $C = D_{b+1}$ with $b \ne n$, and we have

$$C\pi^*(t) = CG = \begin{cases} CD_a = 1, & \text{if } C = D_{a-1}, \\ CD_b = 1, & \text{if } C = D_{b+1}. \end{cases}$$

Thus, C is a section of π, and $T \simeq \mathbb{P}^1$. □

Corollary 2.31. *In the situation of Lemma* 2.3(2)*, if* $M \subset X_{\mathrm{reg}}$ *for a normal Moishezon surface* X*, then* X *is a projective rational surface and there exists a* $\mathbb{P}^1$*-fibration* $\pi\colon X \to T \simeq \mathbb{P}^1$ *with a point* $t \in T$*, such that*

(1) $\pi^{-1}(t) = \sum_{i=a+1}^{l} D_i + C + G_1 + G_2$,
(2) D_a *is a double section of* π,
(3) $(X/T, D_a, \pi^{-1}(t))$ *is a standard PDS configuration,*

for the integer $1 \le a < l$ *in Lemma* 2.3(2)*. In particular,* $(\overline{X}/T, \varpi(D_a), \bar{\pi}^{-1}(t))$ *is a basic PDS configuration for the contraction morphism* $\varpi\colon X \to \overline{X}$ *of* $\sum_{i=a+1}^{l} D_i$ *and for the induced* $\mathbb{P}^1$*-fibration* $\bar{\pi}\colon \overline{X} \to T$ *such that* $\pi = \bar{\pi} \circ \varpi$*.*

Proof. The morphism $\psi \circ \phi\colon M \to \widehat{N}$ in the proof of Lemma 2.3(2) extends to a bimeromorphic morphism $\varphi\colon X \to Y$ to a normal Moishezon surface Y such that $\widehat{N} \subset Y_{\mathrm{reg}}$ and that φ is an isomorphism over $Y \setminus \widehat{E}$ for $\widehat{E} = \varphi_*(\sum_{i=1}^{l} D_i)$. Prime divisors $\widehat{C}$ and $\widehat{G}_j$ for $j = 1$, 2 on $\widehat{N}$ defined in the proof of Lemma 2.3(2) are regarded as prime divisors $\varphi(C)$ and $\varphi(G_j)$ on Y, respectively, and similarly, every prime component of $\widehat{E}$ is expressed as $\varphi(D_i)$ for some $1 \le i \le l$. The following hold by the proof of Lemma 2.3(2):

(4) $\widehat{E} + \widehat{C} + \widehat{G}_1 + \widehat{G}_2 \subset Y_{\mathrm{reg}}$;
(5) $\widehat{C}$ is a (-1)-curve, and $\widehat{G}_j$ is a (-2)-curve for $j = 1$, 2;
(6) $\varphi^{-1}(\widehat{C} + \widehat{G}_1 + \widehat{G}_2) = \sum_{i=a+1}^{l} D_i + C + G_1 + G_2$;
(7) D_a is the proper transform in X of the end component $\widehat{E}_k$ of $\widehat{E}$ such that $\widehat{E}_k C = 1$;

By (5) and by Lemma 2.29 applied to the linear chain $\widehat{C}+\widehat{G}_1+\widehat{G}_2$ of rational curves on Y, there is a $\mathbb{P}^1$-fibration $\pi_Y\colon Y \to T$ to a non-singular projective curve T such that $\pi_Y^*(t) = 2\widehat{C} + \widehat{G}_1 + \widehat{G}_2$ for a point $t \in T$. Then $\widehat{E}_k$ is a double section of π_Y

by (7). In particular, $T \simeq \mathbb{P}^1$, and $(Y/T, \widehat{E}_k, \pi_Y^{-1}(t))$ is a basic PDS configuration. The composite $\pi := \hat{\pi} \circ \varphi \colon X \to T \simeq \mathbb{P}^1$ is also a $\mathbb{P}^1$-fibration, where

$$\pi^{-1}(t) = \varphi^{-1}(\hat{\pi}^{-1}(t)) = \varphi^{-1}(\widehat{C} + \widehat{G}_1 + \widehat{G}_2) = \sum\nolimits_{i=a+1}^{l} D_i + C + G_1 + G_2$$

by (6). Thus, X is projective and rational (cf. [18, Prop. 2.33(1)]), and (1) holds. Moreover, we have (2) and (3) by (7). The last assertion for $\overline{X}$ is deduced from the proof of Proposition 2.21(4) or from the following argument: By construction, there is a bimeromorphic morphism $\gamma \colon \overline{X} \to Y$ such that $\varphi = \gamma \circ \varpi$ and that the γ-exceptional locus is contained in $\varpi_*(\sum_{i=2}^{a-1} D_i)$. In particular γ is an isomorphism over an open neighborhood of t in T. Thus, $(\overline{X}/T, \varpi(D_a), \bar{\pi}^{-1}(t))$ is basic as $(Y/T, \widehat{E}_k, \pi_Y^{-1}(t))$ is so. Thus, we are done. □

3. On pseudo-toric surfaces

Let (X, S) be a *pseudo-toric surface* in the sense of [18, Def. 6.1]. Then, by definition and by [18, Lem. 6.3],

- X is a normal projective rational surface with only rational singularities,
- S is a cyclic chain of rational curves and is big,
- (X, S) is log-canonical with $K_X + S \sim 0$.

The *defect* $\boldsymbol{\delta}(X, S) := \boldsymbol{\rho}(X) + 2 - \boldsymbol{n}(S)$ is always non-negative for any pseudo-toric surface (X, S) by [18, Prop. 6.4], in which $\boldsymbol{\delta}(X, S) = 0$ if and only if (X, S) is a toric surface. In Section 3.1, we discuss several properties of a pseudo-toric surface admitting a fibration to $\mathbb{P}^1$. We shall study the universal cover of the open subset $X_{\mathrm{reg}} \setminus (S - B)$ for a prime component B of S under some condition in Section 3.2. As applications, in Section 3.3, we shall prove Theorems 3.13 and 3.14 on endomorphisms and toric surfaces.

3.1. **Pseudo-toric surfaces with a fibration.**

We shall prove some results on pseudo-toric surfaces admitting fibrations to $\mathbb{P}^1$ by applying results in [18].

Lemma 3.1. *Let (Y, Σ) be a pseudo-toric surface with a fibration $\pi \colon Y \to T \simeq \mathbb{P}^1$. Assume that Σ contains two set-theoretic fibers $D_1 = \pi^{-1}(t_1)$ and $D_2 = \pi^{-1}(t_2)$. Then*:

1. *π is a $\mathbb{P}^1$-fibration and $\Sigma = \Theta_1 + \Theta_2 + D_1 + D_2$ for mutually disjoint sections Θ_1 and Θ_2 of π;*
2. *for any point $o \in T \setminus \{t_1, t_2\}$, the fiber $F_o = \pi^*(o)$ is reduced, $(Y, \Sigma + F_o)$ is log-canonical, and $\Theta_1 + F_o + \Theta_2$ is a linear chain of rational curves with end components Θ_1 and Θ_2, where F_o intersects $\Theta_1 + \Theta_2$ transversely.*

Moreover, the following hold for the reducible fibers $G_1, G_2, \ldots, G_b$ of π over $T \setminus \{t_1, t_2\}$:

3. *The equality $\boldsymbol{\delta}(Y, \Sigma) = \sum_{k=1}^{b}(\boldsymbol{n}(G_k) - 1)$ holds.*
4. *For each $1 \leq k \leq b$, let $G_{k,(1)}$ be the end component of G_k intersecting Θ_1. Then there is a birational morphism $\phi \colon Y \to \overline{Y}$ to a normal projective surface $\overline{Y}$ such that the exceptional locus of ϕ equals $\sum_{k=1}^{b}(G_k - G_{k,(1)})$ and that $(\overline{Y}, \overline{\Sigma})$ is a toric surface for $\overline{\Sigma} = \phi_*\Sigma$.*

(5) *In* (4)*, let* $\bar{\pi}\colon \overline{Y} \to T$ *be the induced* $\mathbb{P}^1$*-fibration such that* $\pi = \bar{\pi} \circ \phi$*. Then* $\bar{\pi}\colon (\overline{Y}, \overline{\Sigma}) \to (T, t_1 + t_2)$ *is a* toric morphism (*cf.* [18, §3.1]*,* [19, Def. 4.5])*.*

Proof. Assertion (1) follows from [18, Lem. 5.2]. We have (2) as a consequence of [18, Lem. 5.4], but here we present another proof applying Proposition 2.15: We set $G_o := (F_o)_{\mathrm{red}}$. Since $K_Y + \Sigma \sim 0$, by (1) and Proposition 2.15, G_o is a linear chain of rational curves, $(Y, \Sigma + G_o)$ is log-canonical, $K_Y + \Sigma + G_o \sim G_o$ is Cartier, and

$$\mathcal{O}_Y(K_Y + \Sigma + G_o) \otimes \mathcal{O}_{G_o} \simeq \mathcal{O}_Y(G_o) \otimes \mathcal{O}_{G_o} \simeq \mathcal{O}_{G_o}.$$

By the vanishing $R^1\pi_*\mathcal{O}_Y = 0$ (cf. Remark 2.9) and by the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(G_o) \to \mathcal{O}_Y(G_o) \otimes \mathcal{O}_{G_o} \to 0,$$

we have an isomorphism $\pi^*(\pi_*\mathcal{O}_Y(G_o)) \xrightarrow{\simeq} \mathcal{O}_Y(G_o)$. It implies that $G_o = F_o$, i.e., F_o is reduced. Then, $\Theta_j F_o = 1$ for $j = 1$, 2. Since $(Y, \Theta_j + F_o)$ is log-canonical, it is toroidal at $\Theta_j \cap F_o$ (cf. [19, Fact 2.5]), and we have $\Theta_j \cap F_o \subset Y_{\mathrm{reg}}$. Thus, F_o intersects $\Theta_1 + \Theta_2$ transversely. Moreover, Θ_j is an end component of the linear chain $\Theta_1 + F_o + \Theta_2$. In fact, any prime component Γ of F_o intersects exactly two prime components of $\Theta_1 + F_o + \Theta_2 - \Gamma$ by [18, Prop. 3.29], since $(Y, \Theta_1 + F_o + \Theta_2)$ is log-canonical, $K_Y + \Theta_1 + F_o + \Theta_2$ is Cartier, and

$$(K_Y + \Theta_1 + F_o + \Theta_2)\Gamma = (F_o - D_1 - D_2)\Gamma = 0.$$

Thus, (2) has been proved. The equality in (3) is derived from an equality

$$\boldsymbol{\rho}(Y) - 2 = (\boldsymbol{n}(D_1) - 1) + (\boldsymbol{n}(D_2) - 1) + \sum\nolimits_{k=1}^{b} (\boldsymbol{n}(G_k) - 1)$$

obtained by [18, Prop. 2.32(7)]. Since $G^\dagger := \sum_{k=1}^{b}(G_k - G_{k,(1)})$ is negative definite, we have the contraction morphism $\phi\colon Y \to \overline{Y}$ of $G^\dagger$. Then $(\overline{Y}, \overline{\Sigma})$ is a pseudo-toric surface by [18, Lem. 6.3(7)], and its defect is 0 by (3) applied to $\bar{\pi}\colon \overline{Y} \to T$. Thus, $(\overline{Y}, \overline{\Sigma})$ is a toric surface and $\bar{\pi}$ is a toric morphism $(\overline{Y}, \overline{\Sigma}) \to (T, t_1 + t_2)$ by [18, Prop. 5.3(3)]. This proves (4) and (5), and we are done. □

Proposition 3.2. *Let X be a normal projective surface with two points P_1, P_2 and let S_1 and S_2 be reduced divisors satisfying the following conditions*:

(i) $S_1 \cap S_2 = \{P_1, P_2\}$ *and* $(X, S_1 + S_2)$ *is log-canonical at* $S_1 \cap S_2$*;*

(ii) $-(K_X + S_1 + S_2)$ *is nef;*

(iii) *there exist effective Cartier divisors* B_1 *and* B_2 *satisfying* $\mathrm{Supp}\, B_1 \subset S_1$*,* $\mathrm{Supp}\, B_2 \subset S_2$*,* $\{P_1, P_2\} = \mathrm{Supp}\, B_1 \cap \mathrm{Supp}\, B_2$*, and* $B_1 \sim B_2$*.*

Then S_1 and S_2 are linear chains of rational curves, and $(X, S_1 + S_2)$ is a pseudo-toric surface. Moreover, there exist a toroidal blowing up $\mu\colon Y \to X$ with respect to $(X, S_1 + S_2)$ and a $\mathbb{P}^1$-fibration $\pi\colon Y \to T \simeq \mathbb{P}^1$ such that

- (Y, Σ) *is a pseudo-toric surface for* $\Sigma = \mu^{-1}(S_1 + S_2)$*,*
- $\boldsymbol{\delta}(Y, \Sigma) = \boldsymbol{\delta}(X, S_1 + S_2)$*,*
- *the proper transforms D_1 and D_2 in Y of S_1 and S_2, respectively, are mutually distinct set-theoretic fibers of π,*
- $\Sigma = \Theta_1 + \Theta_2 + D_1 + D_2$ *for two mutually disjoint sections* Θ_1 *and* Θ_2 *of* π*.*

Proof. For $1 \le i, j \le 2$, there exist a unique prime component $S_{i,(j)}$ of S_i containing P_j, and $(X, S_{1,(j)} + S_{2,(j)})$ is toroidal at P_j, by (i) and [19, Fact. 2.5]. Let $\mu^\dagger \colon Y^\dagger \to X$ be the blowing up along the scheme-theoretic intersection $B_1 \cap B_2$ for effective divisors B_1 and B_2 in (iii), and let $\mu^\ddagger \colon Y \to Y^\dagger$ be the normalization. Then, as in the proof of [18, Lem. 4.23], the composite $\mu = \mu^\dagger \circ \mu^\ddagger \colon Y \to X$ is a toroidal blowing up with respect to $(X, S_{1,(j)} + S_{2,(j)})$ at P_j for $j = 1, 2$. Then, for the reduced divisor $\Sigma = \mu^{-1}(S_1 + S_2)$, the pair (Y, Σ) is log-canonical along the μ-exceptional locus $\mu^{-1}\{P_1, P_2\}$, and

$$K_Y + \Sigma = \mu^*(K_X + S_1 + S_2) \tag{III-1}$$

(cf. [18, Def. 4.19]). The surface $Y^\dagger$ is just the graph of the rational map $X \cdots\to \mathbb{P}^1$ associated with the pencil generated by B_1 and B_2. Let $\pi \colon Y \to T$ be a fibration obtained as the Stein factorization of the composite of $\mu^\ddagger$ and the projection $Y^\dagger \to \mathbb{P}^1$. Then any μ-exceptional divisor dominates T, since its image in $Y^\dagger$ is not a point. By construction, there exist a μ-exceptional Cartier divisor Ξ on Y and effective divisors $\mathfrak{e}_1$ and $\mathfrak{e}_2$ on T such that

$$\mu^*(B_1) - \Xi = \pi^*\mathfrak{e}_1, \quad \mu^*(B_2) - \Xi = \pi^*\mathfrak{e}_2, \quad \text{and} \quad \operatorname{Supp}\mathfrak{e}_1 \cap \operatorname{Supp}\mathfrak{e}_2 = \emptyset.$$

We set $D_i := \pi^*(\mathfrak{e}_i)_{\mathrm{red}}$ for $i = 1, 2$. Then $\Sigma = \Xi_{\mathrm{red}} + D_1 + D_2$, $\mu^{-1}S_i = D_i + \Xi_{\mathrm{red}}$ for $i = 1, 2$, and moreover, $-(K_Y + \Sigma)$ is nef by (ii) and (III-1). Hence, π is a $\mathbb{P}^1$-fibration by $K_Y F \le -\Sigma F < 0$ for a general fiber F of π. Thus, $\Xi_{\mathrm{red}} = \Theta_1 + \Theta_2$ for two sections Θ_1 and Θ_2 of π such that $\Theta_j = \mu^{-1}(P_j)$ for $j = 1, 2$. Since μ is a toroidal blowing up, we have $\Theta_1 \simeq \Theta_2 \simeq T \simeq \mathbb{P}^1$. In particular, Y and X are rational surfaces. Moreover, $\operatorname{Supp}\mathfrak{e}_i$ consists of a point t_i and $D_i = \pi^{-1}(t_i)$ for $i = 1, 2$, since $\Theta_j \cap D_i$ consists of one point lying over P_j for $1 \le i, j \le 2$. Consequently, D_i is the proper transform of S_i in Y for $i = 1, 2$.

We shall show that (Y, Σ) is a pseudo-toric surface. Since (Y, Σ) is log-canonical along $\mu^{-1}\{P_1, P_2\}$, $(Y, \Theta_1 + D_i + \Theta_2)$ is also log-canonical along $D_i \cap (\Theta_1 + \Theta_2)$ for any $1 \le i \le 2$. Thus, the following hold for any i, by Proposition 2.15, since $-(K_Y + \Sigma)$ is nef:

- D_i is a linear chain of rational curves;
- $(Y, \Theta_1 + D_i + \Theta_2)$ is log-canonical along D_i;
- $(K_Y + \Theta_1 + D_i + \Theta_2)\Gamma' = 0$ for any prime component Γ' of D_i.

Consequently, (Y, Σ) is log-canonical along Σ and $(K_Y + \Sigma)\Gamma' = 0$ for any prime component Γ' of Σ. Then Y has only rational singularities, (Y, Σ) is log-canonical, Σ is a cyclic chain of rational curves, and $K_Y + \Sigma \sim 0$ by (ii) and by [18, Lem. 4.7] with its remark. Therefore, (Y, Σ) is a pseudo-toric surface (cf. [18, Def. 6.1]).

As a consequence, $(X, S_1 + S_2)$ is a pseudo-toric surface by [18, Lem. 6.3(7)]. Here, $S_i = \mu_* D_i$ is a linear chain of rational curves for $i = 1, 2$. Moreover,

$$\boldsymbol{\delta}(Y, \Sigma) - \boldsymbol{\delta}(X, S_1 + S_2) = \boldsymbol{\rho}(Y) - \boldsymbol{\rho}(X) - (\boldsymbol{n}(\Sigma) - \boldsymbol{n}(S_1 + S_2)) = 0,$$

since the μ-exceptional divisor $\Theta_1 + \Theta_2$ is contained in Σ. Thus, we are done. $\square$

Corollary 3.3. *In Proposition* 3.2, *let* $G_1, G_2, \ldots, G_b$ *be the reducible fibers of* π *different from* D_1 *and* D_2.

(1) *If* $\boldsymbol{n}(G_k) \geq 3$ *for some* $1 \leq k \leq b$*, then there is a negative curve* C *on* X *such that* $C \cap (S_1 + S_2) = \emptyset$*.*

(2) *If* $b \geq 3$ *or if* $b = 2$ *and one of* S_1 *and* S_2 *is reducible, then there is an index* $j \in \{1, 2\}$ *such that* $C \cap S_1 = C \cap S_2 = \{P_j\}$ *for at least* $b - 1$ *negative curves* C *on* X*.*

(3) *For any* $1 \leq j \leq 2$*, assume that* $C^2 \geq 0$ *for any prime divisor* C *on* X *satisfying* $C \cap S_1 = C \cap S_2 \subset \{P_j\}$*. Then* $\boldsymbol{\delta}(X, S_1 + S_2) \leq 2$*, and if the equality holds, then* X *has no negative curve,* $\boldsymbol{\rho}(X) = 2$*, and there exist two* $\mathbb{P}^1$*-fibrations* $X \to \mathbb{P}^1$ *in which* S_1 *and* S_2 *are sections.*

Proof. (1): In this case, a non-end component of the linear chain G_k is a negative curve not intersecting $\Sigma = \mu^{-1}(S_1 + S_2)$. It is enough to take C as its image under $\mu\colon Y \to X$.

(2): For $1 \leq k \leq b$ and for $1 \leq j \leq 2$, let $G_{k,(j)}$ be the end component of the linear chain G_k intersecting Θ_j (cf. Lemma 3.1(2)). Then $\Gamma_{k,(j)} := \mu(G_{k,(j)})$ is a prime divisor on X such that

$$\Gamma_{k,(j)} \cap S_1 = \Gamma_{k,(j)} \cap S_2 = \{P_j\}$$

for any $1 \leq k, j \leq 2$. If $\Gamma_{k,(1)}$ is a negative curve for any $1 \leq k \leq b$, then (2) holds for P_1. Thus, we may assume that $\Gamma_{1,(1)}$ is not negative.

Note that $\Gamma_{k,(2)} \cap \Gamma_{1,(1)} = \emptyset$ for any $2 \leq k \leq b$. Hence, $\sum_{k=2}^{b} \Gamma_{k,(2)}$ is negative semi-definite by the Hodge index theorem. In particular, if $b \geq 3$, then $(\Gamma_{k,(2)})^2 < 0$ for any $2 \leq k \leq b$, since $\sum_{k=2}^{b} \Gamma_{k,(2)}$ is connected and reducible. More directly, if $(\Gamma_{1,(1)})^2 > 0$, then $(\Gamma_{k,(2)})^2 < 0$ for any $2 \leq k \leq b$ by the Hodge index theorem. If S_i is reducible for some $i \in \{1, 2\}$, then

$$\Gamma_{k,(2)} \cap (\Gamma_{1,(1)} + S_{i,(1)}) = \emptyset$$

for any $2 \leq k \leq b$ and for the end component $S_{i,(1)}$ of the linear chain S_i containing P_1; thus, $(\Gamma_{k,(2)})^2 < 0$ by the Hodge index theorem, since $\Gamma_{1,(1)} + S_{i,(1)}$ is big. Therefore, (2) is satisfied by negative curves $\Gamma_{k,(2)}$ for $2 \leq k \leq b$ if $(\Gamma_{1,(1)})^2 > 0$, $b \geq 3$, or if S_i is reducible for some $i \in \{1, 2\}$. This proves (2).

(3): Under the assumption, when $b > 0$, we have the following by (1), (2), and their proofs:

- $\boldsymbol{n}(G_k) = 2$ for any $1 \leq k \leq b$;
- $b \leq 2$;
- $(\Gamma_{k,(j)})^2 = 0$ for any $1 \leq k \leq b$ and $1 \leq j \leq 2$;
- if $b = 2$, then S_1 and S_2 are irreducible.

In particular, $\boldsymbol{\delta}(X, S_1 + S_2) = \boldsymbol{\delta}(Y, \Sigma) = b \leq 2$ by Lemma 3.1(3) and Proposition 3.2. Assume that $b = 2$. Then $\boldsymbol{\rho}(X) = \boldsymbol{n}(S_1 + S_2) + \boldsymbol{\delta}(X, S_1 + S_2) - 2 = 2$. Since $-K_X$ is big, $\Gamma_{k,(j)}$ is semi-ample by [20, Prop. 1.5]. Thus, we have fibrations φ_1, $\varphi_2\colon X \to \mathbb{P}^1$ such that $\Gamma_{1,(1)}$ and $\Gamma_{2,(2)}$ are set-theoretic fibers of φ_1 and that $\Gamma_{1,(2)}$ and $\Gamma_{2,(1)}$ are set-theoretic fibers of φ_2. Then $\overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0}\,\mathrm{cl}(\Phi_1) + \mathbb{R}_{\geq 0}\,\mathrm{cl}(\Phi_2)$ for a general fiber Φ_l of φ_l for $l = 1, 2$. In particular, X has no negative curve, and every fiber of φ_l is irreducible for $l = 1, 2$. Since $K_X + S_1 + S_2 \sim 0$, φ_l is a

$\mathbb{P}^1$-fibration with S_1 and S_2 as sections for $l = 1, 2$. Thus, (3) holds, and we are done. □

3.2. **Universal covers of some open subsets of a pseudo-toric surface.** For the pseudo-toric surface (Y, Σ) (resp. $(X, S_1 + S_2)$) in Lemma 3.1 (resp. Proposition 3.2), the fundamental group of the open subset $Y_{\text{reg}} \setminus (\Sigma - D_2)$ (resp. $X_{\text{reg}} \setminus S_1$) is shown to be finite and cyclic under some extra condition in Proposition 3.7 (resp. 3.9) below. Furthermore, the universal cover of the open subset extends to a cyclic cover to Y (resp. X) from another pseudo-toric surface, on which we shall study some properties. The properties on negative curves in Proposition 3.9 below concern our first crucial idea explained in the introduction.

Lemma 3.4. *For a pseudo-toric surface (X, S), let $\nu\colon \widehat{X} \to X$ be a finite cover from a normal projective surface $\widehat{X}$ which is étale in codimension 1 over $X \setminus S$. Then $(\widehat{X}, \widehat{S})$ is a pseudo-toric surface for $\widehat{S} = \nu^{-1}S$.*

Proof. Now, $K_{\widehat{X}} + \widehat{S} = \nu^*(K_X + S) \sim 0$ by [19, Lem. 1.39], and $(\widehat{X}, \widehat{S})$ is log-canonical by [19, Lem. 2.10(1)]. The divisor $\widehat{S} = \nu^{-1}S$ is big, and each connected component of $\widehat{S}$ is either an elliptic curve or a cyclic chain of rational curves by [18, Cor. 4.6]. If $\widehat{S}$ is disconnected, then a connected component of $\widehat{S}$ is negative definite by the Hodge index theorem, since $\widehat{S}$ is big. Hence, if ν is Galois, then $\widehat{S}$ is connected, and it is a cyclic chain of rational curves, since it covers S. Even if ν is not Galois, by considering the Galois closure of ν, we see that $\widehat{S}$ is also a cyclic chain of rational curves. Hence, $(\widehat{X}, \widehat{S})$ is a pseudo-toric surface by [18, Rem. 6.2]. □

We have the following by Grauert–Remmert's extension theorem (cf. [6], [7, XII, Thm. 5.4]):

Lemma 3.5. *Let X be a normal variety with a non-empty Zariski-open subset U. Then, for a finite étale cover $U' \to U$ from another normal variety U', there exist a finite cover $\nu\colon X' \to X$ from a normal variety X' such that $\nu^{-1}U \simeq U'$ over U, and furthermore, such an extension ν is unique up to isomorphism over X.*

Remark 3.6. By the uniqueness of the extension ν, the category of complex analytic spaces finite étale over U is equivalent to the category of normal complex analytic spaces finite over X and étale over U. In particular, if the cover $U' \to U$ is Galois, then the extension ν is also Galois with the same Galois group.

Convention. A finite surjective morphism $\nu\colon X' \to X$ of normal varieties is said to *giving a universal cover over U* for a non-empty Zariski-open subset $U \subset X_{\text{reg}}$ if $\nu^{-1}U \to U$ is a universal covering map of U, i.e., ν is étale over U and $\nu^{-1}U$ is simply connected.

Proposition 3.7. *In Lemma 3.1, assume that $\boldsymbol{\delta}(Y, \Sigma) > 0$ and that the fiber $D_2 = \pi^{-1}(t_2)$ is irreducible. We set $U := Y_{\text{reg}} \setminus (\Sigma - D_2)$ and $d_2 := \operatorname{mult}_{D_2} \pi^*(t_2)$. Then:*

1. *The fundamental group $\boldsymbol{\pi}_1(U)$ is finite and cyclic, and its order is a multiple of d_2.*

(2) *There is a pseudo-toric surface* $(\widetilde{Y}, \widetilde{\Sigma})$ *with a cyclic cover* $\nu\colon \widetilde{Y} \to Y$ *such that* $\widetilde{\Sigma} = \nu^{-1}\Sigma$, $\boldsymbol{\delta}(\widetilde{Y}, \widetilde{\Sigma}) = d_2\boldsymbol{\delta}(Y, \Sigma)$, *and that* ν *gives a universal cover over* U.

For the cyclic cover ν, *let* $\tilde{\pi}\colon \widetilde{Y} \to \widetilde{T} \simeq \mathbb{P}^1$ *be a* $\mathbb{P}^1$*-fibration with a finite surjective morphism* $\widetilde{T} \to T$ *obtained as the Stein factorization of* $\pi \circ \nu\colon \widetilde{Y} \to T$. *Then*:

(3) *The finite morphism* $\widetilde{T} \to T$ *is a cyclic cover of degree* d_2 *branched only at* t_1 *and* t_2. *In particular,* $\widetilde{D}_1 = \nu^{-1}D_1$ *and* $\widetilde{D}_2 = \nu^{-1}D_2$ *are set-theoretic fibers of* $\tilde{\pi}$. *Here,* $\widetilde{D}_2$ *is a smooth fiber of* $\tilde{\pi}$.

(4) *The number of reducible fibers of* $\tilde{\pi}$ *different from* $\widetilde{D}_1$ *is equal to* $d_2 b$.

(5) *A reducible fiber of* $\tilde{\pi}$ *different from* $\widetilde{D}_1$ *is a connected component of* $\nu^{-1}G_k$ *for some* $1 \leq k \leq b$, *and* $\boldsymbol{n}(G_k)$ *equals the number of prime components of the reducible fiber.*

Proof. We shall prove by the following five steps.

Step 1. We shall show that $\boldsymbol{\pi}_1(U)$ *is a cyclic group.* Let N be the group of 1-parameter subgroups of the toric surface $(\overline{Y}, \overline{\Sigma})$ in Lemma 3.1(4) and let $\triangle$ be the fan defining $\overline{Y}$, i.e., $\overline{Y} = \mathbb{T}_{\mathsf{N}}(\triangle)$ (cf. [18, §3.1], [19, §4.1]). For the birational morphism $\phi\colon Y \to \overline{Y}$, the divisor $\overline{D}_2 = \phi(D_2)$ is a prime component of the boundary divisor $\overline{\Sigma}$, and it corresponds to a 1-dimensional cone $\mathsf{R} \in \triangle$. Then $\overline{U} := \overline{Y} \setminus (\overline{\Sigma} - \overline{D}_2)$ is the open affine toric surface $\mathbb{T}_{\mathsf{N}}(\mathsf{R})$ associated with R. Hence, $\overline{U} \simeq \mathbb{C} \times \mathbb{C}^\star$, and $\boldsymbol{\pi}_1(\overline{U}) \simeq \mathbb{Z}$. Since $\phi^{-1}(\overline{U}) \simeq \overline{U}$ and $\phi^{-1}(\overline{U}) \subset U$, we have a surjection $\mathbb{Z} \simeq \boldsymbol{\pi}_1(\overline{U}) \to \boldsymbol{\pi}_1(U)$. As a consequence, $\boldsymbol{\pi}_1(U)$ is cyclic.

Step 2 (Extension of a finite étale cover over U). Let $\widehat{U} \to U$ be a finite étale cover from a (connected) surface $\widehat{U}$. This is a cyclic cover by Step 1. By Lemma 3.5, there exists uniquely up to isomorphism over Y a cyclic cover $\hat{\nu}\colon \widehat{Y} \to Y$ from a normal projective surface $\widehat{Y}$ such that $\hat{\nu}^{-1}U \simeq \widehat{U}$ over U. Since $Y \setminus \Sigma \subset U$, $(\widehat{Y}, \widehat{\Sigma})$ is a pseudo-toric surface for $\widehat{\Sigma} := \hat{\nu}^{-1}\Sigma$ by Lemma 3.4. By Stein factorization, $\pi \circ \hat{\nu} = \hat{\tau} \circ \hat{\pi}$ for a fibration $\hat{\pi}\colon \widehat{Y} \to \widehat{T}$ and a finite morphism $\hat{\tau}\colon \widehat{T} \to T$, where $\widehat{T} \simeq \mathbb{P}^1$ as $\widehat{Y}$ is rational. By applying Lemma 3.1(1) to $\hat{\pi}\colon (\widehat{Y}, \widehat{\Sigma}) \to \widehat{T}$, we see that $\hat{\pi}$ is a $\mathbb{P}^1$-fibration and that the following hold for $\widehat{\Theta}_j := \hat{\nu}^{-1}\Theta_j$ and $\widehat{D}_i := \hat{\nu}^{-1}D_i$ for any i, $j \in \{1, 2\}$:

(i) $\widehat{\Theta}_j$ is a section of $\hat{\pi}$;

(ii) there is a point $\hat{t}_i \in \widehat{T}$ such that $\hat{\tau}^{-1}(t_i) = \{\hat{t}_i\}$ and $\widehat{D}_i = \hat{\pi}^{-1}(\hat{t}_i)$.

Moreover, for any fiber $\widehat{F}$ of $\hat{\pi}$ over $\widehat{T} \setminus \{\hat{t}_1, \hat{t}_2\}$, we see that

(iii) $\widehat{F}$ is reduced,

(iv) $\widehat{F}$ is a linear chain of rational curves intersecting $\widehat{\Theta}_1 + \widehat{\Theta}_2$ transversely,

by applying Lemma 3.1(2) to $\hat{\pi}\colon (\widehat{Y}, \widehat{\Sigma}) \to \widehat{T}$.

Step 3. We shall prove the following lemma concerning Step 2.

Lemma 3.8. *In the situation of Step* 2, *if* $d_2 = 1$, *then the following hold, where* $\widehat{G}_k := \hat{\nu}^{-1}G_k$ *for* $1 \leq k \leq b$:

(a) D_2 *is a smooth fiber of* π;

(b) $\hat{\tau}$ *is an isomorphism*;

(c) $\hat{\nu}^*\Theta_j = (\deg \hat{\nu})\widehat{\Theta}_j$ *for any* $j = 1$ *and* 2;

(d) $\hat{\pi}$ *is smooth outside* $\widehat{D}_1 + \sum_{k=1}^{b} \widehat{G}_k$;

(e) $\boldsymbol{n}(\widehat{G}_k) = \boldsymbol{n}(G_k)$ *for any* $1 \le k \le b$, *and* $\boldsymbol{\delta}(\widehat{Y}, \widehat{\Sigma}) = \boldsymbol{\delta}(Y, \Sigma)$;

(f) $\boldsymbol{\pi}_1(U)$ *is finite.*

Proof of Lemma 3.8. We have (a) by [18, Prop. 2.33(4)], since $\mathrm{mult}_{D_2} \pi^*(t_2) = d_2 = 1$. If $\deg \hat{\tau} > 1$, then $\hat{\tau}$ is branched at t_2 and $\hat{\nu}$ is branched along D_2 by Step 2(ii), but it contradicts the étaleness of $\hat{\nu}$ over $U \supset D_2 \cap U \neq \emptyset$. This shows (b). Furthermore, (b) and Step 2(i) imply (c). Remaining assertions are shown as follows:

(d): The étaleness of $\hat{\nu}$ over U implies that $\hat{\pi}^*(\hat{t}_2) = \hat{\nu}^* D_2$ is reduced, i.e., $\hat{\nu}^* D_2 = \widehat{D}_2$, and that the induced morphism

$$\widehat{D}_2 \setminus (\widehat{\Theta}_1 \cup \widehat{\Theta}_2) \to D_2 \cap U = D_2 \setminus (\Theta_1 \cup \Theta_2) \simeq \mathbb{C}^\star$$

is étale. Hence, $\widehat{D}_2$ is irreducible, since $\widehat{\Theta}_1 + \widehat{D}_2 + \widehat{\Theta}_2$ is a linear chain of rational curves as a part of $\widehat{\Sigma}$. Therefore, $\widehat{D}_2$ is a smooth fiber of $\hat{\pi}$ by [18, Prop. 2.33(4)]. For the rest, by Step 2(iii) and [18, Prop. 2.33(4)], it suffices to prove that any fiber $\widehat{F}$ of $\hat{\pi}$ over $\widehat{T} \setminus \{\hat{t}_1, \hat{t}_2\}$ different from $\widehat{G}_1, \ldots, \widehat{G}_b$ is irreducible. Here, $\widehat{F} = \hat{\nu}^{-1} F$ for a smooth fiber F of π over $T \setminus \{t_1, t_2\}$. By the same argument as above for D_2, we see that the induced morphism

$$\widehat{F} \setminus (\widehat{\Theta}_1 \cup \widehat{\Theta}_2) \to F \cap U = F \setminus (\Theta_1 \cup \Theta_2) \simeq \mathbb{C}^\star$$

is étale. Hence, F is irreducible by Step 2(iv). Thus, (d) holds.

(e): By (d), $\widehat{G}_1, \ldots, \widehat{G}_b$ are the reducible fibers of $\hat{\pi}$ over $\widehat{T} \setminus \{\hat{t}_1\}$. Thus, the latter equality of (e) on $\boldsymbol{\delta}(\widehat{Y}, \widehat{\Sigma})$ is derived from the first equalities on $\boldsymbol{n}(\widehat{G}_k)$ by Lemma 3.1(3) applied to π and $\hat{\pi}$. In order to prove $\boldsymbol{n}(\widehat{G}_k) = \boldsymbol{n}(G_k)$, we set

$$\Lambda := \operatorname{Sing} G_k, \quad \widehat{\Lambda} := \operatorname{Sing} \widehat{G}_k, \quad \Omega := G_k \cap (\Theta_1 \cup \Theta_2), \quad \widehat{\Omega} := \widehat{G}_k \cap (\widehat{\Theta}_1 \cup \widehat{\Theta}_2).$$

Note that $\Theta_1 + G_k + \Theta_2$ and $\hat{\nu}^{-1}(\Theta_1 + G_k + \Theta_2) = \widehat{\Theta}_1 + \widehat{G}_k + \widehat{\Theta}_2$ are both linear chains of rational curves by Lemma 3.1(2) (cf. Step 2(iv)). In particular, $\#\Lambda = \boldsymbol{n}(G_k) + 1$ and $\#\widehat{\Lambda} = \boldsymbol{n}(\widehat{G}_k) + 1$. The cyclic cover $\hat{\nu}$ is étale over an open neighborhood of $G_k \setminus (\Lambda \cup \Omega)$, and $G_k \setminus (\Lambda \cup \Omega)$ consists of $\boldsymbol{n}(G_k)$ connected components which are all isomorphic to $\mathbb{C}^\star$. Thus, $\hat{\nu}^{-1}\Omega = \widehat{\Omega}$, and $\hat{\nu}$ induces a bijection $\widehat{\Omega} \to \Omega$. In particular, an end component of G_k satisfies the following condition ($\lozenge$) for prime components Γ of G_k:

($\lozenge$) $\widehat{\Gamma} = \hat{\nu}^*\Gamma$ is a prime component of $\widehat{G}_k$ and the cyclic cover $\widehat{\Gamma} \to \Gamma$ induced by $\hat{\nu}$ is étale over $\Gamma \setminus (\Omega \cup \Lambda) \simeq \mathbb{C}^\star$.

If a prime component Γ satisfies ($\lozenge$), then any prime component intersecting Γ also satisfies ($\lozenge$), since $\hat{\nu}^{-1}(\Theta_1 + G_k + \Theta_2)$ is a linear chain of rational curves. Therefore, ($\lozenge$) holds for any prime component Γ of G_k. In particular, $\widehat{\Lambda} = \hat{\nu}^{-1}\Lambda$, and $\hat{\nu}$ induces a bijection $\widehat{\Lambda} \to \Lambda$. Hence, $\boldsymbol{n}(\widehat{G}_k) = \boldsymbol{n}(G_k)$, and (e) holds.

(f): In the proof of (e), (Y, P) is at most a cyclic quotient singularity for any point $P \in \Lambda$. Thus, there is a connected open neighborhood $\mathcal{U}$ of P in Y such that the fundamental group $\boldsymbol{\pi}_1(\mathcal{U} \setminus \{P\})$ is finite. Here, we may assume that $\hat{\pi}$ is étale over $\mathcal{U} \setminus \{P\}$ and that $\hat{\pi}^{-1}(\mathcal{U} \setminus \{P\}) = \hat{\pi}^{-1}\mathcal{U} \setminus \{\widehat{P}\}$ is connected for a point

$\widehat{P} \in \widehat{\Lambda}$ lying over P, since $\widehat{\Lambda} \to \Lambda$ is bijective. This shows that, for any surjection $\boldsymbol{\pi}_1(U) \to G$ to a finite group G, the composite $\boldsymbol{\pi}_1(\mathcal{U} \setminus \{P\}) \to \boldsymbol{\pi}_1(U) \to G$ is also surjective. Hence, $\boldsymbol{\pi}_1(U)$ is finite, since $\boldsymbol{\pi}_1(U)$ is cyclic (cf. Step 1), and its order is divisible by the order of the cyclic quotient singularity (Y, P) for $P \in \Lambda$. Thus, Lemma 3.8 has been proved. □

Step 4 (Toward reduction to the case: $d_2 = 1$). Assume that $d_2 > 1$ and let $\tau \colon T' \simeq \mathbb{P}^1 \to T$ be a cyclic cover of degree d_2 branched at t_1 and t_2. Let $t'_i \in T'$ be the inverse image of t_i for $i = 1, 2$. For the normalization Y' of $Y \times_T T'$, let $\eta \colon Y' \to Y$ and $\pi' \colon Y' \to T'$ be morphisms induced by projections which make a commutative diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{\eta} & Y \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ T' & \xrightarrow{\tau} & T. \end{array}$$

We set $D'_i := \eta^{-1} D_i = \pi'^{-1}(t'_i)$ for $i = 1, 2$. Since $\pi^*(t_2) = d_2 D_2$ and since (Y, D_2) is 1-log-terminal along D_2,

- π' is smooth along $D'_2 = \pi'^*(t_2)$, and
- $\eta^{-1} U \to U$ is an étale cover of degree d_2

by Lemma 2.12. In particular, we have a surjection $\boldsymbol{\pi}_1(U) \to \mathbb{Z}/d_2\mathbb{Z}$. On the other hand, (Y', Σ') is a pseudo-toric surface for $\Sigma' = \eta^{-1}\Sigma$ by Lemma 3.4. We set $\Theta'_j := \eta^{-1}\Theta_j$ for $j = 1, 2$. Then $\Sigma' = \Theta'_1 + \Theta'_2 + D'_1 + D'_2$, and Θ'_1 and Θ'_2 are mutually disjoint sections of π' by Lemma 3.1(1). Since τ is étale over $T \setminus \{t_1, t_2\}$, for any $1 \leq k \leq b$, $\eta^* G_k$ is the disjoint union of reducible fibers of π' lying over G_k, and each fiber of π' in $\eta^* G_k$ is isomorphic to G_k by η. Hence, the number of reducible fibers of π' different from D'_1 equals $d_2 b$, and

$$\boldsymbol{\delta}(Y', \Sigma') = d_2 \boldsymbol{\delta}(Y, \Sigma) > 0 \tag{III-2}$$

by Lemma 3.1(3). Therefore, (Y', Σ'), $\pi' \colon Y' \to T'$, and D'_2 satisfy the same assumptions in Proposition 3.7 required for (Y, Σ), $\pi \colon Y \to T$, and D_2, where $\operatorname{mult}_{D'_2} \pi'^*(t'_2) = 1$. Here, the open subset $U' := Y'_{\mathrm{reg}} \setminus (\Sigma' - D'_2)$ contains $\eta^{-1} U$, and the complement $U' \setminus \eta^{-1} U$ is a finite set contained in $\eta^{-1} \operatorname{Sing} Y$. Thus, $\boldsymbol{\pi}_1(U') \simeq \boldsymbol{\pi}_1(\eta^{-1} U)$.

Step 5 (Final step). We shall prove (1)–(5) of Proposition 3.7. The fundamental group $\boldsymbol{\pi}_1(U')$ of the open subset $U' \subset Y'$ in Step 4 is finite by Lemma 3.8(f) in Step 3. Since $\boldsymbol{\pi}_1(\eta^{-1} U) \simeq \boldsymbol{\pi}_1(U')$ is isomorphic to the kernel of the surjection $\boldsymbol{\pi}_1(U) \to \mathbb{Z}/d_2\mathbb{Z}$, we have (1) by Step 1. The universal covering map of U' extends to a cyclic cover $\widetilde{Y} \to Y'$ as in Step 2, and the composite $\nu \colon \widetilde{Y} \to Y' \to Y$ gives a universal cover over U. Hence, (2) holds by Step 2 except the equality $\boldsymbol{\delta}(\widetilde{Y}, \widetilde{\Sigma}) = d_2 \boldsymbol{\delta}(Y, \Sigma)$, which is shown by (III-2) in Step 4 and by Lemma 3.8(e) applied to Y'. By Step 4, the finite morphism $\widetilde{T} \to T$ in Proposition 3.7 is identified with the morphism $\tau \colon T' \to T$, and we have (3), except the smoothness of $\tilde{\pi}$ along $\widetilde{D}_2$, which is verified by Lemma 3.8(d). The remaining assertions (4) and (5) are shown by Step 4 and by (b) and (d) of Lemma 3.8 applied to Y' and to the cyclic cover $\widetilde{Y} \to Y'$. Thus, we have finished the proof of Proposition 3.7. □

Proposition 3.9. *In Proposition* 3.2*, assume that* $\boldsymbol{\delta}(X, S_1 + S_2) > 0$ *and that* S_2 *is irreducible. Then the following hold for* $U := X_{\mathrm{reg}} \setminus S_1$ *and* $d_2 := \mathrm{mult}_{D_2} \pi^*(t_2)$*, where* $\pi\colon Y \to T$ *is the* $\mathbb{P}^1$*-fibration in Proposition* 3.2*, and* D_2 *is a set-theoretic fiber* $\pi^{-1}(t_2)$ *identified with the proper transform of* S_2 *in* Y:

(1) *The fundamental group* $\boldsymbol{\pi}_1(U)$ *is finite and cyclic. In particular, there is a finite cyclic cover* $\nu\colon \widetilde{X} \to X$ *from a normal projective surface* $\widetilde{X}$ *which gives a universal cover over* U.

(2) *For the cover* ν *in* (1)*,* $(\widetilde{X}, \nu^{-1}(S_1 + S_2))$ *is a pseudo-toric surface and*
$$\boldsymbol{\delta}(\widetilde{X}, \nu^{-1}(S_1 + S_2)) = d_2\boldsymbol{\delta}(X, S_1 + S_2).$$

(3) *If* $\boldsymbol{\delta}(X, S_1 + S_2) = d_2 = 1$*, then there exist two prime divisors* B_1 *and* B_2 *such that* $(X, S_1 + B_1 + B_2)$ *is a toric surface.*

(4) *If* $\boldsymbol{\delta}(X, S_1 + S_2) > 0$ *and if* $d_2 > 1$*, then one of the following holds*:
   (a) *There is a negative curve* C *on* X *such that* $C \not\subset (S_1 + S_2)$.
   (b) *There is a point* $\widetilde{P} \in \nu^{-1}(S_1 \cap S_2)$ *and exist at least two negative curves* $\widetilde{C}$ *such that*
$$\widetilde{C} \cap \nu^{-1}S_1 = \widetilde{C} \cap \nu^{-1}S_2 = \{\widetilde{P}\}.$$
   (c) *There is no negative curve on* $\widetilde{X}$*,* $\boldsymbol{\rho}(\widetilde{X}) = d_2 = 2$*,* $\boldsymbol{\delta}(X, S_1 + S_2) = 1$*, and there exist two* $\mathbb{P}^1$*-fibrations* $\widetilde{X} \to \mathbb{P}^1$ *in both of which* $\nu^{-1}S_1$ *and* $\nu^{-1}S_2$ *are sections.*

Proof. (1): The assertion on $\boldsymbol{\pi}_1(U)$ follows from Proposition 3.7(1) applied to the open subset $U_Y := Y_{\mathrm{reg}} \setminus (\Sigma - D_2)$, since μ is an isomorphism outside $S_1 \cap S_2$ and it induces an isomorphism $U_Y \simeq U$. The rest follows from Lemma 3.5.

(2): Note that $(X, S_1 + S_2)$ and (Y, Σ) are pseudo-toric surfaces with $\boldsymbol{\delta}(X, S_1 + S_2) = \boldsymbol{\delta}(Y, \Sigma)$ by Proposition 3.2. We set $\widetilde{S}_i = \nu^{-1}S_i$ for $i = 1, 2$. Then $(\widetilde{X}, \widetilde{S}_1 + \widetilde{S}_2)$ is a pseudo-toric surface by Lemma 3.4, since ν is étale over $U \supset X_{\mathrm{reg}} \setminus (S_1 \cup S_2)$. Let $\widetilde{Y}$ be the normalization of $\widetilde{X} \times_X Y$ and let $\nu_Y\colon \widetilde{Y} \to Y$ and $\tilde{\mu}\colon \widetilde{Y} \to \widetilde{X}$ be induced morphisms, which make a commutative diagram:
$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\tilde{\mu}} & \widetilde{X} \\ {\scriptstyle \nu_Y}\downarrow & & \downarrow{\scriptstyle \nu} \\ Y & \xrightarrow{\mu} & X. \end{array}$$
We set $\widetilde{\Sigma} := \nu_Y^{-1}\Sigma = \tilde{\mu}^{-1}(\widetilde{S}_1 + \widetilde{S}_2)$. Then $(\widetilde{Y}, \widetilde{\Sigma})$ is a pseudo-toric surface and $\boldsymbol{\delta}(\widetilde{Y}, \widetilde{\Sigma}) = d_2\boldsymbol{\delta}(Y, \Sigma)$ by Lemma 3.4 and Proposition 3.7(2), since ν_Y is étale over $U_Y \supset Y_{\mathrm{reg}} \setminus \Sigma$. On the other hand,
$$\boldsymbol{\delta}(\widetilde{Y}, \widetilde{\Sigma}) - \boldsymbol{\delta}(\widetilde{X}, \widetilde{S}_1 + \widetilde{S}_2) = \boldsymbol{\rho}(\widetilde{Y}) - \boldsymbol{\rho}(\widetilde{X}) - (\boldsymbol{n}(\widetilde{\Sigma}) - \boldsymbol{n}(\widetilde{S}_1) - \boldsymbol{n}(\widetilde{S}_2)) = 0,$$
since the $\tilde{\mu}$-exceptional locus is contained in $\widetilde{\Sigma}$. This shows (2).

(3): By $\boldsymbol{\delta}(Y, \Sigma) = \boldsymbol{\delta}(X, S_1 + S_2) = 1$ and by Lemma 3.1(3), there is a unique reducible fiber G of π different from D_1 and D_2 such that $\boldsymbol{n}(G) = 2$. Here, G is reduced and is a linear chain $G_{(1)} + G_{(2)}$ of rational curves, where $G_{(j)}$ is the prime component intersecting the section $\Theta_j \subset \Sigma$ for $j = 1$, 2, by Lemma 3.1(2). On the other hand, D_2 is irreducible as the proper transform of S_2, and $\pi^*(t_2) = D_2$ by $d_2 = 1$. Thus, D_2 is a smooth fiber of π (cf. [18, Prop. 2.33(4)]). Hence, $G \sim D_2$, and $K_Y + \Sigma^\sharp \sim 0$ for

$$\Sigma^\sharp := \Sigma - D_2 + G = \Theta_1 + \Theta_2 + G_{(1)} + G_{(2)} + D_1.$$

Then $(Y, S_Y^\sharp)$ is a toric surface by [26, Thm. 6.4] (cf. [18, Thm. 1.1]), since $(Y, S_Y^\sharp)$ is log-canonical (cf. Lemma 3.1(2)) and since $\boldsymbol{n}(S_Y^\sharp) = \boldsymbol{n}(S_Y) + 1 = \boldsymbol{\rho}(Y) + 2$. Thus, $(X, S^\sharp)$ is also a toric surface for $S^\sharp := \mu_* S_Y^\sharp$ by [18, Lem. 3.9], since the μ-exceptional divisor $\Theta_1 + \Theta_2$ is contained in $S_Y^\sharp$. Therefore, (3) holds for $B_j = \mu(G_{(j)})$ for $j = 1$, 2.

(4): Let $\mathrm{Gal}(\nu)$ be the Galois group of the cyclic cover $\nu\colon \widetilde{X} \to X$. Assume that there is a negative curve $\widetilde{C}$ on $\widetilde{X}$ satisfying

$$\widetilde{C} \cap (\widetilde{S}_1 + \widetilde{S}_2) = \emptyset. \tag{III-3}$$

Then the transform $\sigma(\widetilde{C})$ is also a negative curve satisfying (III-3) for any $\sigma \in \mathrm{Gal}(\nu)$, and the image $C = \nu(\widetilde{C})$ in X satisfies $C \cap (S_1 + S_2) = \emptyset$, since we have

$$\nu^{-1}(C) = \sum\nolimits_{\sigma \in \mathrm{Gal}(\nu)} \sigma(\widetilde{C}).$$

Thus, C is a negative curve by the Hodge index theorem as $S_1 + S_2$ is big, and as a consequence, (4a) holds in this case.

Assume next that there is a negative curve $\widetilde{C}$ on $\widetilde{X}$ satisfying

$$\widetilde{C} \cap \widetilde{S}_1 = \widetilde{C} \cap \widetilde{S}_2 = \{\widetilde{P}_j\} \tag{III-4}$$

for some $j \in \{1, 2\}$. Then the transform $\sigma(\widetilde{C})$ is also a negative curve satisfying (III-4) for any $\sigma \in \mathrm{Gal}(\nu)$. If $\sigma(\widetilde{C}) \neq \widetilde{C}$ for some $\sigma \in \mathrm{Gal}(\nu)$, then (4b) holds. If $\sigma(\widetilde{C}) = \widetilde{C}$ for any $\sigma \in \mathrm{Gal}(\nu)$, then $\widetilde{C} = \nu^{-1}C$ for a negative curve C on X such that $C \cap S_1 = C \cap S_2 = \{P_j\}$; thus, (4a) holds.

As the Stein factorization of $\pi \circ \nu_Y \colon \widetilde{Y} \to T$, we have a $\mathbb{P}^1$-fibration $\tilde{\pi}\colon \widetilde{Y} \to \widetilde{T} \simeq \mathbb{P}^1$ and a finite morphism $\widetilde{T} \to T$ (cf. Lemma 3.1). By the observation above on negative curves on $\widetilde{X}$ and by Corollary 3.3(3) applied to $(\widetilde{X}, \widetilde{S}_1, \widetilde{S}_2, \tilde{\mu}\colon \widetilde{Y} \to \widetilde{X}, \tilde{\pi}\colon \widetilde{Y} \to \widetilde{T})$ instead of $(X, S_1, S_2, \mu\colon Y \to X, \pi\colon Y \to T)$, we see that if (4a) and (4b) do not hold, then $\boldsymbol{\delta}(\widetilde{X}, \widetilde{S}_1 + \widetilde{S}_2) = d_2 \boldsymbol{\delta}(X, S_1 + S_2) = 2$ (cf. (2)), $\boldsymbol{\rho}(\widetilde{X}) = 2$, and there exist two $\mathbb{P}^1$-fibrations $\widetilde{X} \to \mathbb{P}^1$ in both of which $\widetilde{S}_1$ and $\widetilde{S}_2$ are sections; in particular, $d_2 = 2$, $\boldsymbol{\delta}(X, S_1 + S_2) = 1$, and (4c) holds. Thus, we are done. □

3.3. **Applications to endomorphisms.** We shall prove Theorems 3.13 and 3.14 below on endomorphisms and toric surfaces by applying results in Section 3.2.

Lemma 3.10. *Let f be a surjective endomorphism of a normal projective surface X and let S be an f-completely invariant divisor, where S is allowed to be zero. Assume that the fundamental group of the open subset $U := X_{\mathrm{reg}} \setminus S$ is finite. Let $\nu\colon \widetilde{X} \to X$ be a finite surjective morphism from a normal projective surface $\widetilde{X}$ which gives a universal cover over U. Then there is an endomorphism $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$ such* *that* $\nu \circ \tilde{f} = f \circ \nu$*. In particular,* $\deg \tilde{f} = \deg f$*, and* $S_{\tilde{f}} = \nu^{-1} S_f$ *for* characteristic completely invariant divisors $S_{\tilde{f}}$ *and* S_f (*cf.* [20, Def. 2.16]).

Proof. Since f is finite and $f^{-1}S = S$, there is a finite subset $\Xi \subset X_{\mathrm{reg}}$ such that$X_{\mathrm{reg}} \cap f^{-1}U = U \setminus \Xi$. Let $\widetilde{X}'$ be the normalization of the fiber product $\widetilde{X} \times_X X$ of ν and f over X, and consider the induced commutative diagram

$$\begin{array}{ccc} \widetilde{X}' & \xrightarrow{p} & \widetilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \nu} \\ X & \xrightarrow{f} & X, \end{array}$$

where p and q are finite morphisms and q is étale over $f^{-1}U$. Then q is étale over U, since Ξ is finite. In particular, we have a finite morphism $\nu^{-1}U \to q^{-1}U$ over U from the universal cover $\nu^{-1}U$ of U, and it extends to a finite morphism $\xi \colon \widetilde{X} \to \widetilde{X}'$ over X by Remark 3.6. We set $\tilde{f} := p \circ \xi \colon \widetilde{X} \to \widetilde{X}$. Then $\nu \circ \tilde{f} = f \circ \nu$. It implies that $\deg \tilde{f} = \deg f$, and moreover, we have $S_{\tilde{f}} = \nu^{-1} S_f$ by [20, Lem. 2.19(3)]. □

Proposition 3.11. *Let X be a normal projective surface admitting a non-isomorphic surjective endomorphism f and let B be an effective divisor on X such that*

(i) $K_X + S_f + B \sim 0$,
(ii) $\# S_f \cap \operatorname{Supp} B = 2$,
(iii) $(X, S_f + B)$ *is log-canonical along* $S_f \cap \operatorname{Supp} B$,
(iv) $mB \sim D$ *for a positive integer m and an effective Cartier divisor D such that* $\operatorname{Supp} D \subset S_f$ *and* $\operatorname{Supp} D \cap \operatorname{Supp} B = S_f \cap \operatorname{Supp} B$.

Then $(X, S_f + B)$ is a pseudo-toric surface. Moreover, if

(v) S_f *is reducible with* $\boldsymbol{n}(S_f) \geq \boldsymbol{\rho}(X)$

in addition, then there is a reduced divisor B' such that $\boldsymbol{n}(B') \leq 2$ and that $(X, B' + S_f)$ is a toric surface.

Proof. Note that $B \neq 0$ by (ii), and $K_X + S_f \not\sim 0$ by (i). Since S_f contains all the negative curves on X (cf. [20, Prop. 2.20(3)]), every prime component of B is nef by (ii), and $-(K_X + S_f + B_{\mathrm{red}}) = -(K_X + S_f + B) + (B - B_{\mathrm{red}})$ is also nef by (i). Thus, the required conditions of Proposition 3.2 are satisfied for $S_1 = S_f$ and $S_2 = B_{\mathrm{red}}$, by (ii), (iii), and (iv). As a consequence of Proposition 3.2, $(X, S_f + B_{\mathrm{red}})$ is a pseudo-toric surface. In particular, $K_X + S_f + B_{\mathrm{red}} \sim 0$, and we have $B = B_{\mathrm{red}}$ by (i). This proves the first assertion.

For the rest, assume (v). By Shokurov's criterion for toric surfaces [26, Thm. 6.4] (cf. [18, Thm. 1.3]), we have

$$0 \leq \boldsymbol{\delta}(X, S_f + B) = \boldsymbol{\rho}(X) + 2 - \boldsymbol{n}(S_f) - \boldsymbol{n}(B) \leq 2 - \boldsymbol{n}(B),$$

where the equality $\boldsymbol{\delta}(X, S_f + B) = 0$ holds if and only if $(X, S_f + B)$ is a toric surface. If B is reducible, then $\boldsymbol{n}(B) = 2$, $\boldsymbol{\delta}(X, S_f + B) = 0$, and hence, $(X, S_f + B)$ is a toric surface. Thus, we may assume that B is irreducible and $\boldsymbol{\delta}(X, S_f + B) = 1$.

We can apply Proposition 3.9 to X, $S_1 = S_f$, and $S_2 = B$. If the number d_2 in Proposition 3.9 is equal to 1, then $(X, S_f + B')$ is a toric surface for a reduced divisor B' with $\boldsymbol{n}(B') = 2$ by Proposition 3.9(3). In case $d_2 > 1$, we shall derive a contradiction as follows: Let $\nu\colon \widetilde{X} \to X$ be the finite cyclic cover giving a universal cover over $U = X_{\mathrm{reg}} \setminus S_f$ (cf. Proposition 3.9(1)). Then we have a non-isomorphic surjective endomorphism $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$ such that $\nu \circ \tilde{f} = f \circ \nu$ by Lemma 3.10, where $S_1 = S_f$ (resp. $\nu^{-1}S_1 = \nu^{-1}S_f = S_{\tilde{f}}$) contains all the negative curves on X (resp. $\widetilde{X}$). Since S_f is reducible, no case of Proposition 3.9(4) does not hold. This is a contradiction. Thus, we are done. □

Remark. By applying results on Section 5.3, we can weaken the condition (v) to

(v′) $\boldsymbol{n}(S_f) \geq \boldsymbol{\rho}(X)$.

In fact, it is enough to consider the case where S_f is irreducible and $\boldsymbol{\rho}(X) = 1$, and the assertion holds if $d_2 = 1$ as in the proof above. When $d_2 > 1$, we can derive a contradiction as follows: By Proposition 3.9(4), we may assume that $d_2 = \boldsymbol{\rho}(\widetilde{X}) = 2$ and that there exist two $\mathbb{P}^1$-fibrations $\widetilde{X} \to \mathbb{P}^1$ in both of which $\nu^{-1}S_f = S_{\tilde{f}}$ is a section. Since $\delta_{\tilde{f}} = \delta_f > 1$, $S_{\tilde{f}}$ is a union of fibers of two $\mathbb{P}^1$-fibrations $\widetilde{X} \to \mathbb{P}^1$ by Corollary 5.25. This is a contradiction. The assertion for (v′) can be applied to the study of endomorphisms in the case of Picard number 1, but we do not proceed it in this article.

Lemma 3.12. *Let X be a normal projective surface with a reduced reducible connected divisor S such that*

- *(X, S) is log-canonical,*
- *$-(K_X + S)$ is nef, and*
- *$(K_X + S)C < 0$ for two prime components C of S.*

Then S is a linear chain of rational curves such that $(K_X + S)C_1 < 0$ and $(K_X + S)C_2 < 0$ for the end components C_1 and C_2 of S, and there is an effective divisor B such that $K_X + S + B \sim 0$, $(X, S+B)$ is log-canonical along S, and $S \cap \operatorname{Supp} B = \{P_1, P_2\}$ for two points $P_1 \in C_1$, $P_2 \in C_2$.

Proof. The assertion except on B follows from [18, Lem. 4.5]. To show the existence of B, first we consider the case where $\boldsymbol{n}(S) \geq 3$, and set $S^\natural := S - C_1 - C_2$. Then $K_X + S$ is Cartier along $S^\natural$ and

$$\mathcal{O}_X(K_X + S) \otimes_{\mathcal{O}_X} \mathcal{O}_{S^\natural} \simeq \mathcal{O}_{S^\natural}$$

by [18, Lem. 4.5(3)]. Since $(K_X + S)C_i < 0$ for $i = 1, 2$, $t(C_1 + C_2) - 2(K_X + S)$ is nef and big for some $0 < t < 1$ by [21, Lem. 4.4]. Thus,

$$H^1(X, \mathcal{O}_X(-K_X - S - S^\natural)) = 0$$

by a version of Kawamata–Viehweg's vanishing theorem [24, Thm. (5.1)] (cf. [19, Prop. 2.15]), since

$$K_X + \ulcorner t(C_1 + C_2) - 2(K_X + S) \urcorner = -K_X - S - S^\natural.$$

Hence, the restriction homomorphism

$$H^0(X, \mathcal{O}_X(-K_X - S)) \to H^0(X, \mathcal{O}_X(-K_X - S) \otimes \mathcal{O}_{S^\natural}) \simeq H^0(S^\natural, \mathcal{O}_{S^\natural})$$

is surjective, and we have an effective divisor $B \sim -(K_X + S)$ such that $S^\natural \cap \operatorname{Supp} B = \emptyset$. In particular, $BC_i = -(K_X + S)C_i > 0$ for $i = 1, 2$. By [18, Prop. 3.29], the following hold for each $i = 1, 2$:

(a) If $C_i \cap \operatorname{Sing} X \subset S^\natural$, then $(K_X + S)C_i = -1$.
(b) If $C_i \cap \operatorname{Sing} X \not\subset S^\natural$, then $C_i \cap \operatorname{Sing} X \setminus S^\natural = \{P_i\}$ and $(K_X + S)C_i = -1/n_i$ for a cyclic quotient singular point P_i of order n_i.

In case (a), $BC_i = 1$, and hence, $(X, S + B)$ is log-canonical at $B \cap C_i$. In case (b), $B \cap C_i = \{P_i\}$ with $BC_i = 1/n_i$, and $(X, S + B)$ is log-canonical at P_i by Lemma 2.8(1). Thus, we are done in the case where $\boldsymbol{n}(S) \geq 3$.

Next, we treat the case: $\boldsymbol{n}(S) < 3$. Then $S = C_1 + C_2$, since S is reducible. For the intersection point P of C_1 and C_2, we can take a non-isomorphic toroidal blowing up $\varphi\colon Y \to X$ at P with respect to (X, S). We set $S_Y := \varphi^{-1}S$. Then $\boldsymbol{\rho}(Y) \geq 3$, (Y, S_Y) is log-canonical, $K_Y + S_Y = \varphi^*(K_X + S)$, and S_Y is a linear chain of rational curves whose end components are proper transforms C_1' and C_2' of C_1 and C_2, respectively. In particular, $-(K_Y + S_Y)$ is nef on S_Y and $(K_Y + S_Y)C_i' < 0$ for $i = 1, 2$. Applying the previous argument to (Y, S_Y), we can find an effective divisor B_Y on Y such that $K_Y + S_Y + B_Y \sim 0$, $(Y, S_Y + B_Y)$ is log-canonical along S_Y, $\varphi^{-1}(P) \cap \operatorname{Supp} B_Y = \emptyset$, and that $S_Y \cap \operatorname{Supp} B_Y = \{P_1', P_2'\}$ for points $P_1' \in C_1'$ and $P_2' \in C_2'$. Thus, $B = \varphi_* B_Y$ satisfies the required condition, since φ is an isomorphism over $X \setminus P$. □

Theorem 3.13. *Let X be a normal projective surface admitting a non-isomorphic surjective endomorphism f such that*

(i) $\boldsymbol{\rho}(X) \geq 3$,
(ii) $K_X + S_f$ *is not pseudo-effective,*
(iii) $(K_X + S_f)C < 0$ *for two prime components C of S_f.*

Then $(X, S_f + B)$ is a toric surface for a reduced divisor B with $\boldsymbol{n}(B) \leq 2$.

Proof. By (ii) and [21, Thm. 1.3], we have $\boldsymbol{n}(S_f) \leq \boldsymbol{\rho}(X) + 1$, in which the equality holds if and only if $(X, B + S_f)$ is a toric surface for a prime divisor B. Thus, we may assume that $\boldsymbol{n}(S_f) \leq \boldsymbol{\rho}(X)$. By (i), (ii), and [21, Prop. 4.3], the pair (X, S_f) is an $\mathcal{L}$-*surface* (cf. [21, Def. 4.2]). Thus, the following hold by (iii) and [21, Thm. 4.5]:

(1) X is a rational surface with only rational singularities; in particular, X is $\mathbb{Q}$-factorial and the numerical equivalence coincides with the $\mathbb{Q}$-linear equivalence for $\mathbb{Q}$-divisors on X;
(2) $\overline{\mathrm{NE}}(X)$ is generated by the numerical classes of negative curves on X;
(3) $-K_X$ and S_f are big, and $-(K_X + S_f)$ is semi-ample;
(4) $\boldsymbol{\rho}(X) = \boldsymbol{n}(S_f)$, and S_f is the union of all the negative curves on X;
(5) S_f is a linear chain of rational curves, and $(K_X + S_f)C_1 < 0$ and $(K_X + S_f)C_2 < 0$ for the end components C_1 and C_2 of S_f;
(6) $S_f - C_1 - C_2$ is negative definite.

By (3) and Lemma 3.12, there is an effective divisor B such that $K_X + S_f + B \sim 0$, $(X, S_f + B)$ is log-canonical along S_f, and $S_f \cap \operatorname{Supp} B = \{P_1, P_2\}$ for two points

$P_1 \in C_1$ and $P_2 \in C_2$. Hence, by (1), (2), and (4), we have a positive integer m and an effective Cartier divisor D such that $\operatorname{Supp} D \subset S_f$ and $mB \sim D$.

We shall show that $\operatorname{Supp} D = S_f$. Assume the contrary. Then the numerical classes of prime components of D generate a proper face of the polyhedral cone $\overline{\mathrm{NE}}(X)$ by (2) and (4). In particular, D is not big. Then $B \sim -(K_X + S_f)$ is not big, and $BD = mB^2 = 0$ by (3). Hence, $\operatorname{Supp} D \cap \operatorname{Supp} B = \emptyset$, and $\operatorname{Supp} D \subset S_f - C_1 - C_2$; this contradicts (6).

Therefore, (X, S_f, B) satisfies all the conditions (i)–(v) of Proposition 3.11, and as a consequence, $(X, S_f + B')$ is a toric surface for a reduced divisor B' with $\boldsymbol{n}(B') \leq 2$. □

Theorem 3.14. *Let X be a normal projective surface admitting a non-isomorphic surjective endomorphism f such that $\boldsymbol{\rho}(X) = 2$, S_f is singular, and $-(K_X + S_f)$ is ample. Then $(X, S_f + B)$ is a toric surface for a reduced divisor B with $\boldsymbol{n}(B) \leq 2$.*

Proof. Since $-(K_X + S_f)$ is ample, by [21, Thm. 1.3], We have $\boldsymbol{n}(S_f) \leq \boldsymbol{\rho}(X) + 1 = 3$, in which the equality holds if and only if $(X, S_f + B)$ is a toric surface for a prime divisor B. Hence, we may assume that $\boldsymbol{n}(S_f) \leq 2$.

If a prime component C of S_f is singular, then $(K_X + S_f)C = 0$ by [18, Prop. 3.29], violating the ampleness of $-(K_X + S_f)$. Thus S_f is reducible and connected. Moreover, by the ampleness of $-(K_X + S_f)$ and by Lemma 3.12, S_f is a linear chain $C_1 + C_2$ of rational curves for two prime components C_1 and C_2, and there exists a divisor B such that $K_X + S_f + B \sim 0$, $(X, S_f + B)$ is log-canonical along S_f, and that $S_f \cap \operatorname{Supp} B = \{P_1, P_2\}$ for two points $P_1 \in C_1$ and $P_2 \in C_2$.

Claim. The pseudo-effective cone $\overline{\mathrm{NE}}(X)$ is generated by $\mathrm{cl}(C_1)$ and $\mathrm{cl}(C_2)$. In particular, X is a rational surface with only rational singularities.

Proof. The latter assertion follows from the first. In fact, S_f is big, since $\mathrm{cl}(S_f)$ lies in the interior of $\overline{\mathrm{NE}}(X)$, and it implies the latter assertion by [18, Lem. 4.7]. Since $\boldsymbol{\rho}(X) = \dim \overline{\mathrm{NE}}(X) = 2$ (cf. [20, Prop. C]), the cone $\overline{\mathrm{NE}}(X)$ is fan-shaped. Thus, for the first assertion, it suffices to show that each extremal ray R of $\overline{\mathrm{NE}}(X)$ contains $\mathrm{cl}(C_1)$ or $\mathrm{cl}(C_2)$. By replacing f with f^2 if necessary, we may assume that $f^{-1}C_i = C_i$ for $i = 1$, 2, and that $f^*\mathsf{R} = \mathsf{R}$ for the endomorphism $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$. Since $-(K_X + S_f)$ is ample, by the contraction theorem (cf. [20, Thm. 1.10]), we have the contraction morphism $\pi \colon X \to T$ of R, in which $\dim T > 0$. If π is birational, then R contains $\mathrm{cl}(C_1)$ or $\mathrm{cl}(C_2)$, since S_f contains all the negative curves. Thus, we may assume that $\dim T = 1$. By [20, Lem. 3.16], there is an endomorphism $h \colon T \to T$ such that $\pi \circ f = h \circ \pi$. Let G be the set-theoretic fiber of π passing through the intersection point P of $C_1 \cap C_2$. Since $f^{-1}(P) = \{P\}$, we have $h^{-1}(\pi(P)) = \{\pi(P)\}$ and $f^{-1}G = G$. If C_1 and C_2 are not fibers of π, then the reduced divisor $C_1 + C_2 + G$ is f-completely invariant, but $(X, C_1 + C_2 + G)$ is not log-canonical at P, violating [20, Thm. E]. Therefore, C_1 or C_2 is a fiber of π, and its numerical class belongs to R. □

Proof of Theorem 3.14 *continued.* By the Claim, as in the proof of Theorem 3.13, we can find a positive integer m and an effective Cartier divisor D such that $mB \sim$

D and $\operatorname{Supp} D \subset S_f$. Here, $\operatorname{Supp} D = S_f$, since $D \sim -m(K_X + S_f)$ is ample but C_i is not ample for $i = 1, 2$. Therefore, (X, S_f, B) satisfies all the conditions (i)–(v) of Proposition 3.11, and as a consequence, $(X, S_f + B')$ is a toric surface for a reduced divisor B' with $\boldsymbol{n}(B') \le 2$. □

Remark. In Theorem 3.14, if the *first dynamical degree* λ_f (cf. [20, Def. 3.1]) is equal to $\delta_f = (\deg f)^{1/2}$, then we have another proof by applying Theorem 3.13 to a toroidal blowing up $X' \to X$ at the intersection point of C_1 and C_2 with an endomorphism $X' \to X'$ as a lift of f (cf. [19, Prop. 5.6]).

4. On $\mathcal{V}$-surfaces

Definition 4.1. Let V be a normal projective surface and let Λ_1 and Λ_2 be two prime divisors on V. The triplet $(V, \Lambda_1, \Lambda_2)$ is called a *$\mathcal{V}$-surface* if

(i) V is rational and $\boldsymbol{\rho}(V) = 2$,
(ii) the pair $(V, \Lambda_1 + \Lambda_2)$ is log-canonical,
(iii) $(\Lambda_1)^2 \le 0$, $(\Lambda_2)^2 < 0$, $(K_V + \Lambda_1 + \Lambda_2)\Lambda_1 < 0$, and $(K_V + \Lambda_1 + \Lambda_2)\Lambda_2 = 0$.

For a $\mathcal{V}$-surface $(V, \Lambda_1, \Lambda_2)$, we set

$$\Sigma_i := (\Lambda_i \setminus (\Lambda_1 \cap \Lambda_2)) \cap \operatorname{Sing} Y$$

for $i = 1, 2$, and call Σ_1 (resp. Σ_2) the first (resp. second) *external singular locus.*

In Section 4, we shall study the structure of $\mathcal{V}$-surfaces and give an application to the study of non-isomorphic surjective endomorphism concerning (R3) in the introduction. After giving some remarks on *half-toric surfaces* (cf. [18, §7]) in Section 4.1, we shall explain basic properties of $\mathcal{V}$-surfaces in Section 4.2, where two subclasses $\mathcal{V}_{\mathrm{A}}$-*surfaces* and $\mathcal{V}_{\mathrm{B}}$-*surfaces* are defined (cf. Definition 4.5). Section 4.3 is devoted to proving Theorem 4.10 which asserts that any $\mathcal{V}_{\mathrm{A}}$-surface becomes a half-toric surface by adding a prime divisor, i.e., $(V, \Lambda_1 + \Lambda_2 + B)$ is an half-toric surface for a prime divisor B. Our second crucial ideal explained in the introduction concerns the proof of Theorem 4.10. In Section 4.4, we introduce ordinary $\mathcal{V}_{\mathrm{B}}$-*surfaces* and extraordinary $\mathcal{V}_{\mathrm{B}}$-*surfaces* (cf. Definition 4.18) as subclasses of $\mathcal{V}_{\mathrm{B}}$-surfaces, and prove that any ordinary $\mathcal{V}_{\mathrm{B}}$-surface also becomes a half-toric surface by adding a prime divisor, but this is not true for any extraordinary $\mathcal{V}_{\mathrm{B}}$-surface (cf. Proposition 4.19 and Theorem 4.21). Moreover, in Section 4.4, as an application of Theorems 4.10 and 4.21, we shall prove Theorem 4.23 on the structure of a normal projective surface X of Picard number ≥ 3 admitting a non-isomorphic surjective endomorphism f such that $K_X + S_f$ is not pseudo-effective, i.e., X belonging to (R3). In Section 4.5, we shall prove Theorem 4.29 on the structure of an extraordinary $\mathcal{V}_{\mathrm{B}}$-surface with the notion of $(2n+1, 2)$-*blowings up* (cf. Definition 4.24).

4.1. Remarks on half-toric surfaces.

The half-toric surfaces are defined and studied in [18, §7]. We shall give two additional results. One is Lemma 4.2 below on negative curves and the pseudo-effective cone. The other is Lemma 4.3 below on endomorphisms, which is applied to the proof of Theorem 1.2 (cf. Section 6).

Lemma 4.2. *Let (X, D) be a half-toric surface. Then*:

(1) *Every negative curve on X is contained in D.*
(2) *The pseudo-effective cone $\overline{\mathrm{NE}}(X)$ is generated by numerical classes of prime components of D.*

Proof. (1): There is a double cover $\tau\colon Y \to X$ étale in codimension 1 such that (Y, τ^*D) is a toric surface: this double cover is unique up to isomorphism and is called the *characteristic double cover* (cf. [18, §7.1]). Let Γ be a negative curve on X. Then every prime component of $\tau^*\Gamma$ is a negative curve. On the other hand, every negative curve on Y is contained in the boundary divisor τ^*D, since it is preserved by the action of open torus $Y \setminus \tau^*D$. Hence, $\tau^*\Gamma \subset \tau^*D$, and $\Gamma \subset D$.

(2): Let $\nu\colon X' \to X$ be a non-isomorphic toroidal blowing up with respect to (X, D). Then $\boldsymbol{\rho}(X') \geq 3$, and $(X', \nu^{-1}D)$ is also a half-toric surface (cf. [18, Lem. 7.4(2)]). In particular, $-K_{X'} \approxeq \nu^{-1}D$ is big (cf. [18, Def. 7.1(i), Lem. 7.2(2)]), and $\overline{\mathrm{NE}}(X')$ is generated by numerical classes of negative curves on X' by [20, Thm. 1.13]. By (1), negative curves on X' are all contained in $\nu^{-1}D$. Thus, (2) holds by $\overline{\mathrm{NE}}(X) = \nu_* \overline{\mathrm{NE}}(X')$. □

Lemma 4.3. *For a half-toric surface (X, D), let f be a non-isomorphic surjective endomorphism of X such that $S_f = D - B$ for an end component B of D. Then every prime component of τ^*B is nef for the characteristic double cover $\tau\colon Y \to X$.*

Proof. Since S_f contains all the negative curves on X (cf. [20, Prop. 2.20(3)]), the prime divisor B is nef. Thus, we may assume that τ^*B is reducible. By [18, Prop. 4.18(3)], $(B \setminus (D-B)) \cap \operatorname{Sing} X$ consists of one point Q of type $\mathcal{D}$, i.e., (X, B) is not 1-log-terminal at Q, and $\tau^*B = \Gamma_1 + \Gamma_2$ for two prime divisors Γ_1 and Γ_2 such that $\Gamma_1 \cap \Gamma_2 = \tau^{-1}(Q) = \{P\}$ for a point P. Here, Y is a toric surface expressed as $\mathbb{T}_{\mathsf{N}}(\triangle)$ for a complete fan $\triangle$ of a free abelian group N of rank 2, and τ^*D is the boundary divisor, the complement of the open torus $\mathbb{T}_{\mathsf{N}}(\{0\})$.

For the complement U of $D - B$ in X, the inverse image $\tau^{-1}U$ is expressed as $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ for the two-dimensional cone $\boldsymbol{\sigma} \in \triangle$ corresponding to P. For $i = 1, 2$, let $\mathsf{R}_i \in \triangle$ be a ray corresponding to the prime component Γ_i of τ^*D, and let W_i be the toric open subset $\mathbb{T}_{\mathsf{N}}(\mathsf{R}_i)$. Then $\boldsymbol{\sigma} = \mathsf{R}_1 + \mathsf{R}_2$, and $W_1 \cup W_2 = \tau^{-1}U \setminus \{P\}$. We shall show that the fundamental group $\boldsymbol{\pi}_1(\tau^{-1}U \setminus \{P\})$ is finite. By inclusions $Y \setminus \tau^*D \subset W_i \subset W_1 \cup W_2$ for $i = 1, 2$, we have surjections

$$\mathsf{N} \simeq \boldsymbol{\pi}_1(Y \setminus \tau^*D) \to \boldsymbol{\pi}_1(W_i) \to \boldsymbol{\pi}_1(W_1 \cup W_2),$$

where $\boldsymbol{\pi}_1(W_i) \simeq \mathsf{N}/(\mathsf{N} \cap \mathsf{R}_i)$ (cf. [4, §3.2]). This implies that $\boldsymbol{\pi}_1(W_1 \cup W_2)$ is finite, since $(\mathsf{N} \cap \mathsf{R}_1) + (\mathsf{N} \cap \mathsf{R}_2)$ is a finite index subgroup of N and it is contained in the kernel of $\mathsf{N} \to \boldsymbol{\pi}_1(W_1 \cap W_2)$. As a consequence, the fundamental group $U \setminus \{Q\}$ is also finite.

Let $\nu\colon \widetilde{X} \to X$ be the finite surjective morphism from a normal projective surface $\widetilde{X}$ which gives a universal cover over $U_{\mathrm{reg}} = U \setminus \{Q\} = X_{\mathrm{reg}} \setminus (D - B) = X_{\mathrm{reg}} \setminus S_f$. By Lemma 3.10, there is a non-isomorphic surjective endomorphism $\tilde{f}$ of $\widetilde{X}$ such that $\nu \circ \tilde{f} = f \circ \nu$ and $S_{\tilde{f}} = \nu^{-1}S_f = \nu^{-1}(D - B)$. In particular, every prime component $\nu^{-1}B$ is nef. On the other hand, ν factors through τ, since $\tau^{-1}(U \setminus \{Q\}) = \tau^{-1}U \setminus \{P\} = W_1 \cup W_2$ is connected and étale cover $U \setminus \{Q\}$ (cf. Lemma 3.5). Then

Γ_i is nef for $i = 1, 2$, since the inverse image of Γ_i in $\widetilde{X}$ is contained in $\nu^{-1}B$. Thus, we are done. □

4.2. **Basic properties of $\mathcal{V}$-surfaces.** Some basic properties of $\mathcal{V}$-surfaces are obtained in Lemma 4.4 below, and $\mathcal{V}_{\mathrm{A}}$-surfaces and $\mathcal{V}_{\mathrm{B}}$-surfaces are introduced in Definition 4.5 below. Moreover, we discuss sufficient conditions for a $\mathcal{V}$-surface to become a half-toric surface by adding a prime divisor.

Lemma 4.4. *The following hold for any $\mathcal{V}$-surface $(V, \Lambda_1, \Lambda_2)$:*

(1) *The cone $\overline{\mathrm{NE}}(V)$ is generated by the numerical classes of Λ_1 and Λ_2. In particular, V contains no negative curves other than Λ_1 or Λ_2.*
(2) *The divisor $-(K_V + \Lambda_1 + \Lambda_2)$ is nef and big. In particular, $-K_V$ is big and any nef $\mathbb{Q}$-divisor on V is semi-ample.*
(3) *The surface V has only rational singularities, and the numerical equivalence coincides with the $\mathbb{Q}$-linear equivalence for $\mathbb{Q}$-divisors on V.*
(4) *The divisor $\Lambda_1 + \Lambda_2$ is a linear chain of rational curves. In particular, $\Lambda_1 \simeq \Lambda_2 \simeq \mathbb{P}^1$, $\Lambda_1 \cap \Lambda_2$ consists of one point P_V, and $\Lambda_1\Lambda_2 = 1/n_V$ for a positive integer n_V: If $n_V = 1$, then $P_V \in V_{\mathrm{reg}}$, and if $n_V > 1$, then (V, P_V) is a cyclic quotient singularity of order n_V.*
(5) *The pair (V, Λ_1) is 1-log-terminal along $\Lambda_1 \setminus \{P_V\}$, and $(K_V + \Lambda_1 + \Lambda_2)\Lambda_1 = -1/m_V$ for a positive integer m_V. If $m_V = 1$, then the first external singular locus Σ_1 is empty, and if $m_V > 1$, then Σ_1 consists of one point at which V has a cyclic quotient singularity of order m_V.*
(6) *One of the following holds for the second external singular locus Σ_2:*
    (a) *Σ_2 consists of two A_1-singular points at which (V, Λ_2) is 1-log-terminal;*
    (b) *Σ_2 consists of one point at which (V, Λ_2) is log-canonical of* type $\mathcal{D}$ *in the sense of* [18, Def. 3.23]; *in other words, (V, Λ_2) is not* 1-log-terminal *at the point* (*cf.* [19, Def. 2.1, Fact 2.5]).
(7) *The divisor $K_V + \Lambda_1 + \Lambda_2$ is not Cartier along Σ_2 but $2(K_V + \Lambda_1 + \Lambda_2)$ is Cartier along Λ_2 with an isomorphism*

$$\mathcal{O}_V(2(K_V + \Lambda_1 + \Lambda_2)) \otimes \mathcal{O}_{\Lambda_2} \simeq \mathcal{O}_{\Lambda_2}.$$

Proof. (1): For $i = 1, 2$, the ray $\mathsf{R}_i = \mathbb{R}_{\geq 0}\,\mathrm{cl}(\Lambda_i)$ of $\overline{\mathrm{NE}}(V)$ is extremal, since $\Lambda_i^2 \leq 0$ (cf. Definition 4.1(iii)) and since $\overline{\mathrm{NE}}(V)$ is fan-shaped by $\boldsymbol{\rho}(V) = 2$. Here, $\mathsf{R}_1 \neq \mathsf{R}_2$: For, otherwise, Λ_1 is also a negative curve, and we have $\Lambda_1 = \Lambda_2$, a contradiction. For a negative curve on V, its numerical class generates an extremal ray of $\overline{\mathrm{NE}}(X)$; hence the negative curve is either Λ_1 or Λ_2. This shows (1).

(2): The divisor $-(K_V + \Lambda_1 + \Lambda_2)$ is nef by (1) and Definition 4.1(iii). If it is not big, then its numerical class belongs to an extremal ray R of $\overline{\mathrm{NE}}(V)$, but we have $\mathsf{R} \neq \mathsf{R}_1$ and $\mathsf{R} \neq \mathsf{R}_2$ by $(K_V + \Lambda_1 + \Lambda_2)\Lambda_2 = 0$ and $\Lambda_2^2 < 0$. Thus, $-(K_V + \Lambda_1 + \Lambda_2)$ is big. In particular, $-K_V$ is big, and the rest of (2) follows from [20, Prop. 1.5].

(3): This follows from [18, Lem. 2.31] and the bigness of $-K_V$, since V is rational with $H^2(V, \mathcal{O}_V) = H^0(V, \mathcal{O}_V(K_V)) = 0$.

(4): For the contraction morphism $\phi\colon V \to V'$ of the negative curve Λ_2, V' is a normal projective surface with $\boldsymbol{\rho}(V') = 1$ and $\phi(\Lambda_1)^2 > 0$. Hence, $\Lambda_1 \cap \Lambda_2 \neq \emptyset$

by $\Lambda_1^2 \leq 0$. In particular, $\Lambda_1 + \Lambda_2$ is connected. Then $\Lambda_1 + \Lambda_2$ is a linear chain of rational curves by [18, Lem. 4.5], since $(V, \Lambda_1 + \Lambda_2)$ is log-canonical and since $-(K_V + \Lambda_1 + \Lambda_2)$ is nef with $(K_V + \Lambda_1 + \Lambda_2)\Lambda_1 < 0$. The assertion for n_V follows from a well-known property of 2-dimensional log-canonical pairs (cf. [19, Fact. 2.5]).

(5)–(7): There are derived from [18, Prop. 3.29] on cases (E)–(H) there. □

Definition 4.5. Let $(V, \Lambda_1, \Lambda_2)$ be a $\mathcal{V}$-surface.

(1) The integer n_V in Lemma 4.4(4) is called the *internal index*, and the point P_V is called the *central point*.
(2) The integer m_V in Lemma 4.4(5) are called the *first external index*.
(3) If Lemma 4.4(6a) (resp. 4.4(6b)) holds, then $(V, \Lambda_1, \Lambda_2)$ is called a $\mathcal{V}_{\mathrm{A}}$-*surface* (resp. $\mathcal{V}_{\mathrm{B}}$-*surface*).

Remark. The internal index n_V equals the *numerical factorial index* of V at P_V (cf. [19, Def. 1.26]). The first external index m_V is the smallest positive integer m such that $m(K_V + \Lambda_1)$ (resp. $m\Lambda_1$) is Cartier along $\Lambda_1 \setminus \{P_V\}$ (cf. Fact 2.5, Lemma 2.8). If we consider the *second external index* as an analogy of the first external index, then it should be 2 by Lemma 4.4(7).

Lemma 4.6. *For a $\mathcal{V}$-surface $(V, \Lambda_1, \Lambda_2)$, there is an effective divisor E such that $\Lambda_2 \cap \operatorname{Supp} E = \emptyset$ and $E \sim -2(K_V + \Lambda_1 + \Lambda_2)$. In this case, $(X, \Lambda_1 + \Lambda_2 + (1/2)E)$ is log-canonical along $\Lambda_1 + \Lambda_2$.*

Proof. There is a positive number $\varepsilon < 1$ such that $\varepsilon\Lambda_1 - 3(K_V + \Lambda_1 + \Lambda_2)$ is nef and big by Definition 4.1(iii) and Lemma 4.4(2) and by [21, Lem. 4.4]. Since

$$K_V + \ulcorner \varepsilon\Lambda_1 - 3(K_V + \Lambda_1 + \Lambda_2) \urcorner = -2(K_V + \Lambda_1 + \Lambda_2) - \Lambda_2,$$

we have $H^1(V, \mathcal{O}_V(-2(K_V + \Lambda_1 + \Lambda_2) - \Lambda_2)) = 0$ by a version of Kawamata–Viehweg's vanishing theorem [24, Thm. (5.1)] (cf. [19, Prop. 2.15]). Hence, the restriction homomorphism

$$H^0(V, \mathcal{O}_V(-2(K_V + \Lambda_1 + \Lambda_2)) \to H^0(V, \mathcal{O}_X(-2(K_V + \Lambda_1 + \Lambda_2)) \otimes \mathcal{O}_{\Lambda_2}) \simeq H^0(\Lambda_2, \mathcal{O}_{\Lambda_2}) \simeq \mathbb{C}$$

is surjective (cf. Lemma 4.4(6)), and we can find an effective divisor E such that $E \sim -2(K_V + \Lambda_1 + \Lambda_2)$ and $\Lambda_2 \cap \operatorname{Supp} E = \emptyset$. Thus, we have proved the first assertion.

For the latter assertion, since $E \cap \Lambda_2 = \emptyset$, it is enough to prove the log-canonicity of $(V, \Lambda_1 + (1/2)E)$ along $\Lambda_1 \cap \operatorname{Supp} E$. Now, $E\Lambda_1 = 2/m_V$ and $\#\Sigma_1 \leq 1$ by Lemma 4.4(5). Hence, one of the following holds:

(1) $\#\Lambda_1 \cap \operatorname{Supp} E \geq 2$;
(2) $\Lambda_1 \cap \operatorname{Supp} E$ consists of one point of $\Lambda_1 \setminus \Sigma_1$.
(3) $\Lambda_1 \cap \operatorname{Supp} E = \Sigma_1$.

If (1) holds, then $m_V = 1$, $\Sigma_1 = \emptyset$, and E intersects Λ_1 transversely at two points; thus, $(V, \Lambda_1 + E)$ and $(V, \Lambda_1 + (1/2)E)$ are log-canonical along $\Lambda_1 \cap \operatorname{Supp} E$. If (2) holds and if $m_V > 1$, then $m_V = 2$ and E intersects Λ_1 transversely at one point; thus, $(V, \Lambda_1 + E)$ and $(V, \Lambda_1 + (1/2)E)$ are log-canonical along $\Lambda_1 \cap \operatorname{Supp} E$. If

(2) holds and if $m_V = 1$, then E intersects Λ_1 tangentially at one point $\{P'\} = \Lambda_1 \cap \operatorname{Supp} E$ by $E\Lambda_1 = 2$; in this case, $(V, \Lambda_1 + (1/2)E)$ is log-canonical at P' by "inversion of adjunction" (cf. [14, Thm. 17.7]), since $P' \in X_{\mathrm{reg}}$ and Λ_1 is non-singular. If (3) holds, then $m_V > 1$ by $\Sigma_1 \neq \emptyset$, and $(V, \Lambda_1 + (1/2)E)$ is log-canonical along $\Lambda_1 \cap \operatorname{Supp} E$ by Lemma 2.8(2). Thus, we are done. □

Proposition 4.7. *For a $\mathcal{V}$-surface $(V, \Lambda_1, \Lambda_2)$, let B be an effective divisor on V such that $K_V + \Lambda_1 + \Lambda_2 + B \approx 0$ and $\Lambda_2 \cap \operatorname{Supp} B = \emptyset$. Then B is a prime divisor, $(V, \Lambda_1 + \Lambda_2 + B)$ is a half-toric surface, and B and Λ_2 are end components of the linear chain $B + \Lambda_1 + \Lambda_2$ of rational curves.*

Proof. By Lemma 4.4(5), $B\Lambda_1 = 1/m_V$ and $B \cap \Lambda_1$ consists of one point $Q_1 \neq P_V$. Let Γ be a prime component of B. Then $\Gamma\Lambda_1 > 0$ by $\Gamma\Lambda_2 = 0$ and $\boldsymbol{\rho}(V) = 2$, and we have $m_V\Gamma\Lambda_1 \in \mathbb{Z}$, since $m_V\Lambda_1$ is Cartier along $\Lambda_1 \setminus \{P_V\}$. Hence, $(B - \Gamma) \cap (\Lambda_1 + \Lambda_2) = \emptyset$, and it implies that $\Gamma = B$. Therefore, B is a prime divisor. Moreover, $(V, \Lambda_1 + \Lambda_2 + B)$ is log-canonical at Q_1 by Lemma 2.8(1).

By (1) and (3) of Lemma 4.4, there exist an effective Cartier divisor A on V and a positive integer m such that $\operatorname{Supp} A \subset \Lambda_1 + \Lambda_2$ and $mB \sim A$. Then $\operatorname{Supp} A = \Lambda_1 + \Lambda_2$ as B is nef and big (cf. Lemma 4.4(2)), and $B \cap \operatorname{Supp} A = \{Q_1\}$. Let $\sigma \colon V' \to V$ be the normalization of the blowing up along the scheme-theoretic intersection $mB \cap A$. Let Λ'_1, Λ'_2, B' and A' be the proper transforms of Λ_1, Λ_2, B, and A in V', respectively, and we set $\Theta := \sigma^{-1}(Q_1)$ and $S := \sigma^{-1}(\Lambda_1 + \Lambda_2) = \Theta + \Lambda'_1 + \Lambda'_2$. Then σ is a toroidal blowing up at Q_1 with respect to $(V, \Lambda_1 + \Lambda_2 + B)$, and the following hold as in the proof of Proposition 3.2:

- $\Theta \simeq \mathbb{P}^1$, $\sigma^{-1}\Lambda_1 = \Theta_1 + \Lambda'_1$, $\sigma^*\Lambda_2 = \Lambda'_2$, and S is a linear chain of rational curves with end components Θ_1 and Λ'_2;
- $K_{V'} + S + B' = \sigma^*(K_V + \Lambda_1 + \Lambda_2 + B) \approx 0$;
- $(V', S + B')$ is log-canonical along Θ;
- there is a positive integer e such that $e\Theta$ is Cartier and that $mB' = \sigma^*(mB) - e\Theta$ and $A' = \sigma^*(A) - e\Theta$;
- there exist a fibration $\pi \colon V' \to T \simeq \mathbb{P}^1$ and points $t_B \neq t_A$ of T such that $B' = \pi^{-1}(t_B)$ and $\operatorname{Supp} A' = \pi^{-1}(t_A)$.

A general fiber F of π is rational and $\Theta F = 2$ by $(K_{V'} + \Theta)F = (K_{V'} + S + B')F = 0$. Then $B' \simeq \mathbb{P}^1$ and $(V', S + B')$ is log-canonical along B' by Lemma 2.16. Therefore, $(V, \Lambda_1 + \Lambda_2 + B)$ is log-canonical along $\Lambda_1 + \Lambda_2 + B$. Now, $\boldsymbol{n}(\Lambda_1 + \Lambda_2 + B) = 3 = \boldsymbol{\rho}(V) + 1$, $K_V + \Lambda_1 + \Lambda_2 + B \approx 0$, and $B + \Lambda_1 + \Lambda_2$ is a linear chain of rational curves with end components B and Λ_2. Thus, $(V, \Lambda_1 + \Lambda_2 + B)$ is a half-toric surface by [18, Thm. 1.3]. □

Corollary 4.8. *If the effective divisor E in Lemma 4.6 is not a prime divisor, then $(V, \Lambda_1 + \Lambda_2 + B)$ is a half-toric surface for any prime component B of E.*

Proof. Let B be a prime component of E. Then B and $E - B$ intersect Λ_1 by $B\Lambda_2 = (E - B)\Lambda_2 = 0$ and $\boldsymbol{\rho}(V) = 2$. Now, $E\Lambda_1 = 2/m_V$ by Lemma 4.4(5), and $m_V\Lambda_1$ is Cartier along $\Lambda_1 \cap \operatorname{Supp} E$. Thus, $B\Lambda_1 = (E - B)\Lambda_1 = 1/m_V$. Hence, $K_V + \Lambda_1 + \Lambda_2 + B \approx 0$, and the assertion follows from Proposition 4.7. □

Corollary 4.9. *Let $(V, \Lambda_1, \Lambda_2)$ be a $\mathcal{V}$-surface admitting a numerically trivial divisor L such that $K_V + \Lambda_1 + \Lambda_2 + L$ is Cartier along Λ_2. Then $(V, \Lambda_1 + \Lambda_2 + B)$ is a half-toric surface for a prime divisor B.*

Proof. By the assumption for L and by Lemma 4.4(7), we have an isomorphism

$$\mathcal{O}_V(K_V + \Lambda_1 + \Lambda_2 + L) \otimes \mathcal{O}_{\Lambda_2} \simeq \mathcal{O}_{\Lambda_2},$$

since $\Lambda_2 \simeq \mathbb{P}^1$. We know that $\varepsilon\Lambda_1 - 2(K_V + \Lambda_1 + \Lambda_2) - L$ is nef and big for $0 < \varepsilon \ll 1$ by [21, Lem. 4.4] and Lemma 4.4(2) and by $(K_V + \Lambda_1 + \Lambda_2)\Lambda_1 < 0$. Then

$$H^1(V, \mathcal{O}_V(-\Lambda_2 - (K_V + \Lambda_1 + \Lambda_2 + L))) = 0$$

by a version of Kawamata–Viehweg's vanishing theorem [24, Thm. (5.1)] (cf. [19, Prop. 2.15]), since

$$K_V + \ulcorner \varepsilon\Lambda_1 - 2(K_V + \Lambda_1 + \Lambda_2) - L \urcorner = -\Lambda_2 - (K_V + \Lambda_1 + \Lambda_2 + L).$$

Hence, the restriction homomorphism

$$\begin{aligned} H^0(V, \mathcal{O}_V(-(K_V + \Lambda_1 + \Lambda_2 + L))) \to H^0(V, \mathcal{O}_V(-(K_V + \Lambda_1 + \Lambda_2 + L)) \otimes \mathcal{O}_{\Lambda_2}) \\ \simeq H^0(\Lambda_2, \mathcal{O}_{\Lambda_2}) \simeq \mathbb{C} \end{aligned}$$

is surjective and we have an effective divisor B on V such that $B \sim -(K_V + \Lambda_1 + \Lambda_2 + L)$ and $\Lambda_2 \cap \operatorname{Supp} B = \emptyset$. Therefore, B is a prime divisor and $(V, \Lambda_1 + \Lambda_2 + B)$ is a half-toric surface by Proposition 4.7, since $K_V + \Lambda_1 + \Lambda_2 + B \sim -L \approx 0$. □

4.3. **Structure of a $\mathcal{V}_{\mathrm{A}}$-surface.** We shall prove the following:

Theorem 4.10. *For any $\mathcal{V}_{\mathrm{A}}$-surface $(V, \Lambda_1, \Lambda_2)$, there is a prime divisor B such that $(V, \Lambda_1 + \Lambda_2 + B)$ is a half-toric surface.*

The proof is given at the end of this section. First, we treat an easy case.

Lemma 4.11. *Theorem* 4.10 *holds if $\Lambda_1^2 = 0$.*

Proof. Let $\pi \colon V \to T$ be the contraction morphism of the extremal ray $\mathbb{R}_{\geq 0} \operatorname{cl}(\Lambda_1)$, where $(K_V + \Lambda_1 + \Lambda_2)\Lambda_1 < 0$. Then $T \simeq \mathbb{P}^1$ and π is a $\mathbb{P}^1$-fibration with only irreducible fibers. Moreover, Λ_2 is a section of π by $(K_V + \Lambda_1 + \Lambda_2)F = (K_V + \Lambda_2)F < 0$ for a general fiber F of π. The second external locus Σ_2 consists of two A_1-singular points $P_{A,1}$ and $P_{A,2}$ at which (V, Λ_2) is 1-log-terminal (cf. Lemma 4.4(6a) and Definition 4.5(3)). For $i = 1$, 2, we set $t_i := \pi(P_{A,i})$, $F_i := \pi^*(t_i)$, and $G_i := \pi^{-1}(t_i)$. Then $G_i \cap \Lambda_2 = \{P_{A,i}\}$. We can show that $F_i = 2G_i$ for $i = 1$, 2. In fact, $F_i = m_i G_i$ for an integer m_i, and $\Lambda_2 G_i = 1/m_i$. Here $m_i > 1$ by [18, Prop. 2.33(4)], and we have $m_i = 2$, since the numerical factorial index of an A_1-singularity is equal to 2. As a consequence, $(V, \Lambda_2 + G_i)$ is log-canonical at $P_{A,i}$ by Lemma 2.8(1). We can prove also that $L := G_1 - G_2$ satisfies conditions of Corollary 4.9. In fact, $2L \sim 0$ by $2G_1 \sim 2G_2 \sim F$, and $K_V + \Lambda_1 + \Lambda_2 + L$ is Cartier along Λ_2 by the log-canonicity of $(V, \Lambda_2 + G_i)$. Therefore, $(V, \Lambda_1 + \Lambda_2 + B)$ is a half-toric surface for a prime divisor B by Corollary 4.9. □

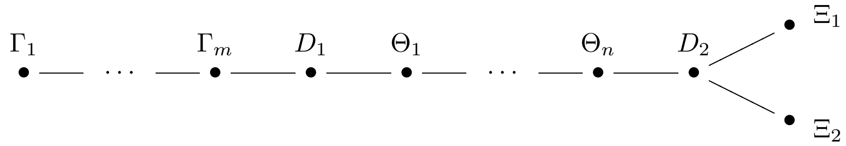
In what follows in Section 4.3, we fix a $\mathcal{V}_{\mathrm{A}}$-surface $(V, \Lambda_1, \Lambda_2)$ such that $\Lambda_1^2 < 0$. We also fix an effective divisor E on V such that $\Lambda_2 \cap \operatorname{Supp} E = \emptyset$ and that $E \sim -2(K_V + \Lambda_1 + \Lambda_2)$. This divisor E exists by Corollary 4.6, and we may assume that E is a prime divisor by Corollary 4.8.

Definition 4.12. Let $\mu \colon Y \to V$ be the minimal resolution of singularities lying on $\Lambda_1 + \Lambda_2$ and set $D := \mu^{-1}(\Lambda_1 + \Lambda_2)$; in particular, $Y \setminus D \simeq V \setminus (\Lambda_1 \cup \Lambda_2)$ by μ. We set D_i to be the proper transform of Λ_i in Y for $i = 1, 2$, and set E_Y to be the proper transform of E in Y. For two A_1-singular points $P_{A,1}$ and $P_{A,2}$ in Σ_2 (cf. Definition 4.5(3), Lemma 4.4(6a)), we set $\Xi_k := \mu^{-1}(P_{A,k})$ for $k = 1, 2$, which is a (-2)-curve.

Lemma 4.13. *The divisor D is a simple normal crossing divisor on Y_{reg} and has a prime decomposition*

$$\sum\nolimits_{i=1}^{m} \Gamma_i + D_1 + \sum\nolimits_{j=1}^{n} \Theta_j + D_2 + \Xi_1 + \Xi_2$$

with a dual graph

(IV-1)


for some integers $m > 0$ and $n \geq 0$. Moreover, the equality

$$K_Y + D = \mu^*(K_V + \Lambda_1 + \Lambda_2) + (1/2)(\Xi_1 + \Xi_2) \tag{IV-2}$$

holds and the following conditions are satisfied:

1. *For the* central point P_V *(cf. Definition* 4.5(1)*), if $P_V \in V_{\mathrm{reg}}$, then $n = 0$; if $P_V \notin V_{\mathrm{reg}}$, then $n > 0$ and $\sum_{j=1}^{n} \Theta_j = \mu^{-1}(P_V)$.*
2. *The first external locus Σ_1 consists of a point Q_V, and $\sum_{i=1}^{m} \Gamma_i = \mu^{-1}(Q_V)$.*
3. *The μ-exceptional locus equals*
$$D - D_1 - D_2 = \sum\nolimits_{i=1}^{m} \Gamma_i + \sum\nolimits_{j=1}^{n} \Theta_j + \Xi_1 + \Xi_2$$
and $\boldsymbol{\rho}(Y) = m + n + 4$.
4. *One has $D_1^2 = -1$ and $D_2^2 \leq -2$.*
5. *There is a positive integer $a \leq m$ such that*
$$D - \sum\nolimits_{i=1}^{a} \Gamma_i = \sum\nolimits_{i=a+1}^{m} \Gamma_i + D_1 + \sum\nolimits_{j=1}^{n} \Theta_j + D_2 + \Xi_1 + \Xi_2$$
is not negative definite but negative semi-definite.
6. *One of the following holds*:
   (a) *$E_Y \Gamma_1 = 2$ and $E_Y \cap (D - \Gamma_1) = \emptyset$.*
   (b) *$m \geq 2$, $\Gamma_1^2 = -2$, $E_Y \Gamma_2 = 1$, and $E_Y \cap (D - \Gamma_2) = \emptyset$.*
   (c) *$m = 1$, $\Gamma_1^2 = -2$, $E_Y D_1 = 1$, and $E_Y \cap (D - D_1) = \emptyset$.*

Proof. We have (IV-2) by a well-known description of the minimal resolution, since $(V, \Lambda_1 + \Lambda_2)$ is toroidal at the central point P_V and is 1-log-terminal along $\Sigma_1 \cup \Sigma_2$, where $\#\Sigma_1 \leq 1$ and $\Sigma_2 = \{P_{A,1}, P_{A,2}\}$ (cf. Lemma 4.4(5), Definition 4.12, [19, Fact 2.5]). By Lemma 4.4(4), we can define $\sum_{j=1}^{n}\Theta_j$ as in (1). By Lemma 4.4(5), we can define the linear chain $\sum_{i=1}^{m}\Gamma_i$ of rational curves as $\mu^{-1}\Sigma_1$ when $\Sigma_1 \neq \emptyset$; we set $m = 0$ when $\Sigma_1 = \emptyset$. Then we have the prime decomposition of D above with the dual graph (IV-1) in which the case: $m = 0$ is allowed. In particular, $D - D_1 - D_2$ is the μ-exceptional locus, and we have (3) by $\boldsymbol{\rho}(Y) - 2 = \boldsymbol{\rho}(Y) - \boldsymbol{\rho}(V) = \boldsymbol{n}(D) - 2 = m + n + 2$.

We shall show (6) assuming that $m > 0$. Then $\Sigma_1 \neq \emptyset$, $m_V > 1$, and $E\Lambda_1 = 2/m_V$ by Lemma 4.4(5). If $\Lambda_1 \cap E \neq \Sigma_1$, then $m_V = 2$, $m = 1$, $E\Lambda_1 = 1$, and (6c) is satisfied. If $\Lambda_1 \cap E = \Sigma_1$, then either (6a) or (6b) is satisfied by Lemma 2.8(2) on minimal resolutions applied to E. Thus, (6) holds true if $m > 0$.

Since (5) implies $m > 0$, it is enough to prove (4) and (5). The divisors Γ_i and Θ_j for $1 \le i \le m$ and $1 \le j \le n$ are non-singular rational curves with self-intersection number ≤ -2 by the minimality of μ. The divisors

$$\mu^{-1}\Lambda_1 = \sum\nolimits_{i=1}^{m}\Gamma_i + D_1 + \sum\nolimits_{j=1}^{n}\Theta_j \quad \text{and} \quad \mu^{-1}\Lambda_2 = \sum\nolimits_{j=1}^{n}\Theta_j + D_2 + \Xi_1 + \Xi_2$$

are negative definite by $\Lambda_1^2 < 0$ and $\Lambda_2^2 < 0$. In particular, $\Xi_1 + D_2 + \Xi_2$ is negative definite, and we have $D_2^2 \le -2$ by the dual graph. Moreover, $D_1^2 = -1$ by Lemma 2.3(1), since $D = \mu^{-1}(\Lambda_1 + \Lambda_2)$ is big. Then we have the expected positive integer a in (5) by Lemma 2.3(2). Thus, (4) and (5) have been proved, and we are done. □

Lemma 4.14. *There is a $\mathbb{P}^1$-fibration $\pi\colon Y \to T \simeq \mathbb{P}^1$ with two points t_1, $t_2 \in T$ such that*

(1) *π is smooth over $T \setminus \{t_1, t_2\}$,*

and the following hold for the integer a in Lemma 4.13(5):

(2) *The prime component Γ_a of D is a double section of π and Γ_a is étale over $T \setminus \{t_1, t_2\}$.*
(3) *The set-theoretic fiber $\pi^{-1}(t_2)$ equals*

$$D - \sum\nolimits_{i=1}^{a}\Gamma_i = \sum\nolimits_{i=a+1}^{m}\Gamma_i + D_1 + \sum\nolimits_{j=1}^{n}\Theta_j + D_2 + \Xi_1 + \Xi_2.$$

(4) *If $a = 1$, then $\pi^{-1}(t_1)$ is irreducible.*
(5) *If $a > 1$, then $a \ge 3$, Lemma* 4.13(6b) *holds, and*

$$\pi^{-1}(t_1) = E_Y + \sum\nolimits_{i=1}^{a-1}\Gamma_i.$$

Proof. By applying Lemma 2.3 and Corollary 2.31 to D (cf. Lemma 4.13(5)), we have a $\mathbb{P}^1$-fibration $\pi\colon Y \to T \simeq \mathbb{P}^1$ with a point $t_2 \in T$ such that Γ_a is a double section of π and that (3) is satisfied for a branched point t_2 of $\pi|_{\Gamma_a}\colon \Gamma_a \to T$. In particular, (2) holds for the other branched point t_1 of $\pi|_{\Gamma_a}$. Every irreducible fiber of π over $T \setminus \{t_1, t_2\}$ is reduced, since it intersects Γ_a transversely at two points; hence, π is smooth along irreducible fibers over $T \setminus \{t_1, t_2\}$ (cf. [18, Prop. 2.33(4)]). Therefore, (1) follows from:

(1′) every fiber over $T \setminus \{t_1, t_2\}$ is irreducible.

In order to show (1′), we apply [18, Prop. 2.33(7)]. Note that

$$\text{(IV-3)} \qquad \boldsymbol{\rho}(Y) = m + n + 4 \quad \text{and} \quad \boldsymbol{n}(\pi^{-1}(t_2)) = m - a + n + 4$$

by Lemma 4.13(3) and by (3). If $a = 1$, then $\boldsymbol{\rho}(Y) - 2 - (\boldsymbol{n}(\pi^{-1}(t_2)) - 1) = 0$ by (IV-3), and every fiber of π over $T \setminus \{t_2\}$ is irreducible by [18, Prop. 2.33(7)]. This shows (1′) and (4) in the case where $a = 1$.

It is enough to prove (1′) and (5) in the case where $a \geq 2$. Then either (6a) or (6b) of Lemma 4.13 is satisfied, since $m \geq a \geq 2$. If Lemma 4.13(6a) is satisfied, then $E_Y + \Gamma_1$ is contained in a fiber of π, since it does not intersect $\pi^{-1}(t_2)$. Moreover, in this case, $E_Y \cap \Gamma_1 \subset D \subset Y_{\mathrm{reg}}$ and $E_Y\Gamma_1 = 2$, contradicting Lemma 2.11(1). Therefore, Lemma 4.13(6b) is satisfied. Let $t_0 \in T \setminus \{t_1\}$ be a point such that $\Gamma_1 \subset \pi^{-1}(t_0)$. Since $\Gamma_1^2 < 0$, $\pi^{-1}(t_0)$ is reducible, i.e., $\boldsymbol{n}(\pi^{-1}(t_0)) \geq 2$. Assume that $a \geq 3$. Then $E_Y + \sum_{i=1}^{a-1} \Gamma_i \subset \pi^{-1}(t_0)$ and $\boldsymbol{n}(\pi^{-1}(t_0)) \geq a$, and we have

$$\boldsymbol{\rho}(Y) - 2 - (\boldsymbol{n}(\pi^{-1}(t_0)) - 1) - (\boldsymbol{n}(\pi^{-1}(t_2)) - 1) = a - \boldsymbol{n}(\pi^{-1}(t_0)) \leq 0$$

by (IV-3). Hence, $\boldsymbol{n}(\pi^{-1}(t_0)) = a$ and every fiver of π over $T \setminus \{t_0, t_2\}$ is irreducible by [18, Prop. 2.33(7)]. In particular, $\pi^{-1}(t_0) = E_Y + \sum_{i=1}^{a-1} \Gamma_i$ and $\#\pi^{-1}(t_0) \cap \Gamma_a = \#\Gamma_{a-1} \cap \Gamma_a = 1$; hence, $t_0 = t_1$. Thus, (1′) and (5) hold when $a \geq 3$.

It remains to show that $a \neq 2$. Assume the contrary. Then $\pi^{-1}(t_0) = \Gamma_1 + \Gamma^\dagger$ for a prime divisor $\Gamma^\dagger$. By Lemma 4.13(6b), $E_Y \cap \pi^{-1}(t_2) = E_Y \cap \Gamma_1 = \emptyset$ and $\#E_Y \cap \Gamma_2 = 1$. Thus, $t_0 \neq t_1$ and $E_Y = \pi^{-1}(t_1)$. In particular, $\pi^*(t_0)$ intersects Γ_2 transversely at two points, and hence, $\pi^*(t_0) = \Gamma_1 + \Gamma^\dagger$ with $\Gamma^\dagger\Gamma_2 = 1$ by $\Gamma_1\Gamma_2 = 1$. On the other hand, $\Gamma_1\Gamma^\dagger = 2$ by $\Gamma_1^2 = -2$. This contradicts the latter half of Lemma 2.11(1), since $\Gamma_1 \cap \Gamma^\dagger \subset D \subset Y_{\mathrm{reg}}$. Therefore, $a \neq 2$. Thus, we are done. □

Lemma 4.15. *Let $\varphi\colon Y \to X$ be the contraction morphism of*

$$D - \Gamma_a - D_2 = \sum\nolimits_{1 \leq i \leq m,\, i \neq a} \Gamma_i + D_1 + \sum\nolimits_{j=1}^{n} \Theta_j + \Xi_1 + \Xi_2$$

and let $\pi_X : X \to T$ be the induced $\mathbb{P}^1$-fibration such that $\pi = \pi_X \circ \varphi$. Then

1. *the double section $C_X := \varphi(\Gamma_a)$ of π_X is a negative curve,*
2. *$(X/T, C_X, \pi_X^{-1}(t_2))$ is an* irreducible PDS configuration of type I_1 *(cf. Definition 2.17),*
3. *$\#(\pi_X^{-1}(t_2) \setminus C_X) \cap \operatorname{Sing} X = 2$.*

Moreover, the integer a equals 1, and $\pi^*(t_1) = 2G$ for a prime divisor G.

Proof. Since Λ_1 is a negative curve, we have (1) by

$$\varphi^{-1}C_X = \sum\nolimits_{i=1}^{m} \Gamma_i + D_1 + \sum\nolimits_{j=1}^{n} \Theta_j = \mu^{-1}\Lambda_1.$$

Assertions (2) and (3) are consequences of Corollary 2.31 applied to D.

Assume that $a \neq 1$. Then $a \geq 3$ and $\pi_X^{-1}(t_1) = \varphi(E_Y)$ by Lemma 4.14(5). We set $E_X := \varphi(E_Y)$ and $\{P_X\} := C_X \cap E_X$. Then (X, P_X) is a cyclic quotient singularity and φ gives the minimal resolution of the singularity (X, P_X), since $\varphi^{-1}(P_X) = \sum_{i=1}^{a-1} \Gamma_i$. By Lemma 2.8, (X, C_X) is 1-log-terminal at P_X, since Γ_a intersects the end component Γ_{a-1} of the linear chain $\varphi^{-1}(P_X)$ of rational curves transversely at one point. For the order $\bar{n}$ of the cyclic quotient singularity (X, P_X), we have $C_X E_X = 2/\bar{n}$ by Lemmas 2.8(3) and 4.13(6b). Hence, $(X/T, C_X, E_X)$ is an irreducible PDS of type II. Then $C_X^2 \geq 0$ by Proposition 2.28(4): This contradicts (1).

Therefore, $a = 1$. Then $G := \pi^{-1}(t_1)$ is irreducible by Lemma 4.14(4). We set $G_X := \varphi(G) = \pi_X^{-1}(t_1)$. Assume that $\pi^*(t_1)$ is reduced, i.e., $\pi^*(t_1) = G$. Then π is smooth along G (cf. Remark 2.9), $G\Gamma_a = 2$ and $(Y/T, \Gamma_a, G)$ is an irreducible PDS configuration of type II_1. Since φ is an isomorphism along G, $(X/T, C_X, G_X)$ is also an irreducible PDS configuration of type II_1. Then $C_X^2 \geq 0$ by Proposition 2.28(4): This contradicts (1). Therefore, $\pi^*(t_1)$ is not reduced, and we have $\pi^*(t_1) = 2G$ and $G\Gamma_1 = 1$ by $\Gamma_1 \subset D \subset Y_{\mathrm{reg}}$. Thus, we are done. □

Now, we are ready to prove Theorem 4.10.

Proof. Let $(V, \Lambda_1, \Lambda_2)$ be a $\mathcal{V}_{\mathrm{A}}$-surface. By Lemma 4.11, we may assume that $(\Lambda_1)^2 < 0$. We fix an effective divisor E in Lemma 4.6, which is assumed to a prime divisor by Corollary 4.8. Considering the birational morphism $\mu\colon Y \to V$ in Definition 4.12 and we shall apply results above on Y and E.

Now, $\mathrm{mult}_{\Gamma_1} \mu^*\Lambda_1 = 1/m_V$ by Fact 2.5, since (V, Λ_1) is 1-log-terminal at $\{Q_V\} = \Sigma_1$ (cf. Lemma 4.4(5), Lemma 4.13(2)). We set $B := \mu(G)$ for the prime divisor $G = \pi^{-1}(t_1)$ in Lemma 4.15. Then $B\Lambda_1 = G\mu^*\Lambda_1 = (1/m_V)G\Gamma_1 = 1/m_V$, and $(K_V + \Lambda_1 + \Lambda_2 + B)\Lambda_1 = 0$. Moreover, $B \cap \Lambda_2 = \emptyset$ by $G \cap \mu^{-1}\Lambda_2 \subset G \cap \pi^{-1}(t_2) = \emptyset$. Hence, $(K_V + \Lambda_1 + \Lambda_2 + B)\Lambda_2 = 0$ (cf. Definition 4.1(iii)). As a consequence, $K_V + \Lambda_1 + \Lambda_2 + B \approx 0$, since $\mathrm{cl}(\Lambda_1)$ and $\mathrm{cl}(\Lambda_2)$ generate $\mathsf{N}(V)$ (cf. Lemma 4.4(1)). Therefore, $(V, \Lambda_1 + \Lambda_2 + B)$ is a half-toric surface by Proposition 4.7. Thus, Theorem 4.10 has been proved. □

4.4. **Structure of an ordinary $\mathcal{V}_{\mathrm{B}}$-surface.** We shall introduce two subclasses of $\mathcal{V}_{\mathrm{B}}$-surfaces: ordinary $\mathcal{V}_{\mathrm{B}}$-surfaces and extraordinary $\mathcal{V}_{\mathrm{B}}$-surfaces in Definition 4.18 below. Then we shall prove in Theorem 4.21 that any ordinary $\mathcal{V}_{\mathrm{B}}$-surface becomes a half-toric surface by adding a prime divisor: This is an analogy of Theorem 4.10, and Proposition 4.19 below is considered as a result on the converse direction. As an application of Theorems 4.10 and 4.21, we shall prove Theorem 4.23 on endomorphisms concerning (R3) in the introduction. We begin with a setup for $\mathcal{V}_{\mathrm{B}}$-surfaces.

Definition 4.16. Let $(V, \Lambda_1, \Lambda_2)$ be a $\mathcal{V}_{\mathrm{B}}$-surface and let $\nu\colon Y \to V$ be the *standard partial resolution* of the singularity at the second external singular locus Σ_2 with respect to the log-canonical pair (V, Λ_2) in the sense of [19, Def. 4.27, Exam. 4.28]. Note that Σ_2 consists of one point at which $(V, \Lambda_1 + \Lambda_2)$ is not 1-log-terminal (cf. Lemma 4.4(6b), Definition 4.5(3)). We define $S := \nu^{-1}(\Lambda_1 + \Lambda_2)$, $E := \nu^{-1}\Sigma_2$, and define $\Lambda_{Y,i}$ as the proper transform of Λ_i in Y for $i = 1, 2$. Then S is a linear chain of rational curves expressed as $\Lambda_{Y,1} + \Lambda_{Y,2} + E$, and $\nu^*\Lambda_1 = \Lambda_{Y,1}$. We set C to be the end component of the linear chain E not intersecting $\Lambda_{Y,2}$ and set $\Sigma_Y := E \cap \mathrm{Sing}\, Y$.

Remark 4.17. By the definition of standard partial resolution, Σ_Y consists of two A_1-singular points contained in C. In particular, K_Y and $2C$ are Cartier along E. Moreover,

$$K_Y + S = \nu^*(K_V + \Lambda_1 + \Lambda_2). \tag{IV-4}$$

Definition 4.18. A $\mathcal{V}_{\mathrm{B}}$-surface $(V, \Lambda_1, \Lambda_2)$ is said to be *ordinary* if any negative curve on Y is contained in S. If not, it is said to be *extraordinary.*

Proposition 4.19. *Let $(V, \Lambda + B)$ be a half-toric surface for a reduced connected divisor Λ and a prime divisor $B \not\subset \Lambda$. Assume that $\boldsymbol{\rho}(V) = 2$ and $B^2 > 0$. Then $(V, \Lambda_1, \Lambda_2)$ is a $\mathcal{V}$-surface for the prime components Λ_1 and Λ_2 of Λ. Moreover, $(V, \Lambda_1, \Lambda_2)$ is either a $\mathcal{V}_{\mathrm{A}}$-surface or an ordinary $\mathcal{V}_{\mathrm{B}}$-surface.*

Proof. Since $(V, \Lambda + B)$ is log-canonical (cf. [18, Thm. 1.7(1)]), the pair (V, Λ) is so. Now, $\Lambda + B$ is a linear chain of rational curves with $\boldsymbol{n}(\Lambda + B) = \boldsymbol{\rho}(V) + 1 = 3$ (cf. [18, Thm. 1.7(1)]). Hence, Λ is a linear chain consisting of two prime components, and B is an end component of $\Lambda + B$. Let Λ_2 be the other end component of $\Lambda + B$ and let Λ_1 be the non-end component of $\Lambda + B$. Then $\Lambda = \Lambda_1 + \Lambda_2$, $\Lambda_1 \cap B \neq \emptyset$, and $\Lambda_2 \cap B = \emptyset$. Hence, $(K_V + \Lambda)\Lambda_1 = -B\Lambda_1 < 0$ and $(K_V + \Lambda)\Lambda_2 = -B\Lambda_2 = 0$ by $2(K_V + \Lambda + B) \sim 0$ (cf. [18, Def. 7.1(i)]). By the Hodge index theorem, we have $\Lambda_2^2 < 0$ by $B\Lambda_2 = 0$. The cone $\overline{\mathrm{NE}}(V)$ is fan-shaped by $\boldsymbol{\rho}(V) = 2$, and the numerical class $\mathrm{cl}(\Lambda_2)$ generates a ray of $\overline{\mathrm{NE}}(V)$. Since $B^2 > 0$, $\mathrm{cl}(B)$ lies in the interior of $\overline{\mathrm{NE}}(V)$. Hence, $\mathrm{cl}(\Lambda_1)$ generates the other ray of $\overline{\mathrm{NE}}(V)$ by Lemma 4.2(2). As a consequence, $\Lambda_1^2 \leq 0$. Therefore, $(V, \Lambda_1, \Lambda_2)$ is a $\mathcal{V}$-surface.

For the last assertion, we may assume that $(V, \Lambda_1, \Lambda_2)$ is a $\mathcal{V}_{\mathrm{B}}$-surface. For the standard partial resolution $\nu\colon Y \to V$ at Σ_2, ν^*B is the proper transform of B in Y, and $(Y, \nu^{-1}(\Lambda_1 + \Lambda_2) + \nu^*B)$ is also a half-toric surface by [18, Lem. 7.4(2)], since $\Lambda_2 \cap B = \emptyset$. Every negative curve on Y is contained in $\nu^{-1}(\Lambda_1 + \Lambda_2)$ by Lemma 4.2(1), since ν^*B is nef. Hence, the $\mathcal{V}_{\mathrm{B}}$-surface $(V, \Lambda_1, \Lambda_2)$ is ordinary. Thus, we are done. □

Lemma 4.20. *Let $(V, \Lambda_1, \Lambda_2)$ be an ordinary $\mathcal{V}_{\mathrm{B}}$-surface. For the standard partial resolution $\nu\colon Y \to V$ and divisors on Y introduced in Definition 4.16, let $\phi\colon Y \to W$ be the contraction morphism of $\nu^{-1}(\Lambda_2) - C = \Lambda_{Y,2} + E - C$, and set $C_1 = \phi(\Lambda_{Y,1})$ and $C_2 := \phi(C)$. Then (W, C_1, C_2) is a $\mathcal{V}_{\mathrm{A}}$-surface, and*

$$K_Y + S = \phi^*(K_W + C_1 + C_2). \tag{IV-5}$$

As a consequence, $\Lambda_1^2 < 0$ for any ordinary $\mathcal{V}_{\mathrm{B}}$-surface $(V, \Lambda_1, \Lambda_2)$.

Proof. By construction, we have $\boldsymbol{\rho}(W) = \boldsymbol{\rho}(Y) - \boldsymbol{n}(\mu^{-1}\Lambda_2) + 1 = \boldsymbol{\rho}(V) = 2$. By (IV-4) in Remark 4.17 and by Lemma 4.4(2), $-(K_Y + S)$ is nef and big, and in particular, $-K_Y$ is big. Then $\overline{\mathrm{NE}}(Y)$ is a polyhedral cone generated by the numerical classes of negative curves on Y by [20, Thm. 1.13], since $\boldsymbol{\rho}(Y) \geq 3$. Thus, $\overline{\mathrm{NE}}(W) = \phi_* \overline{\mathrm{NE}}(Y)$ is generated by the numerical classes of C_1 and C_2, since $(V, \Lambda_1, \Lambda_2)$ is ordinary. In particular, $C_1^2 \leq 0$ and $C_2^2 \leq 0$. Here, we have $C_2^2 < 0$ by $\Lambda_2^2 < 0$, since $\phi^{-1}(C_2) = \nu^{-1}(\Lambda_2)$ is negative definite. Now, $(K_Y + S)\Theta = 0$ for any prime component Θ of $\mu^{-1}\Lambda_2 = \Theta_{Y,2} + E$ by (IV-4) in Remark 4.17 and by $(K_V + \Lambda_1 + \Lambda_2)\Lambda_2 = 0$ (cf. Definition 4.1(iii)). Hence, (IV-5) holds, and as a consequence, $(W, C_1 + C_2)$ is log-canonical. Moreover,

$$(K_W + C_1 + C_2)C_1 = (K_Y + S)\nu^*\Lambda_1 = (K_V + \Lambda_1 + \Lambda_2)\Lambda_1 < 0 \qquad \text{and}$$
$$(K_W + C_1 + C_2)C_2 = (K_Y + S)C = (K_V + \Lambda_1 + \Lambda_2)\nu_*C = 0$$

by (IV-4) and (IV-5). Therefore, (W, C_1, C_2) is a $\mathcal{V}_{\mathrm{A}}$-surface. Now, $\nu^*\Lambda_1$ is a prime component of the reducible linear chain $\phi^{-1}(C_1) = S - C$ of rational curves, and $\phi^{-1}(C_1)$ is negative semi-definite by $C_1^2 \leq 0$. Hence, $\Lambda_1^2 = (\nu^*\Lambda_1)^2 < 0$. □

Theorem 4.21. *For an ordinary $\mathcal{V}_{\mathrm{B}}$-surface $(V, \Lambda_1, \Lambda_2)$, there is a prime divisor B such that $(V, \Lambda_1 + \Lambda_2 + B)$ is a half-toric surface.*

Proof. Applying Theorem 4.10 to the $\mathcal{V}_{\mathrm{A}}$-surface (W, C_1, C_2) in Lemma 4.20, we can find a prime divisor B_W on W such that $(W, C_1 + C_2 + B_W)$ is a half-toric surface. Then $(V, \Lambda_1 + \Lambda_2 + B)$ is also a half-toric surface for $B = \nu_*(\phi^*B_W)$. In fact, we have $B_W \cap C_2 = \emptyset$ by $B_W C_2 = -(K_W + C_1 + C_2)C_2 = 0$, and it implies that ϕ and ν are isomorphisms along $\phi^{-1}B_W = \nu^{-1}B$. Thus, the log-canonicity of $(V, \Lambda_1 + \Lambda_2 + B)$ is deduced from that of $(W, C_1 + C_2 + B_W)$. Moreover, we have $2(K_Y + S + \phi^*B_W) \sim 0$ by $2(K_W + C_1 + C_2 + B_W) \sim 0$ and by (IV-5) in Lemma 4.20. Hence, $2(K_V + \Lambda_1 + \Lambda_2 + B) \sim 0$, and $(V, \Lambda_1 + \Lambda_2 + B)$ is a half-toric surface by Proposition 4.7. □

Lemma 4.22. *Let $(V, \Lambda_1, \Lambda_2)$ be a $\mathcal{V}_{\mathrm{B}}$-surface with a non-isomorphic surjective endomorphism $f\colon V \to V$ such that $S_f = \Lambda_1 + \Lambda_2$. Then $(V, \Lambda_1, \Lambda_2)$ is ordinary.*

Proof. For the standard partial resolution $\nu\colon Y \to V$ of singularities at Σ_2, we have an endomorphism f_Y of Y such that $\nu \circ f_Y = f_V^2 \circ \nu$ by [21, Prop. 5.9]. Here, $S_{f_Y} = \nu^{-1}S_f = \nu^{-1}(\Lambda_1 + \Lambda_2)$ by [20, Lem. 3.15(3)]. Since S_{f_Y} contains all the negative curves on Y (cf. [20, Prop. 2.20(3)]), $(V, \Lambda_1, \Lambda_2)$ is ordinary. □

Theorem 4.23. *Let X be a normal projective surface admitting a non-isomorphic surjective endomorphism $f\colon X \to X$. Assume that $\boldsymbol{\rho}(X) \geq 3$ and $K_X + S_f$ is not pseudo-effective. Then one of the following holds*:

- *$(X, S_f + B)$ is a toric surface for a reduced divisor B such that $1 \leq \boldsymbol{n}(B) \leq 2$;*
- *$(X, S_f + B)$ is a half-toric surface for a prime divisor B, and B is an end component of the linear chain $S_f + B$.*

Proof. The pair (X, S_f) is an $\mathcal{L}$-surface by [21, Prop. 4.3]. Hence, X is rational, $-(K_X + S_f)$ is semi-ample, and S_f is a linear chain of rational curves by [21, Thm. 4.5]. In particular, if $(X, S_f + B)$ is a half-toric surface, then B is an end component of $S_f + B$. We have known the following:

- If $\boldsymbol{n}(S_f) \neq \boldsymbol{\rho}(X)$ or if the union $S_f^\natural$ of non-end components of S_f is not negative definite, then $(X, S_f + B)$ is a toric surface for a prime divisor B by [21, Thm. 4.5].
- If $-(K_X + S_f)$ is not big, then $(X, S_f + B)$ is a toric surface or a half-toric surface for a prime divisor B by [21, Cor. 4.6].

- If $(K_X + S_f)C < 0$ for two end components C of S_f, then $(X, S_f + B)$ is a toric surface for a reduced divisor B with $1 \leq \boldsymbol{n}(B) \leq 2$ by Theorem 3.13.

Therefore, we may assume that $\boldsymbol{\rho}(X) = \boldsymbol{n}(S_f)$, $-(K_X + S_f)$ is big, and $S_f^\natural$ is negative definite. Moreover, for end components C_1 and C_2 of S_f, we may assume that $C_1^2 < 0$, $(K_X + S_f)C_1 < 0$, and $(K_X + S_f)C_2 = 0$ (cf. [21, Thm. 4.5(6)]).

Let $\sigma\colon X \to V$ be the contraction morphism of $S_f^\natural$ and set $\Lambda_1 := \sigma(C_1)$ and $\Lambda_2 := \sigma(C_2)$. We shall show that $(V, \Lambda_1, \Lambda_2)$ is a $\mathcal{V}_{\mathrm{B}}$-surface. Now, $\boldsymbol{\rho}(V) = \boldsymbol{\rho}(X) - (\boldsymbol{n}(S_f) - 2) = 2$, and

$$K_X + S_f = \sigma^*(K_V + \Lambda_1 + \Lambda_2)$$

by $\sigma_* S_f = \Lambda_1 + \Lambda_2$ and by $\mathcal{O}_X(K_X + S_f)|_{S_f^\natural} \simeq \mathcal{O}_{S_f^\natural}$ (cf. [21, Thm. 4.5(7)]). Hence, $(V, \Lambda_1 + \Lambda_2)$ is log-canonical, $-(K_V + \Lambda_1 + \Lambda_2)$ is nef and big,

$$(K_V + \Lambda_1 + \Lambda_2)\Lambda_1 = (K_X + S_f)C_1 < 0, \quad (K_V + \Lambda_1 + \Lambda_2)\Lambda_2 = (K_X + S_f)C_2 = 0,$$

and $\Lambda_2^2 < 0$ by the Hodge index theorem. By [21, Thm. 4.5], $\overline{\mathrm{NE}}(X)$ is a polyhedral cone generated by numerical classes of prime components of S_f. Hence, $\overline{\mathrm{NE}}(V)$ is generated by $\mathrm{cl}(\Lambda_1)$ and $\mathrm{cl}(\Lambda_2)$, and we have $\Lambda_1^2 \leq 0$. Therefore, $(V, \Lambda_1, \Lambda_2)$ is a $\mathcal{V}$-surface (cf. Definition 4.1).

The endomorphism f descends to an endomorphism $f_V\colon V \to V$ with $S_{f_V} = \sigma_* S_f = \Lambda_1 + \Lambda_2$ by [20, Lems. 3.14 and 3.15(3)]. Hence, $(V, \Lambda_1, \Lambda_2)$ is an $\mathcal{V}_{\mathrm{A}}$-surface or an ordinary $\mathcal{V}_{\mathrm{B}}$-surface, by Lemma 4.22. There is a prime divisor B_V such that $(V, \Lambda_1 + \Lambda_2 + B_V)$ is a half-toric surface by Theorems 4.10 and 4.21. For the proper transform B of B_V in X, $(X, S_f + B)$ is also a half-toric surface by [18, Prop. 7.5], since σ is an isomorphism over $V \setminus \Lambda_2 \supset B$. Thus, we are done. ☐

4.5. **Structure of an extraordinary $\mathcal{V}_{\mathrm{B}}$-surface.** We shall determine the structure of an extraordinary $\mathcal{V}_{\mathrm{B}}$-surface in Theorem 4.29 below by using the notion of $(2n+1, 2)$-blowings up defined as follows:

Definition 4.24. Let X be a non-singular surface and let C be a non-singular curve on X with a point $P \in C$. For an integer $n \geq 0$, a bimeromorphic morphism $\mu\colon Y \to X$ is called a $(2n+1, 2)$-*blowing up* at P with respect to (X, C), if it is the blowing up along the following $\mathcal{O}_X$-ideal J_n, where $\mathfrak{m}$ stands for the maximal ideal at P:

- If $n = 0$, then $\mathfrak{m}^2 \subset J_0 \neq \mathfrak{m}$ and $\mathcal{O}_X(-C) + J_0 = \mathfrak{m}$;
- If $n > 0$, then

$$J_n = \mathfrak{m}^{2n+1} + \mathfrak{m}^{n+1}\mathcal{O}_X(-C) + \mathcal{O}_X(-2C).$$

Example 4.25. We shall give an example of $(2n+1, 2)$-blowings up as a toric morphism. Let X is the affine toric surface $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) = \operatorname{Specan} \mathbb{C}[\boldsymbol{\sigma}^\vee \cap \mathsf{M}] \simeq \mathbb{C}^2$ for a free abelian group $\mathsf{N} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ of rank 2 with a free basis (e_1, e_2) and for the standard cone $\boldsymbol{\sigma} = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2$ in $\mathsf{N}_{\mathbb{R}} = \mathsf{N} \otimes \mathbb{R}$, where $\mathsf{M} = \mathrm{Hom}(\mathsf{N}, \mathbb{Z})$ and $\boldsymbol{\sigma}^\vee$ is the dual cone of $\boldsymbol{\sigma}$. Let (m_1, m_2) be the basis of M dual to (e_1, e_2), i.e., $\langle m_i, e_j \rangle = \delta_{i,j}$ for the canonical pairing $\langle\ ,\ \rangle\colon \mathsf{M} \times \mathsf{N} \to \mathbb{Z}$. Then $\boldsymbol{\sigma}^\vee = \mathbb{R}_{\geq 0}m_1 + \mathbb{R}_{\geq 0}m_2$. For $i = 1$, 2, let t_i be the function on X corresponding to m_i in the semi-group ring $\mathbb{C}[\boldsymbol{\sigma}^\vee \cap \mathsf{M}]$, and let D_i be the prime divisor on X corresponding to the ray $\mathbb{R}_{\geq 0}e_i$, which is a face of $\boldsymbol{\sigma}$. Then (t_1, t_2) is a coordinate of $X \simeq \mathbb{C}^2$, and $D_i = \{t_i = 0\}$ for $i = 1, 2$.

We set $v_n := (2n+1)e_1 + 2e_2$, and consider cones

$$\boldsymbol{\sigma}_1 = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}v_n \quad \text{and} \quad \boldsymbol{\sigma}_2 = \mathbb{R}_{\geq 0}v_n + \mathbb{R}_{\geq 0}e_2$$

in $\mathsf{N}_{\mathbb{R}}$. Then $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ give a subdivision of $\boldsymbol{\sigma}$ and the faces of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ form a fan $\triangle$ of N. Let Y be the toric surface $\mathbb{T}_{\mathsf{N}}(\triangle) = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_1) \cup \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_2)$ with a proper birational toric morphism $\mu\colon Y \to X$.

For $i = 1, 2$, let Q_i be the point of $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_i)$ forming the 0-dimensional orbit of the torus $\mathbb{T}_{\mathsf{N}} = \operatorname{Specan} \mathbb{C}[\mathsf{M}]$. We have the following:

(1) (Y, Q_1) is an A_1-singularity;
(2) if $n = 0$, then $\operatorname{Sing} Y = \{Q_1\}$;
(3) if $n > 0$, then $\operatorname{Sing} Y = \{Q_1, Q_2\}$, and (Y, Q_2) is a cyclic quotient singularity of type $(2n+1, 2)$.

These are shown by [23, Prop 1.24] applied to the affine toric surface $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_1)$ or $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_2)$. In fact, $(e_1, ne_1 + e_2)$ is a basis of N and $v_n = e_1 + 2(ne_1 + e_2)$, which imply (1). Similarly, (e_2, e_1) is a basis of N and $v_n = 2e_2 + (2n+1)e_1$, which imply (3), since $2 < 2n+1$ when $n > 0$. When $n = 0$, $(v_0, e_2) = (e_1 + 2e_2, e_2)$ is a basis of N, and hence, $Q_2 \notin \operatorname{Sing} Y$. Thus, (2) holds.

Let Γ be the prime divisor on Y corresponding to the ray $\mathbb{R}_{\geq 0}v_n$, which is a unique μ-exceptional prime divisor. For $i = 1, 2$, let D'_i be the proper transform of D_i in Y, which corresponds to the ray $\mathbb{R}_{\geq 0}e_i \in \triangle$. Then $D'_i \cap \Gamma = \{Q_i\}$, and by (1)–(3), we see that $2(2n+1)\Gamma$ is Cartier and that $D'_1\Gamma = 1/2$ and $D'_2\Gamma = 1/(2n+1)$. In particular, the μ-ample invertible sheaf $\mathcal{O}_Y(-2(2n+1))$ is generated by $I_n := H^0(Y, \mathcal{O}_Y(-2(2n+1)\Gamma))$ (cf. [4, §3.4]). Here, the ideal $I_n \subset H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ is generated by monomials $t_1^{a_1}t_2^{a_2}$ for $a_1, a_2 \in \mathbb{Z}_{\geq 0}$ such that

$$\langle a_1m_1 + a_2m_2, v_n\rangle = (2n+1)a_1 + 2a_2 \geq 2(2n+1).$$

Hence, I_n is generated by t_1^2, $t_1t_2^{n+1}$, and t_2^{2n+1}. If $n > 0$, then

$$I_n = \mathfrak{m}^{2n+1} + \mathfrak{m}^{n+1}\mathcal{O}_X(-D_1) + \mathcal{O}_X(-2D_1),$$

for the maximal ideal $\mathfrak{m}$ at the origin. If $n = 0$, then $I_0 = \mathfrak{m}^2 + \mathcal{O}_X(-D_2)$, and we have $\mathfrak{m}^2 \subset I_0 \neq \mathfrak{m}$ and $I_0 + \mathcal{O}_X(-D_1) = \mathfrak{m}$. Therefore, $\mu\colon Y \to X$ is a $(2n+1, 2)$-blowing up at the origin with respect to (X, D_1).

Lemma 4.26. *Let X be a non-singular surface and let C be a non-singular curve on X with a point $P \in C$. Let $\mu\colon Y \to X$ be a $(2n+1, 2)$-blowing up at P with respect to (X, C) for an integer $n \geq 0$. Then Y is a normal surface, μ has a unique exceptional divisor Γ isomorphic to $\mathbb{P}^1$, and the following hold on* $\operatorname{Sing} Y$:

(1) *For the proper transform C' of C in Y, the intersection $C' \cap \Gamma$ consists of one point Q_1, in which (Y, Q_1) is an A_1-singularity and (Y, C') is 1-log-terminal at Q_1.*
(2) *If $n = 0$, then* $\operatorname{Sing} Y = \{Q_1\}$. *If $n > 1$, then* $\operatorname{Sing} Y = \{Q_1, Q_2\}$ *for a point $Q_2 \in \Gamma \setminus \{Q_1\}$, and (Y, Q_2) is a cyclic quotient singularity of type $(2n+1, 2)$.*

Moreover, $\mu^*C = C' + (2n+1)\Gamma$*,* $C'\Gamma = 1/2$*, and* $K_Y + C' = \mu^*(K_X + C) + \Gamma$*.*

Proof. If $n > 0$, then J_n is defined only by $\mathfrak{m}$ and $\mathcal{O}_X(-C)$. If $n = 0$, then $J_0 = \mathfrak{m}^2 + \mathcal{O}_X(-C^\dagger)$ for a non-singular curve $C^\dagger$ defined locally on X such that $C + C^\dagger$ is normal crossing at P. By embedding an open neighborhood of P into $\mathbb{C}^2$, we may assume that $X = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$, $C = D_1$, and $\mu\colon Y \to X$ is the toric morphism $\mathbb{T}_{\mathsf{N}}(\triangle) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ in Example 4.25. We proceed the argument in Example 4.25.

Assertions (1) and (2), and the equalities $C'\Gamma = D'_1\Gamma = 1/2$ have been shown in Example 4.25. By considering principal divisors of functions t_1 and t_2, we have

$$\begin{aligned}\mu^*D_1 &= \langle m_1, e_1\rangle D'_1 + \langle m_1, v_n\rangle\Gamma = D'_1 + (2n+1)\Gamma,\\ \mu^*D_2 &= \langle m_2, e_2\rangle D'_2 + \langle m_2, v_n\rangle\Gamma = D'_2 + 2\Gamma\end{aligned}$$

(cf. [4, §3.3, Lem.]). On the other hand, $K_Y + D'_1 + \Gamma + D'_2 = \mu^*(K_X + D_1 + D_2) \sim 0$, since μ is a toric morphism. Hence,

$$K_Y + D'_1 = \mu^*(K_X + D_1) + \mu^*D_2 - D'_2 - \Gamma = \mu^*(K_X + D_1) + \Gamma.$$

Since C is identified with D_1, the required properties are all verified. □

Proposition 4.27. *Let* X *be a non-singular surface and let* C *be a non-singular curve on* X *with a point* $P \in C$*. Let* $\mu\colon Y \to X$ *be a bimeromorphic morphism from a normal surface* Y *with a unique* μ*-exceptional prime divisor* Γ *such that, for the proper transform* C' *of* C *in* Y*,*

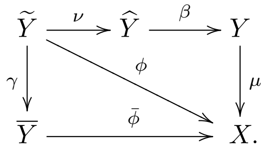
- $C' \cap \Gamma$ *consists of one point, at which* Y *has an* A_1*-singularity, and*
- $\Gamma C' = 1/2$*.*

Then μ *is a* $(2n+1, 2)$*-blowing up at* P *with respect to* (X, C) *for an integer* $n \geq 0$*.*

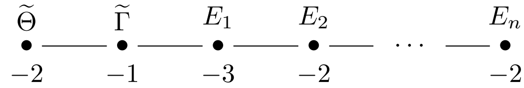
Proof. We set Q_1 to be the intersection point of C' and Γ. Let $\beta\colon \widehat{Y} \to Y$ be the blowing up at Q_1. Then $\widehat{Y}$ is non-singular along the exceptional (-2)-curve $\Theta := \beta^{-1}Q_1$, and we have $\widehat{C} \cap \widehat{\Gamma} = \emptyset$ and $\widehat{C}\Theta = \widehat{\Gamma}\Theta = 1$ for proper transforms $\widehat{C}$ and $\widehat{\Gamma}$ of C' and Γ in $\widehat{Y}$, respectively. In particular, $\operatorname{Sing}\widehat{Y} \subset \widehat{\Gamma}$. Let $\nu\colon \widetilde{Y} \to \widehat{Y}$ be the minimal resolution of singularities and let $\widetilde{C}$, $\widetilde{\Theta}$, and $\widetilde{\Gamma}$ be proper transforms in $\widetilde{Y}$ of $\widehat{C}$, Θ, and $\widehat{\Gamma}$, respectively. Then the composite $\beta \circ \nu\colon \widetilde{Y} \to Y$ is the minimal resolution of singularities, the further composite $\phi := \mu \circ \beta \circ \nu\colon \widetilde{Y} \to X$ is a bimeromorphic morphism of non-singular surfaces, and

$$\phi^{-1}(P) = \nu^{-1}(\beta^{-1}\Gamma) = \widetilde{\Theta} + \widetilde{\Gamma} + E$$

for a ν-exceptional reduced divisor E. Here, $E = 0$ if and only if ν is an isomorphism. By construction, $\widetilde{\Gamma}$ is a unique (-1)-curve in $\phi^{-1}(P)$. Let $\gamma\colon \widetilde{Y} \to \overline{Y}$ be the blowdown of $\widetilde{\Gamma}$. Then we have a bimeromorphic morphism $\bar{\phi}\colon \overline{Y} \to X$ of non-singular surfaces such that $\phi = \bar{\phi} \circ \gamma$. Now, we have a commutative diagram:



The image $\gamma(\widetilde{\Theta})$ in $\overline{Y}$ is a unique (-1)-curve in $(\bar{\phi})^{-1}(P)$, and it intersects the proper transform $\gamma(\widetilde{C})$ of C in $\overline{Y}$. By Lemma 2.4, $\gamma_*(\widetilde{\Theta}+E)$ is a linear chain of rational curves with $\gamma(\widetilde{\Theta})$ as an end component, and any prime component of γ_*E is a (-2)-curve when $E \neq 0$. Hence, if $E \neq 0$, then E has a prime decomposition $\sum_{i=1}^n E_i$ for some integer $n \geq 0$ such that



is a dual graph of $\phi^{-1}(P)$, where -1, -2, and -3 indicate self-intersection numbers. Note that if $n = 1$, then $E = E_1$ is a (-3)-curve. Note also that if $n = 0$, then the linear chain $\phi^{-1}(P)$ is the union of the (-2)-curve $\widetilde{\Theta}$ and the (-1)-curve $\widetilde{\Gamma}$.

We shall prove the assertion by toric descriptions of bimeromorphic morphisms ϕ, $\bar{\phi}$, and μ. Let $\mathsf{N} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, $\boldsymbol{\sigma} = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2$, v_n, and $\triangle$ be as in Example 4.25. We set $u_i = ie_1 + e_2$ for $i \geq 0$, and consider cones

$$\boldsymbol{\sigma}_{(i)} = \mathbb{R}_{\geq 0}u_i + \mathbb{R}_{\geq 0}u_{i+1}, \qquad \boldsymbol{\sigma}_{n+1,\infty} = \mathbb{R}_{\geq 0}u_{n+1} + \mathbb{R}_{\geq 0}e_1,$$
$$\boldsymbol{\sigma}_{(n)1} = \mathbb{R}_{\geq 0}u_n + \mathbb{R}_{\geq 0}v_n, \qquad \boldsymbol{\sigma}_{(n)2} = \mathbb{R}_{\geq 0}v_n + \mathbb{R}_{\geq 0}u_{n+1},$$

where $v_n = u_n + u_{n+1} = (2n+1)e_1 + 2e_2$. Then cones $\boldsymbol{\sigma}_{(i)}$ for $0 \leq i \leq n$ and $\boldsymbol{\sigma}_{n+1,\infty}$ give a subdivision of $\boldsymbol{\sigma}$, and cones $\boldsymbol{\sigma}_{(n)1}$ and $\boldsymbol{\sigma}_{(n)2}$ give a subdivision of $\boldsymbol{\sigma}_{(n)}$. Let $\overline{\triangle}$ be the fan of N consisting of faces of cones $\boldsymbol{\sigma}_{(i)}$ for $0 \leq i \leq n$ and $\boldsymbol{\sigma}_{n+1,\infty}$. Let $\widetilde{\triangle}$ be the fan of N consisting of faces of cones $\boldsymbol{\sigma}_{(i)}$ for $0 \leq i \leq n-1$, $\boldsymbol{\sigma}_{(n)1}$, $\boldsymbol{\sigma}_{(n)2}$, and $\boldsymbol{\sigma}_{n+1,\infty}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathbb{T}_{\mathsf{N}}(\widetilde{\triangle}) & \longrightarrow & \mathbb{T}_{\mathsf{N}}(\triangle) \\ \downarrow & & \downarrow \\ \mathbb{T}_{\mathsf{N}}(\overline{\triangle}) & \longrightarrow & \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \end{array}$$

of associated proper birational toric morphisms. By replacing X with an open neighborhood of P, we may assume that X is an open subset of $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ in which $C = D_1|_X$ and $\{P\} = D_1 \cap D_2$ for the boundary prime divisors D_1 and D_2 corresponding to rays $\mathbb{R}_{\geq 0}e_1$ and $\mathbb{R}_{\geq 0}e_2$, respectively. Then $\bar{\phi} \colon \overline{Y} \to X$ is isomorphic to the base change of $\mathbb{T}_{\mathsf{N}}(\overline{\triangle}) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ by $X \hookrightarrow \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$. In fact, $\bar{\phi}$ is a succession of blowings up at points of the proper transforms of C lying over P.

For describing γ, we first assume that $n = 0$. Then $\bar{\phi}$ is just the blowing up at P. Let $\overline{P}$ be the center of the blowing up $\gamma \colon \widehat{Y} \to \overline{Y}$, which is contained in $\gamma(\widetilde{\Theta}) \setminus \gamma(\widetilde{C})$. For the open immersion $X \hookrightarrow \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ and for the prime divisor D_2 corresponding to the ray $\mathbb{R}_{\geq 0}e_2$, we may assume that $\overline{P}$ is just the intersection point of proper transforms of $C = D_1|_X$ and $D_2|_X$ in $\overline{Y}$. Then γ is isomorphic to the base change of $\mathbb{T}_{\mathsf{N}}(\widetilde{\triangle}) \to \mathbb{T}_{\mathsf{N}}(\overline{\triangle})$ by the open immersion $\overline{Y} \hookrightarrow \mathbb{T}_{\mathsf{N}}(\overline{\triangle})$. Next, assume that $n > 0$. Then the center of $\gamma \colon \widetilde{Y} \to \overline{Y}$ is just the intersection point of $\gamma(\widetilde{\Theta})$ and $\gamma(E_1)$. By the open immersion $\overline{Y} \hookrightarrow \mathbb{T}_{\mathsf{N}}(\overline{\triangle})$, this point corresponds to the cone $\boldsymbol{\sigma}_{(n)} = \boldsymbol{\sigma}_{(n)1} \cup \boldsymbol{\sigma}_{n(2)} \in \overline{\triangle}$. Thus, γ is isomorphic to the base change of $\mathbb{T}_{\mathsf{N}}(\widetilde{\triangle}) \to \mathbb{T}_{\mathsf{N}}(\overline{\triangle})$ by the open immersion $\overline{Y} \hookrightarrow \mathbb{T}_{\mathsf{N}}(\overline{\triangle})$.

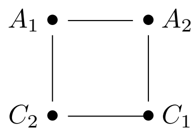
By the description of γ, we see that ϕ is isomorphic to the base change of $\mathbb{T}_{\mathsf{N}}(\widetilde{\triangle}) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ by the open immersion $X \hookrightarrow \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$. The morphism $\beta \circ \nu$ is just the contraction morphism of all the prime components of $\phi^{-1}(P)$ except $\widetilde{\Gamma}$. Thus, $\mu\colon Y \to X$ is also isomorphic to base change of $\mathbb{T}_{\mathsf{N}}(\triangle) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ by $X \hookrightarrow \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$. Hence, we have toric descriptions of μ, ϕ, and $\bar{\phi}$. As a consequence, μ is a $(2n+1, 2)$-blowing up at P by Example 4.25. □

By the notion of $(2n+1, 2)$-blowing up, we shall construct a $\mathcal{V}_B$-surface from a certain projective toric surface.

Lemma 4.28. *Let (X, D) be a projective toric surface of Picard number 2 with a prime decomposition $D = A_1 + A_2 + C_1 + C_2$ of the boundary divisor D such that*

(i) $A_1 \cap C_1 = A_2 \cap C_2 = \emptyset$,
(ii) $C_1C_2 = 1/2$,
(iii) $A_1^2 \leq 0$ *and* $A_2^2 < 0$.

In particular, D is a cyclic chain of rational curves with



as a dual graph. Let n be a non-negative integer satisfying

$$-A_2^2(n + 1/2 - C_1^2) > (A_2C_1)^2. \tag{IV-6}$$

For a point $P \in C_1 \setminus (A_2 + E_2)$, let $\mu\colon Y \to X$ be a birational morphism such that

- *μ is an isomorphism over $X \setminus \{P\}$*
- *μ is a $(2n+1, 2)$-blowing up at P with respect to (X, C_1) over an open neighborhood of P.*

Then the proper transform C_1' of C_1 in Y is negative definite, and one can consider the contraction morphism $Y \to V$ of C_1'. Let Λ_i be the proper transform of A_i in V for $i = 1$, 2. Then $(V, \Lambda_1, \Lambda_2)$ is an extraordinary $\mathcal{V}_{\mathrm{B}}$-surface.

Proof. By Lemma 4.26, $\mu^*C_1 = C_1' + (2n+1)\Gamma$ and $C_1'\Gamma = 1/2$ for the μ-exceptional prime divisor Γ. Hence, $(C_1')^2 = C_1^2 - (2n+1)/2 < 0$ by (iii) and (IV-6). Let $\zeta\colon Y \to V$ be the contraction morphism of C_1'. Then $\boldsymbol{\rho}(V) = \boldsymbol{\rho}(Y) - 1 = \boldsymbol{\rho}(X) = 2$, and μ^*A_i is the proper transform of Λ_i in Y for $i = 1$, 2. Here, $\zeta^*\Lambda_1 = \mu^*A_1$, and we have $\Lambda_1^2 = A_1^2 \leq 0$ by (iii). The divisor $\zeta^{-1}\Lambda_2 = \mu^*A_2 + C_1'$ is negative definite by (iii) and (IV-6), since $(\mu^*A_2)^2 = A_2^2 < 0$, $(C_1')^2 < 0$, and

$$(\mu^*A_2)^2(C_1')^2 - ((\mu^*A_2)C_1')^2 = A_2^2(C_1^2 - (2n+1)/2) - (A_2C_1)^2 > 0.$$

In particular, $\Lambda_2^2 < 0$. Since $K_X + D \sim 0$, we have

$$K_Y + \mu^*A_1 + \mu^*A_2 + C_1' = \mu^*(K_X + A_1 + A_2 + C_1) + \Gamma \sim \Gamma - \mu^*C_2$$

by Lemma 4.26. Moreover, since $\Gamma C_1' = (\mu^*C_2)C_1' = C_1C_2 = 1/2$ (cf. (ii)), we have

$$K_Y + \mu^*A_1 + \mu^*A_2 + C_1' = \zeta^*(K_V + \Lambda_1 + \Lambda_2). \tag{IV-7}$$

In particular, $(V, \Lambda_1 + \Lambda_2)$ is log-canonical, since $(Y, \mu^* A_1 + \mu^* A_2 + C_1')$ is log-canonical (cf. Lemma 4.26). Moreover, for $i = 1$, 2, we have

$$\begin{aligned}(K_V + \Lambda_1 + \Lambda_2)\Lambda_i &= (K_Y + \mu^* A_1 + \mu^* A_2 + C_1')\mu^* A_i \\ &= (K_X + A_1 + A_2 + C_1)A_i = -C_2 A_i.\end{aligned}$$

Hence, $(K_V + \Lambda_1 + \Lambda_2)\Lambda_1 \le 0$ and $(K_V + \Lambda_1 + \Lambda_2)\Lambda_2 = 0$ by (i) and (iii). Therefore, $(V, \Lambda_1, \Lambda_2)$ is a $\mathcal{V}_{\mathrm{B}}$-surface.

We shall show that $(V, \Lambda_1, \Lambda_2)$ is extraordinary. Note that

$$C_1' \cap \operatorname{Sing} Y \subset (C_1' \cap \mu^* A_2) \cup (C_1' \cap \mu^* C_2) \cup (C_1' \cap \Gamma),$$

where $C_1' \cap \mu^* C_2$ (resp. $C_1' \cap \Gamma$) consists of an A_1-singular point by (ii) (resp. by Lemma 4.26(1)). Let $\widetilde{Y} \to Y$ be the minimal resolution of singularities at the intersection point of C_1' and $\mu^* A_2$. Then the composite $\widetilde{Y} \to V$ is just the standard partial resolution of the singularity at the second external singular locus Σ_2 of $(V, \Lambda_1, \Lambda_2)$, by (IV-7). The proper transform of Γ in $\widetilde{Y}$ is a negative curve not contained in the inverse image of $\Lambda_1 + \Lambda_2$ by $\widetilde{Y} \to V$. Therefore, $(V, \Lambda_1, \Lambda_2)$ is extraordinary. □

Theorem 4.29. *Let* $(V, \Lambda_1, \Lambda_2)$ *be an extraordinary* $\mathcal{V}_{\mathrm{B}}$*-surface. Then it is obtained by the method in Lemma* 4.28 *from a projective toric surface of Picard number* 2 *and a non-negative integer satisfying conditions* (i)–(iii) *and the inequality* (IV-6) *in Lemma* 4.28.

Proof. Let $\nu \colon Y \to V$ be the standard partial resolution of the second external singular locus Σ_2 of $(V, \Lambda_1, \Lambda_2)$. Then

$$K_Y + S = \nu^*(K_V + \Lambda_1 + \Lambda_2) \tag{IV-8}$$

for $S := \nu^{-1}(\Lambda_1 + \Lambda_2)$. Let $\Lambda_{Y,i}$ be the proper transform of Λ_i in Y for $i = 1$, 2, and set $E := \nu^{-1}\Sigma_2$. Then E is a linear chain of rational curves with $S = \Lambda_{Y,1} + \Lambda_{Y,2} + E$, and $\Sigma_Y := E \cap \operatorname{Sing} Y$ consists of two A_1-singular points, which are contained $C \cap E_{\mathrm{reg}}$ for the end component C of E not intersecting $\Lambda_{Y,2}$. By assumption, there is a negative curve Γ on Y not contained in $\nu^{-1}(\Lambda_1 + \Lambda_2)$. The image $\nu(\Gamma)$ is not a point nor a negative curve by Lemma 4.4(1). By the Hodge index theorem and by (IV-8) and Lemma 4.4(2), we have

$$(K_Y + S)\Gamma = (K_V + \Lambda_1 + \Lambda_2)\nu(\Gamma) < 0. \tag{IV-9}$$

Then $\Gamma \cap S = \{P\}$ for a point $P \in E$ by [18, Lem. 2.18] and by $\Sigma_2 \subset \nu(\Gamma)$. Let $\gamma \colon Y \to X$ be the contraction morphism of Γ, and set $S_X := \gamma(S)$, $C_X := \gamma(C)$, and $P_X := \gamma(P)$. Then (X, S_X) is log-canonical and is 1-log-terminal at P_X by (IV-9) and [19, Lem. 2.4]. In particular, $P_X \in (S_X)_{\mathrm{reg}}$, and hence, $P \in S_{\mathrm{reg}}$. By (IV-8), there is a positive rational number α such that

$$K_Y + S = \gamma^*(K_X + S_X) + \alpha\Gamma. \tag{IV-10}$$

Note that $-(K_X + S_X)$ is nef and big by (IV-8) and Lemma 4.4(2).

Let Δ be the unique prime component of S containing P. Then $(K_X + S_X)\gamma(\Delta) < 0$ by (IV-10), and $\gamma(\Delta)$ is an end component of the linear chain S_X of rational curves by [18, Prop. 3.29]. Therefore, $\Delta = C$, and $P \in C \cap E_{\mathrm{reg}}$. Let Q be a point in $\Sigma_Y \setminus \Gamma = \Sigma_Y \setminus \{P\}$. Then $Q_X := \gamma(Q)$ is an A_1-singular point of X, which is contained in the end component C_X of S_X. Since $(K_X + S_X)C_X < 0$, we have $\{Q_X\} = C_X \cap (S_X)_{\mathrm{reg}} \cap \operatorname{Sing} X$ and $(K_X + S_X)C_X = -1/2$ by [18, Prop. 3.29(F)]. Hence, $P_X \in X_{\mathrm{reg}}$ and we have

$$\alpha C\Gamma = 1/2 \tag{IV-11}$$

by (IV-10) and by $C(K_Y + S) = C\nu^*(K_V + \Lambda_1 + \Lambda_2) = 0$ (cf. (IV-8)). Since $2C$ is Cartier along $C \cap E_{\mathrm{reg}}$, we have $\alpha^{-1} = 2C\Gamma \in \mathbb{Z}$. On the other hand, there is a positive integer d such that $\gamma^*(S_X) = S + d\Gamma$, since S_X is Cartier at P_X. Then

$$K_Y = \gamma^* K_X + (\alpha + d)\Gamma$$

by (IV-10). Here, $\alpha + d \in \mathbb{Z}$, since K_X is Cartier at P_X. Therefore, $\alpha = 1$ and $C\Gamma = S\Gamma = 1/2$. As a consequence, $\{P, Q\} = \Sigma_Y$.

By Proposition 4.27, $\gamma \colon Y \to X$ is a $(2n+1, 2)$-blowing up at P_X with respect to (X, S_X) for an integer $n \geq 0$. Now, $\boldsymbol{\rho}(Y) = \boldsymbol{\rho}(V) + \boldsymbol{n}(E) = \boldsymbol{n}(S)$, and $\boldsymbol{\rho}(X) = \boldsymbol{\rho}(Y) - 1 = \boldsymbol{n}(S) - 1 = \boldsymbol{n}(S_X) - 1$. Since (X, S_X) is log-canonical and since $-(K_X + S_X)$ is nef and big, there is a prime divisor $C_X^{\dagger}$ such that $(X, S_X + C_X^{\dagger})$ is a toric surface, by [18, Thm. 1.3].

We set $\Lambda_{X,i} := \gamma(\Lambda_{Y,i})$ for $i = 1$, 2, and set $E_X := \gamma_* E$. Then $S_X = \Lambda_{X,1} + \Lambda_{X,2} + E_X$ and $C_X^{\dagger} \sim -(K_X + S_X)$. We shall show that

(a) $\Lambda_{X,1}^2 \leq 0$, $\Lambda_{X,1} \cap E_X = \emptyset$, and $\Lambda_{X,1} \cap \Lambda_{X,2} \neq \emptyset$;
(b) $\Lambda_{X,1} \cap C_X^{\dagger} \neq \emptyset$ and $\Lambda_{X,2} \cap C_X^{\dagger} = \emptyset$;
(c) $C_X C_X^{\dagger} = 1/2$ and $C_X \cap C_X^{\dagger} = \{Q_X\}$;
(d) $\Lambda_{X,2} + E_X - C_X$ is negative definite.

We have (a), since $\Lambda_1^2 \leq 0$ and $\Lambda_1 \cap \Lambda_2 \neq \emptyset$ and since ν and γ are isomorphisms around $\Lambda_{Y,1}$. By (IV-8) and (IV-10), we have

$$C_X^{\dagger} \Lambda_{X,i} = -(K_V + \Lambda_1 + \Lambda_2)\Lambda_i$$

for $i = 1$, 2. This shows (b) by Definition 4.1(iii). Moreover,

$$C_X^{\dagger} C_X = (\Gamma - (K_Y + S_Y))C = 1/2$$

by (IV-10) with $\alpha = 1$. This implies (c), since $\{Q_X\} = C_X \cap (S_X)_{\mathrm{reg}} \cap \operatorname{Sing} X$. We have (d) by $(\Lambda_2)^2 < 0$ and by

$$\gamma^{-1}(\Lambda_{X,2} + E_X - C_X) = \Lambda_{Y,2} + E - C = \nu^{-1}\Lambda_2 - C.$$

Let $Y \to \overline{Y}$ (resp. $X \to \overline{X}$) be the contraction morphism of $E - C$ (resp. $E_X - C_X$). Then ν and γ induce birational morphisms $\bar{\nu} \colon \overline{Y} \to V$ and $\bar{\gamma} \colon \overline{Y} \to \overline{X}$ with a commutative diagram

$$\begin{array}{ccccc} Y & \longrightarrow & \overline{Y} & \xrightarrow{\ \bar{\nu}\ } & V \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\bar{\gamma}} & & \\ X & \longrightarrow & \overline{X}. & & \end{array} \qquad (\nu = \bar{\nu} \circ (Y \to \overline{Y}))$$

Let A_i be the proper transform of Λ_i in $\overline{X}$ for $i = 1$, 2, and let C_1 (resp. C_2) be the images of C (resp. $C^{\dagger}$) under $Y \to X \to \overline{X}$. Then $(\overline{X}, A_1 + A_2 + C_1 + C_2)$ is a toric surface of Picard number 2 satisfying conditions (i)–(iii) of Lemma 4.28, by (a)–(d) above. The integer n satisfies (IV-6) in Lemma 4.28, since $(\Lambda_2)^2 < 0$ and since $\nu^{-1}\Lambda_2$ is the union of the proper transforms of A_2 and C_1 in $\overline{Y}$. By construction, $\bar{\gamma}$ is a $(2n+1, 2)$-blowing up and $\bar{\nu}$ is the contraction morphism of the proper transform of C_1. Therefore, $(V, \Lambda_1, \Lambda_2)$ is obtained by the method of Lemma 4.28 from $(\overline{X}, A_1 + A_2 + C_1 + C_2)$. Thus, we are done. □

5. The case of Picard number 2

We shall determine the structure of a normal projective surface X admitting a non-isomorphic surjective endomorphism f such that $\boldsymbol{\rho}(X) = 2$ and $K_X + S_f$ is not pseudo-effective. We consider the following three cases for X:

(I) X is irrational or $-K_X$ is not big;
(II) X is rational, $-K_X$ is big, and X contains a negative curve;
(III) X is rational, $-K_X$ is big, and X contains no negative curve.

Note that (R2) in the introduction is divided into (II) and (III). The cases (I), (II), and (III) are treated separately in Sections 5.1, 5.2, and 5.3 below. Theorem 5.1 in Section 5.1 is a structure theorem for (I). Theorem 5.6 in Section 5.2 is a structure theorem for (II) and (III). We shall prove another structure theorem for (III) as Theorem 5.17 in Section 5.3, in which we do not assume that $K_X + S_f$ is not pseudo-effective. Theorem 5.17 implies Theorem 5.6 in the case (III).

5.1. **Case (I).** We shall prove:

Theorem 5.1. *Let X be a normal projective surface such that $\boldsymbol{\rho}(X) = 2$. Assume either that X is irrational or that $-K_X$ is not big. Then there is a non-isomorphic surjective endomorphism f of X such that $K_X + S_f$ is not pseudo-effective if and only if there is a finite Galois cover $\nu\colon V \to X$ étale in codimension* 1 *from a normal projective surface V satisfying one of the following conditions*:

(1) *$V \simeq \mathbb{P}^1 \times T$ for a non-singular projective curve T of genus* > 0;
(2) *V is a $\mathbb{P}^1$-bundle over an elliptic curve associated with an indecomposable locally free sheaf of rank* 2 *degree* 0;
(3) *V is a $\mathbb{P}^1$-bundle over an elliptic curve having a negative section.*

Moreover, for a non-isomorphic surjective endomorphism $f\colon X \to X$ with $K_X + S_f$ being not pseudo-effective, there exist an endomorphism f_V of V and a positive integer k such that $\nu \circ f_V = f^k \circ \nu$.

Note that Theorem 5.1 is not deduced from [20, Thm. 4.16] and [21, Thm. 4.7]. After showing preliminary results, which are similar to some results in [21], we shall prove Theorem 5.1 at the end of Section 5.1. We begin with:

Lemma 5.2. *Let G be a finite group acting on $\mathbb{P}^1$. Then there is a non-isomorphic surjective endomorphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that f is G-equivariant and* $\deg S_f \leq 1$.

Proof. We may assume that G is a subgroup of $\mathrm{SL}(2,\mathbb{C})$. We first consider the case where G is a cyclic group or a dihedral group, i.e., $G \simeq \mathbb{Z}/m\mathbb{Z}$ or $G \simeq \mathbb{Z}/m \rtimes \mathbb{Z}/2\mathbb{Z}$ for an integer $m \geq 2$. For a homogeneous coordinate $(\mathtt{x}, \mathtt{y})$ of $\mathbb{P}^1$, we may assume that a generator of $\mathbb{Z}/m\mathbb{Z}$ acts as $(\mathtt{x}:\mathtt{y}) \mapsto (\zeta\mathtt{x}:\zeta^{-1}\mathtt{y})$ for a primitive m-th root ζ of unity and that when G is dihedral, a generator of $\mathbb{Z}/2\mathbb{Z}$ acts as $(\mathtt{x}:\mathtt{y}) \mapsto (-\mathtt{y}:\mathtt{x})$. Let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism defined by

$$(\mathtt{x}:\mathtt{y}) \mapsto (\mathtt{x}^{2m+1} + \mathtt{x}\mathtt{y}^{2m} : \mathtt{x}^{2m}\mathtt{y} + \mathtt{y}^{2m+1}).$$

Then f is G-equivariant of degree $2m+1$. If the ramification index of f at a point is $2m+1$, then

$$a(\mathtt{x}^{2m+1} + \mathtt{x}\mathtt{y}^{2m}) + b(\mathtt{x}^{2m}\mathtt{y} + \mathtt{y}^{2m+1}) = (\alpha\mathtt{x} + \beta\mathtt{y})^{2m+1}$$

for some (a, b), $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, and by comparing coefficients, we have

$$a = \alpha^{2m+1} = (2m+1)\alpha\beta^{2m} \quad \text{and} \quad b = \beta^{2m+1} = (2m+1)\alpha^{2m}\beta :$$

This is impossible. This implies that $S_f = 0$. In fact, if $P \in S_f$, then $f^*(f(P)) = (\deg f)P$. Thus, the assertion holds if G is cyclic or dihedral.

It suffices to prove that G is cyclic or dihedral if there is a G-equivariant endomorphism f satisfying $\deg S_f \geq 2$. Here, $\deg S_f = 2$ by the equality $K_X + S_f = f^*(K_X + S_f) + \Delta_f$ (cf. [20, Lem. 2.17]). Thus, $f^*(f(P)) = (\deg f)P$ for two points $P \in S_f$. By composing f with an automorphism, we may assume that f is given by $(\mathtt{x}:\mathtt{y}) \mapsto (\mathtt{x}^n:\mathtt{y}^n)$ for an integer $n > 1$, where S_f consists of $(1:0)$ and $(0:1)$. If a matrix

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathrm{SL}(2,\mathbb{C})$ is contained in G, then $\sigma \circ f = f \circ \sigma$ implies that

$$(a\mathtt{x}^n + b\mathtt{y}^n : c\mathtt{x}^n + d\mathtt{y}^n) = ((a\mathtt{x} + b\mathtt{y})^n : (c\mathtt{x} + d\mathtt{y})^n),$$

or equivalently,

$$(a\mathtt{x}^n + b\mathtt{y}^n)(c\mathtt{x} + d\mathtt{y})^n = (a\mathtt{x} + b\mathtt{y})^n(c\mathtt{x}^n + d\mathtt{y}^n)$$

as a homogeneous polynomial: By comparing coefficients of monomials $\mathtt{x}^{2n}$, $\mathtt{x}^{2n-1}\mathtt{y}$, $\mathtt{x}\mathtt{y}^{2n-1}$, and $\mathtt{y}^{2n}$, we have

$$ac^n = a^nc, \quad nac^{n-1}d = na^{n-1}bc, \quad nbcd^{n-1} = nab^{n-1}d, \quad bd^n = b^nd.$$

If $ac \neq 0$, then $a^{n-1} = c^{n-1}$ and $ad = bc$, a contradiction. If $bd \neq 0$, then $b^{n-1} = d^{n-1}$ and $ad = bc$, a contradiction. Hence, $ac = bd = 0$, i.e.,

$$\sigma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$$

for some a, $b \in \mathbb{C} \setminus \{0\}$. This implies that G is cyclic or dihedral. Thus, we are done. □

Corollary 5.3. *Let X be a normal projective surface with a finite Galois cover $V \to X$ étale in codimension 1 such that $V \simeq \mathbb{P}^1 \times T$ for a non-singular projective curve T and that the Galois group* $\mathrm{Gal}(V/T)$ *preserves the second projection $V \to T$.*

Then there is a non-isomorphic surjective endomorphism f of X such that $K_X + S_f$ is not pseudo-effective.

Proof. It is enough to construct a non-isomorphic surjective endomorphism g of V such that g is equivariant under $G = \mathrm{Gal}(V/X)$ and that $K_V + S_g$ is not pseudo-effective. In fact, g induces a non-isomorphic surjective endomorphism f of X such that $\nu \circ g = f \circ \nu$ by the G-equivariance, where $\nu \colon V \to X$ is the Galois cover, and we have $\deg f = \deg g > 1$ and $S_g = \nu^* S_f$ (cf. [20, Lem. 2.19(3)]). Moreover, $K_V + S_g = \nu^*(K_X + S_f)$, since ν is étale in codimension 1. Thus, $K_X + S_f$ is not pseudo-effective.

The action of G on $V = \mathbb{P}^1 \times T$ is diagonal by [21, Lem. 2.3]. By Lemma 5.2, there is a non-isomorphic surjective endomorphism $h \colon \mathbb{P}^1 \to \mathbb{P}^1$ such that h is equivariant under the action of G on $\mathbb{P}^1$ and that $\deg S_h \leq 1$. We set $g := h \times \mathrm{id}_T$ as an endomorphism of V. Then g is G-equivariant, $\deg g = \deg h > 1$, and $S_g = p_1^*(S_h)$ for the first projection $p_1 \colon V \to \mathbb{P}^1$ (cf. [20, Lem. 2.19(2)]). Moreover, $(K_V + S_g)F = \deg(K_{\mathbb{P}^1} + S_h) < 0$ for any fiber F of the second projection $V \to T$. Thus, $K_V + S_g$ is not pseudo-effective, and we are done. □

Lemma 5.4. *Let T be an elliptic curve and consider it as a complex torus by fixing an origin. Let $\mu_m \colon T \to T$ be the multiplication map by an integer $m > 1$. Then:*

(1) *For any surjective endomorphism $h \colon T \to T$, there is a surjective endomorphism $h' \colon T \to T$ such that $\mu_m \circ h' = h \circ \mu_m$.*

(2) *If a finite group acts on T, then there is a finite group G' acting on T' with a surjective homomorphism $\rho \colon G' \to G$ such that μ_m is equivariant under actions of G' and G with respect to ρ and that the kernel of ρ is isomorphic to the Galois group of μ_m.*

(3) *Let $\pi \colon V \to T$ be a $\mathbb{P}^1$-bundle and let $\pi' \colon V' \to T$ be the base change of π by $\mu_m \colon T \to T$.*

   (a) *If f is a surjective endomorphism of V, then there is a surjective endomorphism f' of V' such that $p \circ f' = f \circ p$ for the first projection $p \colon V' = V \times_T T \to V$.*

   (b) *If a finite group G acts on V, then there is a finite group G' acting on V' with a surjective homomorphism $\rho \colon G' \to G$ such that $V' \to V$ is equivariant under actions of G' and G with respect to ρ and that $G' \backslash V' \simeq G \backslash V$.*

Proof. (1): The endomorphism h is expressed as the composite $\mathrm{tr}(b) \circ \varphi$ of a surjective group homomorphism $\varphi \colon T \to T$ and the translation morphism $\mathrm{tr}(b)$ by $b \in T$, which is given by $t \mapsto t + b$ for $t \in T$. We take a point $b' \in T$ such that $\mu_m(b') = b$. Then $\mu_m \circ h' = h \circ \mu_m$ for the endomorphism $h' = \mathrm{tr}(b') \circ \varphi \colon T' \to T'$.

(2): Let G' be the set of pairs (ϕ, σ) of an automorphism ϕ of T and an element $\sigma \in G$ such that $\mu_m \circ \phi = L_\sigma \circ \mu_m$, where L_σ stands for the automorphism of T defined as the left action of σ. Then G' is a group by composition $(\phi_1, \sigma_1)(\phi_2, \sigma_2) = (\phi_1 \circ \phi_2, \sigma_1\sigma_2)$, and we have a group homomorphism $\rho \colon G' \to G$ defined by $\rho(\phi, \sigma) = \sigma$. We have another group homomorphism $G' \to \mathrm{Aut}(T')$ by $(\phi, \sigma) \mapsto \phi$. In particular, G' acts on T', and μ_m is equivariant under the actions of G' and G with respect to ρ. By (1), ρ is surjective, and the kernel of ρ is identified with the Galois group of μ_m in $\mathrm{Aut}(T')$.

(3a): The endomorphism f induces a surjective endomorphism h of T such that $\pi \circ f = h \circ \pi$ by a universal property of Albanese morphism. We have a surjective endomorphism h' of T such that $\mu_m \circ h' = h \circ \mu_m$ by (1). Hence, $f \times h' \colon V \times T \to V \times T$ induces a surjective endomorphism f' of $V' = V \times_T T$ such that $p \circ f' = f \circ p$ (cf. [20, Lem. 4.1]).

(3b): By the universality of Albanese morphism, the finite group G acts on T so that π is G-equivariant. By (2), we have a finite group G' acting on T with a surjective homomorphism $\rho \colon G' \to G$ such that μ_m is equivariant under actions of G' and G with respect to ρ. Then G' acts on $V \times T$ diagonally so that the action on the first factor V is given by that of G through ρ and that the action on the second factor T is given as that of G'. It induces an action of G' on $V' = V \times_T T$ such that $p \colon V' \to V$ is equivariant under actions of G' and G with respect to ρ. By (2), we may assume that the kernel of ρ is identified with the Galois group of p. Hence, $G' \backslash V' \simeq G \backslash V$. Thus, we are done. □

Lemma 5.5. *Let X be a $\mathbb{P}^1$-bundle over an elliptic curve T associated with $\mathcal{O}_T \oplus \mathcal{L}$ for an invertible sheaf $\mathcal{L}$ such that* $\deg \mathcal{L} = 0$ *but $\mathcal{L}$ is not a torsion element of* $\mathrm{Pic}(T)$. *Let Θ_1 and Θ_2 be sections of $X \to T$ corresponding to projections $\mathcal{O}_T \oplus \mathcal{L} \to \mathcal{O}_T$ and $\mathcal{O}_T \oplus \mathcal{L} \to \mathcal{L}$, respectively. Then*:

(1) *If a prime divisor Γ on X dominates T and satisfies $\Gamma^2 = 0$, then $\Gamma = \Theta_1$ or Θ_2.*
(2) *If an endomorphism $f \colon X \to X$ is surjective, then $(f^{-1}\Theta_1, f^{-1}\Theta_2) = (\Theta_1, \Theta_2)$ or $(f^{-1}\Theta_1, f^{-1}\Theta_2) = (\Theta_2, \Theta_1)$.*
(3) *If $f \colon X \to X$ is a non-isomorphic surjective endomorphism, then $S_f = \Theta_1 + \Theta_2$ and $K_X + S_f \sim 0$.*

Proof. We have $\Theta_1^2 = \Theta_2^2 = 0$ and $\mathcal{O}_X(\Theta_2) \simeq \mathcal{O}_X(\Theta_1) \otimes \pi^*\mathcal{L}$ for the structure morphism $\pi \colon X \to T$. The pseudo-effective cone $\overline{\mathrm{NE}}(X)$ of X is generated by the numerical class of a fiber of π and by $\mathrm{cl}(\Theta_1) = \mathrm{cl}(\Theta_2)$. In particular, X has no negative curve.

(1): There exist a positive integer m and a divisor $\mathfrak{e}$ on T such that $\Gamma \sim m\Theta_2 + \pi^*\mathfrak{e}$, and we have $\deg \mathfrak{e} = 0$ by

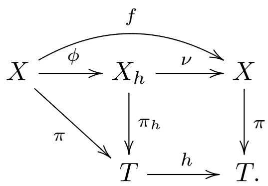
$$0 = \Gamma^2 = m^2\Theta_2^2 + 2m\Theta_2\pi^*\mathfrak{e} = 2m \deg \mathfrak{e}.$$

Hence, $\Gamma\Theta_1 = \Gamma\Theta_2 = 0$. Assume that $\Gamma \neq \Theta_1$. Then $\Gamma \cap \Theta_1 = \emptyset$, and $\mathfrak{e} \sim 0$ by $\Gamma|_{\Theta_1} \sim \mathfrak{e}$. Hence, $\Gamma \sim m\Theta_2$, and moreover, $\Gamma = \Theta_2$, since $\mathcal{O}(\Gamma|_{\Theta_2}) \simeq \mathcal{L}^{\otimes m}$ has no non-zero global section. Therefore, $\Gamma = \Theta_1$ or Θ_2.

(2): Let Γ be a prime component of $f^{-1}\Theta_i$ for $i = 1$ or 2. Then $\Gamma^2 = 0$, since $f^{-1}\Theta_i$ is not big and $\Gamma^2 \geq 0$. Thus, $\Gamma = \Theta_1$ or Θ_2 by (1). This proves (2).

(3): By (2), $\Theta_1 + \Theta_2$ is f-completely invariant and $K_X + \Theta_1 + \Theta_2 \sim 0$. Thus, $S_f \leq \Theta_1 + \Theta_2$ by [20, Thm. 2.24]. Since $\Theta_1 \approx \Theta_2$, there is a positive integer d such that $(f^*\Theta_1, f^*\Theta_2) = (d\Theta_1, d\Theta_2)$ or $(f^*\Theta_1, f^*\Theta_2) = (d\Theta_2, d\Theta_1)$. If $d > 1$, then $S_f = \Theta_1 + \Theta_2$ and $K_X + S_f \sim 0$. Therefore, it is enough to prove that $d > 1$.

Let $h: T \to T$ be the surjective endomorphism induced by f, which satisfies $\pi \circ f = h \circ \pi$. Let $\pi_h: X_h \to T$ be the base change of π by h. Then $X_h \simeq \mathbb{P}_T(\mathcal{O}_T \oplus h^*\mathcal{L})$, and we have a commutative diagram



Here, $d > 1$ if and only if $\deg \phi > 1$. Assume that $\deg \phi = 1$. Then $\deg f = \deg \nu = \deg h > 1$, and the pullback homomorphism $h^*\colon \operatorname{Pic}^0(T) \to \operatorname{Pic}^0(T)$ is a surjection whose kernel is a finite group of order $\deg h$. In particular, $h^*\mathcal{L}$ is also not a torsion element of $\operatorname{Pic}^0(T)$. The inverse image $h^*\mathcal{L}$ is not isomorphic to $\mathcal{L}$ nor $\mathcal{L}^{\otimes -1}$. In fact, if $h^*\mathcal{L} \simeq \mathcal{L}$, then $\mathcal{L}$ is contained in the kernel of $h^* - \mathrm{id}\colon \operatorname{Pic}^0(T) \to \operatorname{Pic}^0(T)$, which is also a finite group (cf. [17, II, §7, Cor. 2]). This is a contradiction, since $\mathcal{L}$ is not a torsion element. Similarly, we have a contradiction by assuming that $h^*\mathcal{L} \simeq \mathcal{L}^{\otimes -1}$. Therefore, $X_h \not\simeq X$ as a $\mathbb{P}^1$-bundle over T. This contradicts $\deg \phi = 1$. Thus, $d > 1$, and we are done. □

Now, we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. First, we shall prove the "only if" part and the last assertion. Let X be a normal projective surface such that $\boldsymbol{\rho}(X) = 2$ and that X is irrational or $-K_X$ is not big. Let f be a non-isomorphic surjective endomorphism of X such that $K_X + S_f$ is not pseudo-effective. Then K_X is also not pseudo-effective. Thus, we can apply [20, Thm. 4.16] for irrational X, and can apply [21, Thm. 4.7] for X with non-big $-K_X$. Consequently, there is a finite Galois cover $\nu\colon V \to X$ étale in codimension 1 with an endomorphism f_V of X such that $\nu \circ f_V = f^k \circ \nu$ for some $k > 0$ and that either

- $V \simeq \mathbb{P}^1 \times T$ for an irrational curve T, i.e., Theorem 5.1(1) holds, or
- V is a $\mathbb{P}^1$-bundle over an elliptic curve T.

Note that, in [20, Thm. 4.16(1)], we can take the finite morphism $\nu\colon \mathbb{P}^1 \times T \to X$ as a Galois cover by the proof there which uses [20, Thm. 4.9]. Thus, for the proof of "only if" part and the last assertion, we may assume that V is a non-trivial $\mathbb{P}^1$-bundle over an elliptic curve T. By [21, Fact 2.23], we may assume that one of the following holds for V:

(A) $V = \mathbb{P}_T(\mathcal{E})$ for a stable locally free sheaf $\mathcal{E}$ of rank 2 degree 1;
(B) $V = \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{L})$ for an invertible sheaf $\mathcal{L}$ of degree 0.

If (A) holds, then the base change $V' = V \times_T T \to T$ of $V \to T$ by the multiplication map $\mu_2\colon T \to T$ by 2 is a trivial $\mathbb{P}^1$-bundle by [21, Fact 2.23(C)], where we consider T as a complex Lie group by fixing an origin. Here, the endomorphism f_V lifts to an endomorphism of V' and the composite $V' \to V \to X$ is Galois by Lemma 5.4(3). Thus, in this case, Theorem 5.1(1) holds for $V' \to X$ as well as the last assertion of Theorem 5.1. If (B) holds, then $\mathcal{L}$ is a torsion element by Lemma 5.5(3), and hence, $\mu_m^*\mathcal{L} \simeq \mathcal{O}_T$ for the multiplication map $\mu_m \colon T \to T$ by a certain positive integer m with respect to a complex Lie group structure of T. In particular, the base change $V' \to T$ of $V \to T$ by μ_m is a trivial $\mathbb{P}^1$-bundle. Thus, in this case, Theorem 5.1(1) holds for $V' \to X$ as well as the last assertion of Theorem 5.1, by Lemma 5.4(3), as in the same argument as above. This proves the "only if" part and the last assertion of Theorem 5.1.

Next, we shall prove the "if" part. We have already proved it in Corollary 5.3 in the case where Theorem 5.1(1) is satisfied. As in the argument in the proof of Corollary 5.3, it suffices to construct a non-isomorphic surjective endomorphism g of V such that g is equivariant under the action of the Galois group of $\nu \colon V \to X$ and that $K_V + S_g$ is not pseudo-effective. If Theorem 5.1(3) holds, then the endomorphism g in [21, Cor. 2.27] is an expected one. In fact, S_g is just the negative section and $K_V + S_g$ is not pseudo-effective. If Theorem 5.1(2) holds, then we have a G-equivariant étale non-isomorphic endomorphism g of V by [21, Prop. 2.25(2)], where $K_V + S_g = K_V$ is not pseudo-effective. Thus, we are done. □

5.2. **Case (II).** The following is a structure theorem for cases (II) and (III):

Theorem 5.6. *Let $f \colon X \to X$ be a non-isomorphic surjective endomorphism of a normal projective* rational *surface X such that $\boldsymbol{\rho}(X) = 2$, $-K_X$ is big, and $K_X + S_f$ is not pseudo-effective. Then one of the following holds*:

(1) *There exist a finite Galois cover $\nu \colon X' \to X$ étale in codimension 1 and an endomorphism $f' \colon X' \to X'$ such that*
  - *X' is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ or over an elliptic curve,*
  - *the Galois group of ν preserves the $\mathbb{P}^1$-bundle structure, and*
  - *$\nu \circ f' = f^k \circ \nu$ for some $k > 0$.*

(2) *The pair $(X, S_f + B)$ is a toric surface for a non-zero reduced divisor B having at most two prime components.*

(3) *The pair $(X, S_f + B)$ is a half-toric surface for a prime divisor B, and B is an end component of the linear chain $S_f + B$.*

Remark 5.7. The condition (1) has a meaning only when $X' \simeq \mathbb{P}^1 \times \mathbb{P}^1$. In fact, if $X' \not\simeq \mathbb{P}^1 \times \mathbb{P}^1$, then either X' is a $\mathbb{P}^1$-bundle over an elliptic curve or X' admits a negative section. In both cases, any automorphism of X' preserves the $\mathbb{P}^1$-bundle structure. Moreover, when $X' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, (1) implies that the action of the Galois group on X' is *diagonal*, i.e., it preserves two projections $X' \to \mathbb{P}^1$ (cf. [21, Lem. 2.3]).

Remark 5.8. The divisor S_f contains all the negative curves (cf. [20, Prop. 2.20(3)]), the pair (X, S_f) is log-canonical (cf. [20, Thm. E]), and $-(K_X + S_f)$ is semi-ample (cf. [21, Lem. 5.2]). If $\boldsymbol{n}(S_f) \geq 3$, then $\boldsymbol{n}(S_f) = 3$ and there is a prime divisor B such that $(X, B + S_f)$ is a toric surface by [21, Thm. 1.3]. Therefore, we may assume that $\boldsymbol{n}(S_f) \leq 2$ for the proof of Theorem 5.6.

In Section 5.2, we shall prove Theorem 5.6 in the case where X admits a negative curve; the proof is at the end of Section 5.2. Before the proof, we assume in addition that $\boldsymbol{n}(S_f) \leq 2$ (cf. Remark 5.8). In particular, the number of negative curves on

X is 1 or 2. Proposition 5.9 below is a result in the case where X has two negative curves. When X has a unique negative curve, we have three cases as in Lemma 5.10, and Propositions 5.14, 5.15, and 5.16 below treat these three cases, separately.

Proposition 5.9. *Assume that X has two negative curves. Then S_f is the sum of these two negative curves, and the following hold*:

(1) *If $-(K_X + S_f)$ is ample, then $(X, S_f + B)$ is a toric surface for a reduced divisor B with $\boldsymbol{n}(B) = 2$.*

(2) *If $-(K_X + S_f)$ is not ample, then $(X, S_f + B)$ is a half-toric surface for a prime divisor B.*

Proof. Let C_1 and C_2 be two negative curves. Then $S_f = C_1 + C_2$ and $\overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0}\,\mathrm{cl}(C_1) + \mathbb{R}_{\geq 0}\,\mathrm{cl}(C_2)$ by $\boldsymbol{\rho}(X) = 2$ and by the assumption: $\boldsymbol{n}(S_f) \leq 2$. Hence, $C_1C_2 > 0$, and S_f is connected. Thus, (1) follows from Theorem 3.14. Assume that $-(K_X + S_f)$ is not ample. Since $K_X + S_f$ is not nef, we may assume that $(K_X + S_f)C_1 < 0$. Then $(K_X + S_f)C_2 = 0$, and (X, C_1, C_2) is a $\mathcal{V}$-surface (cf. Definition 4.1, Remark 5.8). Thus, (2) follows from Theorems 4.10 and 4.21 and Lemma 4.22. □

Lemma 5.10. *Assume that X has a unique negative curve C. Then $f^*C = \delta_f C$ for $\delta_f = (\deg f)^{1/2} > 0$, and there exists a $\mathbb{P}^1$-fibration $\pi\colon X \to T \simeq \mathbb{P}^1$ such that $\pi(C) = T$ and that one of the following holds for a general fiber F of π*:

(a) $(K_X + S_f)F < 0$ *and* $(K_X + S_f)C < 0$;

(b) $(K_X + S_f)F < 0$ *and* $(K_X + S_f)C = 0$;

(c) $(K_X + S_f)F = 0$ *and* $(K_X + S_f)C < 0$.

Moreover, there is an endomorphism h of T such that $\pi \circ f = h \circ \pi$ and $\deg h = \delta_f$.

Proof. The equality $f^*C = \delta_f C$ holds by the uniqueness of C and by $(f^*C)^2 = (\deg f)C$. Note that $S_f \geq C$ and that $-(K_X + S_f)$ is semi-ample (cf. Remark 5.8). First, we consider the case where $(K_X + S_f)C = 0$. By the cone theorem (cf. [20, Thm. 1.9]), there is an extremal ray R of $\overline{\mathrm{NE}}(X)$ such that $(K_X + S_f)\mathsf{R} < 0$. Thus, $\overline{\mathrm{NE}}(X) = \mathsf{R} + \mathbb{R}_{\geq 0}\,\mathrm{cl}(C)$, and we have a $\mathbb{P}^1$-fibration $\pi\colon X \to T \simeq \mathbb{P}^1$ as the contraction morphism of R (cf. [20, Thm. 1.10(2)]). Then $\pi(C) = T$ by $CF > 0$ for a general fiber F of π, and (b) holds.

Second, we consider the case where $(K_X + S_f)C < 0$. Then $-(K_X + S_f)$ is either ample or $(K_X + S_f)^2 = 0$. In fact, if $-(K_X + S_f)$ is big but not ample, then $(K_X + S_f)C' = 0$ for a negative curve C', which does not exist by assumption. When $-(K_X + S_f)$ is ample, we have a $\mathbb{P}^1$-fibration $\pi\colon X \to T \simeq \mathbb{P}^1$ as the contraction morphism of an extremal ray R of $\overline{\mathrm{NE}}(X)$ such that $\overline{\mathrm{NE}}(X) = \mathsf{R} + \mathbb{R}_{\geq 0}\,\mathrm{cl}(C)$; hence, $\pi(C) = T$ and (a) holds. When $(K_X + S_f)^2 = 0$, the semi-ample divisor $-(K_X + S_f)$ defines a $\mathbb{P}^1$-fibration $\pi\colon X \to T \simeq \mathbb{P}^1$ such that $K_X + S_f$ is π-numerically trivial; hence, $\pi(C) = T$ and (c) holds.

In any case above, $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ is a scalar map and $f^*\mathsf{R} = \mathsf{R}$ for the ray R of $\overline{\mathrm{NE}}(X)$ generated by the numerical class of a general fiber F of π by [20, Lem. 3.7], since $f^*C = \delta_f C$. Hence, the last assertion is a consequence of [20, Lem. 3.16]. □

Definition 5.11. For the fibration $\pi \colon X \to T$ in Lemma 5.10, let F denote a general fiber. For $t \in T$, we set $G_t := \pi^{-1}(t)$, which is a prime divisor by $\boldsymbol{\rho}(X) = 2$ (cf. [18, Prop. 2.33(7)]). We define a positive integer m_t by $\pi^*(t) = m_t G_t$, and set $\mathcal{S}_\pi := \{t \in T \mid m_t > 1\}$.

Remark 5.12. The $\mathbb{P}^1$-fibration π is smooth over $T \setminus \mathcal{S}_\pi$ by [18, Prop. 2.33(4)]. Moreover, the horizontal part S_f^{hor} of S with respect to π is non-singular, (X, S_f^{hor}) is 1-log-terminal, and $(X, S_f^{\mathrm{hor}} + G_t)$ is log-canonical for any $t \in T$, by [20, Prop. 3.17].

Lemma 5.13. *In Lemma* 5.10*, the following hold*:

(1) *If $(K_X + S_f)F < 0$, then C is a section of π and is a unique prime component of S_f dominating T.*
(2) *If a prime component C' of S_f is a section of π, then*
$$(K_X + C')|_{C'} = K_{C'} + \sum\nolimits_{t \in \mathcal{S}_\pi} (1 - 1/m_t) Q_t,$$
where $\{Q_t\} := G_t \cap C'$.

Proof. Since $S_f \geq C$, we have $-2 + CF = (K_X + C)F \leq (K_X + S_f)F$. This implies (1). We have (2) by (II-6) in Lemma 2.14 and by Remark 5.12. □

Proposition 5.14. *Assume that Lemma* 5.10(a) *is satisfied.*

(1) *If $S_f \neq C$, then $S_f = C + G_o$ for a set-theoretic fiber $G_o = \pi^{-1}(o)$ of π, and $(X, S_f + B)$ is a toric surface for a divisor B.*
(2) *If $S_f = C$ and $\#\mathcal{S}_\pi \leq 2$, then $(X, S_f + B)$ is a toric surface for a divisor B.*
(3) *If $S_f = C$ and $\#\mathcal{S}_\pi \geq 3$, then $\#\mathcal{S}_\pi = 3$, and there exist a finite Galois cover $\nu \colon X' \to X$ étale in codimension* 1 *and an endomorphism $f' \colon X' \to X'$ such that $\nu \circ f' = f \circ \nu$ and that X' is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with a negative section.*

Proof. (1): We can write $S_f = C + G_o$ for some $o \in T$ by $\boldsymbol{n}(S_f) \leq 2$ and by Lemma 5.13(1), and the assertion follows from Theorem 3.14.

(2): We choose two points t_1 and t_2 of T such that $\mathcal{S}_\pi \subset \{t_1, t_2\}$. Then
$$(K_X + C + G_{t_1} + G_{t_2})F = -1 \quad \text{and} \quad (K_X + C + G_{t_1} + G_{t_2})C = 0$$
by Lemma 5.13(2). Since $(X, C + G_{t_1} + G_{t_2})$ is log-canonical (cf. Remark 5.12), $(X, C + G_{t_1} + G_{t_2} + \Theta)$ is a toric surface for a prime divisor Θ by [18, Thm. 1.3]. Thus, it is enough to take $B = G_{t_1} + G_{t_2} + \Theta$.

(3): Since $(K_X + C)C < 0$, we have $\sum_{t \in \mathcal{S}_\pi} (1 - 1/m_t) < 2$ by Lemma 5.13(2). Thus, $\#\mathcal{S}_\pi = 3$. Moreover, if $\mathcal{S}_\pi = \{t_1, t_2, t_3\}$ with $m_{t_1} \leq m_{t_2} \leq m_{t_3}$, then $(m_{t_1}, m_{t_2}, m_{t_3})$ is one of
$$(2, 2, m \geq 2) \quad (2, 3, 3), \quad (2, 3, 4), \quad (2, 3, 5).$$
We have a finite Galois cover $\tau \colon T' \simeq \mathbb{P}^1 \to T$ such that $\tau^*(t_i) = m_{t_i} \tau^{-1}(t_i)$ for any $1 \leq i \leq 3$ and that τ is étale over $T \setminus \mathcal{S}_\pi$. In fact, we can express τ as the quotient morphism $\mathbb{P}^1 \to \mathbb{P}^1/\mathfrak{G}$ by the action of a polyhedral subgroup $\mathfrak{G}$ of $\mathrm{Aut}(\mathbb{P}^1)$. For the normalization X' of $X \times_T T'$, the induced finite morphism $\nu \colon X' \to X$ is étale in codimension 1 and the induced $\mathbb{P}^1$-fibration $\pi' \colon X' \to T'$ has only reduced fibers

by [20, Lem. 4.2]. Furthermore, π' is smooth by Lemmas 2.12 and 2.13(2) and Remark 5.12. Hence, X' is a $\mathbb{P}^1$-bundle over $T' \simeq \mathbb{P}^1$ with a negative section ν^*C. Since X' is simply connected, by applying Lemma 3.10 to the case where $S = 0$, we have an endomorphism $f' \colon X' \to X'$ such that $\nu \circ f' = f \circ \nu$. Thus, (3) has been proved, and we are done. □

Proposition 5.15. *Assume that Lemma* 5.10(b) *is satisfied.*

(1) *If $S_f \neq C$, then $S_f = C + G_o$ for some $o \in T$, and $(X, S_f + B)$ is a half-toric surface for a section B of π such that $B \cap C = \emptyset$.*
(2) *If $S_f = C$, then there exist a finite cyclic cover $\nu \colon X' \to X$ étale in codimension* 1 *and an endomorphism $f' \colon X' \to X'$ such that $\nu \circ f' = f \circ \nu$ and that X' is a $\mathbb{P}^1$-bundle over an elliptic curve with a negative section.*

Proof. (1): We can write $S_f = C + G_o$ by $\boldsymbol{n}(S_f) \leq 2$ and by Lemma 5.13(1). Then (X, G_o, C) is a $\mathcal{V}_{\mathrm{A}}$-*surface* with $G_o^2 = 0$ (cf. Definitions 4.1 and 4.5(3)) by Lemma 5.10(b), since (X, S_f) is log-canonical and (X, C) is 1-log-terminal (cf. Remarks 5.8 and 5.12). Then $(X, S_f + B)$ is a half-toric surface for a prime divisor B by Theorem 4.10 (cf. Lemma 4.11). Here, $B \cap C = \emptyset$ and B is a section of π by $BC = -(K_X + S_f)C = 0$ and $BF = -(K_X + S_f)F = -(K_X + C)F = 1$.

(2): Since $(K_X + S_f)C = 0$ and $f^*C = \delta_f C$, we have $C \cap \operatorname{Supp} \Delta_f = 0$ by [20, Prop. 2.20(5)]. Then any prime component of Δ_f dominates T, and any prime component of R_f dominates T by $\operatorname{Supp} R_f \subset S_f \cup \operatorname{Supp} \Delta_f$ (cf. [20, Lem. 2.17(4)]). By Lemma 5.13(2) and by $(K_X + C)C = 0$, we have

$$\sum\nolimits_{t \in \mathcal{S}_\pi} (1 - 1/m_t) = 2.$$

In particular, $\mathcal{S}_\pi \neq \emptyset$. By replacing the endomorphism $h \colon T \to T$ in Lemma 5.10 with a power h^k and by [20, Lem. 4.2 and Prop. 4.3], we can find a finite cyclic cover $\tau \colon T' \to T$ from an *elliptic curve* T' and an endomorphism $h' \colon T' \to T'$ such that $\tau \circ h' = h \circ \tau$ and the following are satisfied for the normalization X' of $X \times_T T'$:

- The morphism τ is étale over $T \setminus \mathcal{S}_\pi$ and $\tau^*(t) = m_t \tau^{-1}(t)$ for any $t \in \mathcal{S}_\pi$.
- The induced finite cyclic cover $\nu \colon X' \to X$ is étale in codimension 1.
- The induced $\mathbb{P}^1$-fibration $\pi' \colon X' \to T'$ has only reduced fibers.
- There is an endomorphism $f' \colon X' \to X'$ such that $\nu \circ f' = f \circ \nu$ and $\pi' \circ f' = h' \circ \pi'$.

Then any fiber of π' is isomorphic to $\mathbb{P}^1$ by Lemmas 2.12 and 2.13(2) and by Remark 5.12. Thus, π' is a $\mathbb{P}^1$-bundle over the elliptic curve T', and ν^*C is a negative section of π'. □

Proposition 5.16. *Assume that Lemma* 5.10(c) *is satisfied.*

(1) *If $S_f \geq C + C'$ for another prime divisor C' dominating T, then C and C' are sections of π such that $C'^2 > 0$, $C \cap C' = \emptyset$, and $S_f = C + C'$.*
(2) *If $S_f = C + C'$ for another prime divisor C' dominating T and if $\#\mathcal{S}_\pi \leq 2$, then $(X, S_f + B)$ is a toric surface for a union B of two set-theoretic fibers of π.*

(3) *If* $S_f = C + C'$ *for another prime divisor* C' *dominating* T *and if* $\#\mathcal{S}_\pi \geq 3$, *then* $\#\mathcal{S}_\pi = 3$, *and there exist a finite Galois cover* $\nu\colon X' \to X$ *étale in codimension* 1 *and an endomorphism* $f'\colon X' \to X'$ *such that* $\nu \circ f' = f \circ \nu$ *and that* X' *is a* $\mathbb{P}^1$*-bundle over* $\mathbb{P}^1$ *with a negative section.*

(4) *If* C *is a unique prime component of* S_f *dominating* T*, then* C *is a double section of* π*, and* $S_f = C + G_{t_1}$ *for a branched point* $t_1 \in T$ *of the double cover* $\pi|_C\colon C \to T$*. Moreover,* $(X, S_f + G_{t_2})$ *is a half-toric surface for the other branched point* t_2 *of* $\pi|_C$*.*

Proof. (1): Since $(K_X + S_f)F = 0$, C and C' are sections of π, and $S_f = C + C'$ by $\boldsymbol{n}(S_f) \leq 2$. Here, $C + C'$ is non-singular by Remark 5.12, and it implies that $C \cap C' = \emptyset$. Moreover, $C'^2 > 0$, since $\mathrm{cl}(C')$ does not belong to $\mathbb{R}_{\geq 0}\,\mathrm{cl}(C)$ nor $\mathbb{R}_{\geq 0}\,\mathrm{cl}(F)$.

(2): We set $B := G_{t_1} + G_{t_2}$ for two points t_1 and t_2 such that $\mathcal{S}_\pi \subset \{t_1, t_2\}$. Then $(K_X + S_f + B)F = 0$, and $(K_X + S_f + B)C = (K_X + C + B)C = 0$ by (1) and Lemma 5.13(2). In particular, $K_X + S_f + B \approx 0$. Since $(X, S_f + B)$ is log-canonical (cf. Remark 5.12), $(X, S_f + B)$ is a toric surface by [26, Thm. 6.4] (cf. [18, Thm. 1.1]).

(3): Now, $(K_X + C)C = (K_X + S_f)C < 0$ by (1). Thus, (3) is shown by the same argument as in the proof of Proposition 5.14(3).

(4): Since $(K_X + S_f)F = 0$, C is a double section of π. Let t_1 and $t_2 \in T$ be the branched points of $\pi|_C$. For each $i = 1$, 2, m_{t_i} is even and $(X/T, C, G_{t_i})$ is an irreducible PDS configuration of type I_{n_i} for $n_i = m_{t_i}/2$ by Lemma 2.19, since $(X, C + G_{t_i})$ is log-canonical (cf. Remark 5.12) and since $CG_{t_i} = 2/m_{t_i}$. In particular, $\{t_1, t_2\} \subset \mathcal{S}_\pi$. If $t \in T \setminus \{t_1, t_2\}$, then G_t intersects C transversely at two points, and hence, $m_t = 1$. Thus, π is smooth over $T \setminus \{t_1, t_2\}$ and $\mathcal{S}_\pi = \{t_1, t_2\}$. Therefore, $(X, C + G_{t_1} + G_{t_2})$ is a half-toric surface by Proposition 2.28(2). This finishes the proof. □

Remark. An example satisfying Proposition 5.16(3) is provided in [21, Exam. 2.19].

Proof of Theorem 5.6 *in the case where* X *contains a negative curve.* We may assume that $\boldsymbol{n}(S_f) \leq 2$ by Remark 5.8. There exist at most two negative curves on X by $\boldsymbol{\rho}(X) = 2$. If two negative curves exist, then either (2) or (3) of Theorem 5.6 holds by Proposition 5.9. If X has a unique negative curve, then we have three cases (a), (b), and (c) in Lemma 5.10, and in each case, Theorem 5.6 holds true by Propositions 5.14, 5.15, and 5.16, respectively. □

5.3. **Case (III).** We shall prove:

Theorem 5.17. *Let* f *be a non-isomorphic surjective endomorphism of a normal projective rational surface* X *such that* $\boldsymbol{\rho}(X) = 2$, $-K_X$ *is big, and that* X *has no negative curve. Then*:

(1) *There is a finite Galois cover* $\nu\colon X' \to X$ *étale in codimension* 1 *from* $X' = \mathbb{P}^1 \times \mathbb{P}^1$ *with an endomorphism* $f'\colon X' \to X'$ *such that*
  - *the action of the Galois group of* ν *on* X' *is* diagonal, *i.e., it preserves two projections* $X' \to \mathbb{P}^1$ (*cf. Remark* 5.7), *and*

- *$\nu \circ f' = f^k \circ \nu$ for some $k > 0$.*

(2) *If $K_X + S_f$ is not pseudo-effective and if S_f is connected and reducible, then $(X, S_f + B)$ is a toric surface for a divisor B.*

Theorem 5.6 in the case where X admits no negative curve is a consequence of Theorem 5.17. The proofs of Theorems 5.17 and 5.6 are given at the end of Section 5.3. Lemma 5.18 below is a key result concerning our third crucial idea explained in the introduction.

Lemma 5.18. *Let $\tau \colon X \to Y$ be a finite surjective morphism of normal projective surfaces and let $f \colon X \to X$ and $g \colon Y \to Y$ be non-isomorphic surjective endomorphisms such that $\tau \circ f = g \circ \tau$. Assume that $\lambda_f < \deg f$ for the* first dynamical degree λ_f *of f (cf.* [20, Def. 3.1]). *Then there exist positive integers k and n satisfying*

(V-1) $$\operatorname{Supp} R_\tau \subset f^k(\tau^{-1} \operatorname{Supp} R_{g^n})$$

for the ramification divisors R_τ and R_{g^n} of $\tau \colon X \to Y$ and $g^n \colon Y \to Y$, respectively.

Proof. For any $n \geq 1$, we have an equality

(V-2) $$R_{f^n} + (f^n)^* R_\tau = R_\tau + \tau^* R_{g^n}$$

by $\tau \circ f^n = g^n \circ \tau$. Let $\mathcal{S}$ be the set of prime components Θ of R_τ such that $(f^k)^*\Theta$ has no common prime component with $\tau^{-1} \operatorname{Supp} R_{g^n}$ for any $k \geq 1$ and $n \geq 1$. Assume that $\mathcal{S} = \emptyset$. Then, for any prime component Θ of R_τ, there is a common prime component Θ' of $(f^k)^*\Theta$ and $\tau^{-1} \operatorname{Supp} R_{g^n}$ for some k and n. Here, we can take k and n independently of the choice of prime components Θ of R_τ. In fact, we may replace (k, n) with $(k + m, n + m)$ for any $m \geq 1$, because

$$\tau^{-1} \operatorname{Supp} R_{g^{m+n}} \supset \tau^{-1}(g^m)^{-1} \operatorname{Supp} R_{g^n} = (f^m)^{-1}\tau^{-1} \operatorname{Supp} R_{g^n}$$

by $R_{g^{m+n}} = R_{g^m} + (g^m)^* R_{g^n}$. Thus, (V-1) holds by

$$\Theta = f^k(\Theta') \subset f^k(\tau^{-1} \operatorname{Supp} R_{g^n}).$$

Therefore, it is enough to prove that $\mathcal{S} = \emptyset$.

If $\Theta \in \mathcal{S}$, then $(f^k)^*\Theta \leq R_\tau$ for any $k \geq 1$ by (V-2), and hence, any prime component of $f^*\Theta$ belongs to $\mathcal{S}$. Let $\mathbb{V} = \mathbb{V}_{\mathcal{S}}$ be the free $\mathbb{R}$-vector space generated by elements of $\mathcal{S}$, and we define

$$\mathbb{V}^{\geq 0} := \sum\nolimits_{\Theta \in \mathcal{S}} \mathbb{R}_{\geq 0}\Theta$$

as a polyhedral cone of $\mathbb{V}$. Then $\Theta \mapsto f^*\Theta$ gives rise to an $\mathbb{R}$-linear endomorphism $f^* \colon \mathbb{V} \to \mathbb{V}$ preserving $\mathbb{V}^{\geq 0}$, and the diagram

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{f^*} & \mathbb{V} \\ {\scriptstyle \mathrm{cl}}\downarrow & & \downarrow{\scriptstyle \mathrm{cl}} \\ \mathsf{N}(X) & \xrightarrow{f^*} & \mathsf{N}(X) \end{array}$$

is commutative for the class map $\mathrm{cl} \colon \mathbb{V} \to \mathsf{N}(X)$ given by $\Theta \mapsto \mathrm{cl}(\Theta)$. By a version of Perron–Frobenius theorem (cf. [3]), we can find a non-zero vector D in $\mathbb{V}^{\geq 0}$ such that $f^*D = \lambda D$ for the spectral radius λ of $f^* \colon \mathbb{V} \to \mathbb{V}$. Since $\mathrm{cl}(D) \neq 0$,

λ is also an eigenvalue of $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$. Hence, $\lambda \in \{\delta_f, \lambda_f, \deg f/\lambda_f\}$ by [20, Prop. 3.3(4)], and we have $\lambda > 1$ by the assumption: $\lambda_f < \deg f$. We write $D = \sum r_j \Theta_j$ for prime divisors $\Theta_j \in \mathcal{S}$ and $r_j \geq 0$. Then

$$\lambda^k D = (f^k)^* D = \sum r_j (f^k)^* \Theta_j \leq (\sum r_j) R_\tau$$

for any $k \geq 1$: This contradicts $D \neq 0$ and $\lambda > 1$. Therefore, $\mathcal{S} = \emptyset$, and we are done. □

Lemma 5.19. *In Theorem* 5.17*,* $-K_X$ *is ample, and there exist an integer* $k \in \{1, 2\}$*, two* $\mathbb{P}^1$*-fibrations* $\pi_1\colon X \to T_1 \simeq \mathbb{P}^1$ *and* $\pi_2\colon X \to T_2 \simeq \mathbb{P}^1$*, and endomorphisms* $h_1\colon T_1 \to T_1$ *and* $h_2\colon T_2 \to T_2$ *such that*

- $\phi := (\pi_1, \pi_2)\colon X \to T_1 \times T_2$ *is a finite surjective morphism, and*
- $\pi_i \circ f^k = h_i \circ \pi$ *for any* $i = 1$, 2.

In particular, $(\deg f)^k = \deg h_1 \deg h_2$*, and* $(\lambda_f)^k = \max\{\deg h_1, \deg h_2\}$*.*

Proof. The absence of negative curves implies that $-K_X$ is ample. In fact, $-K_X$ is nef and big, since the negative part of the Zariski-decomposition is 0, and if $-K_X$ not ample, then $(-K_X)C = 0$ for a prime divisor C, and in this case, $C^2 < 0$ by the Hodge index theorem, contradicting the assumption. Since $\boldsymbol{\rho}(X) = 2$, $\overline{\mathrm{NE}}(X) = \mathsf{R}_1 + \mathsf{R}_2$ for extremal rays R_1 and R_2, and for each $i = 1$, 2, the contraction morphism of R_i is a $\mathbb{P}^1$-fibration $\pi_i\colon X \to T_i \simeq \mathbb{P}^1$. The induced morphism $\phi = (\pi_1, \pi_2)\colon X \to T_1 \times T_2$ is finite and surjective as $\boldsymbol{\rho}(X) = \boldsymbol{\rho}(T_1 \times T_2) = 2$.

If $\lambda_f > \delta_f = (\deg f)^{1/2}$, then $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ has two distinct eigenvalues and each R_i is generated by an eigenvector; hence, $f^*\mathsf{R}_i = \mathsf{R}_i$ for $i = 1$, 2 (cf. [20, Lem. 3.7(2)]). Even if $\lambda_f = \delta_f$, $(f^2)^*\mathsf{R}_i = \mathsf{R}_i$ for $i = 1$, 2, and $(f^2)^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ is the multiplication map by $\delta_f^2 = \deg f$ (cf. [20, Lem. 3.7(1)]). Thus, we may assume that $f^*\mathsf{R}_i = \mathsf{R}_i$ for $i = 1$, 2, by replacing f with f^2 if necessary. Then, for each $i = 1$, 2, we have an endomorphism $h_i\colon T_i \to T_i$ such that $\pi_i \circ f = h_i \circ \pi_i$, by [20, Lem. 3.16]. Here, $\phi \circ f = g \circ \phi$ for $g := h_1 \times h_2\colon T_1 \times T_2 \to T_1 \times T_2$, and $\deg f = \deg g = \deg h_1 \deg h_2$. If $\lambda_f > \delta_f$, then $\deg h_1 \neq \deg h_2$ and $\lambda_f = \max\{\deg h_1, \deg h_2\}$. If $\lambda_f = \delta_f$, then $\lambda_f = \deg h_1 = \deg h_2$. □

Lemma 5.20. *Theorem* 5.17 *holds if* $\deg h_1 = 1$ *or* $\deg h_2 = 1$*. Moreover, in this case, if* $S_f \neq 0$*, then* S_f *is non-singular: In particular, the assumption of Theorem* 5.17(2) *is not satisfied.*

Proof. We may assume that $\deg h_2 = 1$. By [20, Thm. 4.9] applied to the $\mathbb{P}^1$-fibration $\pi_2\colon X \to T_2$, there exist a finite Galois cover $\nu\colon X' \to X$ étale in codimension 1 and an endomorphism $f'\colon X' \to X'$ such that $X' \simeq \mathbb{P}^1 \times T'$ for a non-singular projective curves T' with a finite Galois cover $T' \to T_2$ and that $\nu \circ f' = f^l \circ \nu$ for some $l > 0$. Here, $T' \simeq \mathbb{P}^1$ as $-K_{X'} = \nu^*(-K_X)$ is ample. Since the action of the Galois group of ν preserves the second projection $X' \to T'$, the action is diagonal by [21, Lem. 2.3]. Thus, Theorem 5.17(1) holds. The last assertion on S_f follows from [20, Lem. 4.4(1)]. □

By Lemma 5.20 and by replacing f with f^2 if necessary, we may assume Condition 5.21 below for the proof of Theorem 5.17.

Condition 5.21. The morphisms $\phi = (\pi_1, \pi_2)\colon X \to T_1 \times T_2$ in Lemma 5.19 is not an isomorphism. Moreover, for each $i = 1, 2$, the endomorphism $h_i\colon T_i \to T_i$ in Lemma 5.19 satisfies $\pi_i \circ f = h_i \circ \pi_i$ and $\deg h_i > 1$.

Definition 5.22. For $i = 1, 2$, we set $\mathcal{S}_i$ to be the set of points $t \in T_i$ such that $\pi_i^*(t)$ is not reduced. We regard $\mathcal{S}_i$ also as a reduced divisor on T_i. Moreover, for $t \in T_i$, we set $G_t^{(i)} := \pi_i^{-1}(t)$ and set $m_t^{(i)}$ to be the multiplicity of $\pi_i^*(t)$. Thus, $\pi_i^*(t) = m_t^{(i)} G_t^{(i)}$ for any $t \in T_i$ and $\mathcal{S}_i = \{t \in T_i \mid m_t^{(i)} > 1\}$. We also define

$$G := \sum\nolimits_{i=1}^{2} \sum\nolimits_{t \in \mathcal{S}_i} G_t^{(i)} = \pi_1^{-1}(\mathcal{S}_1) + \pi_2^{-1}(\mathcal{S}_2).$$

Remark. Every fiber of $\pi_i\colon X \to T_i$ is irreducible. If $t \in T_i \setminus \mathcal{S}_i$, then $G_t^{(i)} = \pi_i^*(t)$ is a smooth fiber of π_i by [18, Prop. 2.33(4)].

Lemma 5.23. *For any $t_1 \in T_1$ and $t_2 \in T_2$, one has*

$$\deg \phi = m_{t_1}^{(1)} m_{t_2}^{(2)} G_{t_1}^{(1)} G_{t_2}^{(2)}.$$

In particular, $m_t^{(i)} \mid \deg \phi$ for any $i = 1, 2$ and any $t \in \mathcal{S}_i$.

Proof. Let $F^{(i)}$ be a general fiber of $\pi_i\colon X \to T_i$. Then $F^{(1)}F^{(2)} = \deg \phi$, and $F^{(i)} \sim m_t^{(i)} G_t^{(i)}$ for any $t \in T_i$. This implies the first assertion. The second assertion is deduced from $m^{(i)} F^{(j)} G_t^{(i)} = \deg \phi$ and $F^{(j)} G_t^{(i)} \in \mathbb{Z}$ for $\{i, j\} = \{1, 2\}$ and $t \in T_i$. □

Proposition 5.24. *Under Condition* 5.21*, the ramification divisor R_ϕ of $\phi\colon X \to T_1 \times T_2$ is expressed as*

$$R_\phi = \sum\nolimits_{i=1}^{2} \sum\nolimits_{t \in \mathcal{S}_i} (m_t^{(i)} - 1) G_t^{(i)}, \tag{V-3}$$

and the following equality holds for any $i = 1$ and 2:

$$\sum\nolimits_{t \in \mathcal{S}_i} 1/m_t^{(i)} = \#\mathcal{S}_i - 2 + 2/\deg \phi. \tag{V-4}$$

In particular, $2 \leq \#\mathcal{S}_i \leq 3$. Moreover, the pair (X, G) is log-canonical and

$$K_X + G = \pi_1^*(K_{T_1} + \mathcal{S}_1) + \pi_2^*(K_{T_2} + \mathcal{S}_2). \tag{V-5}$$

As a consequence, $(X, G_t^{(i)})$ is 1*-log-terminal for any $t \in \mathcal{S}_i$ and for $i = 1, 2$.*

Proof. By Lemma 5.19 and Condition 5.21, we have $\lambda_f < \deg f$, since $\deg h_1 > 1$, $\deg h_2 = \deg f / \deg h_1 > 1$, and $\lambda_f = \max\{\deg h_1, \deg h_2\}$. By Lemma 5.18, there exist positive integers k and n such that

$$\begin{aligned}\operatorname{Supp} R_\phi \subset f^k(\phi^{-1} \operatorname{Supp} R_{g^n}) &= f^k(\pi_1^{-1} \operatorname{Supp} R_{h_1^n}) \cup f^k(\pi_2^{-1} \operatorname{Supp} R_{h_2^n}) \\ &\subset \pi_1^{-1}(h_1^k(\operatorname{Supp} R_{h_1^n})) \cup \pi_2^{-1}(h_2^k(\operatorname{Supp} R_{h_2^n}))\end{aligned}$$

for the endomorphism $g = h_1 \times h_2\colon T_1 \times T_2 \to T_1 \times T_2$. In particular, $\operatorname{Supp} R_\phi$ is contained in a union of fibers of π_1 and π_2. For $i = 1, 2$ and for any $t \in T_i$, note that $\phi(G_t^{(i)}) = p_i^{-1}(t)$ for the i-th projection $p_i\colon T_1 \times T_2 \to T_i$ and that

$$\phi^*(\phi(G_t^{(i)})) = \pi_i^*(t) = m_t^{(i)} G_t^{(i)}.$$

Hence, the ramification index of ϕ along $G_t^{(i)}$ equals $m_t^{(i)}$. Thus, we have (V-3), and the ramification formula of ϕ is equivalent to $K_X + G = \phi^*(K_{T_1\times T_2} + H)$ for the normal crossing divisor

$$H = \sum\nolimits_{i=1}^{2} \sum\nolimits_{t\in\mathcal{S}_i} p_i^{-1}(t) = p_1^{-1}\mathcal{S}_1 + p_2^{-1}\mathcal{S}_2$$

on $T_1 \times T_2$. Hence, (V-5) holds and (X, G) is log-canonical by [19, Lem. 2.10(1)]. Since π_i is smooth over $T_i \setminus \mathcal{S}_i$ for $i = 1, 2$, we have

$$G_{t_1}^{(1)} \cap \operatorname{Sing} X \subset G_{t_1}^{(1)} \cap \pi_2^{-1}\mathcal{S}_2 \quad \text{and} \quad G_{t_2}^{(2)} \cap \operatorname{Sing} X \subset G_{t_2}^{(2)} \cap \pi_1^{-1}\mathcal{S}_1$$

for any $t_1 \in \mathcal{S}_1$ and $t_2 \in \mathcal{S}_2$. Hence, $(X, G_t^{(i)})$ is 1-log-terminal for any $t \in \mathcal{S}_i$ (cf. [19, Fact 2.5]).

If (V-4) holds, then $2 \le \#\mathcal{S}_i \le 3$ by $\deg\phi > 1$ and

$$\sum\nolimits_{t\in\mathcal{S}_i} 1/m_t^{(i)} \le (1/2)\#\mathcal{S}_i.$$

Hence, for the rest of Proposition 5.24, it is enough to prove (V-4) for $i = 2$. Let $F = F^{(1)}$ be a general fiber of π_1. Then $F \simeq \mathbb{P}^1$ and $\pi_2|_F : F \to T_2$ is a finite surjective morphism of degree ϕ. By the ramification formula $K_X = \phi^*K_{T_1\times T_2} + R_\phi$ with adjunction $K_F = (K_X + F)|_F \sim K_X|_F$ and by (V-3), we have

$$K_F = (\pi_2|_F)^*K_{T_2} + \sum\nolimits_{t\in\mathcal{S}_2} (m_t^{(2)} - 1)G_t^{(2)}|_F = (\pi_2|_F)^*(K_{T_2} + \sum\nolimits_{t\in\mathcal{S}_2} (1 - 1/m_t^{(2)})t).$$

Consequently, $\pi_2|_F$ is étale over $T_2 \setminus \mathcal{S}_2$ and

$$-2 = (\deg\phi)(-2 + \sum\nolimits_{t\in\mathcal{S}_2} (1 - 1/m_t^{(2)})),$$

which is equivalent to (V-4) for $i = 2$. Thus, we are done. □

Corollary 5.25. *Under Condition* 5.21*, one has*

$$S_f = \pi_1^{-1}S_{h_1} + \pi_2^{-1}S_{h_2}.$$

Proof. By the equality $R_f + f^*R_\phi = R_\phi + \tau^*R_{h_1\times h_2}$, we have

$$\operatorname{Supp} R_f \subset \operatorname{Supp} R_\phi \cup \tau^{-1} \operatorname{Supp} R_{h_1\times h_2}.$$

Thus, $\operatorname{Supp} R_f$ is contained in a union of fibers of π_1 and π_2 by Proposition 5.24. Since $S_f \subset \operatorname{Supp} R_{f^k}$ for some $k > 0$ (cf. [20, Lem. 2.17(4)]), each prime component of S_f is a fiber of π_1 or π_2. Thus, the required equality for S_f holds by [20, Lem. 2.19(2)]. □

Corollary 5.26. *Under Condition* 5.21*, for each $i \in \{1, 2\}$, the following conditions are equivalent*:

(i) $\#\mathcal{S}_i = 2$;

(ii) $m_t^{(i)} = \deg\phi$ *for any* (*resp. some*) $t \in \mathcal{S}_i$;

(iii) $G_t^{(i)}$ *is a section of* π_j *for any* (*resp. some*) $t \in \mathcal{S}_i$*, where* $\{j\} = \{1, 2\} \setminus \{i\}$.

Proof. We have (i) ⇒ (ii) by (V-4) in Proposition 5.24 and by $m_t^{(i)} \mid \deg\phi$ shown in Lemma 5.23. We have (ii) ⇔ (iii) by Lemma 5.23, since $m_t^{(j)}G_{t'}^{(j)} \sim F_{t'}^{(j)}$ for

$t' \in T_j$. It suffices to show: (ii) $\Rightarrow$ (i). Assume that $m_b^{(i)} = \deg \phi$ for some $b \in \mathcal{S}_i$. Then

$$(1/2)(\#\mathcal{S}_i - 1) \geq \sum\nolimits_{t \in \mathcal{S}_i \setminus \{b\}} 1/m_t^{(i)} = \#\mathcal{S}_i - 2 + 1/\deg \phi$$

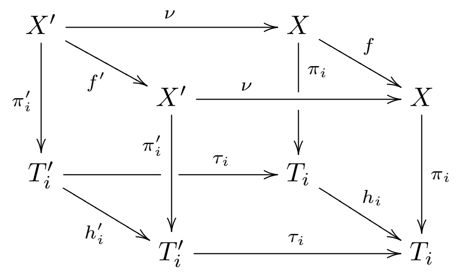
by (V-4) in Proposition 5.24, and it implies that $\#\mathcal{S}_i = 2$, since $\#\mathcal{S}_i \geq 2$. Thus, we are done. □

Proposition 5.27. *Under Condition* 5.21*, assume that* $\#\mathcal{S}_1 < 3$ *or* $\#\mathcal{S}_2 < 3$*. Then* $\#\mathcal{S}_1 = \#\mathcal{S}_2 = 2$*, and* (X, G) *is a toric surface. Moreover, there exist a cyclic cover* $\nu \colon T'_1 \times T'_2 \to X$ *and a cyclic cover* $\tau_i \colon T'_i \simeq \mathbb{P}^1 \to T_i$ *with an endomorphism* $h'_i \colon T'_i \to T'_i$ *for* $i = 1, 2$*, such that*

- ν *is étale in codimension* 1*,*
- $\tau_1 \times \tau_2 = \phi \circ \nu$ *as a morphism* $T'_1 \times T'_2 \to T_1 \times T_2$*,*
- $\deg \tau_1 = \deg \tau_2 = \deg \nu = \deg \phi$*,*
- *the Galois group of* ν *is isomorphic to that of* τ_i *and the* i*-th projection* $\pi'_i \colon T'_1 \times T'_2 \to T'_i$ *is equivariant under the action for each* $i = 1, 2$*,*
- $\nu \circ (h'_1 \times h'_2) = f \circ \nu$*, and*
- $\tau_i \circ h'_i = h_i \circ \tau_i$ *for* $i = 1, 2$*.*

In particular, the cubic diagram

(V-6)



is commutative for $X' := T'_1 \times T'_2$ *and* $f' := h'_1 \times h'_2$*.*

Proof. We may assume that $\#\mathcal{S}_2 = 2$, since $2 \leq \#\mathcal{S}_i \leq 3$ (cf. Proposition 5.24). Let $\tau_2 \colon T'_2 \simeq \mathbb{P}^1 \to T_2$ be the cyclic cover of degree $\deg \phi$ branched at $\mathcal{S}_2$. For the normalization X' of $X \times_{T_2} T'_2$, the induced cyclic cover $\nu \colon X' \to X$ is étale in codimension 1 and the induced $\mathbb{P}^1$-fibration $\pi'_2 \colon X' \to T'_2$ has only reduced fibers, by Corollary 5.26(ii) and [20, Lem. 4.2]. By construction, the Galois group $\mathrm{Gal}(\nu)$ of ν is identified with the Galois group $\mathrm{Gal}(\tau_2)$ of τ_2, and $\pi'_2 \colon X' \to T'_2$ is $\mathrm{Gal}(\nu)$-equivariant.

A smooth fiber $F^{(1)}$ of π_1 lies in X_{reg}, and hence, $\nu^{-1}(F^{(1)})$ is a disjoint union of $\deg \phi$ copies of $\mathbb{P}^1$ which are all sections of π'_2, since $\deg F^{(1)}/T_2 = \deg \phi$ (cf. Lemma 5.23). Hence, the Stein factorization of $\pi_1 \circ \nu \colon X' \to T_1$ consists of a finite cover $\tau_1 \colon T'_1 \to T_1$ of degree ϕ and a fibration $\pi'_1 \colon X' \to T'_1$, where X' is also isomorphic to the normalization of $X \times_{T_1} T'_1$. Moreover, τ_1 is a cyclic cover, since $\nu \colon X' \to X$ is so. Similarly to the case of τ_2, we can identify $\mathrm{Gal}(\nu)$ with the Galois group $\mathrm{Gal}(\tau_1)$ of τ_1, and $\pi'_1 \colon X' \to T'_1$ is $\mathrm{Gal}(\nu)$-equivariant. As a consequence, $(\pi'_1, \pi'_2) \colon X' \to T'_1 \times T'_2$ is an isomorphism over $T_1 \times T_2$ by

$$\deg X'/(T_1 \times T_2) = (\deg \phi)^2 = \deg(T'_1 \times T'_2)/(T_1 \times T_2).$$

In particular, $X' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and π'_i corresponds to the i-th projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ for $i = 1, 2$. By applying [20, Lem. 4.2] to $\pi_1 \colon X \to T_1$ and $\tau_1 \colon T'_1 \to T_1$, we see that τ_1 is étale over $T_1 \setminus \mathcal{S}_1$ and that $\tau_1^*(t) = m_t^{(1)} \tau_1^{-1}(t)$ for any $t \in \mathcal{S}_1$. Then we have $\#\mathcal{S}_1 = 2$ by considering the action of the cyclic group $\mathrm{Gal}(\tau_1)$ on $T'_1 \simeq \mathbb{P}^1$.

We shall show that (X, G) is a toric surface. Now, ϕ induces a finite étale morphism

$$X \setminus G \to (T_1 \setminus \mathcal{S}_1) \times (T_2 \setminus \mathcal{S}_2) \simeq \mathbb{G}_{\mathtt{m}}^2,$$

where $\mathbb{G}_{\mathtt{m}}$ stands for the 1-dimensional algebraic torus. Assume that the toric surface $(T_1 \times T_2, \mathcal{S}_1 \times \mathcal{S}_2)$ is defined by a fan $\triangle$ of the free abelian group $\mathsf{N} = \boldsymbol{\pi}_1(T_1 \times T_2 \setminus (\mathcal{S}_1 \times \mathcal{S}_2)) \simeq \mathbb{Z}^{\oplus 2}$, i.e., $T_1 \times T_2 = \mathbb{T}_{\mathsf{N}}(\triangle)$. Then, by Lemma 3.5, (X, G) is isomorphic over $T_1 \times T_2$ to the toric surface $\mathbb{T}_{\mathsf{N}'}(\triangle)$ for the finite index subgroup $\mathsf{N}' = \pi_1(X \setminus G)$ of N.

Finally, we shall construct the required endomorphism $h'_i \colon T'_i \to T'_i$ for $i = 1, 2$. Since X' is simply connected, by Lemma 3.10 applied to the case where $S = 0$, there is an endomorphism $f' \colon X' \to X'$ such that $\nu \circ f' = f \circ \nu$. By the i-th projection $\pi'_i \colon X' \to T'_i$ and the Stein factorization of $\pi'_i \circ f' \colon X' \to T'_i$, we have an endomorphism $h'_i \colon T'_i \to T'_i$ such that $\pi'_i \circ f' = h'_i \circ \pi'_i$ and $\tau_i \circ h'_i = h_i \circ \tau_i$. As a consequence, $f' = h'_1 \times h'_2$, and we have the cubic commutative diagram (V-6). Thus, we are done. □

Now, we shall finish the proof of Theorem 5.17:

Proof of Theorem 5.17. By Lemma 5.20, we may assume Condition 5.21. For the proof of Theorem 5.17(1), by Proposition 5.27, we may assume that $\#\mathcal{S}_1 = \#\mathcal{S}_2 = 3$. Now,

$$\sum\nolimits_{t \in \mathcal{S}_2} (1 - 1/m_t^{(2)}) = 2 - 2/\deg \phi < 2$$

by (V-4) in Proposition 5.24. Then, as in the proof of Proposition 5.14(3), we have a possible list of the collection $(m_t^{(2)})_{t \in \mathcal{S}_2}$, and there is a finite Galois cover $\tau_2 \colon T'_2 \simeq \mathbb{P}^1 \to T_2$ such that τ_2 is étale over $T_2 \setminus \mathcal{S}_2$ and that $\tau_2^*(t) = m_t^{(2)} \tau_2^{-1}(t)$ for any $t \in \mathcal{S}_2$. For the normalization X' of $X \times_{T_2} T'$, the induced finite morphism $\nu \colon X' \to X$ is étale in codimension 1 and the induced $\mathbb{P}^1$-fibration $\pi'_2 \colon X' \to T'$ has only reduced fibers by [20, Lem. 4.2]. By construction, the Galois group $\mathrm{Gal}(\nu)$ of ν is identified with the Galois group $\mathrm{Gal}(\tau_2)$ of τ_2, and $\pi'_2 \colon X' \to T'$ is $\mathrm{Gal}(\nu)$-equivariant. Since $(X, G_t^{(2)})$ is 1-log-terminal for any $t \in T_2$ (cf. Proposition 5.24), the $\mathbb{P}^1$-fibration π'_2 is smooth by Lemma 2.12. Hence, X' is a $\mathbb{P}^1$-bundle over T'_2 and it is a trivial bundle $\mathbb{P}^1 \times T'_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, since it has another ruling induced by $\pi_1 \circ \nu \colon X' \to X \to T_1$. Then the action of $\mathrm{Gal}(\nu)$ is diagonal by [21, Lem. 2.3]. There is an endomorphism $f' \colon X' \to X'$ satisfying $\nu \circ f' = f \circ \nu$ by Lemma 3.10, since X' is simply connected. This shows Theorem 5.17(1).

For the proof of Theorem 5.17(2), we may assume that $\boldsymbol{n}(S_f) = 2$ by the same reason as in Remark 5.8. Then $S_f = C_1 + C_2$ for two prime components C_1 and C_2, where $C_i = G_{t_i}^{(i)}$ for a point $t_i \in S_{h_i} \subset T_i$ for $i = 1, 2$, by Corollary 5.25. Since $K_X + S_f$ is not nef, $(K_X + S_f)F^{(i)} < 0$ for $i = 1$ or 2, for a general fiber $F^{(i)}$ of π_i. Hence, we may assume that C_1 is a section of π_2. Then $\#\mathcal{S}_1 = 2$ by Corollary 5.26.

Moreover, $\#\mathcal{S}_2 = 2$, C_2 is also a section of π_1, and (X, G) is a toric surface, by Proposition 5.27, where $G \supset S_f = C_1 + C_2$. This shows Theorem 5.17(2), and we are done. □

Finally in Section 5, we finish the proof of Theorem 5.6.

Proof of Theorem 5.6. It has been proved in Section 5.2 in the case where X contains a negative curve. When X contains no negative curve, Theorem 5.6(1) holds by Theorem 5.17(1). Thus, we are done. □

6. Proofs of Theorems in the introduction

Finally, we shall prove Theorems 1.1, 1.2, 1.3, and 1.4 in the introduction.

Proof of Theorem 1.1. We are given a normal projective surface X with a non-isomorphic surjective endomorphism f such that $K_X + S_f$ is not pseudo-effective. If X has a non-quotient singularity, then Theorem 1.1(4) holds with a lift of f^k to V by [21, Thm. 1.2]. Thus, we may assume that X has only quotient singularities.

If $\boldsymbol{\rho}(X) = 1$, then X is a log del Pezzo surface, since K_X is not pseudo-effective; thus, Theorem 1.1(1) holds. If $\boldsymbol{\rho}(X) = 2$ and $-K_X$ is not big, then (2) or (3) of Theorem 1.1 holds with a lift of f^k by Theorem 5.1.

If $\boldsymbol{\rho}(X) = 2$ and $-K_X$ is big, then one of (3), (5), (6), and (7) of Theorem 1.1 holds, by Theorem 5.6, where the existence of a lift of f^k to V is also proved in cases (3) and (5). If $\boldsymbol{\rho}(X) \geq 3$, then either (6) or (7) of Theorem 1.1 holds by Theorem 4.23. Thus, we are done. □

Before going to the proof of Theorem 1.2, we shall show:

Lemma 6.1. *Let X be a normal projective surface having a finite cover $\nu \colon V \to X$ étale in codimension 1 from a projective cone V over an elliptic curve. Then there exists a non-isomorphic surjective endomorphism f of X such that $K_X + S_f$ is not pseudo-effective.*

Proof. We may assume ν to be a Galois cover by an argument in the proof of [21, Lem. 3.3(5)]. Let G be the Galois group. It suffices to construct a G-equivariant non-isomorphic surjective endomorphism $g \colon V \to V$ such that $S_g = 0$. In fact, g induces a non-isomorphic surjective endomorphism f of X such that $\nu \circ g = f \circ \nu$, and we have $S_f = 0$ by [20, Lem. 2.19]; as a consequence, $K_X + S_f = K_X$ is not pseudo-effective, since $K_V = \nu^* K_X$ is so.

Let $\mu \colon W \to V$ be the minimal resolution of singularities. Then we have a $\mathbb{P}^1$-bundle structure $\pi \colon W \to T$ over an elliptic curve T in which the μ-exceptional locus Θ is a negative section of π. Moreover, the action of G lifts to W and T so that morphisms μ and π are G-equivariant and that the divisor Θ is G-invariant. By [21, Lem. 2.14], there is another G-invariant section Θ' of π such that $\Theta \cap \Theta' = \emptyset$. Then we can find a G-equivariant non-isomorphic surjective endomorphism $g_W \colon W \to W$ such that $S_{g_W} = \Theta$ by [21, Cor. 2.27]. It induces a G-equivariant non-isomorphic surjective endomorphism g of V such that $\mu \circ g_W = g \circ \mu$, and we have $S_g = \mu_* S_{g_W} = 0$ by [20, Lem. 3.15(3)]. Thus, we are done. □

Proof of Theorem 1.2. The "only if" part follows from Theorem 1.1 and Lemma 4.3. It is enough to construct a non-isomorphic surjective endomorphism f of X such that $K_X + S_f$ is not pseudo-effective assuming one of conditions (1), (2), and (3) of Theorem 1.2.

If Theorem 1.2(1) holds, i.e., one of (2)–(5) in Theorem 1.1 holds, then we have an expected endomorphism f by Theorem 5.1 and Lemma 6.1. If Theorem 1.2(2) holds, then, for the prime component Γ of D which is not a negative curve, there is a non-isomorphic surjective endomorphism f of X such that $S_f = D - \Gamma$ by [22, Thm. 6.1], where $K_X + S_f \sim -\Gamma$ is not pseudo-effective. If Theorem 1.2(3) holds, then there is a non-isomorphic surjective endomorphism f of X such that $S_f = D - \Gamma$ for the end component Γ by [22, Thm. 6.2], where $K_X + S_f \approx -\Gamma$ is not pseudo-effective. Thus, we are done. □

Proof of Theorem 1.3. If there is a finite Galois cover $V \to X$ satisfying one of conditions (1)–(7) of Theorem 1.3 except (6), then X admits a non-isomorphic surjective endomorphism by [21, Thm. 1.1]. Even in the exceptional case (6), we have a non-isomorphic surjective endomorphism of V equivariant under the action of the Galois group of $V \to X$ by [21, Lem. 2.3 and Cor. 2.18], and it induces a non-isomorphic surjective endomorphism of X. For the rest, by [20, Thm. A] and [21, Thm. 1.1], it is enough to consider a normal projective rational surface X admitting a non-isomorphic surjective endomorphism $f \colon X \to X$ such that X has only quotient singularities, $-K_X$ is big, $\boldsymbol{\rho}(X) \geq 2$, and that $K_X + S_f$ is not pseudo-effective. Then one of conditions (2), (3), (5), (6), and (7) of Theorem 1.1 is satisfied. Hence, one of conditions (3), (4), (6), and (7) of Theorem 1.3 is satisfied. Thus, we are done. □

Proof of Theorem 1.4. By [20, Thms. D and 3.25], one of the following holds:

- (i) There is a finite Galois cover $A \to X$ étale in codimension 1 from an abelian surface A with an endomorphism $f_A \colon A \to A$ as a lift of A.
- (ii) There is a finite Galois cover $C \times T \to X$ étale in codimension 1 for an elliptic curve C and a non-singular projective curve T of genus ≥ 2, where $\lambda_f = \deg f$ and some power f^k lifts to an endomorphism of $C \times T$.
- (iii) There is a $\mathbb{P}^1$-fibration $\pi \colon X \to T$ to a non-singular projective curve T with an endomorphism $h \colon T \to T$ such that $\pi \circ f = h \circ \pi$, X has no negative curves, $\boldsymbol{\rho}(X) = 2$, and $\lambda_f = \max\{\deg h, \deg f / \deg h\} \in \mathbb{Z}$.

In the case (i) (resp. (ii)), Theorem 1.4 holds for V in (1) (resp. (2)) in the list. Thus, we may assume (iii). In particular, K_X is not pseudo-effective and λ_f is an integer dividing into $\deg f$. Note that if the genus of T in (iii) is greater than 1, then $\deg h = 1$ and $\lambda_f = \deg f$. We can consider the following four conditions:

- (a) $K_X + S_f$ is not pseudo-effective, and either X is irrational or $-K_X$ is not big;
- (b) $K_X + S_f$ is not pseudo-effective, X is rational, and $-K_X$ is big;
- (c) $K_X + S_f$ is pseudo-effective and $K_X + S_f \not\sim_{\mathbb{Q}} 0$;
- (d) $K_X + S_f \sim_{\mathbb{Q}} 0$.

First, assume (a). Then, by Theorem 5.1 and [21, Lem. 4.11], there is a finite Galois cover $\mathbb{P}^1 \times T' \to X$ étale in codimension 1 for a non-singular projective curve T' of genus > 0 with an endomorphism g of $\mathbb{P}^1 \times T'$ as a lift of a power f^k. Here, $\lambda_g = \lambda_{f^k} = (\lambda_f)^k$ and $\deg g = \deg f^k = (\deg f)^k$ by [20, Cor. 3.5], and the condition (iii) above holds also for $g\colon \mathbb{P}^1 \times T' \to \mathbb{P}^1 \times T'$. In particular, in this case, Theorem 1.4 holds for V in (3) or (4) in the list. Here, if (3) holds, then $\lambda_f = \deg f$ by (iii) for g, since $\lambda_g = \deg g$.

We can prove the assertion in cases (b) and (c) by arguments similar to the above: Assume (b). Then, by Theorem 5.17, there is a finite Galois cover $\mathbb{P}^1 \times \mathbb{P}^1 \to X$ étale in codimension 1 and some power f^k lifts to an endomorphism g of $\mathbb{P}^1 \times \mathbb{P}^1$, which satisfies (iii), since we have $\lambda_g = (\lambda_f)^k > (\delta_f)^k = \delta_g$ by [20, Cor. 3.5]. Thus, in this case, Theorem 1.4 holds for V in (5). Assume (c). Then, since K_X is not pseudo-effective, by [20, Thm. A], there is a finite Galois cover $\mathbb{P}^1 \times T \to X$ étale in codimension 1 for a non-singular projective curve T of genus ≥ 2 and some power f^k lifts to an endomorphism g of $\mathbb{P}^1 \times T$, which satisfies (iii) by the same argument as above. Thus, in this case, Theorem 1.4 holds for V in (3), where $\lambda_f = \deg f$ holds by (iii) for g, since $\lambda_g = \deg g$.

Finally, assume (d). Then, by [20, Thm. A], there is a finite Galois cover $\nu\colon V \to X$ étale in codimension 1, f lifts to an endomorphism $g\colon V \to V$, and one of the following conditions is satisfied:

(d-1) V is a $\mathbb{P}^1$-bundle over an elliptic curve T' and $S_g = \nu^* S_f$ is a disjoint union of two sections of $V \to T'$;

(d-2) V is a toric surface with $S_g = \nu^* S_f$ as the boundary divisor.

Since $\lambda_f = \lambda_g$ and $\deg f = \deg g$ (cf. [20, Cor. 3.5]), $g\colon V \to V$ satisfies the condition (iii) above. In particular, $\boldsymbol{\rho}(V) = 2$ and V contains no negative curves.

If (d-1) holds, then $V \simeq \mathbb{P}_{T'}(\mathcal{O}_{T'} \oplus \mathcal{L})$ for an invertible sheaf $\mathcal{L}$ on T' of degree 0; thus, Theorem 1.4 holds for V in (4). If (d-2) holds, then, by Theorem 5.17(1), there is a finite cover $\nu'\colon V' \to V$ étale in codimension 1 from $V' = \mathbb{P}^1 \times \mathbb{P}^1$ and some power g^k lifts to an endomorphism of V'. Here, the composite $\nu \circ \nu'\colon V' \to X$ is Galois, since it is étale in codimension 1 and since V' is non-singular and simply connected. Hence, Theorem 1.4 holds for V in (5) by replacing V with V'. Thus, we have completed the proof of Theorem 1.4. □

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