RIMS-1946

The reduced Dijkgraaf–Witten invariant of double twist knots in the Bloch group of $\mathbb{F}_p$

By

Hiroaki KARUO

March 2021

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

THE REDUCED DIJKGRAAF–WITTEN INVARIANT OF DOUBLE TWIST KNOTS IN THE BLOCH GROUP OF $\mathbb{F}_p$

HIROAKI KARUO

Abstract. In 2004, W.D. Neumann showed that the complex hyperbolic volume of a hyperbolic 3-manifold M can be obtained as the image of the Dijkgraaf–Witten invariant of M by a certain 3-cocycle. After that, C.K. Zickert gave an analogue of the Neumann's work for free fields containing finite fields. The author formulated a simple method to calculate a weaker version of the Zickert's analogue, called the reduced Dijkgraaf–Witten invariant, for finite fields and gave a formula for twist knot complements and $\mathbb{F}_p$ in his previous work. In this paper, we show concretely how to calculate the reduced Dijkgraaf–Witten invariants of double twist knot complements and $\mathbb{F}_p$, and give a formula of them for $p = 7$.

1. Introduction

1.1. **Background.** In 1990, Dijkgraaf and Witten [2] introduced a topological invariant for oriented closed 3-manifolds, called the Dijkgaaf–Witten invariant, from Chern–Simons theory. Let M be an oriented closed 3-manifold, G be a finite group, α be a 3-cocycle of G. For a representation $\rho\colon \pi_1(M) \to G$, the DW invariant is given as $(\rho^*\alpha)[M]$, where $[M]$ is the fundamental class of M, and $\rho^*\alpha$ is the pull back of α by ρ. Later, Wakui [17] reconstructed the DW invariant of M combinatorially from a 3-cocycle of G using a triangulation of M. Although there are some calculations of the DW invariants of Seifert 3-manifolds [4] and hyperbolic knot complements [6] the DW invariant is not calculated easily in general since a concrete presentation of a non-trivial 3-cocycle is often complicated. In the following, we also regard $\rho_*[M] \in H_3(G)$ as the DW invariant since the DW invariant and $\rho_*[M] \in H_3(G)$ are equivalent in the sense of $(\rho^*\alpha)[M] = \alpha(\rho_*[M])$, where $\rho_*\colon H_3(M) \to H_3(G)$ is the map induced by ρ.

In 2004, Neumann [9] showed that if M is hyperbolic, then the complex hyperbolic volume of M can be obtained as the image of $\rho_*[M]$ by a particular 3-cocycle of $\mathrm{PSL}_2\mathbb{C}$ to the extended Bloch group of $\mathbb{C}$, where the 3-cocycle is a map similar to the Bloch–Wigner map $H_3(\mathrm{SL}_2F;\mathbb{Z}) \to \mathcal{B}(\mathbb{C})$. Later, Zickert [21] gave the extended Bloch group for free fields (including finite fields) and an analogue of the Neumann's work. However, there is no known explicit calculations of the analogue. In 2013, Hutchinson [3] gave a concrete construction of the Bloch–Wigner map $H_3(\mathrm{SL}_2F;\mathbb{Z}) \to \mathcal{B}(F)$ for any finite field F. After that, the author [5] reformulated a weaker version of Zickert's analogue, called the reduced DW invariant, for finite fields using the Bloch–Wigner map given by Hutchinson. Here, the reduced DW invariant of an oriented closed 3-manifold and $\mathbb{F}_p$ is the sum of moduli of tetrahedra of a triangulation of the 3-manifold. Note that the reduced DW invariant can be defined for oriented cusped 3-manifolds, especially knot complements. The reduced DW invariant of a knot complement and $\mathbb{F}_p$ is the sum of moduli of ideal tetrahedra of a topological ideal triangulation of the knot complement. The author also gave a simple method to calculate the reduced DW invariant of a knot complement and $\mathbb{F}_p$ using solutions of hyperbolicity equations of a 1-tangle diagram of the knot in $\mathbb{F}_p$. In particular, he gave a formula of the reduced DW invariants of twist knot complements and $\mathbb{F}_p$ of order $p = 7, 11, 13$.

Date: Keywords: the Dijkgraaf–Witten invariant, double twist knots, Bloch groups, finite fields
Mathematics Subject Classification 2020: 57K10, 57K31, 57K32.

1.2. **A simple method to calculate the reduced DW invariants of double twist knot complements.** In the paper, we give a simple method to calculate the reduced DW invariants of the complements of the (m, n)-double twist knots ($m, n \geq 4$, m is even) and the finite field $\mathbb{F}_p$ of order p. This method works for any prime number $p > 4$. In particular, we give a formula of the reduced DW invariants of the complements of the (m, n)-double twist knots and $\mathbb{F}_7$ (Theorem 3.1).

As I mentioned, in general, calculation of the DW invariant is complecated. However, calculation of the reduced DW invariant avoids such complicated calculation and it is an advantage that the reduced DW invariant is obtained more easily using $\mathbb{P}^1(\mathbb{F}_p)$-labelings of ideal vertices of an ideal triangulation. In particular, our method allows us to calculate the reduced DW invariants of the (m, n)-double twist knot and $\mathbb{F}_p$ as the sum of the moduli obtained from the solutions of polynomials, which describe the hyperbolicity equations, in $\mathbb{F}_p$. In the case of $\mathbb{C}$, it is known that the representation obtained from one of the solutions of hyperbolicity equations gives the complete hyperbolic structure of the knot complement.

Let us explain an outline of the proof of the theorem. When an ideal triangulation of a knot complement with uncollapsed moduli is given, it is known that the reduced DW invariant and the sum of moduli of ideal tetrahedra are equal (Proposition 4.1). Although we have an ideal triangulation of the complement of the (m, n)-double twist knot with a $\mathbb{P}^1(\mathbb{F}_p)$-labeling from a method of Yokota [19], there are some ideal tetrahedra whose moduli can not be defined, i.e. the labels of $\mathbb{P}^1(\mathbb{F}_p)$ to the four ideal vertices duplicate. It is the biggest difference between the cases of $\mathbb{C}$ and $\mathbb{F}_p$. To avoid this problem, we replace the ideal triangulation with an ideal triangulation with uncollapsed moduli appropriately. From the uncollapsed moduli, we can obtain a $\mathrm{PGL}_2\mathbb{F}_p$-representation similarly to [12]. We take a lift of the representation, an $\mathrm{SL}_2\mathbb{F}_p$-representation, and show that it is conjugate to the parabolic representation obtained from a solution of hyperbolicity equations in $\mathbb{F}_p$. This implies that the sum of moduli in the replaced ideal triangulation is equal to the reduced DW invariant, i.e. we can calculate the reduced DW invariant using the moduli combinatorially. In particular, the reduced DW invariant is equal to the sum of uncollapsed moduli in the ideal triangulation obtained by the Yokota's method (Proposition 8.1).

In [5], a formula of the reduced DW invariants of the n-twist knot complements have periodicity with respect to n for each p. In our case, the reduced DW invariants the (m, n)-double twist knots have more complicated periodicity with respect to m and n for each p. Moreover, the reduced DW invariant can recover the number of conjugacy classes of parabolic $\mathrm{SL}_2\mathbb{F}_p$-representations of the fundamental group of the (m, n)-double twist knots. In the sense, the reduced DW invariant is a generalization of the number of conjugacy classes of parabolic $\mathrm{SL}_2\mathbb{F}_p$-representations.

1.3. **Organization.** In Section 2, we review some notations, especially the reduced DW invariant. In Section 3, as the main result, we give a formula of the reduced DW invariants of the complements of the (m, n)-double twist knots and $\mathbb{F}_7$. In Section 4, we review a useful fact that, under an assumption, the reduced DW invariant of a knot complement is obtained as the sum of moduli of an ideal triangulation, proved in [5]. In Section 5, we give polynomials whose zeros correspond to the conjugacy classes of $\mathrm{SL}_2\mathbb{F}_p$-representations of the fundamental groups of the (m, n)-double twist knots. In Section 6, we describe hyperbolicity equations using the polynomials introduced in Section 5. In Section 8, we show that the reduced DW invariant is equal to the sum of moduli in the ideal triangulation by Yokota's method. In Section 9.1, we give a proof of the main theorem using sequences obtained from the polynomials introduced in Section 5.

1.4. **Acknowledgement.** The author would like to thank Kevin Hutchinson for helpful comments. The author also would like to thank Naoki Kimura for sharing his recent work [6]. The author was supported by JSPS KAKENHI Grant Number JP20J10108.

2. Notations, preliminaries

We fix notations in Section 2.1 and review the Bloch group of $\mathcal{B}(\mathbb{F}_p)$ in Section 2.2, parabolic representations and related facts in Section 2.3, the reduced DW invariant in Section 2.4.

2.1. **Notations, assumptions.**

We denote by $\mathbb{F}_p$, $\mathbb{F}_p^\times$ respectively the finite field of order p, and the set of units of $\mathbb{F}_p$.

Let $\mathbb{P}^1(\mathbb{F}_p)$ denote the 1-dimensional projective space of $\mathbb{F}_p$. Note that $\mathbb{P}^1(\mathbb{F}_p) = \mathbb{F}_p \cup \{\infty\}$.

Throughout the paper, we assume that the 3-dimensional sphere S^3 is oriented and $p > 4$ is an odd prime number unless otherwise noted.

2.2. **The Bloch group of $\mathbb{F}_q$.**

In this subsection, we assume $q \geq 4$ and review the Bloch group of $\mathbb{F}_q$. The *pre-Bloch group* $\mathcal{P}(\mathbb{F}_q)$ of $\mathbb{F}_q$ is the free $\mathbb{Z}$-module $\mathbb{Z}(\mathbb{F}_q^\times \setminus \{1\})$ subject to the following relation:

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0 \quad (x \neq y \in \mathbb{F}_q^\times \setminus \{1\}). \tag{1}$$

The *Bloch group* $\mathcal{B}(\mathbb{F}_q)$ is the kernel of the map

$$\mathcal{P}(\mathbb{F}_q) \to \mathbb{F}_q^\times \tilde{\wedge} \mathbb{F}_q^\times \cong \mathbb{Z}/2\mathbb{Z}, \quad [z] \mapsto z \tilde{\wedge} (1-z).$$

Here, $\mathbb{F}_q^\times \tilde{\wedge} \mathbb{F}_q^\times = \mathbb{F}_q^\times \otimes_{\mathbb{Z}} \mathbb{F}_q^\times / \langle x \otimes y + y \otimes x | x, y \in \mathbb{F}_q^\times \rangle$.

Example 2.1 (Hutchinson [3])**.** It is known that

$$\mathcal{B}(\mathbb{F}_q) \cong \mathbb{Z}/\tfrac{q+1}{2}\mathbb{Z}$$

for any finite field $\mathbb{F}_q$ ($q \geq 4$) of odd characteristic by Hutchinson; see also [18].

Let $\check{\mathcal{P}}(\mathbb{F}_q)$ be the pre-Bloch group $\mathcal{P}(\mathbb{F}_q)$ subject to the following relation, see, e.g. Lemma 5.4 in [18].

$$[x] = \left[1 - \frac{1}{x}\right] = \left[\frac{1}{1-x}\right] = -\left[\frac{1}{x}\right] = -\left[\frac{x}{x-1}\right] = -[1-x] \quad (x \in \mathbb{F}_p \setminus \{0,1\}) \tag{2}$$

Let $\check{\mathcal{B}}(\mathbb{F}_q)$ be the image of the Bloch group $\mathcal{B}(\mathbb{F}_q) \subset \mathcal{P}(\mathbb{F}_q)$ by the projection $\mathcal{P}(\mathbb{F}_q) \to \check{\mathcal{P}}(\mathbb{F}_q)$ induced by the relation (2).

Example 2.2. The following isomorphisms are given in [5] concretely; see also [11]:

$$\check{\mathcal{P}}(\mathbb{F}_7) \cong \mathbb{Z}/4\mathbb{Z}, \ \check{\mathcal{B}}(\mathbb{F}_7) \cong \mathbb{Z}/2\mathbb{Z}.$$

2.3. **Parabolic representations.**

We review some facts about parabolic representations and Dehn filling.

For $A,\ B \in \mathrm{SL}_2\mathbb{F}_p$, A and B are *conjugate* if there exists $P \in \mathrm{GL}_2\mathbb{F}_p$ such that

$$P^{-1}AP = B.$$

Let $K \subset S^3$ be a knot and $\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{SL}_2\mathbb{F}_p$ be a non-abelian representation. The representation ρ is *parabolic* if the image of each meridian is conjugate to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, i.e. the image of each meridian can be presented as $Q^{-1}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} Q$ for some $Q \in \mathrm{GL}_2\mathbb{F}_p$.

Let $K \subset S^3$ be a knot and $N(K)$ be a tubular neighborhood of K in S^3. We consider $S^3 \setminus N(K)$ and take an essential simple closed curve c on $\partial(S^3 \setminus N(K))$. Then, one can obtain a closed 3-manifold by Dehn filling along c; see [5].

Note that $\partial(S^3 \setminus N(K))$ is homeomorphic to a torus $S^1 \times S^1$. We fix a meridian ($S^1 \times \{\text{a point}\}$) and a longitude ($\{\text{a point}\} \times \mathrm{S}^1$) as a basis of $\pi_1(\partial(S^3 \setminus N(K)))$. Then, $\pi_1(\partial(S^3 \setminus N(K))) \cong \mathbb{Z} \oplus \mathbb{Z}$. Each essential simple closed curve C on $\partial(S^3 \setminus N(K))$ is presented by $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ using the meridian and the longitude, where a and b are coprime. We call C the *(a, b)-curve*. Let $M_{(a,b)}(K)$ denote the closed 3-manifold obtained from $S^3 \setminus N(K)$ by Dehn filling along the (a, b)-curve; see, e.g. [5].

Consider a parabolic representation $\rho\colon \pi_1(S^3 \setminus N(K)) \to \mathrm{SL}_2\mathbb{F}_p$. Since $\pi_1(S^3 \setminus K)$ is an infinite group and $\mathrm{SL}_2\mathbb{F}_p$ is a finite group, the kernel of ρ is non-trivial. One can take a non-trivial element from the kernel and regard it as an (a, b)-curve on $\partial(S^3 \setminus N(K))$ as the above, where a and b are coprime. For the closed 3-manifold $M_{a,b}(K)$ obtained by Dehn filling along the (a, b)-curve, ρ induces a representation $\pi_1(M_{a,b}(K)) \to \mathrm{SL}_2\mathbb{F}_p$, also denoted by ρ. Then, the following diagram commutes; see [5] for details.

$$\begin{array}{ccc} \pi_1(S^3 \setminus N(K)) & \longrightarrow & \mathrm{SL}_2\mathbb{F}_p \\ \downarrow & \nearrow & \\ \pi_1(M_{(a,b)}(K)) & & \end{array}$$

2.4. **The reduced Dijkgraaf–Witten invariant.** In the subsection, we review the reduced Dijkgraaf–Witten invariant of a knot complement and $\mathbb{F}_p$ introduced in [5].

For a parabolic representation $\rho\colon \pi_1(S^3 \setminus N(K)) \to \mathrm{SL}_2\mathbb{F}_p$, consider the induced representation $\pi_1(M_{a,b}(K)) \to \mathrm{SL}_2\mathbb{F}_p$ by ρ as in Section 2.3, also denoted by ρ. Consider the induced homomoprhism $\rho_*\colon H_3(M_{a,b}(K)) \to H_3(B\mathrm{SL}_2\mathbb{F}_p) = H_3(\mathrm{SL}_2\mathbb{F}_p)$ from ρ. Then, consider the composite of ρ_* and the Bloch–Wigner map $H_3(\mathrm{SL}_2\mathbb{F}_p; \mathbb{Z}) \to \check{\mathcal{B}}(\mathbb{F}_p)$ given by Hutchinson [3] and the restriction $\mathcal{B}(\mathbb{F}_p) \to \check{\mathcal{B}}(\mathbb{F}_p)$ of the projection $\mathcal{P}(\mathbb{F}_p) \to \check{\mathcal{P}}(\mathbb{F}_p)$ to $\mathcal{B}(\mathbb{F}_p)$. It is known that the image of the fundamental class $[M_{a,b}(K)]$ by the composite does not depend on the choice of the (a, b)-curve even though $M_{a,b}(K)$ depends on the choice of the (a, b)-curve in general; see Lemma F.1 in [5]. The image is called the *reduced Dijkgraaf–Witten invariant* of (K, ρ), denoted by $\widehat{\mathrm{DW}}(K, \rho)$. Furthermore, it is known that if parabolic representations $\rho, \rho'\colon \pi_1(S^3 \setminus K) \to \mathrm{SL}_2\mathbb{F}_p$ are conjugate, then $\widehat{\mathrm{DW}}(M_{a,b}(K), \rho) = \widehat{\mathrm{DW}}(M_{a,b}(K), \rho')$; see [5] for more details.

The reduced DW invariant $\widehat{\mathrm{DW}}(K, \mathbb{F}_p)$ of K and $\mathbb{F}_p$ is the sum of $\widehat{\mathrm{DW}}(K, \rho)$ over all conjugacy classes of parabolic representations in $\mathbb{Z}[\check{\mathcal{B}}(\mathbb{F}_p)]$.

3. Main result

3.1. **Double twist links.** Recall that, for $m, n \geq 1$, the *(m, n)-double twist link* $\mathcal{T}_{m,n}$ is the link in S^3 presented as in Figure 1. Note that $\mathcal{T}_{m,n}$ is a knot if and only if mn is even. In the following, we suppose that m is even and call $\mathcal{T}_{m,n}$ the (m, n)-double twist knot. It is a hyperbolic knot if $m, n \geq 2$.

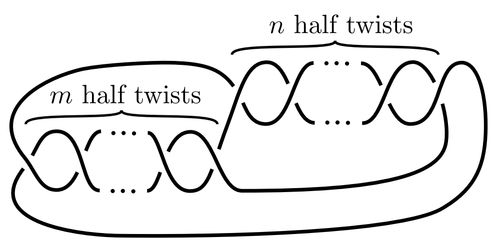

Figure 1. The (m, n)-double twist link

3.2. **Main result.** We will write $(x,y) \equiv (x',y') \bmod (p,q)$ if $x \equiv x' \bmod p$ and $y \equiv y' \bmod q$.

Theorem 3.1. *Suppose $m = 2k$, $k,n \geq 2$ and $(m,n) \not\equiv (6,1),(8,5) \bmod (14,6)$. By identifying $\check{\mathcal{B}}(\mathbb{F}_7) \cong \mathbb{Z}/2\mathbb{Z}$ and the multiplicative group $\langle t \,|\, t^2 = 1\rangle$ naturally, the reduced Dijkgraaf–Witten invariant of the (m,n)-double twist knot and $\mathbb{F}_7$ is given as follows:*

$$\widehat{\mathrm{DW}}(\mathcal{T}_{m,n},\mathbb{F}_7) = \sum_{r_1} \widehat{\mathrm{DW}}(\mathcal{T}_{m,n},\rho_{r_1}) = A_{m,n}+B_{m,n}+C_{m,n}+D_{m,n}+E_{m,n} \in \mathbb{Z}[\check{\mathcal{B}}(\mathbb{F}_7)] = \mathbb{Z}[\langle t \,|\, t^2=1\rangle],$$

where

$$A_{m,n} = \begin{cases} t^{(n+5)/8} & \text{if } m \equiv 12 \pmod{14} \text{ and } n \equiv 3 \pmod 8, \\ t^{(n+3)/8} & \text{if } m \equiv 4 \pmod{14} \text{ and } n \equiv 5 \pmod 8, \\ t^{(n-5)/8} & \text{if } m \equiv 2 \pmod{14} \text{ and } n \equiv 5 \pmod 8, \\ t^{(n-3)/8} & \text{if } m \equiv 10 \pmod{14} \text{ and } n \equiv 3 \pmod 8, \\ 0 & \text{otherwise}, \end{cases}$$

$$B_{m,n} = \begin{cases} t^{(m+n)/8} & \text{if } m \equiv 2 \pmod 8 \text{ and } n \equiv 6 \pmod 8, \\ t^{(m+n)/8} & \text{if } m \equiv 6 \pmod 8 \text{ and } n \equiv 2 \pmod 8, \\ 0 & \text{otherwise}, \end{cases}$$

$$C_{m,n} = \begin{cases} t^{(n+10)/8} & \text{if } m \equiv 4 \pmod 6 \text{ and } n \equiv 6 \pmod 8, \\ t^{(n+6)/8} & \text{if } m \equiv 2 \pmod 6 \text{ and } n \equiv 2 \pmod 8, \\ t^{(n-1)/8} & \text{if } m \equiv 2 \pmod 6 \text{ and } n \equiv 1 \pmod 8, \\ t^{(n+1)/8} & \text{if } m \equiv 4 \pmod 6 \text{ and } n \equiv 7 \pmod 8, \\ 0 & \text{otherwise}, \end{cases}$$

$$D_{m,n} = \begin{cases} t^{(m+14)/8} & \text{if } m \equiv 2 \pmod 8 \text{ and } n \equiv 2 \pmod 6, \\ t^{(m+10)/8} & \text{if } m \equiv 6 \pmod 8 \text{ and } n \equiv 4 \pmod 6, \\ 0 & \text{otherwise}, \end{cases}$$

$$E_{m,n} = \begin{cases} t & \text{if } m \equiv 2 \pmod 4 \text{ and } n \equiv 3 \pmod 6, \\ 0 & \text{otherwise}. \end{cases}$$

We will mention an equivalent condition of the condition $(m,n) \not\equiv (6,1),(8,5) \bmod (14,6)$ in Section 9.1.

Remark 3.2. If we substitute 1 for t, the reduced DW invariant of $\mathcal{T}_{m,n}$ and $\mathbb{F}_7$ turns to be the number of the conjugacy classes of parabolic representations $\pi_1(S^3 \setminus \mathcal{T}_{m,n}) \to \mathrm{SL}_2\mathbb{F}_7$.

4. Modulus and the reduced DW invariant

4.1. **Modulus and the reduced DW invariant.** In the subsection, we review a modulus of a labeled tetrahedron and the useful proposition that the reduced DW invariant can be obtained from an ideal triangulation of a knot complement and a $\mathbb{P}^1(\mathbb{F}_p)$-labeling of ideal vertices (Proposition 4.1). It is known a fact which is similar to Proposition 4.1 in the case of $\mathbb{C}$ and a hyperbolic 3-manifold M by Neumann and Yang [10].

We review a modulus of an oriented ideal tetrahedron with a $\mathbb{P}^1(\mathbb{F}_p)$-labeling following W.Thurston [16], see also [5]. Although W.Thurston [16] mentioned only the case of $\mathbb{C}$, the case obtained by replacing $\mathbb{C}$ with $\mathbb{F}_p$ also holds. First, we review a $\mathrm{PGL}_2\mathbb{F}_p$-action on $\mathbb{P}^1(\mathbb{F}_p)$. For $v \in \mathbb{F}_p \cup \{\infty\} = \mathbb{P}^1(\mathbb{F}_p)$ and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PGL}_2\mathbb{F}_p$, we define

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} v = \frac{\alpha v + \beta}{\gamma v + \delta}.$$

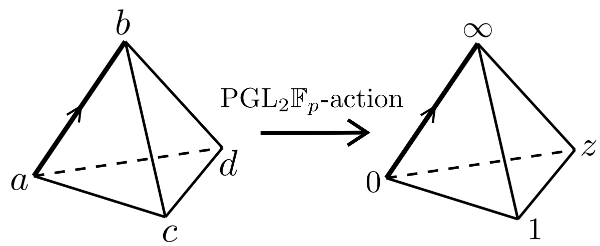

FIGURE 2. A $\mathrm{PGL}_2\mathbb{F}_p$-action

We consider a $\mathbb{P}^1(\mathbb{F}_p)$-labeled oriented ideal tetrahedron, i.e. an oriented ideal tetrahedron whose ideal vertices are labeled by distinct elements of $\mathbb{P}^1(\mathbb{F}_p) = \mathbb{F}_p \cup \{\infty\}$. We fix an edge of the ideal tetrahedron. For convenience, we also fix an orientation of the edge. For example, in Figure 2, consider the left ideal tetrahedron whose ideal vertices are labeled by $a, b, c, d \in \mathbb{F}_p \cup \{\infty\} = \mathbb{P}^1(\mathbb{F}_p)$ and the edge connecting a and b. We set $z = \dfrac{(a-d)(b-c)}{(a-c)(b-d)} \in \mathbb{F}_p \setminus \{0, 1\}$ and call it the *modulus* of the $\mathbb{P}^1(\mathbb{F}_p)$-labeled oriented ideal tetrahedron with respect to the edge. One can show that, by concrete calculation, the equivalent classes of labeled tetrahedra by the action of $\mathrm{PGL}_2\mathbb{F}_p$ are parametrized by z. Furthermore, one can verify that z does not depend on the choice of the orientation of the edge and the moduli with respect the opposite edges are the same. Hence, there are three possibilities of moduli of an edge for the above z: z, $\dfrac{z}{1-z}$ and $1 - \dfrac{1}{z}$. Moreover, for the ideal tetrahedron with modulus z, the modulus of the tetrahedron with reversed orientation with respect to the same edge is $\dfrac{1}{z}$.

For convenience, if we cannot define the modulus, i.e. any two of a, b, c, d duplicate, then we say that the modulus collapses.

In [5], the following useful proposition is proved.

Proposition 4.1 (Proposition 4.8 in [5])**.** *Let S^3 be oriented and K be a knot in S^3. We fix an ideal triangulation of $S^3 \setminus K$ with moduli. We assume that the moduli ($\in \mathbb{F}_p \setminus \{0,1\}$) of the ideal tetrahedra satisfies hyperbolicity equations and that the representation $\pi_1(S^3 \setminus K) \to \mathrm{GL}_2\mathbb{F}_p$ obtained from the moduli is lifted to a parabolic $\mathrm{SL}_2\mathbb{F}_p$-representation ρ. Then, we obtain*

$$\widehat{\mathrm{DW}}(K, \rho) = \sum_{\Delta} \varepsilon_\Delta [\textit{modulus of an ideal tetrahedron } \Delta \textit{ of } S^3 \setminus K] \in \check{\mathcal{P}}(\mathbb{F}_p),$$

where the sum on the right-hand side is over all the ideal tetrahedra and

$$\varepsilon_\Delta = \begin{cases} 1 & \textit{if the orientations of } S^3 \setminus K \textit{ and } \Delta \textit{ correspond}, \\ -1 & \text{otherwise.} \end{cases}$$

Here, hyperbolicity equations are explained in Section 6.

5. Parabolic representations of $\pi_1(S^3 \setminus \mathcal{T}_{m,n})$ and zeros of a polynomial

In this section, we prove that there is a one-to-one correspondence between the conjugacy classes of parabolic representations $\pi_1(S^3 \setminus \mathcal{T}_{m,n}) \to \mathrm{SL}_2\mathbb{F}_p$ and zeros of a polynomial in $\mathbb{F}_p$.

5.1. **the (m,n)-double twist knot group.** Recall that $\mathcal{T}_{m,n}$ denotes the (m,n)-double twist knot in S^3, where $m = 2k$ and $m, n \geq 4$. We will consider an $\mathrm{SL}_2\mathbb{F}_p$-representation of $\pi_1(S^3 \setminus \mathcal{T}_{m,n})$. For later convenience, we consider the 1-tangle diagram of $\mathcal{T}_{m,n}$ depicted in Figure 3. The arrows in Figure 3 denote elements of $\pi_1(S^3 \setminus \mathcal{T}_{m,n}) \to \mathrm{SL}_2\mathbb{F}_p$. Then, $\pi_1(S^3 \setminus \mathcal{T}_{m,n})$ has the following presentation:

$$\begin{aligned} \pi_1(S^3 \setminus \mathcal{T}_{m,n}) &\cong \langle X, Y, W_1, \dots W_m, Z_1, \dots, Z_n |\text{ the relation (4) } \rangle \\ &\cong \begin{cases} \langle X, Y |\text{ the relation (5) } \rangle & \text{if } n \text{ is even} \\ \langle X, Y |\text{ the relation (6) } \rangle & \text{if } n \text{ is odd} \end{cases}. \end{aligned} \tag{3}$$

where the relations are given as follows:

$$W_1 = Y^{-1}XY,\ W_2 = W_1 Y^{-1} W_1^{-1},\ W_{i+1} = W_i W_{i-1} W_i^{-1}\ (i = 2, 3, \dots, m-1),\ W_m = Z_1, \tag{4}$$
$$Z_2 = Z_1^{-1} Y Z_1,\ Z_{j+1} = Z_j^{-1} Z_{j-1} Z_j\ (j = 2, 3, \dots, n-1),\ W_{m-1} = Z_n^{-1} Z_{n-1}^{-1} Z_n,\ Z_n = X,$$

$$\{(Y^{-1}X)^k Y (X^{-1}Y)^{k-1} X^{-1}\}^{\frac{n}{2}} Y \{(X(Y^{-1}X)^{k-1} Y^{-1} (X^{-1}Y)^k\}^{\frac{n}{2}} = X, \tag{5}$$

$$\{(Y^{-1}X)^k Y (X^{-1}Y)^{k-1} X^{-1}\}^{\frac{n-1}{2}} (Y^{-1}X)^k Y^{-1} (X^{-1}Y)^k \{X(Y^{-1}X)^{k-1} Y^{-1} (X^{-1}Y)^k\}^{\frac{n-1}{2}} = X. \tag{6}$$

Here, the second isomorphism is obtained as follows. First, a presentation of W_{m-1} in terms of only $X^{\pm 1}$ and $Y^{\pm 1}$ is obtained by substituting $W_1 = Y^{-1}XY$ for W_2 and then by substituting W_{i-1} and W_i for W_{i+1} $(i = 2, 3, \dots, m-1)$ repeatedly. Then, a presentation of Z_{n-1} in terms of only $X^{\pm 1}$ and $Y^{\pm 1}$ is obtained by substituting $Z_1 = W_m = W_{2k} = (Y^{-1}X)^k Y^{-1} (X^{-1}Y)^k$ for Z_2 and then by substituting Z_{j-1} and Z_j for Z_{j+1} $(j = 2, 3, \dots, n-1)$ repeatedly. Finally, one can verify that the left-hand sides of the relations (5) and (6) are equal to the right-hand side of

$$X = Z_{n-1}^{-1} X (Y^{-1}X)^{k-1} Y^{-1} (X^{-1}Y)^k$$

obtained from

$$(Y^{-1}X)^{k-1} Y^{-1} XY (X^{-1}Y)^{k-1} = W_{2k-1} = W_{m-1} = Z_n^{-1} Z_{n-1}^{-1} Z_n = X^{-1} Z_{n-1}^{-1} X.$$

Note that $\pi_1(S^3 \setminus \mathcal{T}_{m,n})$ has a presentation with two generators which are conjugate to each other and one relation. Double twist knots are special cases of two-bridge knots, and it is known that the knot group of a two-bridge knot also has a presentation with two generators and one relation; see, e.g [1].

5.2. **Parabolic representations of the (m,n)-double twist knot group.** To describe the set of conjugacy classes of parabolic representations $\rho\colon \pi_1(S^3 \setminus \mathcal{T}_{m,n}) \to \mathrm{SL}_2\mathbb{F}_p$, we define a map

$$\phi\colon \{\text{parabolic representations } \rho\colon \pi_1(S^3 \setminus \mathcal{T}_{m,n}) \to \mathrm{SL}_2\mathbb{F}_p\}/\text{conjugation} \longrightarrow \mathbb{F}_p \tag{7}$$

by $\phi([\rho]) = \operatorname{trace} \rho(XY) - 2$. Here $[\rho]$ denotes the conjugacy class of ρ. Note that $\phi([\rho])$ depends only on the conjugacy class of ρ since trace is an invariant under conjugation. Although the following lemma is showed in the case of twist knots in [5], the proof can apply verbatim to the case of two-bridge knots containing double twist knots; see also [13] in the case of $\mathrm{SL}_2\mathbb{C}$.

Lemma 5.1. *For a parabolic representation $\rho\colon \pi_1(S^3 \setminus \mathcal{T}_{m,n}) \to \mathrm{SL}_2\mathbb{F}_p$ and generators X and Y of $\pi_1(S^3 \setminus \mathcal{T}_{m,n})$ in Figure 3, there is $P \in \mathrm{GL}_2\mathbb{F}_p$ such that*

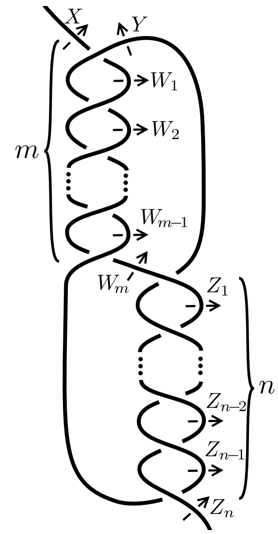

FIGURE 3. Elements of $\pi_1(S^3 \setminus \mathcal{T}_{m,n})$

$$P^{-1}\rho(X)P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ P^{-1}\rho(Y)P = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix},$$

where $u \in \mathbb{F}_p^\times$. In particular, u is $\phi([\rho])$.

One can show the lemma similarly to the proof in Appendix B of [5]. We omit the proof.

We will specify the image of (7). We consider the polynomials $f_i(c)$ with integral coefficients defined by

(8) $$f_{i+1}(c) = (2-c)f_i(c) - f_{i-1}(c)\ (i \geq 1),\ f_0(c) = 1,\ f_1(c) = 1 - c.$$

For $m = 2k \geq 2$, we also consider the polynomials $F_{m,\ell}(c)$ defined by

(9) $$F_{m,i+1}(c) = (f_k(c) - f_{k-1}(c))F_{m,i}(c) + F_{m,i-1}(c)\ (i \geq 1),\ F_{m,0}(c) = 1,\ F_{m,1}(c) = f_k(c).$$

Proposition 5.2. *The conjugacy classes of parabolic representations $\pi_1(S^3 \setminus \mathcal{T}_{m,n}) \to \mathrm{SL}_2\mathbb{F}_p$ and the solutions of $F_n(x) = 0$ over $\mathbb{F}_p$ are one-to-one correspondence by ϕ. Namely, the map*

(10) $$\{\text{parabolic representations } \pi_1(S^3 \setminus \mathcal{T}_{m,n}) \to \mathrm{SL}_2\mathbb{F}_p\}/\text{conjugation} \longrightarrow \{u \in \mathbb{F}_p \mid F_{m,n}(u) = 0\}$$

induced by ϕ is bijective.

One can show the lemma similarly to the case of twist knots in [5] with Lemma 5.1, the presentation (3) and following two lemmas (Lemmas 5.3 and 5.4); see also [14] in the case of $\mathrm{SL}_2\mathbb{C}$.

We set $\widehat{X},\ \widehat{Y},\ \widehat{W}_i\ (i = 1, \ldots, k)$ and $\widehat{Z}_j\ (j \geq 3)$ as follows:

$$\widehat{X} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \widehat{Y} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},$$

$$\widehat{W_1} = \widehat{Y}^{-1}\widehat{X}\widehat{Y},\ \widehat{W_2} = \widehat{W_1}\widehat{Y}^{-1}\widehat{W_1}^{-1},\ \widehat{W}_{i+1} = \widehat{W}_i\widehat{W}_{i-1}\widehat{W}_i^{-1}\ (i = 2, 3, \ldots, m-1),$$

$$\widehat{W}_m = \widehat{Z}_1,\ \widehat{Z}_1 = \widehat{W}\widehat{Y}^{-1}\widehat{W}^{-1},\ \widehat{Z}_2 = \widehat{Z}_1^{-1}\widehat{Y}\widehat{Z}_1,\ \widehat{Z}_{j+1} = \widehat{Z}_j^{-1}\widehat{Z}_{j-1}\widehat{Z}_j\ (j \geq 2).$$

We consider the polynomials $g_i(c)$ defined by

(11) $$g_{i+1}(c) = (2-c)g_i(c) - g_{i-1}(c)\ (i \geq 1),\ g_0(c) = 1,\ g_1(c) = 2 - c.$$

In addition, for $m = 2k$, we consider the polynomials $G_{m,i}(c)$ and $H_{m,i}(c)$ defined by the same recurrence relation

$$I_{m,i+1}(c) = (f_k(c) - f_{k-1}(c))I_{m,i}(c) + I_{m,i-1}(c) \ (I = G, H)$$

with

$$G_{m,0}(c) = 0,\ G_{m,1}(c) = f_k(c) - f_{k-1}(c),\ H_{m,0}(c) = -1,\ H_{m,1}(c) = f_{k-1}(c).$$

Note that the polynomials $F_{m,i}(c)$, defined just before Proposition 5.2, also satisfy the recurrence relation.

We need the following two lemmas to prove Proposition 5.2. We will prove them in Sections 5.3 and 5.4.

Lemma 5.3. *For the above $\widehat{X}$ and $\widehat{W}_\ell$, we have the following equalities:*

$$\widehat{W}_{2i+1} - \widehat{X} = -cg_i(c)\begin{pmatrix} -f_{i-1}(c) & g_{i-1}(c) \\ cg_i(c) & f_{i-1}(c) \end{pmatrix} \quad (i \geq 0), \tag{12}$$

$$\widehat{W}_{2i} - \widehat{X} = -f_i(c)\begin{pmatrix} cg_{i-1}(c) & f_{i-1}(c) \\ Fcf_i(c) & -cg_{i-1}(c) \end{pmatrix} \quad (i \geq 1). \tag{13}$$

Lemma 5.4. *For $m = 2k \geq 2$, it follows*

$$\widehat{Z}_i = (-1)^i F_{m,i}(c)\begin{pmatrix} -G_{m,i}(c) & H_{m,i}(c) \\ cF_{m,i}(x) & G_{m,i}(c) \end{pmatrix} + \widehat{X}. \tag{14}$$

5.3. **Proof of Lemma 5.3.** First, we introduce Lemmas 5.5 and 5.7. Using them, we will prove Lemma 5.3.

Lemma 5.5. *For any $i \geq 0$,*

$$f_{i+2}(c) = (1-c)g_{i+1}(c) - g_i(c), \tag{15}$$

$$g_{i+1}(c) - f_{i+1}(c) = g_i(c). \tag{16}$$

Proof. First, we show (15). In the cases of $i = 0, 1$, we have

$$f_2(c) = 1 - 3c + c^2 = (1-c)(2-c) - 1 = (1-c)g_1(c) - g_0(c),$$

$$f_3(c) = 1 - 6c + 5c^2 - c^3 = (1-c)(3 - 4c + c^2) - (2-c) = (1-c)g_2(c) - g_1(c).$$

We suppose $i \geq 2$. Assume that (15) holds for $i = j, j-1 \geq 2$. Then, we will show that (15) holds for $i = j+1$. From the defining relations of f_k and g_k and the assumption, we have

$$\begin{aligned} f_{j+1}(c) &= (2-c)f_j(c) - f_{j-1}(c) = (2-c)((1-c)g_{j-1}(c) - g_{j-2}(c)) - ((1-c)g_{j-2}(c) - g_{j-3}(c)) \\ &= (1-c)\{(2-c)g_{j-1}(c) - g_{j-2}(c)\} - \{(2-c)g_{j-2}(c) - g_{j-3}(c)\} = (1-c)g_j(c) - g_{j-1}(c). \end{aligned}$$

Next, we show (16). From (15) and the defining relation of g_k, we have

$$g_{i+1}(c) - f_{i+1}(c) = \{(2-c)g_i(c) - g_{i-1}(c)\} - \{(1-c)g_i(c) - g_{i-1}(c)\} = g_i.$$

□

Remark 5.6. From Lemma 5.5, we have

$$f_i(c) - f_{i+1}(c) = cg_i(c), \tag{17}$$

$$cg_{i+1}(c) = (c-1)f_{i+1}(c) + f_i(c). \tag{18}$$

Lemma 5.7. *For $i \geq 0$, we have*

$$g_{i+1}(c)^2 = (2-c)g_i(c)g_{i+1}(c) - g_i(c)^2 + 1. \tag{19}$$

Proof. It is easy to see that the case of $i = 0$ follows.

We assume $i \geq 1$. From the defining relation of $g_i(c)$, we have the following equalities:

$$g_{i+1}(c)^2 = g_{i+1}(c)\{(2-c)g_i(c) - g_{i-1}(c)\} = (2-c)g_i(c)g_{i+1}(c) - g_{i-1}(c)g_{i+1}(c), \tag{20}$$

$$\begin{aligned} g_i(c)^2 - g_{i-1}(c)g_{i+1}(c) &= g_i(c)^2 - g_{i-1}(c)\{(2-c)g_i(c) - g_{i-1}(c)\} \\ &= \{g_i(c) - (2-c)g_{i-1}(c)\}g_i(c) + g_{i-1}(c)^2 \\ &= g_{i-1}(c)^2 - g_{i-2}(c)g_i(c) \\ &\vdots \\ &= g_1(c)^2 - g_0(c)g_2(c) = (2-c)^2 - (3-4c+c^2) = 1. \end{aligned} \tag{21}$$

By combining (20) and (21), the claim holds. □

Proof of Lemma 5.3. For $\widehat{W}_1$ and $\widehat{W}_2$, by concrete calculation, we have

$$\widehat{W}_1 = c\begin{pmatrix} 1 & 0 \\ -c & -1 \end{pmatrix} + \widehat{X}, \quad \widehat{W}_2 = (-1+c)\begin{pmatrix} c & 1 \\ -(-1+c)c & -c \end{pmatrix} + \widehat{X}.$$

First, we consider the case of $\ell = 2i+1$. Assume that the claim holds for $\ell = 2i-1, 2i$. Then, we will show that the claim holds for $\ell = 2i+1$. Since $\widehat{W}_\ell \in \mathrm{SL}_2\mathbb{F}_p$ for all $\ell \geq 1$,

$$\widehat{W}_{2i}^{-1} = f_i(c)\begin{pmatrix} g_{i-1}(c) & f_{i-1}(c) \\ cf_i(c) & -g_{i-1}(c) \end{pmatrix} + \widehat{X}^{-1}$$

By concrete calculation with (15), (16), (19) and the defining relation of $g_i(c)$, one can show

$$\widehat{W}_{2i+1} - \widehat{X} = \widehat{W}_{2i}\widehat{W}_{2i-1}\widehat{W}_{2i}^{-1} - \widehat{X} = -cg_i(c)\begin{pmatrix} -f_{i-1}(c) & g_{i-1}(c) \\ cg_i(c) & f_{i-1}(c) \end{pmatrix}.$$

Similar to the case of $\ell = 2i+1$, one can show

$$\widehat{W}_{2i} - \widehat{X} = \widehat{W}_{2i-1}\widehat{W}_{2i-2}\widehat{W}_{2i-1}^{-1} - \widehat{X} = -f_i(c)\begin{pmatrix} cg_{i-1}(c) & f_{i-1}(c) \\ cf_i(c) & -cg_{i-1}(c) \end{pmatrix}.$$

□

5.4. **Proof of Lemma 5.4.**

Before the proof of Lemma 5.4, we introduce the following three lemmas.

Lemma 5.8. *For $i \geq 0$, we have*

$$(f_{i+1}(c))^2 = (2-c)f_i(c)f_{i+1}(c) - (f_i(c))^2 + c. \tag{22}$$

One can show the lemme similarly to Lemma 5.7.

Lemma 5.9. *For $i \geq 1$, we have*

$$F_{m,i}(c)G_{m,i}(c) - F_{m,i-1}(c)G_{m,i+1}(c) = (-1)^{i+1}(H_{m,2}(c) + 1) \tag{23}$$

$$F_{m,i}(c)G_{m,i}(c) - F_{m,i+1}(c)G_{m,i-1}(c) = (-1)^{i+1}(F_{m,2}(c) - 1). \tag{24}$$

Proof. We will show (23). We have

$$\begin{aligned} F_{m,i}(c)G_{m,i}(c) - F_{m,i-1}(c)G_{m,i+1}(c) &= -\{F_{m,i-1}(c)G_{m,i-1}(c) - F_{m,i-2}(c)G_{m,i}(c)\} \\ &\vdots \\ &= (-1)^{i-1}\{F_{m,1}(c)G_{m,1}(c) - F_{m,0}(c)G_{m,2}(c)\} \\ &= (-1)^{i-1}\{f_k(c)f_{k-1}(c)) - (f_{k-1}(c))^2\} \\ &= (-1)^{i-1}(H_{m,2}(c)+1). \end{aligned}$$

Similar to the above, one can show (24). □

Lemma 5.10. *For any $i \geq 0$,*

$$F_{m,i}(c)H_{m,i}(c) + (G_{m,i}(c))^2/c = (-1)^{i+1}. \tag{25}$$

For any $i \geq 1$,

$$F_{m,i}(c)H_{m,i-1}(c) + G_{m,i}(c)G_{m,i-1}(c)/c = (-1)^{i}F_{m,1}(c), \tag{26}$$

$$F_{m,i-1}(c)H_{m,i}(c) + G_{m,i}(c)G_{m,i-1}(c)/c = (-1)^{i-1}H_{m,1}(c). \tag{27}$$

Proof. By concrete calculation,

$$F_{m,0}(c)H_{m,0}(c) + (G_{m,0}(c))^2/c = -1,$$

$$\begin{aligned} F_{m,1}(c)H_{m,1}(c) + (G_{m,1}(c))^2/c &= f_k(c)f_{k-1}(c) + (f_k(c) - f_{k-1}(c))^2/c \\ &= f_k(c)f_{k-1}(c) + (-cf_k(c)f_{k-1}(c) + c)/c = 1, \end{aligned}$$

where the second equality follows from Lemma 5.8. By concrete calculation,

$$F_{m,1}(c)H_{m,0}(c) + G_{m,1}(c)G_{m,0}(c)/c = -F_{m,1}(c)$$

$$F_{m,0}(c)H_{m,1}(c) + G_{m,1}(c)G_{m,0}(c)/c = H_{m,1}(c)$$

Suppose that, for some $j \geq 1$, (25), (26) and (27) hold for any $i \leq j$. Then, we will show the claim for $i = j+1$. First, we will show (25). We have

$$\begin{aligned} &F_{m,j+1}(c)H_{m,j+1}(c) + (G_{m,j+1}(c))^2/c \\ = \; &(G_{m,1}(c)F_{m,j}(c) + F_{m,j-1}(c))(G_{m,1}(c)H_{m,j}(c) + H_{m,j-1}(c)) + (G_{m,1}(c)G_{m,j}(c) + G_{m,j-1}(c))^2/c \\ = \; &(G_{m,1}(c))^2F_{m,j}(c)H_{m,j}(c) + G_{m,1}(c)F_{m,j}(c)H_{m,j-1}(c) + G_{m,1}(c)F_{m,j-1}(c)H_{m,j}(c) \\ &+F_{m,j-1}(c)H_{m,j-1}(c) + \{(G_{m,1}(c))^2(G_{m,j}(c))^2 + 2G_{m,1}(c)G_{m,j}(c)G_{m,j-1}(c) + (G_{m,j-1}(c))^2\}/c \\ = \; &(-1)^{j+1}(G_{m,1}(c))^2 + (-1)^jG_{m,1}(c)(F_{m,1}(c) - H_{m,1}(c)) + (-1)^j = (-1)^j, \end{aligned}$$

where the second equality follows from (25), (26) and (27), and the last equality follows from $F_{m,1}(c) - H_{m,1}(c) = G_{m,1}(c)$.

Next, we will show (26). We have

$$\begin{aligned} &F_{m,j+1}(c)H_{m,j}(c) + G_{m,j+1}(c)G_{m,j}(c)/c \\ = \; &(G_{m,1}(c)F_{m,j}(c) + F_{m,j-1}(c))H_{m,j}(c) + (G_{m,1}(c)G_{m,j}(c) + G_{m,j-1}(c))G_{m,j}(c)/c \\ = \; &(-1)^{j+1}G_{m,1}(c) + (-1)^{j-1}H_{m,1}(c) = (-1)^{j+1}F_{m,1}(c), \end{aligned}$$

where the second equality follows from (26) and (27).

Similar to the proof of (26), one can show (27). We omit details. □

Proof of Lemma 5.4. By concrete calculation, the claim holds for $i = 1, 2$.

Assume that the claim holds for $i = 1, 2, \ldots, j$. Then, we will show the claim for $i = j + 1$.

By the definition of Z_i $(i \geq 2)$,,

$$\begin{aligned} Z_i Z_{i+1} &= Z_{i-1} Z_i = \cdots = Z_1 Z_2 = Y Z_1 \\ &= \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} F_{m,1}(c) G_{m,1}(c) + 1 & -F_{m,1}(c) H_{m,1}(c) + 1 \\ -c(F_{m,1}(c))^2 & -F_{m,1}(c) G_{m,1}(c) + 1 \end{pmatrix} = \begin{pmatrix} F_{m,2}(c) & G_{m,2}(c)/c \\ G_{m,2}(c) & -H_{m,2}(c) \end{pmatrix}, \end{aligned}$$

where the last equality follows from the definitions of $F_{m,i}(c), G_{m,i}(c), H_{m,i}(c)$ and Lemma 5.8.

Note that, from $\widehat{Z}_j \in \mathrm{SL}_2\mathbb{F}_p$, $\widehat{Z}_j^{-1}$ is described as follows:

$$\widehat{Z}_j^{-1} = (-1)^{j+1} F_{m,j}(c) \begin{pmatrix} -G_{m,j}(c) & H_{m,j}(c) \\ cF_{m,j}(c) & G_{m,j}(c) \end{pmatrix} + \widehat{X}^{-1}. \tag{28}$$

From the definition of $\widehat{Z}_i$,

$$\widehat{Z}_{j+1} = \widehat{Z}_j^{-1} \widehat{Z}_{j-1} \widehat{Z}_j = \widehat{Z}_j^{-1} \begin{pmatrix} F_{m,2} & G_{m,2}/c \\ G_{m,2} & -H_{m,2}. \end{pmatrix} \tag{29}$$

First, we consider the case when j is odd. Then, from (28) and (29),

$$\widehat{Z}_{j+1} = \begin{pmatrix} -F_{m,j}(c) G_{m,j}(c) + 1 & F_{m,j}(c) H_{m,j}(c) + 1 \\ c(F_{m,j}(c))^2 & F_{m,j}(c) G_{m,j}(c) + 1 \end{pmatrix} \begin{pmatrix} F_{m,2} & G_{m,2}/c \\ G_{m,2} & -H_{m,2}. \end{pmatrix}.$$

We will show that the following matrix is the zero matrix:

$$\widehat{Z}_{j+1} - (-1)^{j+1} F_{m,j}(c) \begin{pmatrix} -G_{m,j+1}(c) & H_{m,j+1}(c) \\ cF_{m,j+1}(x) & G_{m,j+1}(c) \end{pmatrix} - \widehat{X} = \begin{pmatrix} I_{1,j}(c) & I_{2,j} \\ I_{3,j} & I_{4,j} \end{pmatrix},$$

where

$$\begin{aligned} I_{1,j}(c) &= -1 + F_{m,2}(c) + G_{m,2}(c)(-1 + F_{m,j}(c) H_{m,j}(c)) - F_{m,2}(c) F_{m,j}(c) G_{m,j}(c) + F_{m,j+1}(c) G_{m,j+1}(c), \\ I_{2,j}(c) &= -G_{m,2}(c)(-1 + F_{m,j}(c) G_{m,j}(c))/c - (1 + H_{m,2}(c)(-1 + F_{m,j}(c) H_{m,j}(c)) + F_{m,j+1}(c) H_{m,j+1}(c)), \\ I_{3,j}(c) &= G_{m,2}(c) + G_{m,2}(c) F_{m,j}(c) G_{m,j}(c) + cF_{m,2}(c)(F_{m,j}(c))^2 - c(F_{m,j+1}(c))^2, \\ I_{4,j}(c) &= -1 + G_{m,2}(c)(F_{m,j}(c))^2 - H_{m,2}(c)(1 + F_{m,j}(c) G_{m,j}(c)) - F_{m,j+1}(c) G_{m,j+1}(c). \end{aligned}$$

Namely, we will show that $I_{1,j}(c) = I_{2,j}(c) = I_{3,j}(c) = I_{4,j}(c) = 0$.

First, we will show $I_{1,j}(c)$. We have

$$\begin{aligned} I_{1,j}(c) &\overset{(25)}{=} -1 + F_{m,2}(c) + G_{m,2}(c)(-(G_{m,j}(c))^2/c) - F_{m,2}(c) F_{m,j}(c) G_{m,j}(c) + F_{m,j+1}(c) G_{m,j+1}(c) \\ &\overset{(24)}{=} G_{m,2}(c)(-(G_{m,j}(c))^2/c) - F_{m,2}(c) F_{m,j}(c) G_{m,j}(c) + F_{m,j+2}(c) G_{m,j}(c) \\ &= G_{m,j}(c)\{-G_{m,2}(c) G_{m,j}(c)/c - F_{m,2}(c) F_{m,j}(c) + F_{m,j+2}(c)\}. \end{aligned}$$

We have to show

$$I_{1,j}(c) = G_{m,j}(c)\{-G_{m,2}(c) G_{m,j}(c)/c - F_{m,2}(c) F_{m,j}(c) + F_{m,j+2}(c)\} = 0. \tag{30}$$

It is enough to show that the second factor is zero. By concrete calculation, we have

$$-G_{m,2}(c) G_{m,0}(c)/c - F_{m,2}(c) F_{m,0}(c) + F_{m,2}(c) = 0$$

and

$$\begin{aligned}
&-G_{m,2}(c)G_{m,1}(c)/c - F_{m,2}(c)F_{m,1}(c) + F_{m,3}(c)\\
= \ & -(f_k(c) - f_{k-1}(c))^3/c - \{(f_k(c) - f_{k-1}(c))f_k(c) + 1\}f_k(c)\\
& +(f_k(c) - f_{k-1}(c))\{(f_k(c) - f_{k-1}(c))f_k(c) + 1\} + f_k(c)\\
= \ & -(f_k(c) - f_{k-1}(c))^3/c - f_{k-1}(c)\{(f_k(c) - f_{k-1}(c))f_k(c) + 1\} + f_k(c)\\
\overset{(22)}{=} \ & -c(f_k(c)f_{k-1}(c) + 1)(f_k(c) - f_{k-1}(c))/c - f_{k-1}(c)\{(f_k(c) - f_{k-1}(c))f_k(c) + 1\} + f_k(c) = 0
\end{aligned}$$

Suppose that, for $j = l-1, l$,

$$-G_{m,2}(c)G_{m,j}(c)/c - F_{m,2}(c)F_{m,j}(c) + F_{m,j+2}(c) = 0.$$

Then,

$$\begin{aligned}
&-G_{m,2}(c)G_{m,l+1}(c)/c - F_{m,2}(c)F_{m,l+1}(c) + F_{m,l+3}(c)\\
= \ & -G_{m,2}(c)\{G_{m,1}(c)G_{m,l}(c) + G_{m,l-1}(c)\}/c - F_{m,2}(c)\{G_{m,1}(c)F_{m,l}(c) + F_{m,l-1}(c)\}\\
& +G_{m,1}(c)F_{m,l+2}(c) + F_{m,l+1}(c) = 0
\end{aligned}$$

Hence, $I_{1,j}(c) = 0$.

Next, we will show $I_{2,j}(c) = 0$.

$$\begin{aligned}
I_{2,j}(c) &= \{-G_{m,2}(c)(-1 + F_{m,j}(c)G_{m,j}(c)) + H_{m,2}(c)(G_{m,j}(c))^2 + (G_{m,j+1}(c))^2\}/c\\
&= G_{m,j}(c)G_{m,j+2}(c) - G_{m,2}(c)F_{m,j}(c)G_{m,j}(c) + H_{m,2}(c)(G_{m,j}(c))^2\\
&= G_{m,j}(c)\{G_{m,j+2}(c) - G_{m,2}(c)F_{m,j}(c) + H_{m,2}(c)G_{m,j}(c)\} = 0,
\end{aligned}$$

where the last equality follows similarly to the proof of (30). Hence, $I_{2,j}(c) = 0$.

Next, we will show $I_{3,j}(c) = 0$.

$$\begin{aligned}
I_{3,j}(c) &= G_{m,2}(c)(1 + F_{m,j}(c)G_{m,j}(c)) + cF_{m,2}(c)(F_{m,j}(c))^2 - c(F_{m,j+1}(c))^2\\
&= G_{m,2}(c)(F_{m,2}(c) - F_{m,j+1}(c)G_{m,j-1}(c)) + cF_{m,2}(c)(F_{m,j}(c))^2 - c(F_{m,j+1}(c))^2\\
&= F_{m,2}(c)(G_{m,2}(c) + c(F_{m,j}(c))^2) - F_{m,j+1}(c)(G_{m,2}(c)G_{m,j-1}(c) + cF_{m,j+1}(c))
\end{aligned}$$

Here, we have the following equalities:

$$G_{m,2}(c)G_{m,j-1}(c) + cF_{m,j+1}(c) = (-1)^j cF_{m,2}(c)F_{m,j-1}(c),$$
$$G_{m,2}(c) + c(F_{m,j}(c))^2 = (-1)^j cF_{m,j-1}(c)F_{m,j+1}(c),$$

where the first equality follows similarly to the proof of (30), and the second equality holds as follows:

$$\begin{aligned}
c(F_{m,j}(c))^2 - cF_{m,j-1}(c)F_{m,j+1}(c) &= cF_{m,j-2}(c)F_{m,j}(c) - c(F_{m,j-1}(c))^2\\
&\ \ \vdots\\
&= cF_{m,0}(c)F_{m,2}(c) - c(F_{m,1}(c))^2\\
&= -c(f_{k-1}(c)f_k(c) + 1) \overset{(22)}{=} G_{m,2}(c).
\end{aligned}$$

Hence, $I_{3,j}(c) = 0$.

Finally, we will show $I_{4,j}(c) = 0$. We have

$$\begin{aligned}
I_{4,j}(c) &= G_{m,2}(c)(F_{m,j}(c))^2 - H_{m,2}(c)F_{m,j}(c)G_{m,j}(c) - F_{m,j}(c)G_{m,j+2}(c)\\
&= F_{m,j}(c)(G_{m,2}(c)(F_{m,j}(c)) - H_{m,2}(c)G_{m,j}(c) - G_{m,j+2}(c)) = 0,
\end{aligned}$$

where the last equality follows similarly to the proof of (30).

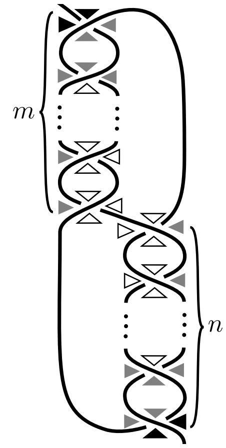

FIGURE 4. An ideal triangulation of $S^3 \setminus \mathcal{T}_{m,n}$ obtained by Yokota's method.

One can show the claim in the case when j is even similarly to the case when j is odd. We omit details. □

6. Hyperbolicity equations of double twist knots

In the section, we review how to obtain an ideal triangulation from a 1-tangle diagram and hyperbolicity equations.

6.1. **Ideal triangulation of $S^3 \setminus \mathcal{T}_{m,n}$.** In the subsection, we will consider a reduced ideal triangulation obtained from an ideal triangulation of $S^3 \setminus \mathcal{T}_{m,n}$ obtained by Yokota's method [20]; see also D. Thurston [15].

First, we review the method to give an ideal triangulation of $S^3 \setminus \mathcal{T}_{m,n}$ by Yokota; see [19], [8] for more details. Consider the 1-tangle diagram of $\mathcal{T}_{m,n}$ in Figure 4. We assign a tetrahedron to each triangle in Figures 4. These tetrahedra form a polyhedron by gluing each other appropriately. It is known that an ideal triangulation of $S^3 \setminus \mathcal{T}_{m,n}$ is obtained from the polyhedron by collapsing one of the faces of the tetrahedron of a black triangle to a point. By collapsing the face, the tetrahedra of the black and gray triangles collapse, i.e. only the tetrahedra of the white triangles do not collapse. Then, the tetrahedra of the white triangles are glued appropriately. From the resulting polyhedron, we remove the point obtained by collapsing the face in the above. Note that, by the operation, the tetrahedra of the white triangles turn to be ideal tetrahedra. The resulting open 3-manifold is homeomorphic to $S^3 \setminus \mathcal{T}_{m,n}$. Here, a pair of the two ideal tetrahedra of the gray triangles in each bigon in the twist part in Figure 5 are glued by each other so that one can apply $(0, 2)$-Pachner move to them; see [19] for details. Here any two ideal triangulation of the same manifold are related to each other by a finite sequence of Pachner moves; see [7] for more details.

We will reduce the ideal triangulation in Figure 4. In Figure 5, the triangles denote ideal tetrahedra, which are denoted by white triangles, in Figure 4. We will remove ideal tetrahedra of the gray triangles in Figure 5 as follows. For each region which forms a bigon in the twist part in Figure 5, two ideal tetrahedra of the gray triangles are glued to each other along just two faces such that one can apply $(0, 2)$-Pachner move. The two ideal tetrahedra are removed by applying $(0, 2)$-Pachner move to them. By applying the same operation for all bigons containing gray triangles, we obtain a reduced ideal triangulation of $S^3 \setminus \mathcal{T}_{m,n}$ in Figure 5.

Note that the orientation of each of the tetrahedra in Figure 6 is compatible with that of $S^3 \setminus \mathcal{T}_{m,n}$.

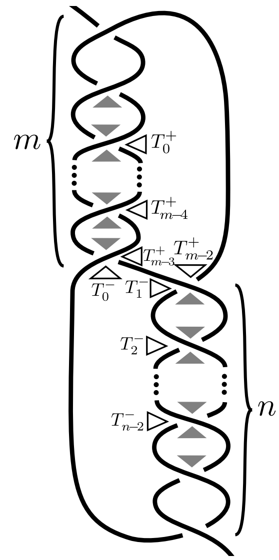

FIGURE 5. A reduced version of the ideal triangulation in Figure 4

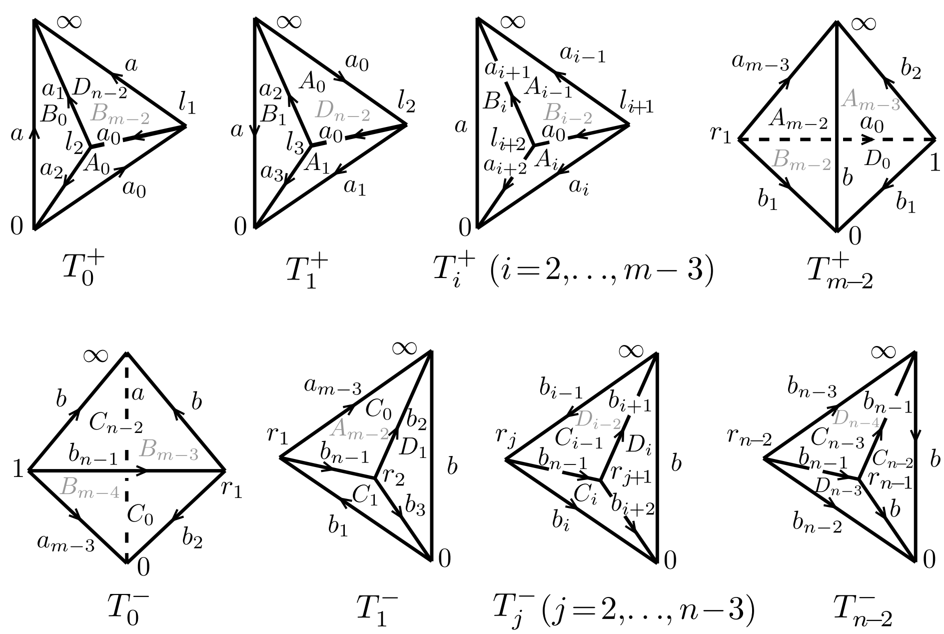

FIGURE 6. An ideal triangulation of the complement of the (m,n)-double twist knot: each of T_i^+ $(0 \leq i \leq m-2)$, and T_i^- $(0 \leq i \leq n-2)$ denotes an ideal tetrahedron, each of a_i $(i = 0, 1, \ldots, m-3)$, $a_{m-2} = b$, $a_{m-1} = b_2$, $a = b_1$, b_j $(j = 3, \ldots, n-1)$ denotes an edge, and each of A_i, B_i $(i = 0, 1, \ldots, m-2), C_j, D_j$ $(j = 0, 1, \ldots, n-2)$ denotes a face. The gray characters mean that the faces are on back side. The faces with the same labels are glued to each other so that the orientations are compatible.

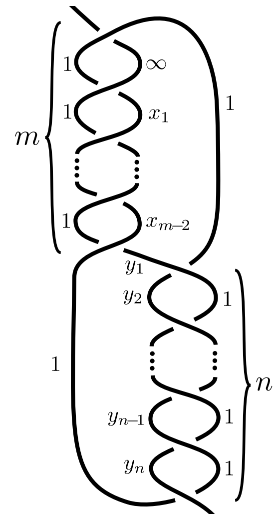

FIGURE 7. A parametrized 1-tangle diagram

6.2. **Hyperbolicity equations for knot complements.** For a knot $K \subset S^3$, we fix an ideal triangulation of $S^3 \setminus K$. Suppose that each ideal tetrahedron is assigned a modulus. For each edge of the ideal triangulation, consider the moduli of the ideal tetrahedra around the edge with respect to the edge and the condition that the product of them is equal to 1 gives a system of equations. We call it *hyperbolicity equations* of the ideal triangulation of the knot complement. It is know that the hyperbolicity equations are equivalent to hyperbolicity equations for a 1-tangle knot diagram; see [15], [5] for details.

When K is a hyperbolic knot, it is known that one of the solutions of hyperbolicity equations in $\mathbb{C}$ gives a complete hyperbolic structure of $S^3 \setminus K$ by D. Thurston, see [15], [8] for more details. We will consider solutions of hyperbolic equations in $\mathbb{F}_p$.

6.3. **Hyperbolicity equations for 1-tangle diagrams.** In this subsection, we review hyperbolicity equations of a 1-tangle diagram following D. Thurston [15] and Yokota [20]. It is known that $\mathrm{SL}_2\mathbb{C}$-representations are obtained from solutions of the hyperbolicity equations and one of the representations is the holonomy representation, which allows the complete hyperbolic structure of the knot complement. By replacing $\mathbb{C}$ with $\mathbb{F}_p$, we obtain $\mathrm{SL}_2\mathbb{F}_p$-representations from solutions of hyperbolicity equations in $\mathbb{F}_p$. We will describe hyperbolicity equations using $F_{m,\ell}(x)$ defined in Section 7.1.

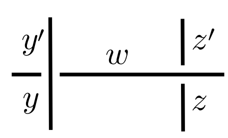

FIGURE 8. Local picture of a knot diagram

We review the hyperbolicity equations of a 1-tangle diagram of the (m, n)-double twist knot. As in Figure 7, we present the (m, n)-double twist knot by a 1-tangle diagram and parametrize each semi-arc of the diagram, where a semi-arc is each connected component of the result obtained from a 1-tangle diagram by eliminating small neighborhoods of all the crossings. Let us explain how to parametrize the 1-tangle diagram. We parametrize semi-arcs adjacent to unbounded regions by 1. We parametrize a semi-arc next to the terminal edges by ∞ (resp. 0) if it is connected to the terminal semi-arc by an underpath (resp. an overpath). We parametrize the other semi-arcs located as in Figure 31 so that they satisfy the equation

$$(1 - w/y)(1 - z'/w) = (1 - w/y')(1 - z/w). \tag{31}$$

In particular, for the 1-tangle diagram of the (m, n)-double twist knot in Figure 7, the following equations are obtained using the above relations (31):

$$\begin{cases} x_2 = x_1 + 1, \\ x_{i+1} = 1 - x_i/x_{i-1} + x_i \quad (i = 2, 3, \dots, m-2), \\ y_1 = x_{m-1}, \\ (1 - y_1/x_{m-2})(1 - y_2/y_1) = (1 - y_1)(1 - 1/y_1), \\ y_{j+1} = 1 - x_j/x_{j-1} + x_j \quad (j = 2, 3, \dots, n-1), \\ y_n = 0. \end{cases} \tag{32}$$

We call the equations *hyperbolicity equations* of the 1-tangle diagram of $\mathcal{T}_{m,n}$ in Figure 7.

It is known that the hyperbolicity equations of an ideal triangulation of $S^3 \setminus \mathcal{T}_{m,n}$ and the hyperbolicity equations of the 1-tangle diagram of $\mathcal{T}_{m,n}$ in Figure 7 are equivalent; see [15], [5] for more details.

7. Hyperbolicity equations and polynomials

7.1. **Hyperbolicity equations and polynomials.** In the subsection, we will describe hyperbolic equations with polynomials defined in Section 3.

Lemma 7.1. *When we identify x_1 with c, for $i \geq 1$, $x_{2i-1} = f_i(c)/f_{i-1}(c)$ and $x_{2i} = g_i(c)/g_{i-1}(c)$. In particular, $y_1 = x_{2k-1} = f_k(c)/f_{k-1}(c)$.*

Proof. In the case of $i = 1$,

$$f_1(c) = c = x_1, \quad g_1(c) = 1 + c = x_2.$$

Hence, it is enough to show the case of $i \geq 2$.

We have to show that $f_i(c)$ and $g_i(c)$ satisfy the following two equalities:

$$f_i(c)/f_{i-1}(c) = 1 - (g_{i-1}(c)/g_{i-2}(c))(f_{i-2}(c)/f_{i-1}(c)) + g_{i-1}(c)/g_{i-2}(c),$$
$$g_i(c)/g_{i-1}(c) = 1 - (f_i(c)/f_{i-1}(c))(g_{i-2}(c)/g_{i-1}(c)) + f_i(c)/f_{i-1}(c).$$

By clearing the numerators in each equations, it is enough to show the following two equalities:

$$(f_i(c) - f_{i-1}(c))g_{i-2}(c) = (f_{i-1}(c) - f_{i-2}(c))g_{i-1}(c), \tag{33}$$
$$(g_i(c) - g_{i-1}(c))f_{i-1}(c) = (g_{i-1}(c) - g_{i-2}(c))f_i(c). \tag{34}$$

First, we show (33). We have

$$\begin{aligned} (f_i(c) - f_{i-1}(c))g_{i-2}(c) &\overset{(16)}{=} (f_i(c) - f_{i-1}(c))(g_{i-1}(c) - f_{i-1}(c)) \\ &= ((1-c)f_{i-1}(c) - f_{i-2}(c))(g_{i-1}(c) - f_{i-1}(c)) \\ &\overset{(18)}{=} (f_{i-1}(c) - f_{i-2}(c))g_{i-1}(c) - cf_{i-1}(c)g_{i-1}(c) - ((1-c)f_{i-1}(c) - f_{i-2}(c))f_{i-1}(c) \\ &\overset{(15)}{=} (f_{i-1}(c) - f_{i-2}(c))g_{i-1}(c). \end{aligned}$$

which is obtained by (15) and (16).

We will show (34). By using (16) twice, we have

$$(g_i(c)-g_{i-1}(c))f_{i-1}(c) = f_i(c)f_{i-1}(c) = (g_{i-1}(c)-g_{i-2}(c))f_i(c).$$

□

By concrete calculation,

$$y_2 = \frac{(-(f_{k+1}(c))^2 g_{k-1}(c) + (f_k(c))^2 g_k(c) - f_{k+1}(c)f_k(c)g_k(c) + (f_{k+1}(c))^2 g_k(c))}{f_k(c)(f_k(c)g_k(c) - f_{k+1}(c)g_{k-1}(c))}.$$

Moreover, we have the following lemma.

Lemma 7.2. *When we identify x_1 with c, $y_2 = F_{m,2}(c)$.*

Proof. First, we consider $f_k(c)g_k(c) - f_{k+1}(c)g_{k-1}(c)$ in the denominator, and we will show

$$f_k(c)g_k(c) - f_{k+1}(c)g_{k-1}(c) = 1. \tag{35}$$

Actually, from the defining relation of $f_k(c)$, we have

$$\begin{aligned} f_k(c)g_k(c) - f_{k+1}(c)g_{k-1}(c) &= f_k(c)g_k(c) - \{(c+1)f_k(c) - f_{k-1}(c)\}g_{k-1}(c) \\ &= f_k(c)(g_k(c) - (c+1)g_{k-1}(c)) + f_{k-1}(c)g_{k-1}(c) \\ &= f_{k-1}(c)g_{k-1}(c) - f_k(c)g_{k-2}(c) \\ &\ \ \vdots \\ &= f_1(c)g_1(c) - f_2(c)g_0(c) = c(c+1) - \{c(c+1)-1\} = 1. \end{aligned} \tag{36}$$

Next, we consider the numerator. We will show that the numerator contains f_k as a factor. From (16),

$$\begin{aligned} &-(f_{k+1}(c))^2 g_{k-1}(c) + (f_k(c))^2 g_k(c) - f_{k+1}(c)f_k(c)g_k(c) + (f_{k+1}(c))^2 g_k(c) \\ =\ & f_k(c)((f_{k+1}(c))^2 + f_k(c)g_k(c) - f_{k+1}(c)g_k(c)). \end{aligned}$$

Finally, we have

$$\begin{aligned} (f_{k+1}(c))^2 + f_k(c)g_k(c) - f_{k+1}(c)g_k(c) &\overset{(16)}{=} f_{k+1}(c)(f_{k+1}(c) - f_k(c)) + f_k(c)g_k(c) - f_{k+1}(c)g_{k-1}(c) \\ &= (f_{k+1}(c) - f_k(c))f_{k+1}(c) + 1 = F_{m,2}(c). \end{aligned}$$

□

More generally, we have the following lemma.

Lemma 7.3. *When we identify x_1 with c, for any $l \geq 2$,*

$$y_l = F_{m,l}(c)/F_{m,l-2}(c). \tag{37}$$

Proof. From Lemma 7.2 and $F_{m,0}(c) = 1$, the claim holds for $l = 2$.

In the case of $l = 3$,

$$\begin{aligned} y_3 &= 1 - y_2/y_1 + y_2 = 1 - F_{m,2}(c)/(f_k(c)/f_{k-1}(c)) + F_{m,2}(c) \\ &= 1 + (f_k(c) - f_{k-1}(c))F_{m,2}(c)/F_{m,1}(c) = 1 + (F_{m,3}(c) - F_{m,1}(c))/F_{m,1}(c) = F_{m,3}(c)/F_{m,1}(c). \end{aligned}$$

Note that, from the defining relation of $F_{m,l}$, we have

$$\begin{aligned} F_{m,l+1}(c)F_{m,l-2}(c) - F_{m,l}(c)F_{m,l-1}(c) &= \{(f_k(c) - f_{k-1}(c))F_{m,l}(c) + F_{m,l-1}(c)\}F_{m,l-2}(c) - F_{m,l}(c)F_{m,l-1}(c) \\ &= -(F_{m,l}(c)F_{m,l-3}(c) - F_{m,l-1}(c)F_{m,l-2}(c)) \\ &\ \ \vdots \\ &= (-1)^l(F_{m,3}(c)F_{m,0}(c) - F_{m,2}(c)F_{m,1}(c)) \qquad (38) \end{aligned}$$

for any $l \in \mathbb{N}$.

We assume that (37) holds for $l = j, j-1 \geq 2$. Then, we show that (37) holds for $l = j+1$. From (38), we have

$$(F_{m,j+1}(c)F_{m,j-2}(c) - F_{m,j}(c)F_{m,j-1}(c)) - (F_{m,j}(c)F_{m,j-3}(c) - F_{m,j-1}(c)F_{m,j-2}(c)) = 0.$$

From the equality and $F_{m,k}(c) \neq 0$ $(k \geq 0)$,

$$\begin{aligned} F_{m,j+1}(c)/F_{m,j-1}(c) &= 1 - (F_{m,j}(c)/F_{m,j-2}(c))(F_{m,j-3}(c)/F_{m,j-1}(c)) + F_{m,j}(c)/F_{m,j-2}(c) \\ &= 1 - y_j/y_{j-1} + y_j = y_{j+1} \end{aligned}$$

□

7.2. **Properties of sequences.** In the subsection, we will show some properties of the sequences defined in Section 7.1.

Lemma 7.4. *There exists no $z \in \mathbb{F}_p$ such that one of the following is satisfied:*
(i) $f_i(z) = f_{i-1}(z) = 0$,
(ii) $g_i(z) = g_{i-1}(z) = 0$,
(iii) $F_{m,i}(z) = F_{m,i-2}(z) = 0$.

Proof. (i) Suppose that there is $z \in \mathbb{F}_p$ such that $f_i(z) = f_{i-1}(z) = 0$. Since $f_{i-2}(z) = (2 - c)f_{i-1}(z) - f_i(z) = 0$ from the definition of $f_i(z) = 0$, we have $f_0(z) = 0$. However, it contradicts $f_0(c) = 1 \neq 0$.

(ii) One can show the claim similarly to (i).

(iii) Suppose that there exists $z \in \mathbb{F}_p$ such that $F_{m,i}(z) = F_{m,i-2}(z) = 0$. Then, $(f_k(z) - f_{k-1}(z))F_{m,i-1}(z) = 0$ by the defining relation (9) of $F_{m,i}(c)$. Then, it follows that $(f_k(z) - f_{k-1}(z)) = 0$ or $F_{m,i-1}(z) = 0$. In the case of $f_k(z) - f_{k-1}(z) = 0$, this contradicts $F_{m,i}(z) = F_{m,i-2}(z) = \cdots = F_{m,j}(z) = 1$ ($j = 0$ or 1) obtained by (9), where $F_{m,0}(z) = 1$ and

$$F_{m,1}(z) = f_k(z) = f_{k-1}(z) = \cdots = f_0(z) = 1$$

obtained using $f_k(z) - f_{k-1}(z) = 0$ and the defining relation of $f_i(c)$. Hence, it follows $F_{m,i-1}(z) = 0$. From (9) and the assumption, $F_{m,i-3}(z) = F_{m,i-1}(z) - (f_k(z) - f_{k-1}(z))F_{m,i-2}(z) = 0$. By applying the same procedure repeatedly, it follows $F_0(z) = 0$. However, this contradicts $F_{m,0}(z) = 1$. Therefore, there exists no $z \in \mathbb{F}_p$ such that $F_{m,i}(z) = F_{m,i-2}(z) = 0$. □

Fix $z \in \{u \in \mathbb{F}_p \mid F_{m,n}(u) = 0\}$. We define $l_i, r_j \in \mathbb{F}_p \cup \{\infty\}$ $(1 \leq i \leq m-1,\ 1 \leq j \leq n)$ by

$$l_1 = f_1(z), \quad l_2 = g_1(z), \quad l_{2i+1} = \begin{cases} f_{i+1}(z)/f_i(z) & \text{if } f_i(z) \neq 0 \\ \infty & \text{otherwise} \end{cases}, \tag{39}$$

$$l_{2i+2} = \begin{cases} g_{i+1}(z)/g_i(z) & \text{if } g_i(z) \neq 0 \\ \infty & \text{otherwise} \end{cases} \quad (2 \leq i \leq k-1),$$

$$r_1 = l_{m-1}, \quad r_k = \begin{cases} F_{m,i}(z)/F_{m,i-2}(z) & \text{if } F_{m,i-2}(c) \neq 0, \\ \infty & \text{if } F_{m,i-2}(z) = 0 \text{ and } F_{m,i}(z) \neq 0, \end{cases} \quad (2 \leq i \leq n) \tag{40}$$

where Lemma 7.4 ensures that there is no other cases.

Remark 7.5. Let $m = 2k \geq 4$. If $r_1 = f_k(z)/f_{k-1}(z) = 1$ for $z \in \mathbb{F}_p$, then z does not give a solution of hyperbolicity equations of the 1-tangle diagram of the (m, n)-double twist knot in Figure 7. Actually, from the defining relation of $F_{m,i}(c)$ and $f_k(z)/f_{k-1}(z) = 1$, $F_{m,i}(z) = 1 \neq 0$ for all $i \geq 1$.

Lemma 7.6. *For an element $z \in \{u \in \mathbb{F}_p \mid F_{m,n}(u) = 0\}$, consider*

$$(z_1, z_2, \dots, z_{m+n-3}) = (l_1, l_2, \dots, l_{m-2}, r_1, r_2, \dots, r_{n-1})$$

obtained from z by (39) and (40).

(a) If $z_i = 0, 1$ or ∞ for some i, then the sequence contains $0, 1, \infty$ in this order and z_i is one of them.

(b) Each of the sequences $(l_1, l_2, \dots, l_{m-2})$ and $(r_1, r_2, \dots, r_{n-1})$ is one of the following forms:
(i) the sequence does not contain $0, 1$ and ∞,
(ii) the sequence contains $0, 1, \infty$ in this order, and 0 and ∞ are not adjacent,
(iii) the sequence consists of only $0, 1$ and ∞.

Proof. (a) Suppose that $z_i = l_i = 0$ for some $i \in \{1, 2, \dots, m-3\}$. If i is odd, then $z_i = f_{(i+1)/2}(z)/f_{(i-1)/2}(z) = 0$, i.e. $f_{(i+1)/2}(z) = 0$ and $f_{(i-1)/2}(z) \neq 0$. By the definition of $l_j(c)$, $z_{i+2} = l_{i+2} = \infty$. From (16), $g_{(i+1)/2}(z) = g_{(i-1)/2}(z)$, i.e. $z_{i+1} = l_{i+1} = 1$. One can show the case when i is even with (17).

Suppose that $z_i = l_i = 1$ for some $i \in \{1, 2, \dots, m-2\}$. If i is odd, then $z_i = f_{(i+1)/2}(z)/f_{(i-1)/2}(z) = 1$, i.e. $f_{(i+1)/2}(z) = f_{(i-1)/2}(z)$. From (17), $g_{(i-1)/2}(z) = 0$. Hence, $z_{i-1} = l_{i-1} = 0$ and $z_{i+1} = l_{i+1} = \infty$. One can show the case when i is even with (16).

Suppose that $z_i = l_i = \infty$ for some $i \in \{1, 2, \dots, m-1\}$. If i is odd, then $z_i = f_{(i+1)/2}(z)/f_{(i-1)/2}(z) = \infty$, i.e. $f_{(i+1)/2}(z) \neq 0$ and $f_{(i-1)/2}(z) = 0$. Then, $z_{i-2} = l_{i-2} = 0$. From (16), $g_{(i-1)/2}(z) = g_{(i-3)/2}(z)$, i.e. $z_{i-1} = l_{i-1} = 1$. One can show the case when i is even with (17).

Suppose that $z_{i+m-2} = r_i = 0$ for some $i \in \{1, 2, \dots, n-3\}$. Then, $F_{m,i}(z) = 0$ and $F_{m,i-2}(z) \neq 0$. We have $z_{i+m} = r_{i+2} = \infty$. From the defining relation of $F_{m,j}(c)$, $F_{m,i+1}(z) = F_{m,i-1}(z)$, i.e. $z_{i+m-1} = r_{i+1} = 1$.

Suppose that $z_{i+m-2} = r_i = 1$ for some $i \in \{2, 3, \dots, n-2\}$, i.e. $F_{m,i}(z) = F_{m,i-2}(z)$. From the defining relation of $F_{m,j}(c)$, $(f_k(z) - f_{k-1}(z))F_{m,i-1}(z) = 0$. If $f_k(z) - f_{k-1}(z) = 0$, then $r_1 = f_k(z)/f_{k-1}(z) = 1$. This contradicts Remark 7.5. Hence, $F_{m,i-1}(z) = 0$. Then, $z_{i+m-3} = r_{i-1} = 0$ and $z_{i+m-1} = r_{i+1} = \infty$.

Suppose that $z_{i+m-2} = r_i = \infty$ for some $i \in \{3, 4, \dots, n-1\}$. Then, $F_{m,i-2}(z) = 0$ and $F_{m,i}(z) \neq 0$. We have $z_{i+m-4} = r_{i-2} = 0$. From the defining relation of $F_{m,j}(c)$, $F_{m,i-1}(z) = F_{m,i-3}(z)$, i.e. $z_{i+m-3} = r_{i-1} = 1$.

From Remark 7.5, the proof is completed.

(b) From (a), it is enough to show that if each of the sequences contains $\infty, 0$ or $\infty, 0$ in this order then it consists of only $0, 1, \infty$.

First, we consider $(l_1, l_2, \dots, l_{m-2})$. From (a), it is enough to show that if

$$(l_{i-2}, l_{i-1}, l_i, l_{i+1}, l_{i+2}, l_{i+3}) = (0, 1, \infty, 0, 1, \infty) \tag{41}$$

for some $i \in \{3, 4, \dots, m-5\}$ then $(l_1, l_2, \dots, l_{m-2})$ consists of only $0, 1, \infty$. We suppose (41). If $i > 3$ is odd, then $f_{(i-1)/2}(z) = g_{(i+1)/2}(z) = 0$, $f_{(i+3)/2}(z) = f_{(i+1)/2}(z) \neq 0$ and $g_{(i-1)/2}(z) = g_{(i-3)/2}(z) \neq 0$. From (18), $(z-1)f_{(i+1)/2}(z) = 0$. From (a), $z = 1$. Then,

$$(f_1(z), g_1(z), f_2(z), \dots) = (0, 1, -1, 0, -1, -1, 0, -1, 1, 0, 1, \dots),$$

where the period is 12. By the defining relation of l_j, $(l_1, l_2, \dots, l_{m-2})$ consists of only $0, 1, \infty$.

For $i = 3$, $l_1 = f_1(z) = 0$, i.e. z=1. In this case, $(l_1, l_2, \dots, l_{m-2})$ consists of only $0, 1, \infty$.

One can show the case when i is even similarly to the case when i is even.

Next, we consider $(r_1, r_2, \dots, r_{n-1})$. From (a), it is enough to show that if

$$(F_{m,i-2}(z), F_{m,i-1}(z), F_{m,i}(z), F_{m,i+1}(z), F_{m,i+2}(z), F_{m,i+3}(z)) = (0, 1, \infty, 0, 1, \infty) \tag{42}$$

for some $i \in \{3, 4, \dots, n-4\}$ then $(r_1, r_2, \dots, r_n)$ consists of only $0, 1, \infty$. We suppose (42). Then, $F_{m,i+1}(z) = F_{m,i-2}(z) = 0$, $F_{m,i+2}(z) = F_{m,i}(z)$ and $F_{m,i-1}(z) = F_{m,i-3}(z)$. From the defining relation of $F_{m,j}(z)$,

$$0 = F_{m,i+1}(z) = (f_k(z) - f_{k-1}(z))F_{m,i}(z) + F_{m,i-1}(z) = (f_k(z) - f_{k-1}(z))F_{m,i+2}(z) + F_{m,i-3}(z),$$
$$0 = F_{m,i+1}(z) = -(f_k(z) - f_{k-1}(z))F_{m,i+2}(z) + F_{m,i+3}(z).$$

Then,

$$F_{m,i+3}(z) = (f_k(z) - f_{k-1}(z))F_{m,i+2}(z) = -F_{m,i-3}(z).$$

Similar to this, we also have

$$F_{m,i+2}(z) = (f_k(z) - f_{k-1}(z))F_{m,i-3}(z) = -F_{m,i-4}(z).$$

We have

$$\begin{aligned} 0 &= F_{m,i-2}(z) = (f_k(z) - f_{k-1}(z))F_{m,i-3}(z) + F_{m,i-4}(z) \\ &= -(f_k(z) - f_{k-1}(z))F_{m,i+3}(z) - F_{m,i+2}(z) = F_{m,i+4}(z). \end{aligned}$$

Similar to this, we also have $F_{m,i-5}(z) = F_{m,i+1}(z) = 0$. From (a), by applying the same procedure, $(r_1, r_2, \dots, r_{n-1})$ consists of only $0, 1, \infty$. □

8. Reduced DW invariant as the sum of uncollapsed moduli

In the following, assume that if $r_1 = \infty$ then $r_2 = 2$ and if $r_1 = 0$ then $r_4 = 2$.

The purpose of this section is to prove that the reduced DW invariant of $S^3 \setminus \mathcal{T}_{m,n}$ and a parabolic representation $\rho\colon \pi_1(S^3 \setminus \mathcal{T}_{m,n}) \to \mathrm{SL}_2\mathbb{F}_p$ is equal to the sum of the uncollapsed moduli in the ideal triangulation in Figure 6.

8.1. Reduced DW invariant as the sum of uncollapsed moduli.

Consider a parabolic representation $\rho\colon \pi_1(S^3 \setminus \mathcal{T}_{m,n}) \to \mathrm{SL}_2\mathbb{F}_p$. By the correspondence in Proposition 5.2, there is $z \in \{u \in \mathbb{F}_p \mid F_n(u) = 0\}$ which corresponds to the conjugacy class $[\rho]$. Consider $(z_1, z_2, \dots, z_{m+n-2}) = (l_1, l_2, \dots, l_{m-2}, r_1, \dots, r_n)$ obtained from z by (39) (40).

Let I^+ (resp. I^-) be the set of ideal tetrahedra whose moduli do not collapse among T_i^+ ($i = 0, 1, \dots, m-3$) (resp. T_i^- ($i = 1, 2, \dots, n-2$)). We define

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho) = \sum_{T \in I^+} [\text{modulus of } T] \text{ and } \widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho) = \sum_{T \in I^-} [\text{modulus of } T],$$

where $[x]$ means x regarded as an element of $\check{\mathcal{P}}(\mathbb{F}_p)$. Then, we have the following proposition.

Proposition 8.1. *Consider the ideal triangulation of $S^3 \setminus \mathcal{T}_{m,n}$ in Figure 6. Suppose that if $r_1 = 0$ then $r_2 = 2$ and if $r_1 = 0$ then $r_4 = 2$. Then, we have*

$$\widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho) = \begin{cases} \widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho) + \widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho) + 2[r_1] & \text{if } r_1 \neq 0, \infty \\ \widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho) + \widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho) & \text{if } r_1 = 0, \infty. \end{cases}$$

From Remark 7.5 and the assumption that if $r_1 = \infty$ then $r_2 = 2$ and if $r_1 = 0$ then $r_4 = 2$, the proof is completed by combining Sections 8.2, 8.3 and 8.4.

In particular, from the assumption, $(r_1, r_2, \dots, r_{n-1})$ is not of the form (iii) in Lemma 7.6.

8.2. **The case when both of** $(l_1, l_2, \dots, l_{m-2})$ **and** $(r_1, r_2, \dots, r_n)$ **are of the form (i) in Lemma 7.6.** Suppose that both of $(l_1, l_2, \dots, l_{m-2})$ and $(r_1, r_2, \dots, r_n)$ are of the form (i).

Consider the ideal triangulation of $S^3 \setminus \mathcal{T}_{m,n}$ with the $\mathbb{P}^1(\mathbb{F}_p)$-labeling in Figure 6. Then, all the moduli of the ideal tetrahedra do not collapse. The hyperbolicity equations of the ideal triangulation is equivalent to (32). By Lemmas 7.1 and 7.3 and the definition of z_i $(i = 1, 2, \dots, m+n-2)$, $(x_1, \dots, x_{m-2}, y_1, \dots, y_n) = (z_1, z_2, \dots, z_{m+n-2})$ satisfies the hyperbolicity equations (32) for the ideal triangulation in Figure 6. Then, it is known that one can obtain a $\mathrm{PGL}_2\mathbb{F}_p$-representation from the moduli; see [12].

Since the proof of [5] can apply verbatim to this case, one can take a lift of the representation obtained from the moduli to an $\mathrm{SL}_2\mathbb{F}_p$-representation and show that the lift is a parabolic representation which is conjugate to the representation obtained from $z \in \{u \in \mathbb{F}_p \mid F_{m,n}(u) = 0\}$ by Proposition 5.2. We omit the proof; see also [12].

Recall that we write $(x, y) \equiv (x', y') \bmod (p, q)$ if $x \equiv x' \bmod p$ and $y \equiv y' \bmod q$.

Remark 8.2. For $p = 7$, the condition $(m, n) \not\equiv (6, 1), (8, 5) \bmod (14, 6)$ is equivalent to the condition that if $r_1 = 0$ then $r_4 = 2$ and if $r_1 = \infty$ then $r_2 = 2$.

8.3. **The case when neither of** $(l_1, l_2, \dots, l_{m-2})$ **and** $(r_1, r_2, \dots, r_{n-1})$ **are of the form (iii) and one of them is of the form (ii).** In each case, similar to Section 8.2, the $\mathrm{PGL}_2\mathbb{F}_p$-representation obtained from the moduli can be lifted to an $\mathrm{SL}_2\mathbb{F}_p$-representation and the lift is conjugate to the parabolic representation obtained from $z \in \{u \in \mathbb{F}_p \mid F_{m,n}(u) = 0\}$.

Case 1: $(l_1, l_2, \dots, l_{m-2})$ **is of the form (ii) and** $(r_1, r_2, \dots, r_{n-1})$ **is of the form (i).** Suppose that there is only $i \in \{2, 3, \dots, m-4\}$ such that $l_i = 0$. Then, from Figure 6, the moduli of the four ideal tetrahedra $T^+_{i-1}, T^+_i, T^+_{i+1}, T^+_{i+2}$ collapse. We replace the four ideal tetrahedra with the ideal tetrahedra in Figure 9. Note that the other ideal tetrahedra are not changed.

After replacing, all the moduli of the ideal tetrahedra do not collapse from $l_{i+3} \neq 0, 1, \infty$.

The hyperbolicity equations of the ideal triangulation are rewritten as follows:

$$\begin{cases} l_2 = l_1 + 1, \\ l_{j+1} = 1 - l_j/l_{j-1} + l_j \quad (j = 2, 3, \dots, i-2, i+4, \dots, m-2), \\ 1 - l_{i-1}/l_{i-2} + l_{i-1} = 0, \\ l_{i+4} = l_{i+3} + 1, \\ r_1 = l_{m-1}, \\ (1 - r_1/l_{m-2})(1 - r_2/r_1) = (1 - r_1)(1 - 1/r_1), \\ r_{j+1} = 1 - r_j/r_{j-1} + r_j \quad (j = 3, 4, \dots, n-1), \\ r_n = 0. \end{cases}$$

If there is $i' \neq i$ such that $l_{i'} = 0$, then we replace the four ideal tetrahedra with collapsed moduli with ideal tetrahedra as in Figure 9 and replace $l_{j+1} = 1 - l_j/l_{j-1} + l_j$ $(j = i'-1, i', i'+1, i'+2, i'+3)$ with $1 - l_{i-1}/l_{i-2} + l_{i-1} = 0, l_{i+4} = l_{i+3} + 1$ in the hyperbolicity equations.

One can verify the ideal triangulation with the $\mathbb{P}^1(\mathbb{F}_p)$-labeling satisfies the hyperbolicity equations using Lemma 8.3.

In particular, the sum of the moduli in $\hat{\mathcal{P}}(\mathbb{F}_p)$ is equal to zero since each pair of two ideal tetrahedra which are glued to each other with respect to the horizontal plane, i.e. the plane contains $a_{i-3}, a_{i-1}, a_{i+1}, a_{i+3}$, in Figure 9. Hence, from Proposition 4.1, the reduced DW invariant is equal to the sum of uncollapsed moduli in the ideal triangulation.

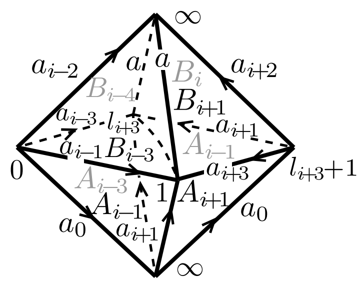

FIGURE 9. Four ideal tetrahedra

Lemma 8.3. (i) *If* $l_i = 0$, $l_{i-1} \neq \infty$ *and* $l_{i+3} \neq 0$ *for some* $i \in \{2, 3, \ldots, m-4\}$, *then* $l_{i-1}l_{i+3} = 1$.
(ii) *If* $r_i = 0$, $r_{i-1} \neq \infty$ *and* $r_{i+3} \neq 0$ *for some* $i \in \{2, 3, \ldots, m-4\}$, *then* $r_{i-1}r_{i+3} = 1$.

We will prove the lemma in Section 8.5.

Case 2: $(l_1, l_2, \ldots, l_{m-2})$ **is of the form (i),** $(r_1, r_2, \ldots, r_n)$ **is of the form (ii) and** $r_1 \neq 0$**.** Similar to Case 1, one can show this case. We omit details.

Case 3: $(l_1, l_2, \ldots, l_{m-2})$ **is of the form (i) and** $r_1 = 0$**.** Suppose that $(l_1, l_2, \ldots, l_{m-2})$ is of the form (i) and $r_1 = 0$. We replace the six ideal tetrahedra T_i^+ $(i = m-3, m-2)$, T_i^- $(i = 0, 1, 2, 3)$. After replacing, all the moduli of the ideal tetrahedra in Figure 10 do not collapse from Lemma 8.4.

The hyperbolicity equations of the ideal triangulation are rewritten as follows and one can verify the $\mathbb{P}^1(\mathbb{F}_p)$-labeling satisfies the equations:

$$\begin{cases} l_2 = l_1 + 1, \\ l_{i+1} = 1 - l_i/l_{i-1} + l_i \quad (i = 2, 3, \ldots, m-3), \\ 1/l_{m-2} = 1/l_{m-3} - 1, \\ r_5 = r_4 + 1, \\ r_{i+1} = 1 - r_i/r_{i-1} + r_i \quad (i = 5, 6, \ldots, n-1), \\ r_n = 0. \end{cases}$$

If there is $i' \in \{5, 6, \ldots, n-4\}$ such that $r_{i'} = 0$, then one can replace the ideal triangulation with that consisting of uncollapsed moduli and replace the hyperbolicity equations similarly to Case 1.

In particular, the sum of the moduli in $\hat{\mathcal{P}}(\mathbb{F}_p)$ is equal to zero since each pair of two ideal tetrahedra which are glued to each other with respect to the horizontal plane in Figure 10. Hence, from Proposition 4.1, the reduced DW invariant is equal to the sum of uncollapsed moduli in the ideal triangulation.

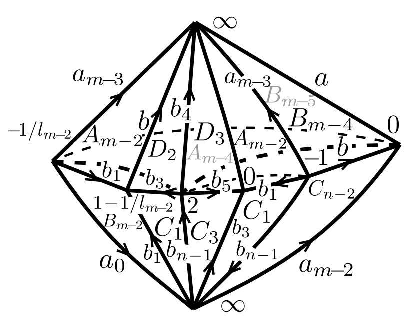

FIGURE 10. Eight ideal tetrahedra

Case 4: both of $(l_1, l_2, \ldots, l_{m-2})$ **and** $(r_1, r_2, \ldots, r_{n-1})$ **are of the form (ii) and** $r_1 \neq \infty$ Suppose that both of $(l_1, l_2, \ldots, l_{m-2})$ and $(r_1, r_2, \ldots, r_{n-1})$ are of the form (ii) and $r_1 \neq \infty$. We replace the six ideal tetrahedra T_i^+ $(i = m-5, m-4, m-3, m-2)$, T_i^- $(i = 0, 1)$ with the ideal tetrahedra in Figure 11. After replacing, all the moduli of the ideal tetrahedra in Figure 11 do not collapse from Lemma 8.4.

The hyperbolicity equations of the ideal triangulation are rewritten as follows and one can verify the $\mathbb{P}^1(\mathbb{F}_p)$-labeling satisfies the equations:

$$\begin{cases} l_2 = l_1 + 1, \\ l_{i+1} = 1 - l_i/l_{i-1} + l_i \quad (i = 2, 3, \ldots, m-5), \\ 1/l_{m-4} = 1/l_{m-5} - 1, \\ r_3 = r_2 + 1, \\ r_{i+1} = 1 - r_i/r_{i-1} + r_i \quad (i = 3, 4, \ldots, n-1), \\ r_n = 0. \end{cases}$$

If there is i' such that $l_{i'} = 0$ or $r_{i'} = 0$, then one can replace the ideal triangulation and the hyperbolicity equations similarly to Case 1.

Lemma 8.4. *(i)If* $r_1 = \infty$ *and* $l_{m-4} \neq \infty$*, then* $-1/l_{m-4} \neq 0, 1, 2, \infty$.
(ii) If $r_1 = 0$, $l_{m-2} \neq \infty$*, then* $-1/l_{m-2} \neq 0, 1, 2, \infty$.

We will show the lemma in Section 8.5.

In particular, the sum of the moduli in $\hat{\mathcal{P}}(\mathbb{F}_p)$ is equal to zero since each pair of two ideal tetrahedra which are glued to each other with respect to the horizontal plane, i.e. the plane contains $a_{m-4}, b, b_3, b_1, a_{m-2}$ in Figure 11. Hence, from Proposition 4.1, the reduced DW invariant is equal to the sum of uncollapsed moduli in the ideal triangulation.

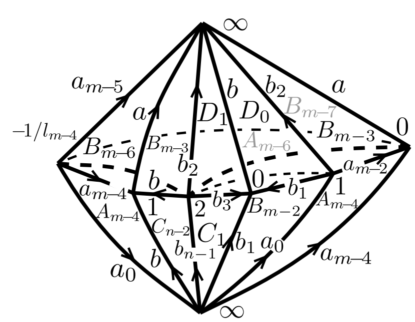

FIGURE 11. Eight ideal tetrahedra

Case 5: the rest By combining Cases 1, 2, 3 and 4, one can show any other cases.

8.4. **The case when one of** $(l_1, l_2, \ldots, l_{m-2})$ **and** $(r_1, r_2, \ldots, r_{n-1})$ **is of the form (iii) in Lemma 7.6.** In each case, similar to Section 8.2, the $\mathrm{PGL}_2\mathbb{F}_p$-representation obtained from the moduli can be lifted to an $\mathrm{SL}_2\mathbb{F}_p$-representation and the lift is conjugate to the parabolic representation obtained from $z \in \{u \in \mathbb{F}_p \mid F_{m,n}(u) = 0\}$ by concrete calculation.

From the assumption that if $r_1 = \infty$ then $r_2 = 2$ and if $r_1 = 0$ then $r_4 = 2$, it is enough to show Cases 6 and 7.

Case 6: $(l_1, l_2, \ldots, l_{m-2})$ **is of the form (iii) and** $r_1 = \infty$**.** Suppose that $(l_1, l_2, \ldots, l_{m-2})$ is of the form (iii) and $r_1 = \infty$. Then, $m = 6i + 4$ $(i \geq 0)$.

First, we consider the case of $m = 4$. In this case, we replace the five ideal tetrahedra T_i^+ ($i = 0, 1, 2$), T_j^- ($j = 0, 1$) with the ideal tetrahedra in Figure 12.

The hyperbolicity equations of the ideal triangulation are rewritten as follows and one can verify the $\mathbb{P}^1(\mathbb{F}_p)$-labeling satisfies the equations:

$$\begin{cases} r_3 = r_2 + 1, \\ r_{i+1} = 1 - r_i/r_{i-1} + r_i \quad (i = 3, 4, \dots, n-1), \\ r_n = 0. \end{cases}$$

If there is $i' \in \{3, 4, \dots, n-4\}$ such that $r_{i'} = 0$, one can replace the ideal tetrahedra and the hyperbolicity equations similar to Case 1.

In particular, the sum of the moduli in $\hat{\mathcal{P}}(\mathbb{F}_p)$ is equal to zero since each pair of two ideal tetrahedra which are glued to each other with respect to the horizontal plane in Figure 11. Hence, from Proposition 4.1, the reduced DW invariant is equal to the sum of uncollapsed moduli in the ideal triangulation.

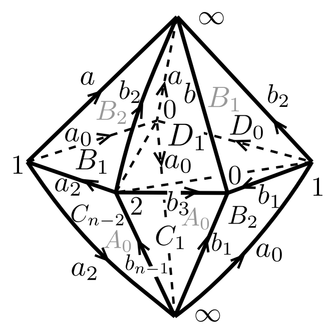

FIGURE 12. Six ideal tetrahedra

Next, we consider the case of $m = 6i + 4 > 4$. In this case, we replace the six ideal tetrahedra T_i^+ ($i = m-5, m-4, m-3, m-2$), T_j^- ($j = 0, 1$) with the ideal tetrahedra in Figure 13. Here, ε_i is defined by

$$\varepsilon_i = \begin{cases} 2 & \text{if } i \equiv 0 \bmod 3, \\ 1/2 & \text{if } i \equiv 1 \bmod 3, \\ -1 & \text{if } i \equiv 2 \bmod 3. \end{cases} \tag{43}$$

The hyperbolicity equations of the ideal triangulation are rewritten as follows and one can verify the $\mathbb{P}^1(\mathbb{F}_p)$-labeling satisfies the equations:

$$\begin{cases} r_3 = r_2 + 1, \\ r_{i+1} = 1 - r_i/r_{i-1} + r_i \quad (i = 3, 4, \dots, n-1), \\ r_n = 0. \end{cases}$$

In particular, the sum of the moduli in $\hat{\mathcal{P}}(\mathbb{F}_p)$ is equal to zero since each pair of two ideal tetrahedra which are glued to each other with respect to the horizontal plane in Figure 13. Hence, from Proposition 4.1, the reduced DW invariant is equal to the sum of uncollapsed moduli in the ideal triangulation.

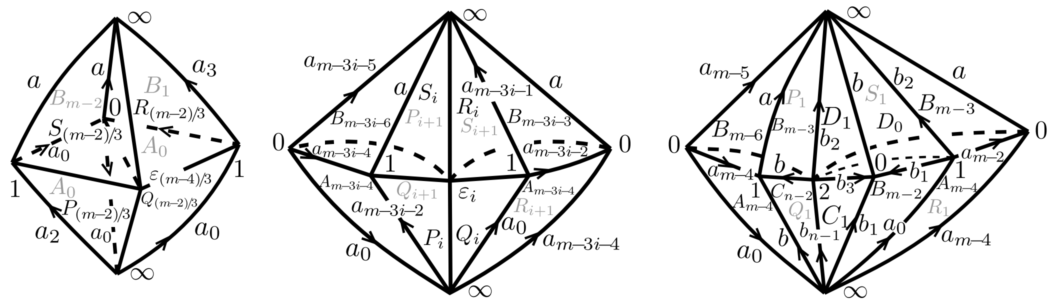

FIGURE 13. In the left, the middle, the right, there are four, four, six ideal tetrahedra respectively.

Case 7: $(l_1, l_2, \ldots, l_{m-2})$ **is of the form (iii) and** $r_1 = 0$**.** Suppose that $(l_1, l_2, \ldots, l_{m-2})$ is of the form (iii) and $r_1 = 0$. Then, $m = 6i + 2$ (≥ 1). In this case, we replace the six ideal tetrahedra T_i^+ $(i = m-3, m-2)$, T_j^- $(j = 0, 1, 2, 3)$ with the ideal tetrahedra in Figure 14. Here, ε_i is defined by (43).

The hyperbolicity equations of the ideal triangulation are rewritten as follows and one can verify the $\mathbb{P}^1(\mathbb{F}_p)$-labeling satisfies the equations:

$$\begin{cases} r_5 = r_4 + 1 \\ r_{i+1} = 1 - r_i/r_{i-1} + r_i \quad (i = 5, 6, \ldots, n-1) \\ r_n = 0 \end{cases}$$

If need, one can replace the ideal triangulation and the hyperbolicity equation similarly to Case 1.

In particular, the sum of the moduli in $\hat{\mathcal{P}}(\mathbb{F}_p)$ is equal to zero since each pair of two ideal tetrahedra which are glued to each other with respect to the horizontal plane in Figure 14. Hence, from Proposition 4.1, the reduced DW invariant is equal to the sum of uncollapsed moduli in the ideal triangulation.

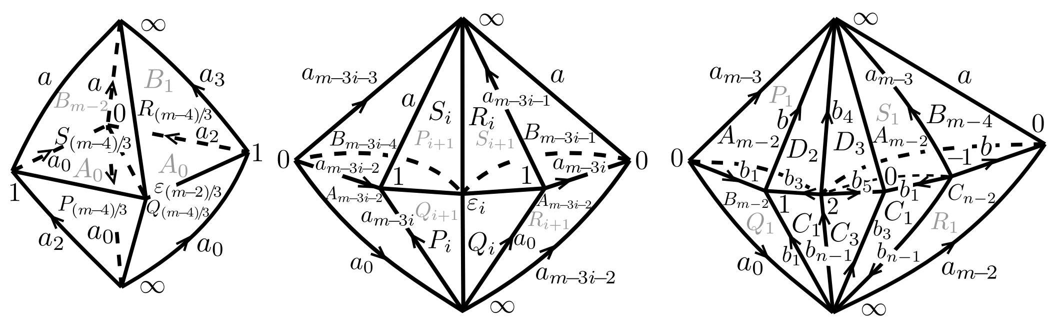

FIGURE 14. In the left, the middle, the right, there are four, four, six ideal tetrahedra respectively.

8.5. **Proof of Lemmas 8.3 and 8.4.**

Proof of Lemma 8.3. (i) Suppose that $l_i = 0$, $l_{i-1} \neq \infty$ and $l_{i+3} \neq 0$ for some $i \in \{2, 3, \ldots, m-4\}$. Then, from (a) in Lemma 7.6, $l_{i+1} = 1$ and $l_{i-1}, l_{i+3} \in \{2, 3, \ldots, p-1\}$.

First, we consider the case when i is odd. Then, from $l_{i+1} = g_{(i+1)/2}(z)/g_{(i-1)/2}(z) = 1$, $g_{(i+1)/2}(z) = g_{(i-1)/2}(z)$. By the definition of $g_j(c)$,

$$g_{(i+3)/2}(z) = (2-z)g_{(i+1)/2}(z) - g_{(i-1)/2}(z) = g_{(i-1)/2}(z) - g_{(i+1)/2}(z) = g_{(i-3)/2}(z).$$

Hence,

$$l_{i-1}l_{i+3} = (g_{(i-1)/2}(z)/g_{(i-3)/2}(z))(g_{(i+3)/2}(z)/g_{(i+1)/2}(z)) = 1.$$

Similar to the above case, one can show the claim in the case when i is even. We omit details.

(ii) Suppose that $r_i = 0$, $r_{i-1} \neq \infty$ and $r_{i+3} \neq 0$ for some $i \in \{2, 3, \dots, m-4\}$, Then, from Lemma 7.6, $r_{i+1} = 1$ and $r_{i-1}, r_{i+3} \in \{2, 3, \dots, p-1\}$. Then, from $r_1 = F_i(z)/F_{i-2}(z) = 0$, $F_i(z) = 0$ and $F_{i-2}(z) \neq 0$. From $r_{i+1} = F_{i+1}(z)/F_{i-1}(z) = 1$, $F_{i+1}(z) = F_{i-1}(z)$. From the definition of $F_j(c)$ and $F_i(z) = 0$,

$$\begin{aligned} F_{i-3}(z) &= -(f_k(z) - f_{k-1}(z))F_{i-2}(z) + F_{i-1}(z) \\ &= -(f_k(z) - f_{k-1}(z))\{-(f_k(z) - f_{k-1}(z))F_{i-1}(z)\} + F_{i-1}(z) \\ &= = -(f_k(z) - f_{k-1}(z))\{-(f_k(z) - f_{k-1}(z))F_{i+1}(z)\} + F_{i+1}(z) \\ &= -(f_k(z) - f_{k-1}(z))F_{i+2}(z) + F_{i+1}(z) = F_{i+3}(z). \end{aligned}$$

Hence,

$$r_{i-1}r_{i+3} = (F_{i-1}(z)/F_{i-3}(z))(F_{i+3}(z)/F_{i+1}(z)) = 1.$$

□

Proof of Lemma 8.4. (i) Suppose that $r_1 = \infty$ and $l_{m-4} \neq \infty$. Then, $l_{m-2} = 1$ and $l_{m-3} = 0$. By the definition of l_i, we have $g_{k-1}(z) = g_{k-2}(z)$, $f_{k-1}(z) = 0$ and $f_{k-2}(z) \neq 0$. From $l_{m-4} = g_{k-2}(z)/g_{k-3}(z) \neq \infty$, $g_{k-3}(z) \neq 0$.

If $g_{k-2}(z) = g_{k-1}(z) = 0$, it contradicts (a) in Lemma 7.6. Hence, $-1/l_{m-4} \in \mathbb{F}_p \setminus \{0\}$.

Suppose that $-1/l_{m-4} = 1$, i.e. $g_{k-3}(z) = -g_{k-2}(z)$. Then,

$$0 = g_{k-1}(z) - (2-z)g_{k-2}(z) + g_{k-3}(z) = -(2-z)g_{k-1}(z).$$

From $g_{k-1}(z) \neq 0$, $z = 2$. However, if $z = 2$, then l_{m-4} must be ∞. It contradicts $l_{m-4} \neq \infty$.

Suppose that $-1/l_{m-4} = 2$, i.e. $g_{k-3}(z) = -2g_{k-2}(z)$. Then,

$$0 = g_{k-1}(z) - (2-z)g_{k-2}(z) + g_{k-3}(z) = -(3-z)g_{k-1}(z).$$

From $g_{k-1}(z) \neq 0$, $z = 3$. However, $f_i(3) \neq 0$ for any i. It contradicts $r_1 = \infty$.

(ii) Suppose that $r_1 = 0$ and $l_{m-2} \neq \infty$. Then, $r_2 = 1$ and $r_3 = \infty$. By the definition of r_1, we have $F_{m,1}(z) = f_k(z) = 0$. From $l_{m-2} = g_{k-1}(z)/g_{k-2}(z) \neq \infty$, $g_{k-2}(z) \neq 0$. From $f_k = 0$, $g_{k-1}(z) \neq 0$. Hence, $-1/l_{m-4} \in \mathbb{F}_p \setminus \{0\}$.

Suppose that $-1/l_{m-2} = s \in \mathbb{F}_p \setminus \{0\}$, i.e. $g_{k-2}(z) = -sg_{k-1}(z)$. From (16),

$$(s+1)g_{k-1}(c) = f_k(c) = 0.$$

If $s \neq -1$, it contradicts (a) in Lemma 7.6. □

9. Proof of theorem

9.1. **Proof of theorem.** Before we prove the theorems, we review the isomorphism between $\check{\mathcal{B}}(\mathbb{F}_7)$ and $\mathbb{Z}/2\mathbb{Z}$.

Lemma 9.1 (A part of Lemma A.2 in [5])**.** *We have the isomorphism*

$$\phi\colon \check{\mathcal{B}}(\mathbb{F}_7) \to \mathbb{Z}/2\mathbb{Z}$$

given by

$$\phi([2]) = \phi([4]) = \phi([6]) = \phi([5]-[3]) = 1.$$

In the following, we identify $\check{\mathcal{B}}(\mathbb{F}_7)$ and $\mathbb{Z}/2\mathbb{Z}$ through the above isomorphism.

We will consider each case of $r_1 \in \mathbb{P}^1(\mathbb{F}_p) \setminus \{1\}$. In the following, let ρ_z denote the parabolic representation of $\pi_1(S^3 \setminus \mathcal{T}_{m,n})$ obtained from the value $z \in \{u \in \mathbb{F}_p \,|\, F_{m,n}(u) = 0\}$ by Proposition 5.2.

The case of $r_1 = 2$. If $r_1 = 2$, then $z = 4$ or 6.

In the case of $z = 4$, $m \equiv 12 \bmod 14$ from the sequence in Section 9.2. Since the period of $(f_1(4), g_1(4), f_2(4), \dots)$ is 28, there are two possible cases: $f_k(4) \equiv 6,\ f_k(4) - f_{k-1}(4) \equiv 3 \bmod 7$ if $m \equiv 12 \bmod 28$, and $f_k(4) \equiv -6,\ f_k(4) - f_{k-1}(4) \equiv -3 \bmod 7$ if $m \equiv 26 \bmod 28$. By the definition of $F_{m,i}(c)$, we have the sequence

$$(F_{m,0}(4), F_{m,1}(4), F_{m,2}(4), F_{m,3}(4), \dots).$$

Although the sequences of $m \equiv 12, 26 \bmod 28$ are different, the sequences

$$(f_k(4)/f_{k-1}(4), F_{m,2}(4)/F_{m,0}(4), F_{m,3}(4)/F_{m,1}(4), \dots) = (2, 5, 0, 1, \infty, 3, 4, 6, 2, 5, \dots) \tag{44}$$

obtained from them are the same since both of $f_k(4)$ and $f_k(4) - f_{k-1}(4)$ are the minus of the other ones. Since the period of (44) is 8, $n \equiv 3 \bmod 8$.

If $m = 14i + 12$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) = (2[4] + 4[5])(m+3)/14 = 0 \in \mathbb{Z}/2\mathbb{Z},$$

where $\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z)$ is defined in Section 8.1.

If $n = 8i + 3$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) = [6] + (2[6] + 2[5])(n-3)/8 = (n+5)/8 \in \mathbb{Z}/2\mathbb{Z},$$

where $\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z)$ is defined in Section 8.1.

Hence, we have

$$\begin{aligned} \widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_z) &= \widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) + 2[r_1] + \widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) \\ &= 2[2] + [6] + (2[4] + 4[5])(m+3)/14 + (2[6] + 2[5])(n-3)/8 \\ &= 1 + (n-3)/8 \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

In the case of $z = 6$, $m \equiv 2 \bmod 8$ from the sequence in Section 9.2. Since the period of $(f_1(6), g_1(6), f_2(6), \dots)$ is 16, there are two possible cases: $f_k(6) \equiv 2,\ f_k(6) - f_{k-1}(6) \equiv 1 \bmod 7$ if $m \equiv 2 \bmod 16$, and $f_k(6) \equiv -2,\ f_k(6) - f_{k-1}(6) \equiv -1 \bmod 7$ if $m \equiv 10 \bmod 16$. By the definition of $F_{m,i}(c)$, we have the sequence

$$(F_{m,0}(6), F_{m,1}(6), F_{m,2}(6), F_{m,3}(6), \dots).$$

Although the sequences of $m \equiv 2, 10 \bmod 16$ are different, the sequences

$$(f_k(6)/f_{k-1}(6), F_{m,2}(6)/F_{m,0}(6), F_{m,3}(6)/F_{m,1}(6), \dots) = (2, 3, 6, 5, 4, 0, 1, \infty, 2, 3, \dots) \tag{45}$$

obtained from them are the same since both of $f_k(6)$ and $f_k(6) - f_{k-1}(6)$ are the minus of the other ones. Since the period of (45) is 8, $n \equiv 6 \bmod 8$.

If $m = 8i + 2$ $(i \geq 1)$, we have

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) = (2[3] + 2[4])(m-2)/8 = (m-2)/8 \in \mathbb{Z}/2\mathbb{Z}.$$

If $n = 8i + 6$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) = (2[2] + 2[5])(n+2)/8 = (n+2)/8 \in \mathbb{Z}/2\mathbb{Z}.$$

Hence, we have

$$\begin{aligned}\widehat{\mathrm{DW}}(\mathcal{T}_{m,n},\rho_z) &= \widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n},\rho_z)+2[r_1]+\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n},\rho_z)\\ &= 2[2]+(2[3]+2[4])(m-2)/8+(2[2]+2[5])(n+2)/8\\ &= (m-2)/8+(n+2)/8\in\mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

The case of $r_1=3$. If $r_1=3$, then $z=3,4$ or 5.

In the case of $z=3$, $m\equiv 4 \bmod 6$ from the sequence in Section 9.2. Then, $f_k(3)\equiv 1$ and $f_k(3)-f_{k-1}(3)\equiv 3 \bmod 7$. By the definition of $F_{m,i}(c)$,

$$(f_k(4)/f_{k-1}(4), F_{m,2}(4)/F_{m,0}(4), F_{m,3}(4)/F_{m,1}(4),\dots)=(3,4,6,2,5,0,1,\infty,3,4,\dots). \tag{46}$$

Since the period of (46) is 8, $n\equiv 6 \bmod 8$.

If $m=6i+4$ $(i\geq 0)$, we have

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n},\rho_z)=2[2](m+2)/6=0.$$

If $n=8i+6$ $(i\geq 0)$, we have

$$\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n},\rho_z)=(2[5]+2[6])(n+2)/8=(n+2)/8.$$

Hence, we have

$$\begin{aligned}\widehat{\mathrm{DW}}(\mathcal{T}_{m,n},\rho_z) &= \widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n},\rho_z)+2[r_1]+\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n},\rho_z)\\ &= 2[3]+2[2](m+2)/6+(2[5]+2[6])(n+2)/8\\ &= 1+(n+2)/8\in\mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

In the case of $z=4$, $m\equiv 4 \bmod 14$ from the sequence in Section 9.2. Similar to the case of $r_1=2$ and $z=4$, for $m\equiv 4,18 \bmod 28$, we have the sequence

$$(f_k(4)/f_{k-1}(4), F_{m,2}(4)/F_{m,0}(4), F_{m,3}(4)/F_{m,1}(4),\dots)=(3,6,5,4,0,1,\infty,2,3,6,\dots). \tag{47}$$

Since the period of (47) is 8, $n\equiv 5 \bmod 8$.

If $m=14i+4$ $(i\geq 0)$, we have

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n},\rho_z)=[4]+[5]+([4]+2[5])(m-4)/14.$$

If $n=8i+5$ $(i\geq 0)$, we have

$$\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n},\rho_z)=2[2]+[5]+(2[2]+2[5])(n-5)/8.$$

Hence, we have

$$\begin{aligned}\widehat{\mathrm{DW}}(\mathcal{T}_{m,n},\rho_z) &= 2[3]+2[2]+[4]+2[5]+([4]+2[5])(m-4)/14+(2[2]+2[5])(n-5)/8\\ &= 1+(n-5)/8\in\mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

In the case of $z=5$, $m\equiv 2 \bmod 8$ from the sequence in Section 9.2. Since the period of $(f_1(5), g_1(5), f_2(5),\dots)$ is 16, there are two possible cases: $f_k(5)\equiv 3$, $f_k(5)-f_{k-1}(5)\equiv 2 \bmod 7$ if $m\equiv 2 \bmod 16$, and $f_k(5)\equiv -3$, $f_k(5)-f_{k-1}(5)\equiv -2 \bmod 7$ if $m\equiv 10 \bmod 16$. By the definition of $F_{m,i}(c)$, we have the sequence

$$(F_{m,0}(5), F_{m,1}(5), F_{m,2}(5), F_{m,3}(5),\dots).$$

Although the sequences of $m\equiv 2,10 \bmod 16$ are different, the sequences

$$(f_k(5)/f_{k-1}(5), F_{m,2}(5)/F_{m,0}(5), F_{m,3}(5)/F_{m,1}(5),\dots)=(3,0,1,\infty,5,6,3,0,\dots) \tag{48}$$

obtained from them are the same since both of $f_k(5)$ and $f_k(5)-f_{k-1}(5)$ are the minus of the other ones. Since the period of (48) is 6, $n\equiv 2 \bmod 6$.

If $m = 8i + 2$ $(i \geq 1)$, we have

$$\widehat{\mathrm{DW}}^{+}(\mathcal{T}_{m,n}, \rho_z) = (2[3] + 2[6])(m + 6)/8.$$

If $n = 6i + 2$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^{-}(\mathcal{T}_{m,n}, \rho_z) = 2[4](n - 2)/6.$$

Hence, we have

$$\begin{aligned}\widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_z) &= \widehat{\mathrm{DW}}^{+}(\mathcal{T}_{m,n}, \rho_z) + 2[r_1] + \widehat{\mathrm{DW}}^{-}(\mathcal{T}_{m,n}, \rho_z) \\ &= 2[3] + (2[3] + 2[6])(m + 6)/8 + 2[4](n - 2)/6 \\ &= 1 + (m + 6)/8 \in \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

The case of $r_1 = 4$. If $r_1 = 4$, then $z = 4$ or 6.

In the case of $z = 4$, $m \equiv 2 \bmod 14$ from the sequence in Section 9.2. Similar to the case of $r_1 = 2$ and $z = 4$, for $m \equiv 2 \bmod 14$, we have the sequence

$$(f_k(4)/f_{k-1}(4), F_{m,2}(4)/F_{m,0}(4), F_{m,3}(4)/F_{m,1}(4), \dots) = (4, 6, 2, 5, 0, 1, \infty, 3, 4, 6, \dots). \tag{49}$$

Since the period of (49) is 8, $n \equiv 5 \bmod 8$.

If $m = 14i + 2$ $(i \geq 1)$, we have

$$\widehat{\mathrm{DW}}^{+}(\mathcal{T}_{m,n}, \rho_z) = (2[4] + 4[5])(m - 2)/14.$$

If $n = 8i + 5$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^{-}(\mathcal{T}_{m,n}, \rho_z) = 2[5] + [6] + (2[5] + 2[6])(n - 5)/8.$$

Hence, we have

$$\begin{aligned}\widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_z) &= 2[4] + 2[5] + [6] + (2[4] + 4[5])(m - 2)/14 + (2[5] + 2[6])(n - 5)/8 \\ &= (n - 5)/8 \in \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

In the case of $z = 6$, $m \equiv 6 \bmod 8$ from the sequence in Section 9.2. Similar to the case of $r_1 = 2$ and $z = 6$, for $m \equiv 6 \bmod 8$, we have the sequence

$$(f_k(6)/f_{k-1}(6), F_{m,2}(6)/F_{m,0}(6), F_{m,3}(6)/F_{m,1}(6), \dots) = (4, 0, 1, \infty, 2, 3, 6, 5, 4, 0, \dots). \tag{50}$$

Since the period of (50) is 8, $n \equiv 2 \bmod 8$.

If $m = 8i + 6$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^{+}(\mathcal{T}_{m,n}, \rho_z) = (2[3] + 2[4])(m + 2)/8.$$

If $n = 8i + 2$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^{-}(\mathcal{T}_{m,n}, \rho_z) = (2[2] + 2[5])(n - 2)/8.$$

Hence, we have

$$\begin{aligned}\widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_z) &= 2[4] + (2[3] + 2[4])(m + 2)/8 + (2[2] + 2[5])(n - 2)/8 \\ &= (m + 2)/8 + (n - 2)/8 \in \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

The case of $r_1 = 5$. If $r_1 = 5$, then $z = 3, 4$ or 5.

In the case of $z = 3$, $m \equiv 2 \bmod 6$ from the sequence in Section 9.2. Similar to the case of $r_1 = 3$ and $z = 3$, for $m \equiv 2 \bmod 6$, we have the sequence

$$(f_k(6)/f_{k-1}(6), F_{m,2}(6)/F_{m,0}(6), F_{m,3}(6)/F_{m,1}(6), \dots) = (5, 0, 1, \infty, 3, 4, 6, 2, 5, 0, \dots) \tag{51}$$

Since the period of (51) is 8, $n \equiv 2 \bmod 8$.

If $m = 6i + 2$ $(i \geq 1)$, we have

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) = 2[2](m-2)/6.$$

If $n = 8i + 2$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) = (2[5] + 2[6])(n-2)/8.$$

Hence, we have

$$\begin{aligned} \widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_z) &= 2[5] + 2[2](m-2)/6 + (2[5] + 2[6])(n-2)/8 \\ &= 1 + (n-2)/8 \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

In the case of $z = 4$, $m \equiv 10 \bmod 14$ from the sequence in Section 9.2. Similar to the case of r_1 and $z = 4$, for $m \equiv 10 \bmod 14$, we have the sequence

$$(f_k(4)/f_{k-1}(4), F_{m,2}(4)/F_{m,0}(4), F_{m,3}(4)/F_{m,1}(4), \dots) = (5, 4, 0, 1, \infty, 2, 3, 6, 5, 4, \dots). \tag{52}$$

Since the period of (52) is 8, $n \equiv 3 \bmod 8$.

If $m = 14i + 10$ $(i \geq 1)$, we have

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) = [4] + 3[5] + (2[4] + 4[5])(m-10)/14.$$

If $n = 8i + 3$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) = [5] + (2[2] + 2[5])(n-3)/8.$$

Hence, we have

$$\begin{aligned} \widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_z) &= [4] + 6[5] + (2[4] + 4[5])(m-10)/14 + (2[2] + 2[5])(n-3)/8 \\ &= (n-3)/8 \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

In the case of $z = 5$, $m \equiv 6 \bmod 8$. Similar to the case of $r_1 = 3$ and $z = 5$, $m \equiv 6 \bmod 8$, we have the sequence

$$(f_k(5)/f_{k-1}(5), F_{m,2}(5)/F_{m,0}(5), F_{m,3}(5)/F_{m,1}(5), \dots) = (5, 6, 3, 0, 1, \infty, 5, 6, \dots). \tag{53}$$

Since the period of (53) is 6, $n \equiv 4 \bmod 6$.

If $m = 8i + 6$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) = (2[3] + 2[6])(m+2)/8.$$

If $n = 6i + 4$ $(i \geq 1)$, we have

$$\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) = 2[4](n+2)/6.$$

Hence, we have

$$\begin{aligned} \widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_z) &= 2[5] + (2[3] + 2[6])(m+2)/8 + 2[4](n+2)/6 \\ &= 1 + (m+2)/8 \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

The case of $r_1 = 6$. If $r_1 = 6$, then $z = 2, 5$ or 6.

In the case of $z = 2$, $m \equiv 2 \bmod 4$. Since the period of the corresponding sequence in Section 9.2 is 8, there are two possible cases: $f_k(2) \equiv 6$, $f_k(2) - f_{k-1}(2) \equiv 5 \bmod 7$ if $m \equiv 2 \bmod 8$, and $f_k(2) \equiv -6$, $f_k(2) - f_{k-1}(2) \equiv -5 \bmod 7$ if $m \equiv 6 \bmod 8$. By the definition of $F_{m,i}(c)$, we have the sequence

$$(F_{m,0}(2), F_{m,1}(2), F_{m,2}(2), F_{m,3}(2), \dots).$$

Although the sequences of $m \equiv 2, 6 \bmod 8$ are different, the sequences

(54) $$(f_k(2)/f_{k-1}(2), F_{m,2}(2)/F_{m,0}(2), F_{m,3}(2)/F_{m,1}(2), \dots) = (6, 3, 0, 1, \infty, 5, 6, 3, \dots)$$

obtained from them are the same since both of $f_k(2)$ and $f_k(2) - f_{k-1}(2)$ are the minus of the other ones. Since the period of (54) is 6, $n \equiv 3 \bmod 6$.

If $m = 4i + 2$ $(i \geq 1)$, we have

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) = 0.$$

If $n = 6i + 3$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) = [4] + 2[4](n-3)/6.$$

Hence, we have

$$\begin{aligned} \widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_z) &= \widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) + 2[r_1] + \widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) \\ &= 2[6] + [4] + 2[4](n-3)/6 = 1 \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

In the case of $z = 5$, $m \equiv 4 \bmod 8$. Similar to the case of $r_1 = 3$ and $z = 5$, for $m \equiv 4 \bmod 8$, we have the sequence

(55) $$(f_k(5)/f_{k-1}(5), F_{m,2}(5)/F_{m,0}(5), F_{m,3}(5)/F_{m,1}(5), \dots) = (6, 5, 4, 0, 1, \infty, 2, 3, 6, 5, \dots).$$

Since the period of (55) is 8, $n \equiv 4 \bmod 8$.

If $m = 8i + 4$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) = [3] + [6] + (2[3] + 2[6])(m-4)/8.$$

If $n = 8i + 4$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) = [2] + [5] + (2[2] + 2[5])(n-4)/8.$$

Hence, we have

$$\begin{aligned} \widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_z) &= [2] + [3] + [5] + 3[6] + (2[3] + 2[6])(m-4)/8 + (2[2] + 2[5])(n-4)/8 \\ &= 1 + (m-4)/8 + (n-4)/8 \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

In the case of $z = 6$, $m \equiv 4 \bmod 8$. Similar to the case of $r_1 = 6$ and $z = 6$, for $m \equiv 4 \bmod 8$, we have the sequence

(56) $$(f_k(6)/f_{k-1}(6), F_{m,2}(6)/F_{m,0}(6), F_{m,3}(6)/F_{m,1}(6), \dots) = (6, 2, 5, 0, 1, \infty, 3, 4, 6, 2, \dots).$$

Since the period of (56) is 8, $n \equiv 4 \bmod 8$.

If $m = 8i + 4$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) = [3] + [4] + (2[3] + 2[4])(m-4)/8.$$

If $n = 8i + 4$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) = [5] + [6] + (2[5] + 2[6])(n-4)/8.$$

Hence, we have

$$\begin{aligned} \widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_z) &= [3] + [4] + [5] + 3[6] + (2[3] + 2[4])(m-4)/8 + (2[5] + 2[6])(n-4)/8 \\ &= 1 + (m-4)/8 + (n-4)/8 \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

The case of $r_1 = 0$. If $r_1 = 0$, then $z = 1, 4$ or 4.

In the case of $z = 1$, $m \equiv 2 \bmod 6$. Since the period of the corresponding sequence in Section 9.2 is 12, there are two possible cases: $f_k(1) \equiv 0$, $f_k(1) - f_{k-1}(1) \equiv -1 \bmod 7$ if $m \equiv 1 \bmod 12$, and $f_k(1) \equiv 0,\ f_k(1) - f_{k-1}(1) \equiv 1 \bmod 7$ if $m \equiv 7 \bmod 12$. By the definition of $F_{m,i}(c)$, we have the sequence

$$(F_{m,0}(1), F_{m,1}(1), F_{m,2}(1), F_{m,3}(1), \dots).$$

Although the sequences of $m \equiv 1, 7 \bmod 12$ are different, the sequences

$$(f_k(1)/f_{k-1}(1), F_{m,2}(1)/F_{m,0}(1), F_{m,3}(1)/F_{m,1}(1), \dots) = (0, 1, \infty, 2, 3, 6, 5, 4, 0, 1, \dots) \tag{57}$$

obtained from them are the same since both of $f_k(1)$ and $f_k(1) - f_{k-1}(1)$ are the minus of the other ones. Since the period of (57) is 8, $n \equiv 1 \bmod 8$.

If $m = 6i + 2$ $(i \geq 1)$, we have

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) = 0.$$

If $n = 8i + 1$ $(i \geq 1)$, we have

$$\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) = (2[2] + 2[5])(n-1)/8.$$

Hence, we have

$$\begin{aligned}\widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_z) &= \widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) + 2[r_1] + \widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) \\ &= (2[2] + 2[5])(n-1)/8 = (n-1)/8 \in \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

In the case of $z = 4$, $m \equiv 6 \bmod 14$. Similar to the case of $r_1 = 2$ and $z = 4$, for $m \equiv 6 \bmod 14$ we have the sequence

$$(f_k(4)/f_{k-1}(4), F_{m,2}(4)/F_{m,0}(4), F_{m,3}(4)/F_{m,1}(4), \dots) = (0, 1, \infty, 5, 6, 3, 0, 1, \dots).$$

By the assumption, we do not treat the case of $m \equiv 6 \bmod 14$ and $n \equiv 1 \bmod 6$.

The case of $r_1 = \infty$. If $r_1 = 6$, then $z = 1$ or 4.

In the case of $z = 1$, $m \equiv 4 \bmod 6$. Similar to the case of $r_1 = 0$ and $z = 1$, for $m \equiv 4 \bmod 6$, we have the sequence

$$(f_k(1)/f_{k-1}(1), F_{m,2}(1)/F_{m,0}(1), F_{m,3}(1)/F_{m,1}(1), \dots) = (\infty, 2, 3, 6, 5, 4, 0, 1, \infty, 2, \dots). \tag{58}$$

Since the period of (58) is 8, $n \equiv 7 \bmod 8$.

If $m = 6i + 4$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^+(\mathcal{T}_{m,n}, \rho_z) = 0.$$

If $n = 8i + 7$ $(i \geq 0)$, we have

$$\widehat{\mathrm{DW}}^-(\mathcal{T}_{m,n}, \rho_z) = (2[2] + 2[5])(n+1)/8.$$

Hence, we have

$$\widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_z) = (2[2] + 2[5])(n+1)/8 = (n+1)/8 \in \mathbb{Z}/2\mathbb{Z}.$$

In the case of $z = 4$, $m \equiv 8 \bmod 14$. Similar to the case of $r_1 = 2$ and $z = 4$, for $m \equiv 8 \bmod 14$, we have the sequence

$$(f_k(4)/f_{k-1}(4), F_{m,2}(4)/F_{m,0}(4), F_{m,3}(4)/F_{m,1}(4), \dots) = (\infty, 5, 6, 3, 0, 1, \infty, 5, \dots).$$

By the assumption, we do not treat the case of $m \equiv 8 \bmod 14$ and $n \equiv 5 \bmod 6$.

By identifying the additive group $\mathbb{Z}/2\mathbb{Z}$ and the multiplicative group $\langle t \,|\, t^2 = 1 \rangle$ naturally, from the above calculations, we have

$$\widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \mathbb{F}_7) = \sum_{r_1} \widehat{\mathrm{DW}}(\mathcal{T}_{m,n}, \rho_{r_1}) = A_{m,n} + B_{m,n} + C_{m,n} + D_{m,n} + E_{m,n} \in \mathbb{Z}[\langle t \,|\, t^2 = 1 \rangle],$$

where

$$A_{m,n} = \begin{cases} t^{(n+5)/8} & \text{if } m \equiv 12 \pmod{14} \text{ and } n \equiv 3 \pmod 8, \\ t^{(n+3)/8} & \text{if } m \equiv 4 \pmod{14} \text{ and } n \equiv 5 \pmod 8, \\ t^{(n-5)/8} & \text{if } m \equiv 2 \pmod{14} \text{ and } n \equiv 5 \pmod 8, \\ t^{(n-3)/8} & \text{if } m \equiv 10 \pmod{14} \text{ and } n \equiv 3 \pmod 8, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{m,n} = \begin{cases} t^{(m+n)/8} & \text{if } m \equiv 2 \pmod 8 \text{ and } n \equiv 6 \pmod 8, \\ t^{(m+n)/8} & \text{if } m \equiv 6 \pmod 8 \text{ and } n \equiv 2 \pmod 8, \\ 0 & \text{otherwise,} \end{cases}$$

$$C_{m,n} = \begin{cases} t^{(n+10)/8} & \text{if } m \equiv 4 \pmod 6 \text{ and } n \equiv 6 \pmod 8, \\ t^{(n+6)/8} & \text{if } m \equiv 2 \pmod 6 \text{ and } n \equiv 2 \pmod 8, \\ t^{(n-1)/8} & \text{if } m \equiv 2 \pmod 6 \text{ and } n \equiv 1 \pmod 8, \\ t^{(n+1)/8} & \text{if } m \equiv 4 \pmod 6 \text{ and } n \equiv 7 \pmod 8, \\ 0 & \text{otherwise,} \end{cases}$$

$$D_{m,n} = \begin{cases} t^{(m+14)/8} & \text{if } m \equiv 2 \pmod 8 \text{ and } n \equiv 2 \pmod 6, \\ t^{(m+10)/8} & \text{if } m \equiv 6 \pmod 8 \text{ and } n \equiv 4 \pmod 6, \\ 0 & \text{otherwise,} \end{cases}$$

$$E_{m,n} = \begin{cases} t & \text{if } m \equiv 2 \pmod 4 \text{ and } n \equiv 3 \pmod 6, \\ 0 & \text{otherwise.} \end{cases}$$

9.2. **Table of sequences.** In the following, we regard $f_i(z), g_i(z) \in \mathbb{F}_7$ for $z \in \mathbb{F}_7$.

The sequences obtained from $(f_1(z), g_1(z), f_2(z), g_2(z), \dots)$ for $z = 0, \dots, 6$ are given as follows:

$$(f_1(z), g_1(z), f_2(z), \dots) = \begin{cases} (1,2,1,3,1,4,1,5,1,6,1,0,1,1,1,2,\dots)(\text{period } 14) & \text{if } c = 0, \\ (0,1,6,0,6,6,0,6,1,0,1,1,\dots)(\text{period } 6)(\times(-1)) & \text{if } c = 1, \\ (6,0,6,6,1,0,1,1,6,0,\dots)(\text{period } 4)(\times(-1)) & \text{if } c = 2, \\ (5,6,1,0,1,1,5,6,1,0,1,1,\dots)(\text{period } 6) & \text{if } c = 3, \\ (4,5,5,3,0,3,2,5,3,1,6,0,6,6,3,2,\dots)(\text{period } 14)(\times(-1)) & \text{if } c = 4, \\ (3,4,4,1,6,0,6,6,4,3,3,6,1,0,\dots)(\text{period } 8)(\times(-1)) & \text{if } c = 5, \\ (2,3,5,1,6,0,6,6,5,4,2,6,1,0,\dots)(\text{period } 8)(\times(-1)) & \text{if } c = 6. \end{cases}$$

From the above sequences, we have the following sequences:

$$(l_1, l_2, l_3, \dots) = \begin{cases} (1,2,1,5,1,6,1,3,1,4,1,0,1,\infty,1,2,\dots)(\text{period } 14) & \text{if } c = 0, \\ (0,1,\infty,0,1,\infty,0,\dots)(\text{period } 3) & \text{if } c = 1, \\ (6,0,1,\infty,6,0,1,\infty,6,\dots)(\text{period } 4) & \text{if } c = 2, \\ (5,6,3,0,1,\infty,5,\dots)(\text{period } 6) & \text{if } c = 3, \\ (4,5,3,2,0,1,\infty,4,\dots)(\text{period } 7) & \text{if } c = 4, \\ (3,4,6,2,5,0,1,\infty,3,\dots)(\text{period } 8) & \text{if } c = 5, \\ (2,3,6,5,4,0,1,\infty,2,\dots)(\text{period } 8) & \text{if } c = 6. \end{cases}$$

References

[1] Burde, G., Zieschang, H., *Knots*, Second edition. De Gruyter Studies in Mathematics, **5** (2003) Walter de Gruyter.

[2] Dijkgraaf, R., Witten, E., *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990) 393–429.

[3] Hutchinson, K., *A Bloch-Wigner complex for SL_2*, J. K-Theory **12** (2013) 15–68.

[4] Jones, J.D.S., Westbury, B.W., *Algebraic K-theory, homology spheres, and the η-invariant*, Topology 34 (1995) 929957.

[5] Karuo, H., *The reduced Dijkgraaf–Witten invariant of twist knots in the Bloch group of a finite field*, to appear in J. Knot Theory Ramifications.
`http://www.kurims.kyoto-u.ac.jp/~karu/reducedDW.twist.pdf`

[6] Kimura, N., *A generalization of the Dijkgraaf-Witten invariants for cusped 3-manifolds*, Preprint, 2018, arXiv:1805.05130 [math.GT].

[7] Matveev, S.V., *Algorithmic topology and classification of 3-manifolds*, Algorithms and Computation in Mathematics **9** (2003) Springer-Verlag.

[8] Murakami, H., Yokota, Y., *Volume conjecture for knots*, SpringerBriefs in Mathematical Physics **30** (2018) Springer, Singapore.

[9] Neumann, W.D., *Extended Bloch group and the Cheeger-Chern-Simons class*, Geom. Topol. **8** (2004) 413–474.

[10] Neumann, W.D., Yang, J., *Bloch invariants of hyperbolic 3-manifolds*, Duke Math. J. **96** (1999) 29–59.

[11] Ohtsuki, T., *On the Bloch groups of finite fields and their quotients by the relation corresponding to a tetrahedral symmetry*, preprint, 2019,
`http://www.kurims.kyoto-u.ac.jp/~tomotada/paper/brff.pdf`.

[12] Ohtsuki, T., Takata, T., *On the Kashaev invariant and the twisted Reidemeister torsion of two-bridge knots*, Geom. Topol. **19** (2015) 853–952.

[13] Riley, R., *Holomorphically parameterized families of subgroups of $SL(2,\mathbb{C})$*, Mathematika **32** (1985) 248–264.

[14] Riley, R., *Nonabelian representations of 2-bridge knot groups*, Quart. J. Math. Oxford Ser. (2) **35** (1984) 191-208.

[15] Thurston, D.P., *Hyperbolic volume and the Jones polynomial*, Notes accompanying lectures at the summer school on quantum invariants of knots and three-manifolds (Joseph Fourier Institute, University of Grenoble, org. C. Lescop, July, 1999),
`http://www.math.columbia.edu/~dpt/speaking/Grenoble.pdf`.

[16] Thurston, W.P., *The geometry and topology of three-manifolds*, lecture notes, Princeton University (1979),
`http://library.msri.org/books/gt3m/`.

[17] Wakui, M., *On Dijkgraaf-Witten invariant for 3-manifolds*, Osaka J. Math. **29** (1992) 675–696.

[18] Weibel, C.A., *The K-book. An introduction to algebraic K-theory*, Graduate Studies in Mathematics **145** (2013) American Mathematical Society.

[19] Yokota, Y., *On the potential functions for the hyperbolic structures of a knot complement*, Invariants of knots and 3-manifolds (Kyoto, 2001), Geom. Topol. Monogr. **4**, Geom. Topol. Publ., Coventry, 2002.

[20] Yokota, Y., *On the complex volume of hyperbolic knots*, J. Knot Theory Ramifications **20** (2011) 955–976.

[21] Zickert, C.K., *The volume and Chern-Simons invariant of a representation*, Duke Math. J. 150 (2009), 489532.

Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto, 606-8502, Japan

E-mail address: `karu@kurims.kyoto-u.ac.jp`