RIMS-1953

A Note on Integrality of Convex Polyhedra Represented by Linear Inequalities with $\{0, \pm 1\}$-coefficients

By

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October 2021

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October 21, 2021

Abstract

We consider a polyhedron P represented by linear inequalities with $\{0, \pm 1\}$-coefficients. We show a condition that guarantees existence of an integral vector in P, which also turns out to be an extreme point of P. We reveal how our polyhedral and geometric approach shows the recent interesting integrality results of Murota and Tamura about subdifferentials of integrally convex functions. Their proofs are algebraic, based on the Fourier-Motzkin elimination for the relevant systems of linear inequalities. Our approach provides further insight into subdifferentials of integrally convex functions to fully appreciate the integrality results of Murota and Tamura from a polyhedral and geometric point of view.

1. Introduction

The present note is motivated by the recent results of Tamura and Murota [9, 10]. They have recently shown interesting integrality properties about subdifferentials of integrally convex functions:

(i) For any integer-valued, integrally convex function its subdifferential at every point in the effective domain contains an integral vector ([9]).

(ii) It still holds true with the addition of any integral box constraint having the nonempty intersection with the subdifferentials ([10]).

Their proofs are algebraic, based on the Fourier-Motzkin elimination for the relevant systems of linear inequalities.

In the present note we show a polyhedral and geometric approach to proving the results (i) and (ii) stated above by focusing our attention on a *greedy point* in the relevant polyhedron, which turns out to be an integral extreme point.

1.1. Definitions

Let n be a positive integer and put $V = \{1, \cdots, n\}$. Let $\mathbb{Z}$ be the set of integers and $\mathbb{R}$ be that of reals. For any positive integer k define $[k] = \{1, \cdots, k\}$. For any two integral vectors $a, b \in \mathbb{Z}^V$ with $a \leq b$ define a box $[a, b]_{\mathbb{R}} = \{z \in \mathbb{R}^V \mid a \leq z \leq b\}$ in $\mathbb{R}^V$ and an integral box $[a, b]_{\mathbb{Z}} = [a, b]_{\mathbb{R}} \cap \mathbb{Z}^V$ in $\mathbb{Z}^V$. For any $x \in \mathbb{R}^V$ and $X \subseteq V$ define $x(X) = \sum_{i \in X} x(i)$. Also define $x^X \in \mathbb{R}^X$ to be $x^X(i) = x(i)$ for $i \in X$.

Denote by 3^V the set of all ordered pairs (X, Y) of disjoint subsets X, Y of V. For any $(X, Y) \in 3^V$ we identify (X, Y) with the $\{0, \pm 1\}$-vector $\chi_{(X,Y)}$ in $\mathbb{Z}^V$ defined by

$$\chi_{(X,Y)}(i) = \begin{cases} \ \ 1 & \text{for } i \in X \\ -1 & \text{for } i \in Y \\ \ \ 0 & \text{for } i \in V \setminus (X \cup Y) \end{cases} \qquad (i \in V). \tag{1.1}$$

Each $(X, Y) \in 3^V$ is called a *signed set*. We also write χ_X as $\chi_{(X,\emptyset)}$ for $X \subseteq V$.

2. Linear Inequalities with $\{0, \pm 1\}$-coefficients

Given a function $f : 3^V \to \mathbb{Z} \cup \{+\infty\}$ with nonempty $\mathcal{F} \equiv \{(X, Y) \mid f(X, Y) < +\infty\}$, consider a system of linear inequalities with $\{0, \pm 1\}$-coefficients given by

$$x(X) - x(Y) \leq f(X, Y) \qquad ((X, Y) \in \mathcal{F}). \tag{2.1}$$

A signed set $(X, Y) \in \mathcal{F}$ is called *tight* in (2.1) if $x(X) - x(Y) = f(X, Y)$ for some x satisfying (2.1). Define a polyhedron $\mathrm{P}(f)$ by

$$\mathrm{P}(f) = \{x \in \mathbb{R}^V \mid \forall (X, Y) \in \mathcal{F} : x(X) - x(Y) \leq f(X, Y)\}. \tag{2.2}$$

We assume the following **(A1)** and **(A2)**:

(A1) $(\emptyset, \emptyset) \in \mathcal{F}$ and $f(\emptyset, \emptyset) = 0$.

(A2) For all $i \in V$, we have $(\{i\}, \emptyset), (\emptyset, \{i\}) \in \mathcal{F}$ and signed sets $(\{i\}, \emptyset)$ and $(\emptyset, \{i\})$ are tight in (2.1).

Theorem 2.1: *Under Assumptions* **(A1)** *and* **(A2)**, *if* $\mathrm{P}(f) \neq \emptyset$, *then there exists at least one integral vector that is an extreme point of* $\mathrm{P}(f)$.

(Proof) Denote by Q the set of points in $(\mathbb{Z}^{V \cup \{n+1\}})^*$ given by

$$Q = \{(\chi_{(X,Y)}, f(X,Y)) \mid (X,Y) \in \mathcal{F}\}. \tag{2.3}$$

By the assumption there exists a vector $\tilde{x} \in \mathrm{P}(f)$. That is, the closed half-space

$$H^+ \equiv \{(y,z) \mid (y,z) \in (\mathbb{R}^{V \cup \{n+1\}})^*, \langle y, \tilde{x} \rangle \leq z\} \tag{2.4}$$

of $(\mathbb{R}^{V \cup \{n+1\}})^*$ includes Q, where $\langle y, \tilde{x} \rangle = \sum_{i \in V} y(i)\tilde{x}(i)$. Hence Q generates a convex cone $\mathrm{Cone}(Q)$ such that $\mathrm{Cone}(Q) \subseteq H^+$. We show that there exists a facet $\hat{F}$ of $\mathrm{Cone}(Q)$ that has an integral normal vector $(\hat{x}, -1) \in \mathbb{Z}^{V \cup \{n+1\}}$. More specifically, we find an n-dimensional simplex S in a facet $\hat{F}$ of $\mathrm{Cone}(Q)$ such that the projection of S to $(\mathbb{R}^V)^*$ is contained in $(\mathbb{R}^V_{\geq 0})^*$. Define

$$Q_{\geq 0} = \{(y,z) \in Q \mid y \in (\mathbb{R}^V_{\geq 0})^*\}. \tag{2.5}$$

Now, let us consider the following greedy-type procedure.

Algorithm Greedy

Step 1: For a sufficiently large integer M put $x \in \mathbb{R}^V$ as $x(i) = -M \quad (\forall i \in V)$.
Step 2: For each $i = 1, 2, \cdots, n$ do the following:

($\dagger$) Compute $\bar{\alpha} = \max\{\alpha \in \mathbb{R}_{\geq 0} \mid \forall (y,z) \in Q_{\geq 0} : \langle y, x + \alpha\chi_{\{i\}} \rangle \leq z\}$.
Put $x(i) \leftarrow x(i) + \bar{\alpha}$.

Step 3: Return $\hat{x} = x$.

When computing ($\dagger$) for current i, we have $x(j) = -M$ for all $j = i+1, \cdots, n$. Note that we have assumed **(A1)** and **(A2)**, so that $(\{k\}, \emptyset)$ belongs to $\mathcal{F}$ and is tight in (2.1) for all $k \in V$ and $\{\chi_{\{k\}} \mid k \in V\}$ generates the cone $(\mathbb{R}^V_{\geq 0})^*$. Hence there exists $(\chi_{X_i}, z_i) \in Q_{\geq 0}$ such that (i) $i \in X_i$ and $X_i \subseteq [i]$ and (ii) after updating x as $x(i) \leftarrow x(i) + \bar{\alpha}$ we have

$$\langle \chi_{X_i}, x \rangle = z_i. \tag{2.6}$$

Consequently, when we finish the nth iteration, the finally obtained $x = \hat{x}$ satisfies (2.6) for all $i \in V$. Here (2.6) for all $i \in V$ is a system of linear equations (with a variable vector x) whose coefficient matrix is an $n \times n$ *triangular* $\{0,1\}$-matrix having all *diagonal entries equal to one*. Hence the obtained $\hat{x}$ must be an integral vector since the right-hand side of (2.6) for each $i \in V$ is an integer z_i. Moreover, putting $X_0 = \emptyset$ and $z_0 = 0$, we have an n-dimensional simplex $S \equiv \{(\chi_{X_i}, z_i) \mid i = 0, 1, \cdots, n\}$ that lies in a facet $\hat{F}$ of cone $\mathrm{Cone}(Q)$. It follows from the convexity of $\mathrm{Cone}(Q)$ that the integral $\hat{x}$ belongs to $\mathrm{P}(f)$ and is an extreme point of $\mathrm{P}(f)$ since $(\hat{x}, -1)$ is the normal vector of facet $\hat{F}$ of $\mathrm{Cone}(Q)$. $\square$

Remark 1: It should be noted that the greedy-type procedure considered in the above proof employs the underlying permutation $(1, \cdots, n)$ and orientation in the orthant $\mathbb{R}^V_{\geq 0}$. We can show the existence of an integral vector in $\mathrm{P}(f)$ associated with any other permutation of $[n](= V)$ and an orthant obtained from $\mathbb{R}^V_{\geq 0}$ by re-orientation of some coordinate axes, *mutatis mutandis*. (Note that each (re-)orientation is identified with a sign vector $\sigma : [n] \to \{+, -\}$; sign vector $\sigma : [n] \to \{+\}$ corresponds to $\mathbb{R}^V_{\geq 0}$.) Hence, under Assumptions **(A1)** and **(A2)** there may exist $n!2^n$ integral extreme points of $\mathrm{P}(f)$ with possible duplication. $\square$

Remark 2: An example of a system of linear inequalities with $\{0, \pm 1\}$-coefficients satisfying Assumptions **(A1)** and **(A2)** appears when we consider what is called a bisubmodular function $f : 3^V \to \mathbb{Z}$ and its associated bisubmodular polyhedron $\mathrm{P}(f)$ (see [2, Sec. 3.5(b)] and [3]). For such a bisubmodular polyhedron $\mathrm{P}(f)$ *every* extreme point of $\mathrm{P}(f)$ is a greedy point obtained by the greedy-type procedure with respect to a permutation of $[n]$ and an orientation $\sigma : [n] \to \{+, -\}$. $\square$

Remark 3: **Algorithm** Greedy described above is a special case of the algorithm to find a lexicographically optimal solution, which is examined in [4] for what is called a *greedy system of linear inequalities* with rational coefficients not necessarily taken from among $\{0, \pm 1\}$. $\square$

3. Implications in Integrally Convex Functions

We show implications of Theorem 2.1 in the recent results obtained by Murota and Tamura [9, 10] about integrally convex functions.

We first give some basic definitions to state their results precisely.

3.1. Discrete convexity

3.1.1. Discrete integral convexity

Consider a function $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ on integer lattice $\mathbb{Z}^V$ such that its effective domain $\mathrm{dom}(f) \equiv \{x \in \mathbb{Z}^V \mid f(x) < +\infty\}$ is nonempty. Such a function f is called *discrete convex* if it is extensible to a convex function $\underline{f} : \mathbb{R}^V \to \mathbb{R} \cup \{+\infty\}$ in such a way that $\underline{f}(x) = f(x)$ for all $x \in \mathrm{dom}(f)$ and the epigraph $\{(x, \alpha) \in \mathbb{R}^{V \cup \{n+1\}} \mid \alpha \geq \underline{f}(x)\}$

of $\underline{f}$ is obtained as the convex hull of the set of halflines $\{(x, \alpha) \mid \alpha \geq f(x)\}$ for all $x \in \mathrm{dom}(f)$.

Favati and Tardella [1] introduced the concept of *integrally convex function*. For a discrete convex function $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ and its convex extension $\underline{f} : \mathbb{R}^V \to \mathbb{R} \cup \{+\infty\}$ suppose that for every integral box $[a, b]_{\mathbb{Z}}$ in $\mathbb{Z}^V$ with $\max\{b(v) - a(v) \mid v \in V\} \leq 1$ and $[a, b]_{\mathbb{Z}} \cap \mathrm{dom}(f) \neq \emptyset$ the following $(*)$ holds:

- $(*)$ the convex extension of the restriction of f on $[a, b]_{\mathbb{Z}}$ coincides with the restriction of $\underline{f}$ on $[a, b]_{\mathbb{R}}$.

(Here, the restriction of f on $[a, b]_{\mathbb{Z}}$ should be defined on $\mathbb{Z}^V$ while its effective domain is within $[a, b]_{\mathbb{Z}}$. We consider the restriction of $\underline{f}$ on $[a, b]_{\mathbb{R}}$ similarly in $\mathbb{R}^V$.) Then we call such a discrete convex function *integrally convex* ([1]). Moreover, any set of integer points in $\mathbb{Z}^V$ is called *integrally convex* if it is the effective domain of an integrally convex function on $\mathbb{Z}^V$.

Informally, a discrete convex function f is integrally convex if and only if its lower-envelope $\underline{f}$ is obtained by pasting the lower-envelopes of f restricted on the unit hyper-cubes $[a, a + \mathbf{1}]$ for all $a \in \mathbb{Z}^V$, where $\mathbf{1}$ is the vector of all ones in $\mathbb{Z}^V$.

See [6] for more details about integral convexity and for a class of integrally convex functions appearing as M-convex functions, L-convex functions, and others.

3.1.2. Subdifferentials of discrete convex functions

Let $f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\}$ be a discrete convex function with $\mathrm{dom}(f) \neq \emptyset$. For any $x \in \mathrm{dom}(f)$ the *subdifferential* of f at x is defined by

$$\partial_{\mathbb{R}} f(x) = \{y \in (\mathbb{R}^V)^* \mid \forall z \in \mathbb{Z}^V : f(x + z) \geq f(x) + \langle y, z \rangle\}, \tag{3.1}$$

which is equal to the subdifferential $\partial_{\mathbb{R}} \underline{f}(x)$ of the lower envelope $\underline{f}$ of f at x in an ordinary sense of convex analysis [11]. Each $y \in \partial_{\mathbb{R}} f(x)$ is called a *subgradient* of f at x. If f is integrally convex, then (3.1) is equivalently represented by

$$\partial_{\mathbb{R}} f(x) = \{y \in (\mathbb{R}^V)^* \mid \forall z \in \{0, \pm 1\}^V : f(x + z) \geq f(x) + \langle y, z \rangle\} \tag{3.2}$$

at any $x \in \mathrm{dom}(f)$. This is a crucial property of integrally convex functions, which characterizes integrally convex functions (cf. [4, Th. 2.2]).

3.1.3. Convex conjugate functions and discrete Fenchel duality

Consider a discrete convex function $f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\}$ and a discrete concave function $g : \mathbb{Z}^V \to \mathbb{Z} \cup \{-\infty\}$ with nonempty effective domains $\mathrm{dom}(f) = \{x \in \mathbb{Z}^V \mid f(x) <$

$+\infty\}$ and $\mathrm{dom}(g) = \{x \in \mathbb{Z}^V \mid f(x) > -\infty\}$. Also let $f^\bullet$ and g°, respectively, be the discrete convex conjugate of f and the discrete concave conjugate of g, i.e.,

$$f^\bullet(y) = \sup\{\langle y, x\rangle - f(x) \mid x \in \mathbb{Z}^V\} \qquad (y \in (\mathbb{Z}^V)^*), \tag{3.3}$$

$$g^\circ(y) = \inf\{\langle y, x\rangle - g(x) \mid x \in \mathbb{Z}^V\} \qquad (y \in (\mathbb{Z}^V)^*). \tag{3.4}$$

Furthermore, define

$$f^{\bullet\bullet}(x) = \sup\{\langle y, x\rangle - f^\bullet(y) \mid y \in (\mathbb{Z}^V)^*\} \qquad (x \in \mathbb{Z}^V), \tag{3.5}$$

$$g^{\circ\circ}(x) = \inf\{\langle y, x\rangle - g^\circ(y) \mid y \in (\mathbb{Z}^V)^*\} \qquad (x \in \mathbb{Z}^V). \tag{3.6}$$

Recently Murota and Tamura [9] have shown that $f^{\bullet\bullet} = f$ and $g^{\circ\circ} = g$ for any integrally convex function f and any integrally concave function g, based on Theorem 3.1 stated in Section 3.2.

It is an interesting subject to investigate conditions on f and g that validate the *discrete Fenchel duality* expressed as

$$\inf\{f(x) - g(x) \mid \mathbb{Z}^V\} = \sup\{g^\circ(y) - f^\bullet(y) \mid (\mathbb{Z}^V)^*\}. \tag{3.7}$$

It is well-known that the discrete Fenchel duality (3.7) holds for L$^\natural$-convex/concave functions and M$^\natural$-convex/concave functions (see [6, 7, 8]). L$^\natural$-convex/concave functions and M$^\natural$-convex/concave functions defined on the integer lattice $\mathbb{Z}^V$ are integrally convex. Very recently it has also been shown by Murota and Tamura [10] that the discrete Fenchel duality (3.7) holds for an integrally convex function f and a separable discrete concave function g, based on Theorem 3.2 stated in Section 3.2.

3.2. Recent results of Murota and Tamura [9, 10]

Murota and Tamura [9, 10] have recently shown the following three theorems for integrally convex functions $f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\}$. Third one is a consequence of the second.

Theorem 3.1 ([9]): *For any $x \in \mathrm{dom}(f)$ the subdifferential $\partial_{\mathbb{R}} f(x)$ of f at x contains an integral vector.*

Theorem 3.2 ([10]): *For any $x \in \mathrm{dom}(f)$ and any integral box $[a, b]_{\mathbb{Z}}$ in $\mathbb{Z}^V$, if we have $\partial_{\mathbb{R}} f(x) \cap [a, b]_{\mathbb{R}} \neq \emptyset$, then we also have $\partial_{\mathbb{R}} f(x) \cap [a, b]_{\mathbb{Z}} \neq \emptyset$.*

Theorem 3.3 ([10]): *The discrete Fenchel duality* (3.7) *holds for any integrally convex function f and separable discreet concave function g such that $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$ and the left-hand side of* (3.7) *is a finite value.*

Remark 4: As noted in [10], the existence of a rational subgradient h of f at some $x_0 \in \mathrm{dom}(f)$ implies $h \in \partial_{\mathbb{R}}(f)(x_0) \cap [\lfloor h \rfloor, \lceil h \rceil]_{\mathbb{R}}$ and Theorem 3.2 with a bounded box $[a, b]_{\mathbb{R}}$ implies Theorem 3.1 (where $\lfloor h \rfloor$ and $\lceil h \rceil$ are, respectively, the rounding down and the rounding up of h to the nearest integral vectors). Also, Theorem 3.2 in [10] is originally stated by using a box $[a, b]_{\mathbb{R}}$ with possibly $a(i) = -\infty$ or $b(i) = +\infty$ for some i's in V but allowing infinite boxes is not essential as seen here. □

Murota and Tamura [9, 10] have shown Theorems 3.1 and 3.2 by means of the Fourier-Motzkin elimination. Their algebraic approach itself is very interesting. We reveal how our polyhedral and geometric approach shows their theorems.

3.3. Proof of Theorem 3.2

Since Theorems 3.1 and 3.3 are implied by Theorem 3.2, we show Theorem 3.2.

For a family $\mathcal{F} \subseteq 3^V$ and $(X_1, Y_1), (X_2, Y_2) \in \mathcal{F}$, we call the pair of (X_1, Y_1) and (X_2, Y_2) is *consistent* if $(X_1 \cup X_2) \cap (Y_1 \cup Y_2) = \emptyset$ and *inconsistent* otherwise. We call $\mathcal{F}$ *consistent* if every pair of $(X_1, Y_1), (X_2, Y_2) \in \mathcal{F}$ is consistent. Also define

$$(X_1, Y_1) \sqcup (X_2, Y_2) = ((X_1 \cup X_2) \setminus (Y_1 \cup Y_2), (Y_1 \cup Y_2) \setminus (X_1 \cup X_2)), \quad (3.8)$$

$$(X_1, Y_1) \sqcap (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2). \quad (3.9)$$

Note that the two binary operations $\sqcup$ and $\sqcap$ on 3^V appear in the definition of bisubmodular function ([2, Sec. 3.5(b)]). We write $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$ if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$.

We first show the following lemma.

Lemma 3.4: *Let $f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\}$ be any integrally convex function. Suppose that for an $x \in \mathrm{dom}(f)$ and an integral box $[a, b]_{\mathbb{Z}}$ in $(\mathbb{Z}^V)^*$ we have $\partial_{\mathbb{R}} f(x) \cap [a, b]_{\mathbb{R}} \neq \emptyset$. Then we have for each $i \in V$*

$$\max\{x(i) \mid x \in \partial_{\mathbb{R}} f(x) \cap [a, b]_{\mathbb{R}}\} \in \mathbb{Z}, \quad \max\{-x(i) \mid x \in \partial_{\mathbb{R}} f(x) \cap [a, b]_{\mathbb{R}}\} \in \mathbb{Z}. \quad (3.10)$$

(Proof) Because of the symmetry associated with $\partial_{\mathbb{R}} f(x) \cap [a, b]_{\mathbb{R}}$ it suffices to show for $i = 1$

$$\max\{x(1) \mid x \in \partial_{\mathbb{R}} f(x) \cap [a, b]_{\mathbb{R}}\} \in \mathbb{Z}. \quad (3.11)$$

Without loss of generality suppose that $x_0 = \mathbf{0}$ and $f(\mathbf{0}) = 0$. Then, from (3.2) the subdifferential of f at $x_0 = \mathbf{0}$ is represented by

$$x(X) - x(Y) \leq f(X, Y) \qquad ((X, Y) \in \mathcal{F}) \quad (3.12)$$

with $\mathcal{F} = \{(X,Y) \in 3^V \mid f(X,Y) < +\infty\}$. Hence for (3.11) consider the problem:

$$
\begin{aligned}
(\mathbf{P}) : \text{Maximize} \quad & x(1) && (3.13)\\
\text{subject to} \quad & x(X) - x(Y) \le f(X,Y) \quad ((X,Y) \in \mathcal{F}), && (3.14)\\
& x(i) \le b(i) \quad (i \in V), && (3.15)\\
& -x(i) \le -a(i) \quad (i \in V). && (3.16)
\end{aligned}
$$

By the LP duality theorem the maximum value of (3.13) is equal to the minimum value of the following dual problem.

$$
\begin{aligned}
(\mathbf{D}) : \text{Minimize} \quad & \sum_{(X,Y)\in\mathcal{F}} \lambda(X,Y) f(X,Y) + \sum_{i\in V} \mu^+(i) b(i) - \sum_{i\in V} \mu^-(i) a(i) && (3.17)\\
\text{subject to} \quad & \sum_{(X,Y)\in\mathcal{F}} \lambda(X,Y) \chi_{(X,Y)} + \sum_{i\in V} (\mu^+(i) - \mu^-(i)) \chi_{\{i\}} = \chi_{\{1\}}, && (3.18)\\
& \lambda(X,Y) \ge 0 \quad ((X,Y) \in \mathcal{F}), && (3.19)\\
& \mu^+(i) \ge 0, \quad \mu^-(i) \ge 0 \quad (i \in V). && (3.20)
\end{aligned}
$$

Since $b(i) - a(i) \ge 0$ for all $i \in V$, we can reduce[1] the objective-function value by putting $\mu^+(i) \leftarrow \mu^+(i) - \min\{\mu^+(i), \mu^-(i)\}$ and $\mu^-(i) \leftarrow \mu^-(i) - \min\{\mu^+(i), \mu^-(i)\}$ while keeping feasibility of the solutions, so that we can assume $\mu^+(i)\mu^-(i) = 0$ for all $i \in V$. Define

$$
V^+ = \{i \in V \setminus \{1\} \mid \mu^+(i) > 0\}, \qquad V^- = \{i \in V \setminus \{1\} \mid \mu^-(i) > 0\}. \tag{3.21}
$$

If $\mu^-(1) > 0$ (and $\mu^+(1) = 0$), then

$$
\tfrac{\mu^-(1)}{1+\mu^-(1)} \Big\{ \sum_{(X,Y)\in\mathcal{F}} \lambda(X,Y) \chi_{(X,Y)} + \sum_{i\in V^+} \mu^+(i) \chi_{\{i\}} - \sum_{i\in V^-} \mu^-(i) \chi_{\{i\}} \Big\} - \mu^-(1) \chi_{\{1\}} = \mathbf{0} \tag{3.22}
$$

and since Problem **(D)** is feasible, we have

$$
\tfrac{\mu^-(1)}{1+\mu^-(1)} \Big\{ \sum_{(X,Y)\in\mathcal{F}} \lambda(X,Y) f(X,Y) + \sum_{i\in V^+} \mu^+(i) b(i) - \sum_{i\in V^-} \mu^-(i) a(i) \Big\} - \mu^-(1) a(1) \ge 0. \tag{3.23}
$$

Hence for $\nu = (1 - \frac{\mu^-(1)}{1+\mu^-(1)})$, putting $\lambda(X,Y) \leftarrow \nu\lambda(X,Y)$ $((X,Y) \in \mathcal{F})$, $\mu^+(i) \leftarrow \nu\mu^+(i)$ $(i \in V^+)$, $\mu^-(i) \leftarrow \nu\mu^-(i)$ $(i \in V^-)$, and $\mu^-(1) \leftarrow 0$, we can reduce the objective-function value while keeping feasibility of the solutions, so that we can assume that $\mu^+(1) \ge 0$ and $\mu^-(1) = 0$. Furthermore, if $\mu^+(i) > 1$, we have

$$
\sum_{(X,Y)\in\mathcal{F}} \lambda(X,Y) \chi_{(X,Y)} + \sum_{i\in V^+} \mu^+(i) \chi_{\{i\}} - \sum_{i\in V^-} \mu^-(i) \chi_{\{i\}} + (\mu^+(1) - 1) \chi_{\{1\}} = \mathbf{0} \tag{3.24}
$$

[1] We use 'reduce' even if the value remains the same.

and since Problem **(D)** is feasible, we have

$$\sum_{(X,Y)\in\mathcal{F}} \lambda(X,Y)f(X,Y) + \sum_{i\in V^+} \mu^+(i)b(i) - \sum_{i\in V^-} \mu^-(i)a(i) + (\mu^+(1)-1)b(1) \geq 0. \tag{3.25}$$

Hence putting $\lambda(X,Y) \leftarrow 0$ $((X,Y) \in \mathcal{F})$, $\mu^+(i) \leftarrow 0$ $(i \in V^+)$, $\mu^-(i) \leftarrow 0$ $(i \in V^-)$, and $\mu^+(1) \leftarrow 1$, we can reduce the objective-function value to $b(1)$ while keeping feasibility of the solutions.

Consequently, we can impose

(F1) $\mu^+(i)\mu^-(i) = 0$ for all $i \in V$, and $\mu^-(1) = 0$ and $\mu^+(i) \leq 1$.

Now, suppose that for a feasible solution $\lambda(X,Y)$ $((X,Y) \in \mathcal{F})$, $\mu^+(i)$ $(i \in V)$ and $\mu^-(i)$ $(i \in V)$ of Problem **(D)** there exists an inconsistent pair of (X_1, Y_1) and (X_2, Y_2) in $\mathcal{F}$ such that $\lambda(X_1, Y_1) > 0$ and $\lambda(X_2, Y_2) > 0$. Then we have

$$\tfrac{1}{2}\{\chi_{(X_1,Y_1)} + \chi_{(X_2,Y_2)}\} = \tfrac{1}{2}\{\chi_{(X_1,Y_1)\sqcap(Y_1,Y_2)} + \chi_{(X_1,Y_1)\sqcup(Y_1,Y_2)}\} \in \mathrm{Conv}(\mathrm{dom}(f)), \tag{3.26}$$

where $\mathrm{Conv}(\mathrm{dom}(f))$ is the convex hull of $\mathrm{dom}(f)$. It follows from (3.26) and the integral convexity of f that there exists an affinely independent set of points $\chi_{(Z^{(i)},W^{(i)})}$ $(i \in I)$ with $(Z^{(i)}, W^{(i)}) \in \mathcal{F}$ $(i \in I)$ such that for all $i \in I$ we have $(X_1, Y_1) \sqcap (Y_1, Y_2) \sqsubseteq (Z^{(i)}, W^{(i)}) \sqsubseteq (X_1, Y_1) \sqcup (Y_1, Y_2)$ and

$$\tfrac{1}{2}\{\chi_{(X_1,Y_1)} + \chi_{(X_2,Y_2)}\} = \sum_{i\in I} \mu(Z^{(i)}, W^{(i)})\chi_{(Z^{(i)},W^{(i)})}, \tag{3.27}$$

$$\tfrac{1}{2}\{f(X_1, Y_1) + f(X_2, Y_2)\} \geq \sum_{i\in I} \mu(Z^{(i)}, W^{(i)})f(Z^{(i)}, W^{(i)}) \tag{3.28}$$

for some $\mu(Z^{(i)}, W^{(i)}) > 0$ $(i \in I)$ with $\sum_{i\in I} \mu(Z^{(i)}, W^{(i)}) = 1$. Hence it follows from (3.27) and (3.28) that for $\alpha = \min\{\lambda(X_1, Y_1), \lambda(X_2, Y_2)\} > 0$ we can reduce the objective-function value by putting

$$\lambda(X_1, Y_1) \leftarrow \lambda(X_1, Y_1) - \alpha, \tag{3.29}$$

$$\lambda(X_2, Y_2) \leftarrow \lambda(X_2, Y_2) - \alpha, \tag{3.30}$$

$$\lambda(Z^{(i)}, W^{(i)}) \leftarrow \lambda(Z^{(i)}, W^{(i)}) + 2\alpha\mu(Z^{(i)}, W^{(i)}) \quad (i \in I). \tag{3.31}$$

For each $i \in V$, while there exists an inconsistent pair of (X_1, Y_1) and (X_2, Y_2) in $\mathcal{F}$ such that $\lambda(X_1, Y_1) > 0$, $\lambda(X_2, Y_2) > 0$, and $i \in (X_1 \cup X_2) \cap (Y_1 \cup Y_2)$, update λ by (3.29)–(3.31). Each such updating of λ for current i reduces the number of signed sets (X, Y) with $i \in X \cup Y$ at least by one while keeping solutions feasible and reducing the objective-function value. Hence we obtain $\lambda(X,Y)$ $((X,Y) \in \mathcal{F})$ such that

(F2) $\{(X,Y) \in \mathcal{F} \mid \lambda(X,Y) > 0\}$ is consistent.
(*See Remark* 5 *at the end of the present section.*)

Moreover, under **(F2)** suppose that there exist signed sets $(X,Y) \in \mathcal{F}$ such that $\lambda(X,Y) > 0$ and $1 \notin X \cup Y$. Putting $\mathcal{G} = \{(X,Y) \in \mathcal{F} \mid 1 \notin X \cup Y, \lambda(X,Y) > 0\}$, define $S = \cup\{X \mid (X,Y) \in \mathcal{G}\}$ and $T = \cup\{Y \mid (X,Y) \in \mathcal{G}\}$, where note that $S \cap T = \emptyset$ due to **(F2)**. Then there exist positive numbers $\hat{\mu}^+(i) \le \mu^+(i)$ for $i \in T$ and $\hat{\mu}^-(i) \le \mu^-(i)$ for $i \in S$ such that

$$\sum_{(X,Y)\in\mathcal{G}} \lambda(X,Y)\chi_{(X,Y)} + \sum_{i\in T} \hat{\mu}^+(i)\chi_{\{i\}} - \sum_{i\in S} \hat{\mu}^-(i)\chi_{\{i\}} = \mathbf{0}. \tag{3.32}$$

It follows from (3.32) and the feasibility of Problem **(D)** that

$$\sum_{(X,Y)\in\mathcal{G}} \lambda(X,Y)f(X,Y) + \sum_{i\in T} \hat{\mu}^+(i)b(i) - \sum_{i\in S} \hat{\mu}^-(i)a(i) \ge 0. \tag{3.33}$$

Hence we can reduce the objective-function value by putting $\lambda(X,Y) \leftarrow 0$ $((X,Y) \in \mathcal{G})$, $\mu^+(i) \leftarrow \mu^+(i) - \hat{\mu}^+(i)$ $(i \in T)$, and $\mu^-(i) \leftarrow \mu^-(i) - \hat{\mu}^-(i)$ $(i \in S)$ while keeping feasibility of the solutions. Consequently, under **(F1)** and **(F2)** we can also impose

(F3) $1 \in X$ for all $(X,Y) \in \mathcal{F}$ with $\lambda(X,Y) > 0$.

It follows from **(F1)**, **(F2)**, and **(F3)** that the minimum value of the objective function of Problem **(D)** is equal to that of the following problem $(\hat{\mathbf{D}})$.

$$(\hat{\mathbf{D}}) : \text{Minimize} \quad \sum_{(X,Y)\in\mathcal{F}} \lambda(X,Y)\{f(X,Y) - a(X \setminus \{1\}) + b(Y)\} + \beta b(1) \tag{3.34}$$

$$\text{subject to} \quad \{(X,Y) \in \mathcal{F} \mid \lambda(X,Y) > 0\} \text{ is consistent,} \tag{3.35}$$

$$1 \in X \quad ((X,Y) \in \mathcal{F} \text{ with } \lambda(X,Y) > 0), \tag{3.36}$$

$$\sum_{(X,Y)\in\mathcal{F}} \lambda(X,Y) + \beta = 1, \tag{3.37}$$

$$\lambda(X,Y) \ge 0 \quad ((X,Y) \in \mathcal{F}), \qquad \beta \ge 0. \tag{3.38}$$

Because of the convex combination in (3.34) we see that the minimum value of Problem $(\hat{\mathbf{D}})$ is given by

$$\min\bigl\{b(1), \min\{f(X,Y) - a(X \setminus \{1\}) + b(Y) \mid (X,Y) \in \mathcal{F}, 1 \in X\}\bigr\}, \tag{3.39}$$

which is an integer and is equal to $\max\{x(1) \mid x \in \mathrm{P}(f) \cap [a,b]_{\mathbb{R}}\} \in \mathbb{Z}$. $\square$

(Proof of Theorem 3.2): Lemma 3.4 implies that the system of linear inequalities (3.12) satisfies Assumptions **(A1)** and **(A2)** by putting

$$\mathcal{F} \leftarrow \mathcal{F} \cup \{(\{i\}, \emptyset) \mid i \in V\} \cup \{(\emptyset, \{i\}) \mid i \in V\}, \tag{3.40}$$

$$f(\{i\}, \emptyset) \leftarrow \max\{x(i) \mid x \in \partial_{\mathbb{R}} f(x) \cap [a, b]_{\mathbb{R}}\} \quad (i \in V), \tag{3.41}$$

$$f(\emptyset, \{i\}) \leftarrow \max\{-x(i) \mid x \in \partial_{\mathbb{R}} f(x) \cap [a, b]_{\mathbb{R}}\} \quad (i \in V). \tag{3.42}$$

(3.40)–(3.42) make the inequalities of (3.15) and (3.16) redundant for (3.12). Hence the present theorem, Theorem 3.2, follows from Theorem 2.1. $\square$

Here we see how the box constraints in Theorem 3.2 make Assumption **(A2)** valid under the integral convexity of f. We also see that Condition **(F2)** is crucial.

Remark 5: Condition **(F2)** can also be understood as follows. For a given $\mathcal{F}_1 \subseteq 3^V$ put $\xi = \sum_{(X,Y)\in\mathcal{F}_1} \chi_{(X,Y)}$. Define $S^+ = \{i \in V \mid \xi(i) > 0\}$ and $S^- = \{i \in V \mid \xi(i) < 0\}$. Then $\mathcal{F}_1$ is consistent if and only if for every $(X, Y) \in \mathcal{F}_1$ we have $(X, Y) \sqsubseteq (S^+, S^-)$. $\square$

4. Concluding Remarks

We have considered a polyhedron P represented by linear inequalities with $\{0, \pm 1\}$-coefficients and have shown conditions (in Theorem 2.1) that guarantee the existence of an integral vector in P which also turns out to be an extreme point of P. We have revealed how our polyhedral and geometric approach shows the recent integrality results of Murota and Tamura [9, 10] about subdifferentials of integrally convex functions.

The greedy point obtained by **Algorithm** Greedy for each subdifferential can also be obtained from the system of linear inequalities resulting from the Fourier-Motzkin elimination adopted by Murota and Tamura [9, 10]. Our approach provides further insight into structures of subdifferentials of integrally convex functions to fully appreciate their integrality results from a polyhedral and geometric point of view.

Murota and Tamura [10] have shown the discrete Fenchel duality for a pair of an integrally convex function and a separable discrete concave function. It is an interesting and challenging problem to find a more general class of discrete convex/concave functions (such as those given in [3]) for which the discrete Fenchel duality holds (also see [5] for related subjects).

Acknowledgements

The author is very grateful to Akihisa Tamura and Kazuo Murota for their careful reading of and useful comments on an earlier version of this note. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. The author's work is supported by JSPS KAKENHI JP26280001.

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