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Abstract

In anabelian geometry, various strong/desired form of Grothendieck Conjecture-type results for hyperbolic curves over relatively small arithmetic fields — for instance, finite fields, number fields, or p-adic local fields — have been obtained by many researchers, especially by A. Tamagawa and S. Mochizuki. Let us recall that, in their proofs, the Weil Conjecture or p-adic Hodge theory plays an essential role. Therefore, to obtain such Grothendieck Conjecture-type results, it appears that the condition that the cyclotomic characters of the absolute Galois groups of the base fields are highly nontrivial is indispensable. On the other hand, in an author's recent joint work with Y. Hoshi and S. Mochizuki, we introduced the notion of TKND-AVKF-field [concerning the divisible subgroups of the groups of rational points of semi-abelian varieties] and obtained the semi-absolute version of the Grothendieck Conjecture for higher dimensional (≥ 2) configuration spaces associated to hyperbolic curves of genus 0 over TKND-AVKF-fields contained in the algebraic closure of the field of rational numbers. For instance, every [possibly, infinite] cyclotomic extension field of a number field is such a TKND-AVKF-field. In particular, this Grothendieck Conjecture-type result suggests that the condition that the cyclotomic character of the absolute Galois group of the base field under consideration is [sufficiently] nontrivial is, in fact, not indispensable for strong/desired form of anabelian phenomena. In the present paper, to pose another evidence for this observation, we prove the relative birational version of the Grothendieck Conjecture for smooth curves over TKND-AVKF-fields with a certain mild condition that every cyclotomic extension field of a number field satisfies. From the viewpoint of the condition on base fields, this result may be regarded as a partial generalization of F. Pop and S. Mochizuki's results on the birational version of the Grothendieck Conjecture for smooth curves.

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Key words and phrases: anabelian geometry; birational Grothendieck Conjecture; function field; smooth curve; abelian variety; divisible element.

Contents

Introduction

Let p be a prime number. For a connected Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. For any field F of characteristic 0 and any algebraic variety [i.e., a separated, of finite type, and geometrically integral scheme] Z over F, we shall write $\overline{F}$ for the algebraic closure [determined up to isomorphisms] of F; $G_F \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$; $K(Z)$ for the function field of Z; $\Delta_Z \stackrel{\text{def}}{=} \Pi_{Z \times_F \overline{F}}$; $\Delta_{K(Z)}$ for the kernel of the natural surjection $G_{K(Z)} \twoheadrightarrow G_F$. For any field F of characteristic 0 and any algebraic varieties Z_1, Z_2 over F, we shall write

$$\mathrm{Isom}_F(Z_1, Z_2) \quad (\text{respectively, } \mathrm{Isom}_F(K(Z_1), K(Z_2)))$$

for the set of F-isomorphisms between Z_1 and Z_2 (respectively, $K(Z_1)$ and $K(Z_2)$);

$$\mathrm{Isom}_{G_F}(\Pi_{Z_1}, \Pi_{Z_2})/\mathrm{Inn}(\Delta_{Z_2}) \quad (\text{respectively, } \mathrm{Isom}_{G_F}(G_{K(Z_1)}, G_{K(Z_2)})/\mathrm{Inn}(\Delta_{K(Z_2)}))$$

for the set of isomorphisms $\Pi_{Z_1} \stackrel{\sim}{\to} \Pi_{Z_2}$ (respectively, $G_{K(Z_1)} \stackrel{\sim}{\to} G_{K(Z_2)}$) of profinite groups that lie over G_F, considered up to compositions with inner automorphisms arising from elements $\in \Delta_{Z_2}$ (respectively, $\in \Delta_{K(Z_2)}$).

In anabelian geometry, the birational version of the Grothendieck Conjecture has been studied intensively [cf. for instance, see [1], [2], [17], [21], [26], [28], [29], [33], [40]]. Roughly speaking, this birational version is a question on the reconstructibility of function fields from their absolute Galois groups. After the pioneering and celebrated works of K. Uchida and F. Pop for the function fields of smooth curves [i.e., smooth and 1-dimensional algebraic varieties] over finitely generated fields [cf. [40], Theorem; [26], Theorem 1], S. Mochizuki obtained the following result, which may be regarded as one of the strongest achievements in characteristic 0:

Theorem 0.1. *Let K be a generalized sub-p-adic field [i.e., a subfield of a finitely generated extension field of the completion of the maximal unramified extension of the field of p-adic numbers $\mathbb{Q}_p$ — cf. [18], Definition 4.11]; X, Y smooth proper curves over K. Then the natural map*

$$\mathrm{Isom}_K(K(Y), K(X)) \longrightarrow \mathrm{Isom}_{G_K}(G_{K(X)}, G_{K(Y)})/\mathrm{Inn}(\Delta_{K(Y)})$$

is bijective.

Note that, if we restrict our attention to the case where the base field K is a sub-p-adic field [i.e., a subfield of a finitely generated extension field of $\mathbb{Q}_p$ — cf. [17], Definition 15.4, (i)], then the stronger Hom-version for the function fields of algebraic varieties over K of arbitrary dimension is also obtained by S. Mochizuki [cf. [17], Theorem 17.1]. Note also that Theorem 0.1 is a corollary of a highly nontrivial result, namely, the Grothendieck Conjecture for hyperbolic curves over generalized sub-p-adic fields:

Theorem 0.2 ([18], Theorem 4.12)**.** *Let K be a generalized sub-p-adic field; X, Y hyperbolic curves over K. Then the natural map*

$$\mathrm{Isom}_K(X, Y) \longrightarrow \mathrm{Isom}_{G_K}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_Y)$$

is bijective.

Moreover, it would be natural to ask

whether or not the semi-absolute analogue of Theorem 0.1 holds.

In the author's knowledge, this is an open question. [Note that the semi-absolute analogue of Theorem 0.2 does not hold even if we assume that the base field is a sub-p-adic field — cf. for instance, [8], Remark 5.6.1.] In this direction, as a corollary of a certain semi-absolute Grothendieck Conjecture-type result for special curves called "quasi-tripods" obtained by Y. Hoshi [cf. [8], Theorem A], one may obtain a partial result for this question. However, in the present paper, we do not pursue this question anymore.

Next, to see another achievement [also obtained by S. Mochizuki] and state our main result, let us recall the notions of Kummer-faithful field and TKND-AVKF-field [cf. Definition 1.1, (ii), (iii), (iv), (v), below]. Let F be a field of characteristic 0. Write F_{div} ($\subseteq \overline{F}$) for the field obtained by adjoining the divisible elements of the multiplicative groups of finite extension fields of F to the field of rational numbers $\mathbb{Q}$. Then we shall say that F is

- *Kummer-faithful* if, for each semi-abelian variety A over a finite extension field E of F, every divisible element $\in A(E)$ is trivial;
- *TKND* [i.e., "torally Kummer-nondegenerate"] if $F_{\mathrm{div}} \subseteq \overline{F}$ is an infinite field extension;
- *AVKF* [i.e., "abelian variety Kummer-faithful"] if, for each abelian variety A over a finite extension field E of F, every divisible element $\in A(E)$ is trivial;

- *TKND-AVKF* if F is both TKND and AVKF.

For instance, every sub-p-adic field is Kummer-faithful, and every [possibly, infinite] cyclotomic extension field of a number field is TKND-AVKF [cf. [21], Remark 1.5.4, (i); [37], Theorem 3.1, and its proof; [37], Remark 3.4.1]. In particular, one obtains many TKND-AVKF-fields whose associated cyclotomic characters totally vanish. On the other hand, it is easy to see that every Kummer-faithful field is TKND-AVKF.

In the context of absolute anabelian geometry, S. Mochizuki proved that the semi-absolute birational version of the Grothendieck Conjecture for smooth curves over Kummer-faithful fields holds [cf. [21], Theorem 1.11]:

Theorem 0.3. *Let K, L be Kummer-faithful fields [of characteristic 0]; X, Y smooth proper curves over K, L, respectively. Write*

$$\mathrm{Isom}(K(Y)/L, K(X)/K)$$

for the set of isomorphisms $K(Y) \xrightarrow{\sim} K(X)$ of fields that induce isomorphisms $L \xrightarrow{\sim} K$;

$$\mathrm{Isom}(G_{K(X)}/G_K, G_{K(Y)}/G_L)/\mathrm{Inn}(G_{K(Y)})$$

for the set of isomorphisms $G_{K(X)} \xrightarrow{\sim} G_{K(Y)}$ of profinite groups that induce isomorphisms $G_K \xrightarrow{\sim} G_L$ via the natural surjections $G_{K(X)} \twoheadrightarrow G_K$ and $G_{K(Y)} \twoheadrightarrow G_L$, considered up to compositions with inner automorphisms that arise from elements $\in G_{K(Y)}$. Then the natural map

$$\mathrm{Isom}(K(Y)/L, K(X)/K) \longrightarrow \mathrm{Isom}(G_{K(X)}/G_K, G_{K(Y)}/G_L)/\mathrm{Inn}(G_{K(Y)})$$

is bijective.

Note that since there exists a generalized sub-p-adic field that is not Kummer-faithful [cf. Proposition 1.7, (ii)], Theorem 0.3 may not be regarded as a generalization of Theorem 0.1. Note also that since Theorem 0.3 is a result on semi-absolute anabelian geometry, and Theorem 0.1 deals with relative anabelian geometry, Theorem 0.1 may not be regarded as a generalization of Theorem 0.3.

On the other hand, in a recent joint work with Y. Hoshi and S. Mochizuki, we obtained a certain semi-absolute Grothendieck Conjecture-type result for higher dimensional (≥ 2) configuration spaces associated to hyperbolic curves of genus 0 over TKND-AVKF-fields [cf. [10], Theorem G, (ii)]:

Theorem 0.4. *Let (m, n) be a pair of positive integers; $K, L \subseteq \overline{\mathbb{Q}}$ TKND-AVKF-fields; X, Y hyperbolic curves over K, L, respectively. Write g_X (respectively, g_Y) for the genus of X (respectively, Y); X_m (respectively, Y_n) for the m-th (respectively, n-th) configuration space associated to X (respectively, Y);*

$$\mathrm{Isom}(\Pi_{X_m}/G_K, \Pi_{Y_n}/G_L)/\mathrm{Inn}(\Pi_{Y_n})$$

for the set of isomorphisms $\Pi_{X_m} \xrightarrow{\sim} \Pi_{Y_n}$ of profinite groups that induce isomorphisms $G_K \xrightarrow{\sim} G_L$ via the natural surjections $\Pi_{X_m} \twoheadrightarrow G_K$ and $\Pi_{Y_n} \twoheadrightarrow G_L$, considered up to compositions with inner automorphisms arising from elements $\in \Pi_{Y_n}$. Suppose that

- *$m \geq 2$ or $n \geq 2$;*
- *$g_X = 0$ or $g_Y = 0$.*

Then the natural map

$$\mathrm{Isom}(X_m, Y_n) \longrightarrow \mathrm{Isom}(\Pi_{X_m}/G_K, \Pi_{Y_n}/G_L)/\mathrm{Inn}(\Pi_{Y_n})$$

is bijective.

It appears to the author that this anabelian geometric result suggests that

> *the condition that the cyclotomic characters of the absolute Galois groups of the base fields are [sufficiently] nontrivial is, in fact, not a necessary condition for [the strong/desired form of] anabelian phenomena.*

[With regard to the weak form of anabelian phenomena [i.e., reconstructions of *isomorphism classes* of geometric objects under considerations from their fundamental groups] for hyperbolic curves over fields whose associated cyclotomic characters vanish, many results have already been obtained by various researchers so far — cf. for instance, see [13], [30], [32], [34], [35], [36], [37], [39].] On the other hand, since the proof of Theorem 0.4 depends heavily on the rich symmetry of the second dimensional configuration space associated to the projective line minus three points, the method of [10] may not be applied to prove general low dimensional (≤ 1) anabelian Grothendieck Conjecture in an evident way. However, it appears that Theorem 0.4 may be regarded as an evidence of the existence of anabelian phenomena for geometric objects over TKND-AVKF-fields [of characteristic 0]. In particular, it is natural to ask

> *whether or not the various anabelian geometric results that have been obtained so far may be generalized to the results in the case where the base fields are TKND-AVKF-fields.*

With regard to this question, in the present paper, we give a partial generalization of Theorems 0.1, 0.3 obtained by S. Mochizuki, namely, the relative birational version of the anabelian Grothendieck Conjecture for smooth curves over TKND-AVKF-fields with a certain mild condition [cf. Theorem 4.7]:

Theorem A. *Let K be a TKND-AVKF-field [of characteristic 0]; X, Y smooth proper curves over K. Suppose that there exists a surjective homomorphism $K^\times \twoheadrightarrow \mathbb{Z}$. Then the natural map*

$$\mathrm{Isom}_K(K(Y), K(X)) \longrightarrow \mathrm{Isom}_{G_K}(G_{K(X)}, G_{K(Y)})/\mathrm{Inn}(\Delta_{K(Y)})$$

is bijective.

The key ingredient of the proof of Theorem A is to establish a criterion for algebricity of certain *set-theoretic* functions on smooth proper curves over algebraically closed fields, which we shall refer to as *quasi-rational functions* [cf. Definition 3.1; Proposition 3.6]. This criterion strengthens the criterion for algebricity of certain set-theoretic automorphism that appears in [10], §1, which may be regarded as one of the key ingredients of the proof of Theorem 0.4. The author expects that such a consideration on augmented geometric objects [compared to usual scheme-theoretic objects] will make a further progress on a deeper understanding of anabelian phenomena. On the other hand, we note that the assumption that $K^{\times}$ admits a surjective homomorphism onto $\mathbb{Z}$ is applied to verify the [partial] compatibility of cyclotomes that arise from X and Y. At the time of writing of the present paper, the author does not know whether or not this assumption may be dropped [even if we assume that K is a Kummer-faithful field — cf. Remark 4.9.2].

Finally, after making an observation on the freeness of the multiplicative groups of certain fields modulo the divisible subgroups [cf. Proposition 4.9], as a corollary of Theorem A and [38], Theorem A, we obtain the following concrete result [cf. Corollary 4.10]:

Corollary B. *Let M be a number field [i.e., a finite extension field of $\mathbb{Q}$]. Write L ($\subseteq \overline{\mathbb{Q}}$) for the field obtained by adjoining all roots of p to M [so L contains all roots of unity, and $M \subseteq L$ is a nonabelian metabelian Galois extension]. Let K be a subfield of a finitely generated extension field of L; X, Y smooth proper curves over K. Then the natural map*

$$\mathrm{Isom}_K(K(Y), K(X)) \longrightarrow \mathrm{Isom}_{G_K}(G_{K(X)}, G_{K(Y)})/\mathrm{Inn}(\Delta_{K(Y)})$$

is bijective.

In the author's knowledge, it appears that Corollary B is the first result concerning [the strong/desired form of] the Grothendieck Conjecture for the function fields of smooth curves over fields whose associated cyclotomic characters totally vanish.

The present paper is organized as follows. In §1, we first recall the definitions of Kummer-faithful field and TKND-AVKF-field. Next, we investigate basic properties of these fields and give some examples and counter-examples. In §2, we reconstruct, from the data of the absolute Galois group of the function field of a smooth curve over a TKND-AVKF-field [of characteristic 0], together with the natural surjection onto the absolute Galois group of the base field, the image of the multiplicative group of the function field via the Kummer map. The discussion that appears in this section is an appropriate modification of S. Mochizuki's argument [for the function fields of smooth curves over Kummer-faithful fields]. In §3, we introduce the notion of quasi-rational functions on smooth curves over algebraically closed fields which may be regarded as a generalized notion of usual rational functions. The quasi-rational functions are certain set-theoretic functions on smooth curves that are "not so far" from the rational functions. Our main result in this section is to give a certain sufficient condition that quasi-rational functions become rational functions automatically. This algebricity criterion may be applied to give a bridge between the Kummer classes of the multiplicative groups of the function fields of smooth curves and the function fields themselves. In §4, we first apply the results obtained in §2, §3, to prove Theorem A. Next, we prove a certain generalization of the result on the freeness of the multiplicative groups of certain fields modulo torsion subgroups obtained by W. May. Finally, by applying Theorem A, together with this generalization, we prove Corollary B.

Notations and Conventions

Sets: Let A, B be sets. Then we shall write $\mathrm{Fn}(A, B)$ for the set of maps from A to B.

Numbers: The notation $\mathbb{Q}$ will be used to denote the group or field of rational numbers. The notation $\mathbb{Z}$ will be used to denote the additive group or ring of integers. We shall refer to a finite extension field of $\mathbb{Q}$ as a *number field*. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of $\mathbb{Z}$. If p is a prime number, then $\mathbb{Q}_p$ will be used to denote the field of p-adic numbers; $\mathbb{Q}_p^{\mathrm{ur}}$ will be used to denote the maximal unramified extension field of $\mathbb{Q}_p$. If A is a commutative ring, then $A^\times$ will be used to denote the group of units $\in A$.

Fields: Let F be a field; n a positive integer; p a prime number. Then we shall write $\overline{F}$ for the algebraic closure [determined up to isomorphisms] of F; F^{sep} ($\subseteq \overline{F}$) for the separable closure of F; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$;

$$\mu_n(F) \stackrel{\mathrm{def}}{=} \{x \in F^\times \mid x^n = 1\}; \quad \mu(F) \stackrel{\mathrm{def}}{=} \bigcup_m \mu_m(F);$$

$$F^{\times\infty} \stackrel{\mathrm{def}}{=} \bigcap_m (F^\times)^m; \quad F^{\times p^\infty} \stackrel{\mathrm{def}}{=} \bigcap_m (F^\times)^{p^m},$$

where m ranges over the positive integers; F_{div} ($\subseteq \overline{F}$) for the field obtained by adjoining the divisible elements of the multiplicative groups of finite extension fields of F to the prime field of F.

Topological groups: Let G be a profinite group; $H \subseteq G$ a subgroup of G. Then we shall write $\overline{H} \subseteq G$ for the closure of H in G; $Z_G(H)$ for the *centralizer* of $H \subseteq G$, i.e.,

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}.$$

[Note that since G is Hausdorff, the centralizer $Z_G(H)$ is automatically closed in G.] We shall write G^{ab} for the quotient of G by $\overline{[G, G]} \subseteq G$; $\mathrm{Aut}(G)$ for the group of continuous automorphisms of G; $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G; $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

Schemes: Let K be a field; $K \subseteq L$ a field extension; X an algebraic variety [i.e., a separated, of finite type, and geometrically integral scheme] over K. Then we shall write $X(L)$ for the set of L-valued points of X; $X_L \stackrel{\mathrm{def}}{=} X \times_K L$.

Fundamental groups: For a connected Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. Let K be a field; X an algebraic variety over K. Then we shall write

$$\Delta_X \stackrel{\mathrm{def}}{=} \Pi_{X_{K^{\mathrm{sep}}}}.$$

In particular, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_K \longrightarrow 1$$

[cf. [5], Exposé IX, Théorème 6.1]. We shall write

$$\Pi_X^{(\mathrm{ab})} \stackrel{\mathrm{def}}{=} \Pi_X/\mathrm{Ker}(\Delta_X \twoheadrightarrow \Delta_X^{\mathrm{ab}}).$$

Curves: Let K be a field. Then we shall write $\mathbb{A}_K^1$ (respectively, $\mathbb{P}_K^1$) for the affine line (respectively, the projective line) over K. Let X be a smooth curve [i.e., a smooth and 1-dimensional algebraic variety] over K. Then we shall write $\overline{X}$ for the smooth compactification of X over K; $K(X)$ for the function field of X;

$$\Delta_{K(X)} \stackrel{\mathrm{def}}{=} G_{K(X)\otimes_K K^{\mathrm{sep}}}.$$

In particular, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta_{K(X)} \longrightarrow G_{K(X)} \longrightarrow G_K \longrightarrow 1.$$

We shall refer to an element $\in \overline{X}_{\overline{K}} \setminus X_{\overline{K}}$ as a *cusp* of X. Then we shall say that X is a *hyperbolic curve* if $2g-2+r>0$, where g denotes the genus of X; r denotes the cardinality of the set of cusps of X.

Next, suppose that X is a hyperbolic curve over K. Then we shall refer to the stabilizer subgroup of Δ_X associated to some pro-cusp of the pro-universal étale covering of $X_{\overline{K}}$ that lies over a cusp $x \in \overline{X}_{\overline{K}} \setminus X_{\overline{K}}$ as a *cuspidal inertia subgroup* of Π_X [or Δ_X] associated to x. Write $\{U_i\}_{i\in I}$ for the family of open subschemes of $X_{\overline{K}}$. Then it follows immediately from the various definitions involved that there exists a natural isomorphism of profinite groups

$$\Delta_{K(X)} \stackrel{\sim}{\to} \varprojlim_{i\in I} \Delta_{U_i},$$

where the transition maps are the [outer] surjections induced by the natural open immersions. We shall refer to an inverse limit of cuspidal inertia subgroups of some cofinal collection of Δ_{U_i}'s as a *cuspidal inertia subgroup* of $G_{K(X)}$ [or $\Delta_{K(X)}$].

1 Kummer-faithful fields and TKND-AVKF-fields

In the present section, we recall [slightly generalized version of] the definitions of Kummer-faithful field and TKND-AVKF-field. Moreover, we give some examples and counter-examples of these fields.

Definition 1.1 ([10], Definition 6.1, (iii); [10], Definition 6.6, (i), (ii), (iii); [21], Definition 1.5)**.** Let F be a field.

(i) If $E^{\times\infty} = \{1\}$ for every finite field extension $F \subseteq E$, then we shall say that F is a *torally Kummer-faithful field*.

(ii) If $F_{\mathrm{div}} \subseteq \overline{F}$ is an infinite field extension, then we shall say that F is a *TKND-field* [i.e., "torally Kummer-nondegenerate field"].

(iii) If F satisfies the following condition, then we shall say that F is a *Kummer-faithful field*:

> Let A be a semi-abelian variety over a finite extension field E of F. Then every divisible element $\in A(E)$ is trivial.

(iv) If F satisfies the following condition, then we shall say that F is an *AVKF-field* [i.e., "abelian variety Kummer-faithful field"]:

> Let A be an abelian variety over a finite extension field E of F. Then every divisible element $\in A(E)$ is trivial.

(v) If F is both a TKND-field and an AVKF-field, then we shall say that F is a *TKND-AVKF-field*.

Remark 1.1.1. It follows immediately from the various definitions involved that every torally Kummer-faithful field (respectively, Kummer-faithful field) is a TKND-field (respectively, TKND-AVKF-field).

Remark 1.1.2. It follows immediately from the various definitions involved that every subfield of a torally Kummer-faithful field (respectively, Kummer-faithful field; AVKF-field) is also a torally Kummer-faithful field (respectively, Kummer-faithful field; AVKF-field). On the other hand, the notion of TKND-field does not satisfy this property [cf. [38], Remark 1.1.1].

Next, by applying a similar argument to the argument applied in [21], Remark 1.5.4, (i), we prove the following result:

Proposition 1.2. *Let K be a field; L a finitely generated extension field of K. Then the following hold:*

(i) Suppose that K is torally Kummer-faithful (respectively, TKND). Then L is also torally Kummer-faithful (respectively, TKND).

(ii) Suppose that K is AVKF. Then L is also AVKF.

(iii) Suppose that K is Kummer-faithful (respectively, TKND-AVKF). Then L is also Kummer-faithful (respectively, TKND-AVKF).

Proof. Note that, if the field extension $K \subseteq L$ is algebraic, then there is nothing to prove. Thus, we may assume without loss of generality that the field extension $K \subseteq L$ is transcendental.

First, we verify assertion (i). Let $L \subseteq L^\dagger$ ($\subseteq \overline{L}$) be a finite field extension. Write $K^\dagger \subseteq L^\dagger$ for the algebraic closure of K in $L^\dagger$. Then it suffices to verify that $(L^\dagger)^{\times\infty} \subseteq K^\dagger$. Let X be a geometrically connected, normal, proper scheme over $K^\dagger$ such that the function field of X coincides with $L^\dagger$. Observe that $\mathbb{Z}$ has no nontrivial divisible element. Then, for each point $x \in X$ of codimension 1, it holds that every element $\in (L^\dagger)^{\times\infty}$ determines a unit of the local ring at x of X. Thus, since X is a geometrically connected, normal, proper scheme over $K^\dagger$, we conclude that $(L^\dagger)^{\times\infty} \subseteq \mathcal{O}_X(X) = K^\dagger$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $L \subseteq L^\dagger$ ($\subseteq \overline{L}$) be a finite field extension; A an abelian variety over $L^\dagger$. Then it suffices to prove that $A(L^\dagger)$ has no nontrivial divisible element. Let U be an algebraic variety over a finite extension field of K such that the function field of U coincides with $L^\dagger$, and the abelian variety A extends to an abelian scheme $\mathcal{A}$ over U. Observe from the properness criterion that any divisible element $\in A(L^\dagger)$ extends to a divisible element $\in \mathcal{A}(U)$. For each closed point $x \in U$, write K_x for the residue field of U at x; $A_x \overset{\mathrm{def}}{=} \mathcal{A} \times_U \operatorname{Spec} K_x$. Then since the field extension $K \subseteq K_x$ is finite, it follows from our assumption that K is AVKF that the point $\in A_x(K_x)$ determined by any divisible element $\in \mathcal{A}(U)$ is the origin of A_x. On the other hand, since U is an algebraic variety, the subset of closed points $\in U$ forms a dense subset of U. Thus, since the image of any section $\in \mathcal{A}(U)$ forms a closed subset of $\mathcal{A}$, we conclude that every divisible element $\in \mathcal{A}(U)$ coincides with the origin. This completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertions (i), (ii), together with the various definitions involved. This completes the proof of Proposition 1.2. $\square$

Lemma 1.3. *Let p be a prime number; R a complete discrete valuation ring of residue characteristic p; G a commutative formal group over R. Write K for the field of fractions of R. Let $K \subseteq L$ be a [possibly, infinite] Galois extension. Write S for the integral closure of R in the Henselian valuation field L; $\mathfrak{m}_S$ for the maximal ideal of S; $G(\mathfrak{m}_S)$ for the group associated to the commutative formal group $G \times_R S$ over S. Suppose that the group of p-power torsion points of $G(\mathfrak{m}_S)$ is finite. [For instance, every [possibly, infinite] tame extension field of K satisfies this assumption.] Then there exists no nontrivial p-divisible element of $G(\mathfrak{m}_S)$.*

Proof. We identify $G(\mathfrak{m}_S)$ with $\mathfrak{m}_S^{\oplus n}$, where n denotes the dimension of G. First, it follows immediately from the definition of formal group law that, if $K \subseteq L$ is a finite field extension, then it holds that

$$p^i G(\mathfrak{m}_S) \subseteq (\mathfrak{m}_S^{i+1})^{\oplus n} \subseteq \mathfrak{m}_S^{\oplus n} = G(\mathfrak{m}_S)$$

for each positive integer i. In particular, there exists no nontrivial p-divisible element of $G(\mathfrak{m}_S)$ under the assumption that $K \subseteq L$ is a finite field extension. Next, we have

$$G(\mathfrak{m}_S) = \bigcup_{K \subseteq K^\dagger \subseteq L} G(\mathfrak{m}_{R^\dagger}),$$

where $K \subseteq K^\dagger$ ($\subseteq L$) ranges over the finite field extensions contained in L; $R^\dagger$ denotes the integral closure of R in $K^\dagger$; $G(\mathfrak{m}_{R^\dagger})$ denotes the group associated to the commutative formal group $G \times_R R^\dagger$ over $R^\dagger$. Let $x \in G(\mathfrak{m}_S)$ be a p-divisible element. To verify Lemma 1.3, it suffices to prove that x is trivial. By replacing K by a finite extension field of K, we may assume without loss of generality that $x \in G(\mathfrak{m}_R)$. For each positive integer m, fix an element $x_m \in G(\mathfrak{m}_S)$ such that $p^m x_m = x$. Write p^c for the cardinality of the group of p-power torsion points of $G(\mathfrak{m}_S)$. Let $\sigma \in \mathrm{Gal}(L/K)$ be an element. Then since $x \in G(\mathfrak{m}_R)$, it holds that $\sigma(x_m) - x_m$ is a p-power torsion point for each positive integer m. In particular, it holds that $\sigma(p^c x_m) = p^c x_m$, hence that $p^c x_m \in G(\mathfrak{m}_R)$ for each positive integer m. This implies that $p^c x$ is a p-divisible element of $G(\mathfrak{m}_R)$. Therefore, it follows from the above argument that $p^c x$ is trivial, hence that x is a p-power torsion point. Thus, since x is a p-divisible element, we conclude from our assumption that $G(\mathfrak{m}_S)$ has finitely many p-power torsion points that x is trivial. This completes the proof of Lemma 1.3. $\square$

Remark 1.3.1. The argument applied in the proof of Lemma 1.3 is similar to the argument applied in the proof of [10], Lemma 6.2, (ii^{AV}), hence also of [23], Proposition 7.

Proposition 1.4. *Let R be a Noetherian local domain whose residue characteristic is positive. Write K for the field of fractions of R; k for the residue field of R. Then the following hold:*

(i) Suppose that k is torally Kummer-faithful. Then K is also torally Kummer-faithful.

(ii) K is TKND [cf. [38], Proposition 2.3].

(iii) Suppose that k is Kummer-faithful. Then K is also Kummer-faithful [cf. [24], Proposition 3.7].

Proof. Recall that there exists a discrete valuation ring S such that S dominates R, and the residue field of S is a finitely generated extension field over k. Then it follows immediately from Remark 1.1.2 and Proposition 1.2, (i), (iii), that, by replacing R by the completion of S, we may assume without loss of generality that R is a complete discrete valuation ring. Let $K \subseteq K^\dagger$ be a finite field extension. Write $R^\dagger$ ($\subseteq K^\dagger$) for the integral closure of R in $K^\dagger$; $k^\dagger$ for the residue field of $R^\dagger$.

Next, we verify assertions (i), (ii). Observe that since $R^\dagger$ is also a complete discrete valuation ring whose residue characteristic is positive, it holds that $(K^\dagger)^{\times\infty}$ is contained in the image of $(k^\dagger)^{\times\infty}$ via the Teichmüller character associated to $R^\dagger$. Thus, by varying $K^\dagger$, we observe the following:

- If k is torally Kummer-faithful, then K is also torally Kummer-faithful.
- K_{div} is contained in the maximal unramified extension field of K. In particular, K is TKND.

This completes the proofs of assertions (i), (ii).

Next, we verify assertion (iii). Let A be an abelian variety over $K^\dagger$. It suffices to verify that $A(K^\dagger)$ has no nontrivial divisible element. By replacing K by a finite extension field of K, we may assume without loss of generality that $K = K^\dagger$, and A has semi-stable reduction over K [cf. [6], Exposé IX, Théorème 3.6]. Write $\mathcal{A}$ for the semi-abelian scheme over R that lies over A; $\mathcal{A}_s$ for the special fiber of $\mathcal{A}$; $\widehat{\mathcal{A}}$ for the formal completion of $\mathcal{A}$ at the origin. Then the reduction map induces a natural exact sequence

$$0 \longrightarrow \widehat{\mathcal{A}}(R) \longrightarrow A(K) = \mathcal{A}(R) \longrightarrow \mathcal{A}_s(k).$$

Note that it follows immediately from Lemma 1.3 that $\widehat{\mathcal{A}}(R)$ has no nontrivial divisible element. On the other hand, since k is Kummer-faithful, it holds that $\mathcal{A}_s(k)$ has no nontrivial divisible element. Note that, for each positive integer n, the group of n-torsion points $\in A(K)$ is finite. Thus, we conclude that $A(K)$ has no nontrivial divisible element. This completes the proof of assertion (iii), hence of Proposition 1.4. □

Remark 1.4.1. Note that a similar assertion for "stably p-$\times\mu$-indivisible field" is proved in [16], Proposition 1.10.

Remark 1.4.2. Let k be a field of characteristic 0. Then the one-parameter formal power series field $k((t))$ over k is not TKND. Indeed, it follows from a direct computation that $1 + tk[[t]] \subseteq k((t))^{\times\infty}$. Then it holds that $k \subseteq \mathbb{Q}(tk[[t]]) = \mathbb{Q}(1 + tk[[t]]) \subseteq \mathbb{Q}(k((t))^{\times\infty})$. On the other hand, observe that k and $1+tk[[t]]$ generate the field $k((t))$. Thus, since every finite extension field of $k((t))$ is isomorphic to $k^\dagger((t))$ for some finite extension field $k^\dagger$ of k, we conclude that $k((t))_{\mathrm{div}} = \overline{k((t))}$. In particular, the assumption that the residue characteristic of R is positive that appears in Proposition 1.4 is indispensable.

Remark 1.4.3. The assumption that the local domain R is Noetherian that appears in Proposition 1.4 is also indispensable. Indeed, let p be a prime number; E a Tate elliptic curve over $\mathbb{Q}_p$. Write $\mathbb{Q}_p(\mu_{p^\infty})$ for the extension field of $\mathbb{Q}_p$ obtained by adjoining all p-power roots of unity to $\mathbb{Q}_p$; R for the integral closure of $\mathbb{Z}_p$ in $\mathbb{Q}_p(\mu_{p^\infty})$. Then it holds that R is not Noetherian, and the residue field of R is finite, hence, in particular, Kummer-faithful. On the other hand, it holds that $E(\mathbb{Q}_p(\mu_{p^\infty}))$ has infinitely many p-power torsion points, hence, in particular, that $\mathbb{Q}_p(\mu_{p^\infty})$ is not AVKF. Moreover, it follows immediately from the various definitions involved that $\mathbb{Q}_p(\mu_{p^\infty})$ is not torally Kummer-faithful.

Remark 1.4.4. The argument applied in the proof of Proposition 1.4, (ii), is a review of Murotani's argument applied in the proof of [24], Proposition 3.7.

Remark 1.4.5. Let p be a prime number; l a prime number $\neq p$; K a p-adic local field. Fix a system of l-power roots of p [compatible with the l-th power map]. Write L for the field obtained by adjoining these roots of p to K. Then L is a Kummer-faithful field. Indeed, let $L^\dagger$ be a finite extension field of L. Observe that the residue field of $L^\dagger$ is finite, and the finite fields are Kummer-faithful. Moreover, it follows immediately from the definition of L that $L^\dagger$ is a subfield of the maximal tame extension field of a finite extension field of K. Thus, in light of Lemma 1.3, by applying a similar argument to the argument applied in the proof of Proposition 1.4, (ii), we conclude that L is a Kummer-faithful field.

Remark 1.4.6. At the time of writing of the present paper, the author does not know whether or not an assertion for AVKF similar to Proposition 1.4, (i), (iii), holds.

Definition 1.5. Let p be a prime number; F a field. Then:

(i) We shall say that F is a *quasi-finite field* if F is perfect, and $G_F \xrightarrow{\sim} \widehat{\mathbb{Z}}$.

(ii) We shall say that F is a *sub-p-adic field* if F is a subfield of a finitely generated extension field of $\mathbb{Q}_p$ [cf. [17], Definition 15.4, (i)].

(iii) We shall say that F is a *generalized sub-p-adic field* if F is a subfield of a finitely generated extension field of the completion of $\mathbb{Q}_p^{\mathrm{ur}}$ [cf. [18], Definition 4.11].

Remark 1.5.1. Let p be a prime number. Then every sub-p-adic field is a Kummer-faithful field [cf. [21], Remark 1.5.4, (i)]. We slightly generalize this fact below [cf. Proposition 1.7, (i)].

Definition 1.6 ([3], Chapter I, §1.1)**.** Let F be a field; d a positive integer. Then:

(i) A structure of *local field of dimension d* on F is a sequence of complete discrete valuation fields $F^{(d)} \stackrel{\mathrm{def}}{=} F, F^{(d-1)}, \dots, F^{(0)}$ such that

- $F^{(0)}$ is a perfect field;
- for each integer $0 \leq i \leq d-1$, $F^{(i)}$ is the residue field of the complete discrete valuation field $F^{(i+1)}$.

(ii) We shall say that F is a *higher local field* if F admits a structure of local field of some positive dimension. With respect to some fixed structure of higher local field, we shall refer to $F^{(0)}$ as the *final residue field* of F.

Proposition 1.7. *Let p be a prime number. Then the following hold:*

(i) Let k be a quasi-finite field of characteristic p that is algebraic over the prime field; K a mixed characteristic or positive characteristic higher local field whose final residue field is k. Let M be a subfield of a finitely generated extension field of K. Then M is a Kummer-faithful field.

(ii) Let E be an elliptic curve over $\mathbb{Q}_p^{\mathrm{ur}}$. Suppose that E has good ordinary reduction and complex multiplication over $\mathbb{Q}_p^{\mathrm{ur}}$. [Note that such an elliptic curve may be constructed as the base extension of the Serre-Tate's canonical lifting of an ordinary elliptic curve over a finite field of characteristic p — cf. [15], Chapter V, Theorem 3.3.] Then, for each prime number l, the elliptic curve E has infinitely many l-power torsion points valued in $\mathbb{Q}_p^{\mathrm{ur}}$. In particular, $\mathbb{Q}_p^{\mathrm{ur}}$ is a generalized sub-p-adic field that is not AVKF, hence not Kummer-faithful [cf. Remark 1.1.1].

Proof. First, we verify assertion (i). It follows immediately from Remark 1.1.2, together with Proposition 1.2, (iii), that we may assume without loss of generality that $M = K$. Then, in light of Proposition 1.4, (ii), it suffices to verify that k is a Kummer-faithful field. Let A be a semi-abelian variety over a finite extension field of k. For each prime number l, write $T_l A$ for the l-adic Tate module associated to A. Since k is algebraic over the prime field, it holds that $A(\overline{k})$ is a torsion group. On the other hand, since k is quasi-finite and algebraic over the prime field, it follows immediately from the various definitions involved that $H^0(G_{k^\dagger}, T_l A) = \{0\}$ for each prime number l and each finite field extension $k \subseteq k^\dagger$ ($\subseteq \overline{k}$). Thus, we conclude from these observations that k is a Kummer-faithful field. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since E has good reduction, it holds that, for each prime number $l \neq p$, every l-power torsion point is a $\mathbb{Q}_p^{\mathrm{ur}}$-valued point. Thus, it suffices to verify that E has infinitely many p-power torsion points valued in $\mathbb{Q}_p^{\mathrm{ur}}$. However, since E has good ordinary reduction and complex multiplication over $\mathbb{Q}_p^{\mathrm{ur}}$, this fact follows immediately from [31], Chapter IV, A.2.4, Theorem. This completes the proof of assertion (ii), hence of Proposition 1.7. □

Theorem 1.8. *Let p be a prime number; F a number field. Write E ($\subseteq \overline{\mathbb{Q}}$) for the field obtained by adjoining all roots of p to F [so E contains all roots of unity, and $F \subseteq E$ is a nonabelian metabelian Galois extension]. Let K be a subfield of a finitely generated extension field of E. Then K is a TKND-AVKF-field.*

Proof. In light of [38], Theorem A; Remark 1.1.2; Proposition 1.2, (iii), it suffices to verify that K is TKND. However, this follows immediately from the argument applied in the proof of Proposition 1.2, (i), together with the fact that E is algebraic over the prime field. This completes the proof of Theorem 1.8. □

Remark 1.8.1. Note that [38], Theorem A is proved by applying Grothendieck's monodromy theorem [cf. [6]] and Ribet's theorem concerning the finiteness of torsion points of abelian varieties valued in the maximal cyclotomic extensions of number fields [cf. [11]], together with some lemmas observed by Kubo-Taguchi and Moon [cf. [12], [23]].

Remark 1.8.2. It follows immediately from the well-known theory of complex multiplication that the maximal abelian extension of $\mathbb{Q}(\mu_4(\overline{\mathbb{Q}}))$ is not AVKF [cf. [38], Proposition C]. In particular, one concludes from [4], Theorem 16.11.3, that Hilbertian fields need not to be AVKF in general.

2 Reconstruction of the Kummer classes of rational functions

Let K be a TKND-AVKF-field of characteristic 0; X a proper hyperbolic curve over K. In the present section, we discuss the [semi-absolute] reconstruction of the Kummer classes of $K(X)^\times$ from the data of the natural surjection $G_{K(X)} \twoheadrightarrow G_K$, together with the data of cuspidal inertia subgroups [cf. Definition 2.4; Proposition 2.6]. The argument applied in the reconstruction of the Kummer classes is similar to the argument applied in [21], §1. After this, we observe that any isomorphism between the data $G_{K(X)} \twoheadrightarrow G_K$ and a similar data "$G_{K(Y)} \twoheadrightarrow G_L$" maps the cuspidal inertia subgroups of $G_{K(X)}$ to the cuspidal inertia subgroups of $G_{K(Y)}$. This implies that any such isomorphism induces an isomorphism between the respective Kummer classes [cf. Corollary 2.7]. Finally, we also discuss a phenomenon of partial cyclotomic rigidity in the relative anabelian geometric situation [cf. Proposition 2.10], which will be applied in the proof of the relative birational version of the Grothendieck Conjecture for smooth curves over TKND-AVKF-fields in §4.

First, we begin by reviewing the synchronization of geometric cyclotomes.

Definition 2.1. We shall write

$$\Lambda_X$$

for the dual $\widehat{\mathbb{Z}}$-module of the second cohomology group $H^2(\Delta_X, \widehat{\mathbb{Z}})$. Note that since X is a smooth proper curve of genus ≥ 2, it holds that $H^2(\Delta_X, \widehat{\mathbb{Z}}) \xrightarrow{\sim} H^2_{\text{ét}}(X_{\overline{K}}, \widehat{\mathbb{Z}})$. In particular, it follows immediately from Poincaré duality that Λ_X is isomorphic to $\widehat{\mathbb{Z}}(1)$, where "(1)" denotes the Tate twist, i.e., $\widehat{\mathbb{Z}}(1) \stackrel{\text{def}}{=} \varprojlim_n \mu_n(\overline{K})$.

Definition 2.2.

(i) Let G_1, G_2 be profinite groups; $\phi : G_1 \twoheadrightarrow G_2$ an outer surjection. Then we shall refer to the quotient profinite group

$$(G_1 \twoheadrightarrow) \quad G_1/\overline{[\mathrm{Ker}(\phi), G_1]}$$

of G_1 as the *maximal cuspidally central quotient* associated to ϕ [cf. [20], Definition 1.1, (i)].

(ii) Let $x \in X(K)$ be a closed point. Then we shall write $X_x \stackrel{\mathrm{def}}{=} X \setminus \{x\}$;

$$\Delta^{\mathrm{cn}}_{X_x}$$

for the maximal cuspidally central quotient associated to the outer surjection $\Delta_{X_x} \twoheadrightarrow \Delta_X$ induced by the natural open immersion $X_x \hookrightarrow X$.

Proposition 2.3. *Let $x \in X(K)$ be a closed point; $I_x \subseteq \Delta_{K(X)}$ a cuspidal inertia subgroup associated to x. Then we have a natural exact sequence of profinite groups*

$$1 \longrightarrow I_x \longrightarrow \Delta^{\mathrm{cn}}_{X_x} \longrightarrow \Delta_X \longrightarrow 1.$$

Moreover, one may reconstruct the natural scheme-theoretic identification

$$\Lambda_X \xrightarrow{\sim} I_x \ (\xrightarrow{\sim} \widehat{\mathbb{Z}}(1))$$

[from the natural outer surjection $\Delta_{X_x} \twoheadrightarrow \Delta_X$], in a purely group-theoretic way, as follows: The Leray-Serre spectral sequence

$$E_2^{i,j} = H^i(\Delta_X, H^j(I_x, I_x)) \ \Rightarrow \ H^{i+j}(\Delta^{\mathrm{cn}}_{X_x}, I_x)$$

associated to the above exact sequence induces a differential

$$H^1(I_x, I_x) = H^0(\Delta_X, H^1(I_x, I_x)) = E_2^{0,1} \longrightarrow E_2^{2,0} = H^2(\Delta_X, H^0(I_x, I_x)) = \mathrm{Hom}(\Lambda_X, I_x).$$

Then the image of the identity automorphism $\in \mathrm{Hom}(I_x, I_x) = H^1(I_x, I_x)$ via the above differential gives us the isomorphism $\Lambda_X \xrightarrow{\sim} I_x$ as desired.

Proof. Observe that the kernel of the natural outer surjection $\Delta_{X_x} \twoheadrightarrow \Delta_X$ is topologically normally generated by the image of the pro-cyclic group I_x via the natural outer surjection $\Delta_{K(X)} \twoheadrightarrow \Delta_{X_x}$. Thus, the former assertion follows immediately from the various definitions involved. Since the construction of $\Delta^{\mathrm{cn}}_{X_x}$ is purely group-theoretic, the latter assertion also follows immediately from the various definitions involved. This completes the proof of Proposition 2.3. □

Next, by applying the synchronization of geometric cyclotomes discussed above, we reconstruct the group of Kummer classes of $K(X)^\times$.

Definition 2.4. We shall construct a subset

$$\overline{K}(X)^\kappa \subseteq \varinjlim_{K \subseteq K^\dagger} H^1(G_{K(X) \otimes_K K^\dagger}, \Lambda_X),$$

— where $K \subseteq K^\dagger$ $(\subseteq \overline{K})$ ranges over the finite field extensions — as follows: Let $S \subseteq X$ be a nonempty finite subset of closed points. Write $U \stackrel{\mathrm{def}}{=} X \setminus S \subseteq X$. By replacing K by a finite extension field of K, we assume that $S \subseteq X(K)$. Let $K \subseteq M$ be a finite field extension. Observe that the natural exact sequence

$$1 \longrightarrow \Delta_U \longrightarrow \Pi_{U_M} \longrightarrow G_M \longrightarrow 1$$

determines an exact sequence

$$0 \longrightarrow H^1(G_M, \Lambda_X) \longrightarrow H^1(\Pi_{U_M}, \Lambda_X) \xrightarrow{r} H^1(\Delta_U, \Lambda_X)^{G_M}.$$

Thus, by allowing the [sufficiently large] finite extension fields $K \subseteq K^\dagger$ to vary, we obtain an exact sequence

$$0 \longrightarrow \varinjlim_{K \subseteq K^\dagger} H^1(G_{K^\dagger}, \Lambda_X) \longrightarrow \varinjlim_{K \subseteq K^\dagger} H^1(\Pi_{U_{K^\dagger}}, \Lambda_X) \longrightarrow \varinjlim_{K \subseteq K^\dagger} H^1(\Delta_U, \Lambda_X)^{G_{K^\dagger}}.$$

Here, we observe that, for any finite field extension $K \subseteq K^\dagger$,

$$H^1(\Delta_U, \Lambda_X)^{G_{K^\dagger}} = H^1(\Delta_U^{\mathrm{ab}}, \Lambda_X)^{G_{K^\dagger}}.$$

Next, for each $x \in S$, let I_x be a cuspidal inertia subgroup of $\Delta_{K(X)}$ $(\subseteq G_{K(X)})$ associated to x. Then we have an exact sequence of G_M-modules

$$\bigoplus_{x \in S} I_x \longrightarrow \Delta_U^{\mathrm{ab}} \longrightarrow \Delta_X^{\mathrm{ab}} \longrightarrow 0,$$

which determines an exact sequence of modules

$$0 \longrightarrow H^1(\Delta_X^{\mathrm{ab}}, \Lambda_X)^{G_M} \longrightarrow H^1(\Delta_U^{\mathrm{ab}}, \Lambda_X)^{G_M} \longrightarrow \bigoplus_{x \in S} H^1(I_x, \Lambda_X).$$

Note that since K is an AVKF-field, it holds that, for any finite field extension $K \subseteq K^\dagger$,

$$H^1(\Delta_X, \Lambda_X)^{G_{K^\dagger}} = \{0\}.$$

Thus, we obtain a natural injection

$$i : H^1(\Delta_U^{\mathrm{ab}}, \Lambda_X)^{G_M} \hookrightarrow \bigoplus_{x \in S} H^1(I_x, \Lambda_X).$$

Write

$$1_x \in H^1(I_x, \Lambda_X) = \mathrm{Hom}(I_x, \Lambda_X)$$

for the isomorphism $I_x \xrightarrow{\sim} \Lambda_X$ of Proposition 2.3;

$$\mathbb{Z}_x \subseteq H^1(I_x, \Lambda_X)$$

for the subgroup generated by 1_x;

$$i_x : G_M \hookrightarrow \Pi_{X_M}$$

for the section of the natural surjection $\Pi_{X_M} \twoheadrightarrow G_M$ determined by the image of $N_{G_{K(X)\otimes_K M}}(I_x)$ via the natural surjection $G_{K(X)\otimes_K M} \twoheadrightarrow \Pi_{X_M}$. Next, we fix $x_0 \in S$. Write

$$D_x \in H^1(G_M, \Delta_X^{\mathrm{ab}})$$

for the element obtained by forming the difference between i_{x_0} and i_x;

$$\mathcal{P}_S \ \subseteq \ \bigoplus_{x\in S} \mathbb{Z}_x \quad \left(\subseteq \ \bigoplus_{x\in S} H^1(I_x, \Lambda_X)\right)$$

for the subgroup consisting of $(n_x)_{x\in S} \in \bigoplus_{x\in S} \mathbb{Z}_x$ such that

$$\sum_{x\in S} n_x \ = \ 0, \qquad \sum_{x\in S} n_x \cdot D_x \ = \ 0 \quad \big(\in H^1(G_M, \Delta_X^{\mathrm{ab}})\big)$$

[where we identify $\mathbb{Z}_x$ with $\mathbb{Z}$ via the unique isomorphism $\mathbb{Z}_x \xrightarrow{\sim} \mathbb{Z}$ that maps 1_x to 1, and note that one verifies immediately that these conditions on $(n_x)_{x\in S}$ are *independent* of the choice of $x_0 \in S$];

$$\mathcal{P}_S^{\kappa}$$

for the image of $(i \circ r)^{-1}(\mathcal{P}_S)$ via the natural homomorphism

$$H^1(\Pi_{U_M}, \Lambda_X) \longrightarrow \varinjlim_{K\subseteq K^\dagger} H^1(\Pi_{U_{K^\dagger}}, \Lambda_X).$$

Let $y \in U(\overline{K})$ be an element; $I_y \subseteq \Delta_{K(X)} \subseteq G_{K(X)}$ a cuspidal inertia subgroup associated to y. Fix a finite field extension $K \subseteq K_y$ such that $y \in U(K_y)$. Write

$$E_y : \varinjlim_{K\subseteq K^\dagger} H^1(\Pi_{U_{K^\dagger}}, \Lambda_X) \longrightarrow \varinjlim_{K\subseteq K^\dagger} H^1(G_{K^\dagger}, \Lambda_X)$$

for the natural restriction homomorphism induced by the section of the natural surjection $\Pi_{U_{K_y}} \twoheadrightarrow G_{K_y}$ determined by the image of $N_{G_{K(X)\otimes_K K_y}}(I_y)$ via the natural surjection $G_{K(X)\otimes_K K_y} \twoheadrightarrow \Pi_{U_{K_y}}$. Then, in light of the [inductive limit of] inflation map

$$\varinjlim_{K\subseteq K^\dagger} H^1(\Pi_{U_{K^\dagger}}, \Lambda_X) \hookrightarrow \varinjlim_{K\subseteq K^\dagger} H^1(G_{K(X)\otimes_K K^\dagger}, \Lambda_X),$$

by allowing the nonempty finite subset $S \subseteq X$ of closed points to vary, we obtain

$$\mathcal{P}^{\kappa} \ \stackrel{\mathrm{def}}{=} \ \bigcup_{S} \mathcal{P}_S^{\kappa} \subseteq \varinjlim_{K\subseteq K^\dagger} H^1(G_{K(X)\otimes_K K^\dagger}, \Lambda_X).$$

Finally, we define

$$\overline{K}(X)^{\kappa} \ \subseteq \ \mathcal{P}^{\kappa} \quad \left(\subseteq \varinjlim_{K\subseteq K^\dagger} H^1(G_{K(X)\otimes_K K^\dagger}, \Lambda_X)\right)$$

as a subset consisting of elements $f \in \mathcal{P}^{\kappa}$ satisfying one of the following conditions:

(i) There exist a nonempty finite subset $S_f \subseteq X$ of closed points and $z_1, z_2 \in (X \setminus S_f)(\overline{K})$ such that

$$f \in \mathcal{P}^{\kappa}_{S_f}, \quad E_{z_1}(f) = 1, \quad E_{z_2}(f) \neq 1$$

[where, if one works additively, then $E_{z_1}(f) = 0$, and $E_{z_2}(f) \neq 0$].

(ii) There exist an element $g \in \mathcal{P}^{\kappa}_{S_g} \subseteq \mathcal{P}^{\kappa}$ satisfying condition (i), and $w \in (X \setminus S_g)(\overline{K})$ such that the image of $E_w(g)$ via the [inductive limit of] inflation map

$$\varinjlim_{K \subseteq K^{\dagger}} H^1(G_{K^{\dagger}}, \Lambda_X) \hookrightarrow \varinjlim_{K \subseteq K^{\dagger}} H^1(G_{K(X) \otimes_K K^{\dagger}}, \Lambda_X)$$

coincides with f.

Proposition 2.5. *We maintain the notation of Definition 2.4. Then the following hold:*

(i) The subgroup $\mathcal{P}_S \subseteq \bigoplus_{x \in S} \mathbb{Z}_x$ $(\xrightarrow{\sim} \bigoplus_{x \in S} \mathbb{Z})$ coincides with the group of principal divisors associated to the rational functions on X whose support are contained in S.

(ii) Let $h \in \mathcal{P}^{\kappa}_S$ be an element. Then there exist a finite field extension $K \subseteq K_h$, a rational function h_f on U_{K_h}, and an element $c \in H^1(G_{K_h}, \Lambda_X)$, such that

$$h = \overline{c} \cdot h^{\kappa}_f,$$

where $\overline{c}$ denotes the image of c via the natural composite homomorphism

$$H^1(G_{K_h}, \Lambda_X) \to \varinjlim_{K \subseteq K^{\dagger}} H^1(G_{K^{\dagger}}, \Lambda_X) \hookrightarrow \varinjlim_{K \subseteq K^{\dagger}} H^1(\Pi_{U_{K^{\dagger}}}, \Lambda_X);$$

h^{κ}_f denotes the image of h_f via the Kummer map. [In the remainder, we shall say that h is nonconstant if the rational function h_f is nonconstant.]

(iii) Let $K \subseteq K^{\dagger}$ be a finite field extension; f a nonconstant rational function on $X_{K^{\dagger}}$. Then there exist points $z_1, z_2 \in X_{K^{\dagger}}(\overline{K})$ such that z_1 and z_2 are neither zeros nor poles of f, and the image of $f(z_1)$ (respectively, $f(z_2)$) via the [inductive limit of] Kummer map

$$\overline{K}^{\times} = \varinjlim_{K \subseteq K^{\dagger}} (K^{\dagger})^{\times} \longrightarrow \varinjlim_{K \subseteq K^{\dagger}} H^1(G_{K^{\dagger}}, \Lambda_X)$$

is trivial (respectively, nontrivial).

(iv) Let $h \in \mathcal{P}^{\kappa}$ be a nonconstant element. Then it holds that $h \in \overline{K}(X)^{\kappa}$ if and only if h coincides with the Kummer class of a nonconstant rational function on $X_{K^{\dagger}}$ for some finite field extension $K \subseteq K^{\dagger}$.

(v) Let $c \in \overline{K}^{\times}$ be an element; g a nonconstant rational function on X. Then there exists a point $z \in X(\overline{K})$ such that $g(z)$ coincides with c.

Proof. Since K is an AVKF-field, assertion (i) follows immediately from an elementary property of Jacobian, together with the definition of $\mathcal{P}_S$. Assertion (ii) follows immediately from assertion (i), together with the various definitions involved. Since K is a TKND-field, assertion (iii) follows immediately from the fact that every nonconstant rational function on $X_{K^\dagger}$ determines a [finite] surjective morphism $X_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$. Assertion (iv) follows immediately from assertions (ii), (iii). Assertion (v) follows immediately from a similar argument to the argument applied in the proof of assertion (iii). This completes the proof of Proposition 2.5. $\square$

Proposition 2.6. *The image of the [inductive limit of] Kummer map*

$$\varinjlim_{K \subseteq K^\dagger} (K(X) \otimes K^\dagger)^\times \longrightarrow \varinjlim_{K \subseteq K^\dagger} H^1(G_{K(X)\otimes_K K^\dagger}, \Lambda_X)$$

— where $K \subseteq K^\dagger$ $(\subseteq \overline{K})$ ranges over the finite field extensions — coincides with $\overline{K}(X)^\kappa$ [cf. Definition 2.4].

Proof. Proposition 2.6 follows immediately from Proposition 2.5, (iv), (v), together with the various constructions involved. $\square$

Corollary 2.7. *Let L be a TKND-AVKF-field of characteristic 0; Y a smooth proper curve over L;*

$$\sigma : G_{K(X)} \xrightarrow{\sim} G_{K(Y)}$$

an isomorphism of profinite groups that induces an isomorphism $G_K \xrightarrow{\sim} G_L$ via the natural surjections $G_{K(X)} \twoheadrightarrow G_K$ and $G_{K(Y)} \twoheadrightarrow G_L$. For each finite field extension $K \subseteq K^\dagger$ $(\subseteq \overline{K})$, write $L \subseteq L^\dagger$ $(\subseteq \overline{L})$ for the corresponding finite field extension via the isomorphism $G_K \xrightarrow{\sim} G_L$. Then σ induces naturally commutative diagrams of groups:

$$\begin{array}{ccc}
\varinjlim_{K \subseteq K^\dagger} (K^\dagger)^\times/(K^\dagger)^{\times\infty} & \longrightarrow & \varinjlim_{K \subseteq K^\dagger} H^1(G_{K^\dagger}, \Lambda_X) \\
\downarrow \wr & & \downarrow \wr \\
\varinjlim_{L \subseteq L^\dagger} (L^\dagger)^\times/(L^\dagger)^{\times\infty} & \longrightarrow & \varinjlim_{L \subseteq L^\dagger} H^1(G_{L^\dagger}, \Lambda_Y),
\end{array}$$

$$\begin{array}{ccc}
\varinjlim_{K \subseteq K^\dagger} (K(X) \otimes_K K^\dagger)^\times/(K^\dagger)^{\times\infty} & \longrightarrow & \varinjlim_{K \subseteq K^\dagger} H^1(G_{K(X)\otimes_K K^\dagger}, \Lambda_X) \\
\downarrow \wr & & \downarrow \wr \\
\varinjlim_{L \subseteq L^\dagger} (K(Y) \otimes_L L^\dagger)^\times/(L^\dagger)^{\times\infty} & \longrightarrow & \varinjlim_{L \subseteq L^\dagger} H^1(G_{K(Y)\otimes_L L^\dagger}, \Lambda_Y),
\end{array}$$

where $K \subseteq K^\dagger$ $(\subseteq \overline{K})$ ranges over the finite field extensions; the horizontal arrows of the above commutative diagrams denote the natural injections induced by the [inductive limits of] Kummer maps [cf. Proposition 2.3].

Proof. First, it follows immediately from Proposition 2.6 that it suffices to verify that σ maps the cuspidal inertia subgroups of $G_{K(X)}$ to the cuspidal inertia subgroups of $G_{K(Y)}$. Note that, if there exists a prime number l such that the l-adic cyclotomic character associated to K [or equivalently, to L — cf. [21], Theorem 1.11, (a)] is open, then such a preservation of cuspidal inertia subgroups follows from a similar argument to the argument applied in the procedure of [21], Theorem 1.11, (b). Thus, we may assume without loss of generality that, for each prime number l, the image of the l-adic cyclotomic character associated to K [or equivalently, to L] is *finite*. Let $I \subseteq G_{K(X)}$ be a cuspidal inertia subgroup.

Next, we verify the following assertion:

> Claim 2.7.A: The image of the closed subgroup $\sigma(I) \subseteq G_{K(Y)}$ via the natural composite homomorphism
>
> $$G_{K(Y)} \twoheadrightarrow \Pi_Y \twoheadrightarrow \Pi_Y^{(\mathrm{ab})}$$
>
> — where the first arrow denotes the natural surjection induced by the natural morphism Spec $K(Y) \to Y$; the second arrow denotes the natural surjection [cf. Notations and Conventions] — is trivial.

Indeed, let l be a prime number. Observe that it follows immediately from our assumption that the l-adic cyclotomic character associated to L is finite that, by replacing L by a finite extension field of L, we may assume without loss of generality that $\sigma(I)^l$ is isomorphic to $\mathbb{Z}_l$ as a G_L-module. On the other hand, since L is an AVKF-field, it holds that the image of any G_L-equivariant homomorphism $(\sigma(I)^l \cong)\ \mathbb{Z}_l \to (\Delta_Y^{\mathrm{ab}})^l$ vanishes. Thus, by varying l, we conclude that the image of $\sigma(I)$ in Δ_Y^{ab} is trivial. This completes the proof of Claim 2.7.A.

Next, we verify the following assertion:

> Claim 2.7.B: Let $J \subseteq G_{K(Y)}$ be a pro-cyclic closed subgroup. Suppose that, for any open subgroup $G_{K(Z)} \subseteq G_{K(Y)}$ [where Z denotes the domain curve of the finite ramified covering of Y corresponding to the open subgroup], the image of the closed subgroup $J \cap G_{K(Z)} \subseteq G_{K(Z)}$ via the natural composite homomorphism
>
> $$G_{K(Z)} \twoheadrightarrow \Pi_Z \twoheadrightarrow \Pi_Z^{(\mathrm{ab})}$$
>
> — where the first arrow denotes the natural surjection induced by the natural morphism Spec $K(Z) \to Z$; the second arrow denotes the natural surjection [cf. Notations and Conventions] — is trivial. Then it holds that $J \subseteq G_{K(Y)}$ is a cuspidal inertia subgroup.

Indeed, Claim 2.7.B follows immediately from [9], Lemma 1.6.

Finally, it follows immediately from the various definitions involved that the intersection of I with any open subgroup of $G_{K(X)}$ is also a cuspidal inertia subgroup of the open subgroup. Then, by applying Claim 2.7.A for such open subgroups, we observe that the assumption of Claim 2.7.B for $J = \sigma(I)$ holds. Thus, we conclude from Claim 2.7.B that $\sigma(I) \subseteq G_{K(Y)}$ is a cuspidal inertia subgroup. This completes the proof of Corollary 2.7. □

Next, we discuss a phenomenon of partial cyclotomic rigidity in the relative anabelian geometric situation.

Lemma 2.8. *It holds that $\mathbb{Q} \cap \widehat{\mathbb{Z}}^\times = \{\pm 1\}$, where we regard $\mathbb{Q}$ and $\widehat{\mathbb{Z}}^\times$ $(\subseteq \widehat{\mathbb{Z}})$ as subsets of $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ via the natural injections $\mathbb{Q} \hookrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $\widehat{\mathbb{Z}}^\times \hookrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.*

Proof. Let m be a positive integer; n a nonzero integer coprime to m; $a \in \widehat{\mathbb{Z}}^\times$ an element such that $n = am$. Observe that, for every prime number l, it holds that the l-adic valuation of m coincides with the l-adic valuation of n. Then since m is coprime to n, we conclude that $m, n \in \{\pm 1\}$. This argument implies that $\mathbb{Q} \cap \widehat{\mathbb{Z}}^\times = \{\pm 1\}$. This completes the proof of Lemma 2.8. $\square$

Remark 2.8.1. In [22], this elementary property is applied to reconstruct the cyclotomic rigidity isomorphism surrounding κ-coric rational functions.

Lemma 2.9. *Write*

$$\kappa : \overline{K}^\times = \varinjlim_{K \subseteq K^\dagger} (K^\dagger)^\times \longrightarrow \varinjlim_{K \subseteq K^\dagger} H^1(G_{K^\dagger}, \widehat{\mathbb{Z}}(1))$$

— where $K \subseteq K^\dagger$ ranges over the finite field extensions — for the [inductive limit of] Kummer map(s). Let $\sigma \in \mathrm{Aut}(\widehat{\mathbb{Z}}(1)) = \widehat{\mathbb{Z}}^\times$ be an element. Write

$$\tau \in \mathrm{Aut}\big(\varinjlim_{K \subseteq K^\dagger} H^1(G_{K^\dagger}, \widehat{\mathbb{Z}}(1))\big)$$

for the automorphism induced by σ. Suppose that

- $\tau(\mathrm{Im}(\kappa)) = \mathrm{Im}(\kappa)$, *and*
- *there exists a surjective homomorphism $K^\times \twoheadrightarrow \mathbb{Z}$.*

Then it holds that $\sigma \in \{\pm 1\}$.

Proof. Fix a surjective homomorphism $\phi : K^\times \twoheadrightarrow \mathbb{Z}$. Write $\tilde{\phi} : \overline{K}^\times \twoheadrightarrow \mathbb{Q}$ for the surjective homomorphism obtained by assigning

$$\overline{K}^\times \ni x \mapsto \frac{\phi(\mathrm{Nm}_{K^\dagger/K}(x))}{[K^\dagger : K]} \in \mathbb{Q},$$

where $K^\dagger$ denotes a finite extension field of K that contains x; $\mathrm{Nm}_{K^\dagger/K} : (K^\dagger)^\times \to K^\times$ denotes the norm map. Then we have the following commutative diagram

$$\begin{array}{ccc} K^\times & \xrightarrow{\phi} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \overline{K}^\times & \xrightarrow{\tilde{\phi}} & \mathbb{Q} \\ {\scriptstyle\kappa}\downarrow & & \downarrow \\ \varinjlim_{K \subseteq K^\dagger} H^1(G_{K^\dagger}, \widehat{\mathbb{Z}}(1)) & \xrightarrow{\hat{\phi}} & \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}, \end{array}$$

where the right-hand vertical arrows denote the natural injections; the lower horizontal arrow denotes the homomorphism induced by $\tilde{\phi}$. Observe that the natural actions of σ on the domain and the codomain of $\hat{\phi}$ is compatible with $\hat{\phi}$. In particular, since $\tau(\mathrm{Im}(\kappa)) = \mathrm{Im}(\kappa)$, the automorphism of $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ determined by $\sigma \in \widehat{\mathbb{Z}}^{\times}$ maps $1 \in \mathbb{Q} \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ to some nonzero rational number $\in \mathbb{Q} \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Thus, we conclude from Lemma 2.8 that $\sigma \in \{\pm 1\}$. This completes the proof of Lemma 2.9. $\square$

Proposition 2.10. *In the notation of Corollary 2.7, suppose that*

- $K = L$,
- *the isomorphism σ lies over G_K, and*
- *there exists a surjective homomorphism $K^{\times} \twoheadrightarrow \mathbb{Z}$.*

Then the natural automorphism $\widehat{\mathbb{Z}}(1) \xrightarrow{\sim} \Lambda_X \xrightarrow{\sim} \Lambda_Y \xleftarrow{\sim} \widehat{\mathbb{Z}}(1)$ induced by σ [cf. Proposition 2.3] is the identity automorphism or the inversion automorphism.

Proof. In light of the first commutative diagram of Corollary 2.7, Proposition 2.10 follows immediately from Lemma 2.9. $\square$

3 Quasi-rational functions on smooth curves over algebraically closed fields

In the present section, we introduce the notion of *quasi-rational functions* on smooth [proper] curves over algebraically closed fields, which may be regarded as a certain generalized notion of rational functions. After introducing quasi-rational functions [cf. Definition 3.1], we give a criterion that quasi-rational functions become rational functions automatically [cf. Proposition 3.6]. This criterion is the main observation of the present paper and will be applied in the proof of the relative birational version of the Grothendieck Conjecture for smooth curves over TKND-AVKF-fields in the next section.

Definition 3.1. Let K be an algebraically closed field; $k \subseteq K$ a subfield; X a smooth proper curve over K; $f \in \mathrm{Fn}(X(K), K \cup \{\infty\})$. Then we shall say that f is a *quasi-rational function* on X associated to k if there exist elements

$$\phi_f \in \mathrm{Fn}(X(K), k^{\times}), \quad g \in K(X)$$

such that $f = \phi_f \cdot g$.

Remark 3.1.1. We maintain the notation of Definition 3.1. Then the following hold:

(i) Every rational function $\in K(X)$ is a quasi-rational function on X associated to any subfield of K.

(ii) Let $f_1, f_2 \in \mathrm{Fn}(X(K), K \cup \{\infty\})$ be quasi-rational functions on X associated to k. Then it follows immediately from the various definitions involved that the product $f_1 \cdot f_2$ is a quasi-rational function on X associated to k.

Remark 3.1.2. In the notation of Definition 3.1, suppose that $k \subsetneq K$, and $f \in \mathrm{Fn}(X(K), K \cup \{\infty\})$ is a quasi-rational function on X associated to k. Then since every nonconstant rational function on X induces a surjective morphism $X \to \mathbb{P}^1_K$, the rational function g is uniquely determined by f up to multiplication by a constant function $\in \mathrm{Fn}(X(K), k^\times)$.

Remark 3.1.3. In the notation of Definition 3.1, suppose that $k \subsetneq K$. Let

$$f_1, f_2 \in \mathrm{Fn}(X(K), K \cup \{\infty\})$$

be quasi-rational functions on X associated to k. Note that, for each $x \in X(K)$, it holds that

$$(f_1 + f_2)|_x \stackrel{\mathrm{def}}{=} \phi_{f_1}(x) \cdot g_1 + \phi_{f_2}(x) \cdot g_2 \in K(X)$$

[cf. Remark 3.1.2]. Then, by assigning x to the evaluation of the rational function $(f_1+f_2)|_x$ at x, we obtain an element $f_1 + f_2 \in \mathrm{Fn}(X(K), K \cup \{\infty\})$.

Lemma 3.2. *Let K be an algebraically closed field; X a smooth proper curve over K; g_1, g_2 nonconstant rational functions on X. For each $i = 1, 2$, write m_i for the geometric degree of g_i [i.e., the degree of the finite morphism $X \to \mathbb{P}^1_K$ determined by g_i]. Suppose that the rational function g_1+g_2 is also a nonconstant function. Then, if we write m for the geometric degree of $g_1 + g_2$, then it holds that $m \leq m_1 + m_2$.*

Proof. Observe that the geometric degrees m_1, m_2, m coincide with the degrees of the pole divisors associated to the nonconstant rational functions g_1, g_2, $g_1 + g_2$, respectively. This observation immediately implies that $m \leq m_1 + m_2$. This completes the proof of Lemma 3.2. □

Lemma 3.3. *In the notation of Remark 3.1.3, suppose that*

- *$k \subseteq K$ is an infinite algebraic field extension,*
- *$g_1 \in K(X)$ is a nonconstant rational function, and*
- *$1 = f_1 + f_2 \in \mathrm{Fn}(X(K), K \cup \{\infty\})$.*

Then there exists an element $x \in X(K)$ such that $(f_1 + f_2)|_x \in K(X)$ is a constant function.

Proof. First, since $g_1 \in K(X)$ is a nonconstant rational function, it holds that $\phi_{f_2} \cdot g_2 = 1 - \phi_{f_1} \cdot g_1$ admits both a value $\in K$ and a pole. In particular, $g_2 \in K(X)$ is also a nonconstant rational function. Next, since $k \subseteq K$ is an infinite algebraic extension, by replacing k by a finite extension field of k, we may assume without loss of generality that the finite morphisms $X \to \mathbb{P}^1_K$ induced by the nonconstant rational functions g_1 and g_2 descend to finite morphisms $X_k \to \mathbb{P}^1_k$ over k. Write m_1, m_2 for the degrees of the finite morphisms associated to g_1, g_2, respectively. Suppose that,

> for every $x \in X(K)$, the rational function $(f_1 + f_2)|_x \in K(X)$ is a nonconstant function.

Note that since $f_1 + f_2 = 1$, it holds that $(f_1 + f_2)|_x(x) = 1 \in \mathbb{P}^1_k(k)$. Then it follows immediately from Lemma 3.2 that, for each $x \in X(K)$, there exists a finite field extension $k \subseteq k_x$ $(\subseteq K)$ of degree $\leq m_1 + m_2$ such that $x \in X_k(k_x) \subseteq X(K)$. On the other hand, since $k \subseteq K$ is an infinite algebraic field extension, there exists an element $y \in K$ such that the finite field extension $k \subseteq k(y)$ is of degree $> m_1 + m_2$. In particular, the residue field of any point $\in g_1^{-1}(y) \subseteq X_k(K)$ is a finite extension field of k of degree $> m_1 + m_2$. This is a contradiction. Thus, we conclude that there exists an element $x \in X(K)$ such that $(f_1 + f_2)|_x \in K(X)$ is a constant function. This completes the proof of Lemma 3.3. $\square$

Lemma 3.4. *In the notation of Remark 3.1.3, suppose that*

- $k \subsetneq K$,
- $1 = f_1 + f_2 \in \mathrm{Fn}(X(K), K \cup \{\infty\})$, *and*
- *there exists an element $x \in X(K)$ such that $(f_1 + f_2)|_x \in K(X)$ is a constant function.*

Write $C \subseteq X(K)$ for the inverse image of $\mathbb{P}^1_K(K) \setminus \mathbb{P}^1_k(k) \subseteq \mathbb{P}^1_K(K)$ via the morphism $X \to \mathbb{P}^1_K$ determined by the rational function g_1 [cf. Remark 3.1.2]. Then there exists a rational function $h \in K(X) \subseteq \mathrm{Fn}(X(K), K \cup \{\infty\})$ such that the restriction of h to $C \subseteq X(K)$ coincides with the restriction of f_1 to $C \subseteq X(K)$.

Proof. Fix an element $x \in X(K)$ such that $(f_1 + f_2)|_x \in K(X)$ is a constant function. Then since $f_1 + f_2 = 1$, it holds that $(f_1 + f_2)|_x = 1$. Next, by replacing g_1, g_2 by $\phi_{f_1}(x)^{-1} \cdot g_1$, $\phi_{f_2}(x)^{-1} \cdot g_2$, respectively, we may assume without loss of generality that $\phi_{f_1}(x) = \phi_{f_2}(x) = 1$, hence that $g_1 + g_2 = 1$. Thus, since $f_1 + f_2 = 1$, it holds that

$$\phi_{f_2} - 1 = (\phi_{f_2} - \phi_{f_1}) \cdot g_1.$$

Note that $\phi_{f_2} - 1$ and $\phi_{f_2} - \phi_{f_1}$ are k-valued [set-theoretic] functions. Then, for each $y \in C$, it holds that $\phi_{f_2}(y) - 1 = \phi_{f_2}(y) - \phi_{f_1}(y) = 0$, hence that $\phi_{f_1}(y) = \phi_{f_2}(y) = 1$. Therefore, one may choose the rational function g_1 as a desired rational function h. This completes the proof of Lemma 3.4. $\square$

Lemma 3.5. *In the notation of Remark 3.1.3, suppose that*

- $k \subseteq K$ *is an infinite algebraic field extension,*
- $1 = f_1 + f_2 \in \mathrm{Fn}(X(K), K \cup \{\infty\})$,
- $g_1 \in K(X)$ *is a nonconstant rational function, and*
- *there exists an element* $a \in K \setminus k$ *such that* $f_1 + a$ *and* $f_2 - a$ $(= -(f_1 + a - 1))$ *are quasi-rational functions on* X *associated to* k.

Then it holds that $f_1 \in K(X)$.

Proof. Fix an element $a \in K \setminus k$ such that $f_1 + a$ and $f_2 - a$ are quasi-rational functions on X associated to k. Then there exist elements

$$\phi_a \in \mathrm{Fn}(X(K), k^{\times}), \quad g_a \in K(X)$$

such that $\phi_{f_1} \cdot g_1 + a = f_1 + a = \phi_a \cdot g_a$. Fix such elements. Note that since g_1 is a nonconstant rational function, it holds that $\phi_{f_1} \cdot g_1 + a$ admits both a value $\in K$ and a pole. In particular, g_a is also a nonconstant rational function. Write $C \subseteq X(K)$ (respectively, $C_a \subseteq X(K)$) for the inverse image of $\mathbb{P}^1_K(K) \setminus \mathbb{P}^1_k(k) \subseteq \mathbb{P}^1_K(K)$ via the finite morphism $X \to \mathbb{P}^1_K$ determined by the nonconstant rational function g_1 (respectively, g_a). Observe that since $a \in K \setminus k$, it follows immediately from the equality $\phi_{f_1} \cdot g_1 + a = \phi_a \cdot g_a$, together with the various definitions involved, that

$$C \bigcup C_a = X(K).$$

On the other hand, it follows immediately from Lemmas 3.3, 3.4, together with our assumption that $f_2 - a$ is a quasi-rational function on X associated to k, that there exists a nonconstant rational function $h \in K(X)$ (respectively, $h_a \in K(X)$) such that the restriction of h (respectively, h_a) to C (respectively, C_a) coincides with the restriction of f_1 to C (respectively, C_a). Therefore, since $C \cup C_a = X(K)$, it suffices to verify that $h = h_a$. Observe that since $k \subseteq K$ is an infinite algebraic field extension, it holds that $C \cap C_a$ is an infinite set. In particular, the rational function $h - h_a$ coincides with 0 on some infinite subset. Thus, we conclude that $h = h_a$. This completes the proof of Lemma 3.5. $\square$

Finally, by applying Lemma 3.5, we obtain the following criterion for algebricity of quasi-rational functions:

Proposition 3.6. *Let* K *be an algebraically closed field;* $k \subseteq K$ *a subfield such that the extension is algebraic and of infinite degree;* X *a smooth proper curve over* K; $f \in \mathrm{Fn}(X(K), K \cup \{\infty\})$ *a quasi-rational function on* X *associated to* k *such that the rational function determined [up to multiplication by a constant function — cf. Remark 3.1.2] by* f *is a nonconstant function. Suppose that, for every* $a \in K$, *the set-theoretic function* $f + a$ *is a quasi-rational function on* X *associated to* k. *Then* f *is a rational function on* X.

Proof. Let $b \in K \setminus k$ be an element. Note that it follows immediately from our assumption that the set-theoretic functions

$$1 - f = -1 \cdot (f - 1), \quad f + b, \quad 1 - f - b = -1 \cdot (f + b - 1)$$

are also quasi-rational functions on X associated to k [cf. Remark 3.1.1, (ii)]. Thus, since the rational function determined [up to multiplication by a constant function] by f is nonconstant, we conclude from Lemma 3.5, together with the various definitions involved, that f is a rational function on X. This completes the proof of Proposition 3.6. $\square$

Remark 3.6.1. One may regard the above criterion for algebricity of quasi-rational functions as a strengthened version of [10], Corollary 1.3.

4 Birational anabelian Grothendieck Conjecture over TKND-AVKF-fields

In the present section, by applying the results obtained in the previous sections, we first prove the relative birational version of the Grothendieck Conjecture for smooth curves over TKND-AVKF-fields of characteristic 0 whose multiplicative groups admit surjective homomorphisms onto $\mathbb{Z}$ [cf. Theorem 4.7]. Next, we prove a certain generalization of the result on the freeness of the multiplicative groups of fields modulo torsion obtained by May [cf. Proposition 4.9]. Finally, as a corollary Theorem 4.7 and Proposition 4.9, we obtain the relative birational version of the Grothendieck Conjecture for smooth curves over subfields of finitely generated extension fields of the field obtained by adjoining all roots of a prime number to a number field [cf. Corollary 4.10].

Definition 4.1. Let K be a field of characteristic 0; X an algebraic variety over K. Then:

(i) We shall write

$$\mathrm{Sect}^{\mathrm{Gal}}(X)$$

for the set of equivalence classes of sections of the natural surjection $\Pi_X \twoheadrightarrow G_K$ that arise from the K-valued points of X, where we consider two such sections to be equivalent if they differ by composition with an inner automorphism induced by an element of Δ_X. In particular, we have a natural surjection

$$X(K) \twoheadrightarrow \mathrm{Sect}^{\mathrm{Gal}}(X).$$

(ii) We shall write

$$\mathrm{Sect}^{\text{Op-Gal}}(X) \stackrel{\text{def}}{=} \varinjlim_{K \subseteq K^{\dagger}} \mathrm{Sect}^{\mathrm{Gal}}(X_{K^{\dagger}}),$$

where $K \subseteq K^\dagger$ ($\subseteq \overline{K}$) ranges over the finite field extensions; the transition maps [that appear in the inductive limit] are the natural maps obtained by restricting the domains of sections. In particular, we have a natural surjection

$$X_{\overline{K}}(\overline{K}) \twoheadrightarrow \mathrm{Sect}^{\text{Op-Gal}}(X).$$

Proposition 4.2. *Let K be an AVKF-field of characteristic 0; X a hyperbolic curve over K. Then the natural surjection*

$$X_{\overline{K}}(\overline{K}) \twoheadrightarrow \mathrm{Sect}^{\text{Op-Gal}}(X)$$

[cf. Definition 4.1, (ii)] is bijective.

Proof. Proposition 4.2 follows immediately from the injectivity portion of the Section Conjecture for arbitrary hyperbolic curves over AVKF-fields [cf. the proofs of [10], Corollary 6.4; [37], Theorem 3.1; [37], Corollary 3.2], together with the various definitions involved. □

Proposition 4.3. *Let K be a field of characteristic 0; X a smooth curve over K. Write*

$$\mathcal{I}_X$$

for the set of conjugacy classes of cuspidal inertia subgroups of $\Delta_{K(X)}$ that are not associated to cusps of $X_{\overline{K}}$. Then the natural surjection

$$X_{\overline{K}}(\overline{K}) \twoheadrightarrow \mathcal{I}_X$$

is bijective.

Proof. Proposition 4.3 follows immediately from [19], Proposition 1.2, (i), together with the various definitions involved. □

Proposition 4.4. *We maintain the notation of Proposition 4.3. Suppose that K is an AVKF-field, and X is a hyperbolic curve over K. Then the natural surjection*

$$\mathcal{I}_X \twoheadrightarrow \mathrm{Sect}^{\text{Op-Gal}}(X)$$

[obtained by forming the images of the normalizers in $G_{K(X)}$ of cuspidal inertia subgroups of $\Delta_{K(X)}$ via the natural [Δ_X-outer] surjection $G_{K(X)} \twoheadrightarrow \Pi_X$ that lies over G_K] is bijective. Moreover, this bijection is compatible with the bijections that appear in Propositions 4.2, 4.3.

Proof. Proposition 4.4 follows immediately from Propositions 4.2, 4.3. □

Lemma 4.5. *Let K_1 be a field; $K_1 \subseteq K_2$ an infinite field extension such that K_2 is an algebraically closed field. Then there exists an intermediate field $K_1 \subseteq K_3 \subseteq K_2$ such that the field extension $K_3 \subseteq K_2$ is an infinite algebraic field extension.*

Proof. Recall that every field extension admits a transcendence basis, and the absolute Galois groups of the purely transcendental extension fields of arbitrary field are infinite. Thus, since K_2 is an algebraically closed field, one may take K_3 as the field obtained by adjoining a transcendental basis of the field extension $K_1 \subseteq K_2$ to K_1. This completes the proof of Lemma 4.5. $\square$

Here, we review a well-known lemma for "anabelian" profinite groups.

Lemma 4.6.

(i) Let G be a profinite group; $H \subseteq G$ a normal open subgroup such that $Z_G(H) = \{1\}$; $\sigma \in \mathrm{Aut}(G)$ an automorphism that induces the identity automorphism on H. Then σ is the identity automorphism.

(ii) Let

$$1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$$

be an exact sequence of profinite groups; $\sigma \in \mathrm{Aut}(G_2)$ an automorphism that induces the identity automorphisms on G_1 and G_3. Suppose that G_1 is center-free. Then σ is the identity automorphism.

Proof. First, we verify assertion (i). Observe that since $Z_G(H) = \{1\}$, the natural homomorphism $G \to \mathrm{Aut}(H)$ obtained by forming conjugations is injective. Moreover, σ and the inner automorphism of $\mathrm{Aut}(H)$ determined by the restriction of σ on H are compatible with this injection. Thus, since σ induces the identity automorphism on H, we conclude that σ is the identity automorphism. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $g_1 \in G_1$, $g_2 \in G_2$ be elements. Then since σ induces the identity automorphism on G_3, it holds that $\sigma(g_2)g_2^{-1} \in G_1$. Moreover, since σ induces the identity automorphism on G_1, it holds that

$$\sigma(g_2^{-1})g_1\sigma(g_2) = \sigma(g_2^{-1}g_1g_2) = g_2^{-1}g_1g_2,$$

hence that g_1 commutes with $\sigma(g_2)g_2^{-1}$. This argument implies that $\sigma(g_2)g_2^{-1} \in Z_{G_1}(G_1)$. Thus, since G_1 is center-free, we conclude that σ is the identity automorphism. This completes the proof of assertion (ii), hence of Lemma 4.6. $\square$

Next, we prove our main theorem [cf. Theorem A]:

Theorem 4.7. *Let K be a TKND-AVKF-field of characteristic* 0*; X, Y smooth proper curves over K. Write*

$$\mathrm{Isom}_K(K(Y), K(X))$$

for the set of K-isomorphisms between $K(Y)$ and $K(X)$;

$$\mathrm{Isom}_{G_K}(G_{K(X)}, G_{K(Y)})/\mathrm{Inn}(\Delta_{K(Y)})$$

for the set of isomorphisms $G_{K(X)} \xrightarrow{\sim} G_{K(Y)}$ of profinite groups that lie over G_K, considered up to compositions with inner automorphisms that arise from elements $\in \Delta_{K(Y)}$. Suppose that there exists a surjective homomorphism $K^\times \twoheadrightarrow \mathbb{Z}$. Then the natural map

$$\mathrm{Isom}_K(K(Y), K(X)) \longrightarrow \mathrm{Isom}_{G_K}(G_{K(X)}, G_{K(Y)})/\mathrm{Inn}(\Delta_{K(Y)})$$

is bijective.

Proof. First, we verify the injectivity. It follows immediately from the various definitions involved that we may assume without loss of generality that $X = Y$. Moreover, by replacing X by a finite ramified Galois covering of X, we may assume without loss of generality that X is a smooth proper curve of genus ≥ 2. Then the desired injectivity follows immediately from the well-known injectivity of the natural map

$$(\mathrm{Aut}_K(K(X)) \ \subseteq) \quad \mathrm{Aut}_{\overline{K}}(K(X) \otimes_K \overline{K}) \ = \ \mathrm{Aut}_{\overline{K}}(X_{\overline{K}}) \ \longrightarrow \ \mathrm{Out}(\Delta_X).$$

Next, we verify the surjectivity. Recall that $\Delta_{K(X)}$ is isomorphic to the inverse limit of a system of free profinite groups whose transition maps are surjective. In particular, for every open subgroup $H \subseteq \Delta_{K(X)}$, it holds that $Z_{\Delta_{K(X)}}(H) = \{1\}$. Thus, to verify the surjectivity, it follows immediately from Galois descent, together with Lemma 4.6, (i), (ii), that we may assume without loss of generality that X and Y have genus ≥ 2. [In particular, one may apply the constructions in §2.] Let

$$\sigma \in \mathrm{Isom}_{G_K}(G_{K(X)}, G_{K(Y)})/\mathrm{Inn}(\Delta_{K(Y)})$$

be an element; $\tilde{\sigma} \in \mathrm{Isom}_{G_K}(G_{K(X)}, G_{K(Y)})$ a lifting of σ. Recall from the proof of Corollary 2.7 that $\tilde{\sigma}$ induces a bijection between the set of cuspidal inertia subgroups of $G_{K(X)}$ and the set of cuspidal inertia subgroups of $G_{K(Y)}$. Then, in light of Proposition 4.4, since K is an AVKF-field of characteristic 0, the isomorphism $\tilde{\sigma}$ determines a commutative diagram

$$\begin{array}{ccccc}
X_{\overline{K}}(\overline{K}) & \xrightarrow{\sim} & \mathcal{I}_X & \xrightarrow{\sim} & \mathrm{Sect}^{\text{Op-Gal}}(X) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
Y_{\overline{K}}(\overline{K}) & \xrightarrow{\sim} & \mathcal{I}_Y & \xrightarrow{\sim} & \mathrm{Sect}^{\text{Op-Gal}}(Y).
\end{array}$$

Write

$$p^{\kappa}_{\tilde{\sigma}} : \varinjlim_{K \subseteq K^\dagger} H^1(G_{K(Y)\otimes_K K^\dagger}, \widehat{\mathbb{Z}}(1)) \xrightarrow{\sim} \varinjlim_{K \subseteq K^\dagger} H^1(G_{K(X)\otimes_K K^\dagger}, \widehat{\mathbb{Z}}(1))$$

— where $K \subseteq K^\dagger$ ($\subseteq \overline{K}$) ranges over the finite field extensions — for the isomorphism induced by $\tilde{\sigma}$;

$$p_{\tilde{\sigma}} : \mathrm{Fn}(Y_{\overline{K}}(\overline{K}), \overline{K} \cup \{\infty\}) \ \xrightarrow{\sim}\ \mathrm{Fn}(X_{\overline{K}}(\overline{K}), \overline{K} \cup \{\infty\})$$

for the bijection induced by the bijection $X_{\overline{K}}(\overline{K}) \xrightarrow{\sim} Y_{\overline{K}}(\overline{K})$ that appears in the above commutative diagram.

Next, we verify the following assertion:

> Claim 4.7.A: The isomorphism $p^{\kappa}_{\tilde{\sigma}}$ induces an isomorphism
>
> $$f_{\tilde{\sigma}} : \varinjlim_{K \subseteq K^\dagger} (K(Y) \otimes_K K^\dagger)^\times/(K^\dagger)^{\times\infty} \ \xrightarrow{\sim}\ \varinjlim_{K \subseteq K^\dagger} (K(X) \otimes_K K^\dagger)^\times/(K^\dagger)^{\times\infty}$$
>
> via the [inductive limits of] Kummer maps.

Indeed, Claim 4.7.A follows immediately from Corollary 2.7.

Next, we verify the following assertion:

> Claim 4.7.B: Let M ($\subseteq \overline{K}$) be a field such that $\varinjlim_{K \subseteq K^\dagger} (K^\dagger)^{\times\infty} \subseteq M$. Then the isomorphism that appears in Claim 4.7.A induces an isomorphism
>
> $$f^M_{\tilde{\sigma}} : (K(Y) \otimes_K \overline{K})^\times/M^\times \ \xrightarrow{\sim}\ (K(X) \otimes_K \overline{K})^\times/M^\times.$$
>
> Moreover, $f^M_{\tilde{\sigma}}$ or the composite of $f^M_{\tilde{\sigma}}$ with the inversion automorphism of the codomain of $f^M_{\tilde{\sigma}}$ is compatible with the bijection
>
> $$\mathrm{Fn}(Y_{\overline{K}}(\overline{K}), (\overline{K}^\times/M^\times) \cup \{0, \infty\}) \ \xrightarrow{\sim}\ \mathrm{Fn}(X_{\overline{K}}(\overline{K}), (\overline{K}^\times/M^\times) \cup \{0, \infty\})$$
>
> induced by the bijection $X_{\overline{K}}(\overline{K}) \xrightarrow{\sim} Y_{\overline{K}}(\overline{K})$ [that appears in the above commutative diagram] via the natural injections
>
> $$(K(X) \otimes_K \overline{K})^\times/M^\times \hookrightarrow \mathrm{Fn}(X_{\overline{K}}(\overline{K}), (\overline{K}^\times/M^\times) \cup \{0, \infty\}),$$
>
> $$(K(Y) \otimes_K \overline{K})^\times/M^\times \hookrightarrow \mathrm{Fn}(Y_{\overline{K}}(\overline{K}), (\overline{K}^\times/M^\times) \cup \{0, \infty\}).$$

Indeed, observe from Proposition 2.10, together with our assumption that $\tilde{\sigma}$ lies over G_K, that $f_{\tilde{\sigma}}$ induces the identity automorphism or the inversion automorphism on $\overline{K}^\times / \varinjlim_{K \subseteq K^\dagger} (K^\dagger)^{\times\infty}$. Then the first assertion follows immediately from this observation. Moreover, in light of the operation of Galois evaluation, the second assertion follows immediately from this observation and the commutativity of the above diagram. This completes the proof of Claim 4.7.B.

Next, we verify the following assertion:

> Claim 4.7.C: The bijection $p_{\tilde{\sigma}}$ induces a field isomorphism
>
> $$K(Y) \otimes_K \overline{K} \xrightarrow{\sim} K(X) \otimes_K \overline{K}$$
>
> over $\overline{K}$. Moreover, this field isomorphism is G_K-equivariant, hence, in particular, induces a field isomorphism
>
> $$K(Y) \xrightarrow{\sim} K(X)$$
>
> over K.

Indeed, to verify the first assertion of Claim 4.7.C, it suffices to prove that $p_{\tilde{\sigma}}$ maps any rational function on $Y_{\overline{K}}$ to a rational function on $X_{\overline{K}}$. Note that it follows immediately from the various definitions involved that $p_{\tilde{\sigma}}$ maps any constant rational function on $Y_{\overline{K}}$ to a constant rational function on $X_{\overline{K}}$. Recall that K is a TKND-field. Then it holds that $K_{\mathrm{div}} \subsetneq \overline{K}$ is an infinite field extension. Fix an intermediate extension field $K_{\mathrm{div}} \subseteq M \subseteq \overline{K}$ such that the field extension $M \subseteq \overline{K}$ is an infinite algebraic field extension [cf. Lemma 4.5]. Note that $\varinjlim_{K \subseteq K^{\dagger}} (K^{\dagger})^{\times \infty} \subseteq K_{\mathrm{div}}$ $(\subseteq M)$. Then it follows immediately from Claim 4.7.B that $p_{\tilde{\sigma}}$ maps any rational function on $Y_{\overline{K}}$ to a quasi-rational function on $X_{\overline{K}}$ associated to M. Let f be such a quasi-rational function on $X_{\overline{K}}$ associated to M that arises as the image of some *nonconstant* rational function on $Y_{\overline{K}}$. Then one may make the following *observations*:

- Since $M \subsetneq \overline{K}$, the rational function determined [up to multiplication by a constant function] by f is *nonconstant*.
- For each $a \in \overline{K}$, it holds that $f + a$ is also a quasi-rational function on $X_{\overline{K}}$ associated to M.

Thus, we conclude from Proposition 3.6, together with the above observations, that $p_{\tilde{\sigma}}$ maps any rational function on $Y_{\overline{K}}$ to a rational function on $X_{\overline{K}}$. This completes the proof of the first assertion of Claim 4.7.C. Since $\tilde{\sigma}$ lies over G_K, the second assertion follows immediately from the first assertion, together with the various constructions involved. This completes the proof of Claim 4.7.C.

Finally, observe that the construction of the bijection $p_{\tilde{\sigma}}$ is functorial with respect to the restrictions of $\tilde{\sigma}$ to the open subgroups of $G_{K(X)}$. Thus, by applying Claim 4.7.C to the isomorphisms between the open subgroups of $G_{K(X)}$ and the open subgroups of $G_{K(Y)}$ induced by $\tilde{\sigma}$, we conclude that $\tilde{\sigma}$ arises from a(n) [unique] field isomorphism $K(Y) \xrightarrow{\sim} K(X)$ over K. This completes the proof of Theorem 4.7. $\square$

Remark 4.7.1. At the time of writing of the present paper, the author does not know whether or not the condition that the multiplicative group of the base field admits a surjective homomorphism onto $\mathbb{Z}$ may be dropped in general [even if we assume that the base field is Kummer-faithful — cf. Remark 4.9.2 below].

Remark 4.7.2. Let K, L be TKND-AVKF-fields of characteristic 0; X, Y hyperbolic curves over K, L, respectively. With regard to further developments of anabelian geometry for geometric objects over TKND-AVKF-fields [of characteristic 0], in light of the historical developments of anabelian geometry [cf. for instance, see [7], [17], [18], [20] [21], [26], [33]], it is natural to pose the following questions:

> Question 1: Can we obtain the semi-absolute analogue of Theorem 4.7? More precisely, in a similar notation to the notation in Theorem 0.3, is the natural map
>
> $$\mathrm{Isom}(K(Y)/L, K(X)/K) \longrightarrow \mathrm{Isom}(G_{K(X)}/G_K, G_{K(Y)}/G_L)/\mathrm{Inn}(G_{K(Y)})$$
>
> bijective?

Question 2: Can we obtain the hyperbolic curve analogue [i.e., usual setting of the Grothendieck Conjecture in anabelian geometry] of Theorem 4.7? More precisely, in the case where $K = L$, is the natural map

$$\mathrm{Isom}_K(X, Y) \longrightarrow \mathrm{Isom}_{G_K}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_Y)$$

bijective?

However, at the time of writing of the present paper, the author does not know whether or not the answer of each question is affirmative or not [even if we assume the preservation of decomposition subgroups associated to the closed points of hyperbolic curves under consideration in Question 2]. If the answer of Question 1 is affirmative, then we obtain a complete generalization of Theorem 0.3. On the other hand, to obtain such a result, one needs to investigate a certain generalization of Uchida's lemma. The author hopes to be able to address such an issue in a future paper.

Remark 4.7.3. Let p be a prime number. Then it would be interesting to investigate to what extent Theorem 4.7 may be generalized to the situation that the base field is of characteristic p. At the time of writing of the present paper, it appears to the author that we need to restrict our attention to the geometrically pro-prime-to-p situation to apply Kummer theory [cf. [27], [29]]. However, if we consider such a situation, then we need to impose a stronger assumption on the base field to execute [certain modifications of] various procedures that appear in the present paper in a reasonable way. The author hopes to be able to address such an issue in a future paper.

Proposition 4.8. *Let F be a field; K a subfield of a finitely generated extension field L of F. Suppose that K is not contained in F. Then there exists a surjective homomorphism $K^\times \twoheadrightarrow \mathbb{Z}$.*

Proof. Suppose that there exists no surjective homomorphism $K^\times \twoheadrightarrow \mathbb{Z}$. Note that every nontrivial subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$. In particular, for each discrete valuation $v : L^\times \twoheadrightarrow \mathbb{Z}$ on L, the restriction of v on $K^\times$ is trivial. Then since L is a finitely generated extension field of F, it holds that $K \subseteq F$ [cf. the proof of Proposition 1.2, (i)]. This is a contradiction. Thus, we conclude that there exists a surjective homomorphism $K^\times \twoheadrightarrow \mathbb{Z}$. This completes the proof of Proposition 4.8. $\square$

Proposition 4.9. *Let F be a number field; $f \in F$. Write E ($\subseteq \overline{\mathbb{Q}}$) for the field obtained by adjoining all roots of f to F [so E contains all roots of unity]. Then it holds that $E^\times/E^{\times\infty}$ is a free abelian group.*

Proof. For each positive integer i, write $E_i \subseteq E$ for the subfield generated by all roots of unity and an i-th root of f over F;

$$\phi_i : E_i^\times / E_i^{\times\infty} \to E^\times / E^{\times\infty}$$

for the natural homomorphism induced by the inclusion $E_i \subseteq E$. In particular, it holds that

$$E^\times / E^{\times\infty} \;=\; \bigcup_{i \geq 1} \mathrm{Im}(\phi_i).$$

On the other hand, it follows immediately from [14], Theorem 2, together with [37], Lemma D, (iii), (iv), that, for each positive integer i, it holds that $E_i^\times / E_i^{\times\infty}$ is a free abelian group. Thus, we may assume without loss of generality that $f \notin \mu(F)$, hence that $\mathrm{Gal}(E/E_i) \cong \widehat{\mathbb{Z}}$.

Next, observe that $\mathrm{Ker}(\phi_i) \cong \mathbb{Z}$. Indeed, since every subgroup of a free abelian group is a free abelian group, it holds that $\mathrm{Ker}(\phi_i)$ is also a free abelian group. Here, we note that since $E^{\times\infty}$ is divisible, and $\mu(\overline{\mathbb{Q}}) \subseteq E^{\times\infty}$, it holds that $E^\times / E^{\times\infty}$ is torsion-free, hence that the subgroup $\mathrm{Im}(\phi_i) \subseteq E^\times / E^{\times\infty}$ is a flat $\mathbb{Z}$-module. Let l be a prime number. Then, in light of the flatness of $\mathrm{Im}(\phi_i)$, we obtain a natural injection $\mathrm{Ker}(\phi_i)/l\mathrm{Ker}(\phi_i) \hookrightarrow E_i^\times / (E_i^\times)^l$ whose image is contained in the kernel of the natural homomorphism $E_i^\times / (E_i^\times)^l \to E^\times / (E^\times)^l$. On the other hand, since $\mathrm{Gal}(E/E_i) \cong \widehat{\mathbb{Z}}$, it follows immediately from Kummer theory that this kernel is isomorphic to $\mathbb{Z}/l\mathbb{Z}$. In addition, since $E_i^{\times\infty} = \mu(\overline{\mathbb{Q}})$ [cf. [37], Lemma D, (iii), (iv)], and $f \notin \mu(F)$, it holds that $\mathrm{Ker}(\phi_i) \neq \{0\}$. Thus, since $\mathrm{Ker}(\phi_i)$ is a free abelian group, we conclude that $\mathrm{Ker}(\phi_i) \cong \mathbb{Z}$.

Next, since

- $\mathrm{Ker}(\phi_i)$ is a finitely generated abelian group;
- $E_i^\times / E_i^{\times\infty}$ is a free abelian group;
- $E^\times / E^{\times\infty}$ is a torsion-free abelian group,

one may observe that $\mathrm{Im}(\phi_i)$ is also a free abelian group. Then, in light of Pontryagin's criterion of freeness for countable torsion-free abelian groups [cf. [25], Lemma 16], to verify Proposition 4.9, it suffices to prove that $\mathrm{Im}(\phi_i)$ is saturated in $E^\times / E^{\times\infty}$. Write $(E_i^\times)^{\mathrm{sat}}$ for the saturation of $E_i^\times$ in $E^\times$. Note that since $E^{\times\infty}$ is divisible, the saturation of $\mathrm{Im}(\phi_i)$ in $E^\times / E^{\times\infty}$ coincides with

$$((E_i^\times)^{\mathrm{sat}} \cdot E^{\times\infty}) / E^{\times\infty}.$$

Let $x \in (E_i^\times)^{\mathrm{sat}} \subseteq E^\times$ be an element; m a positive integer such that $x^m \in E_i^\times$. Now we have the following commutative diagram of exact sequences of abelian groups

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & E_i^\times / E_i^{\times\infty} & \xrightarrow{\phi_i} & E^\times / E^{\times\infty} \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & E_i^\times / (E_i^\times)^m & \longrightarrow & E^\times / (E^\times)^m,
\end{array}$$

where the vertical arrows denote the natural surjections. Then it follows immediately from a diagram chase that $x^m \in \mathrm{Ker}(\phi_i) \cdot (E_i^\times)^m$, hence that $x^m \in (E_i^\times)^m \cdot E^{\times\infty}$. Thus, since $E^{\times\infty}$ is divisible, and $\mu(\overline{\mathbb{Q}}) \subseteq E^{\times\infty}$, we conclude that $x \in E_i^\times \cdot E^{\times\infty}$. This argument implies that $\mathrm{Im}(\phi_i)$ is saturated in $E^\times/E^{\times\infty}$. This completes the proof of Proposition 4.9. $\square$

Remark 4.9.1. Let l be a prime number; F a number field that contains a primitive l-th root of unity. Write K ($\subseteq \overline{\mathbb{Q}}$) for the maximal pro-$l$ extension field of F. Then since $K^\times$ is l-divisible, there exists no surjective homomorphism $K^\times \twoheadrightarrow \mathbb{Z}$. On the other hand, it follows immediately from [37], Lemma D, (iii), (vi), that K is TKND.

Remark 4.9.2. We retain the notation of Remark 1.4.5. Then there exists no surjective homomorphism $L^\times \twoheadrightarrow \mathbb{Z}$. Indeed, observe from the construction of L that the value group of L is l-divisible. Then it suffices to prove that the unit group of the ring of integers of L does not admit any surjective homomorphism onto $\mathbb{Z}$. Moreover, since L may be written as the union of p-adic local fields, it suffices to prove that a similar assertion for the p-adic local fields holds. However, this follows immediately from the [easily verified] fact that $\mathbb{Z}_p$ does not admit any surjective homomorphism onto $\mathbb{Z}$. In particular, in light of Remark 1.4.5, we obtain an example of Kummer-faithful field whose multiplicative group does not admit any surjective homomorphism onto $\mathbb{Z}$.

Corollary 4.10. *Let p be a prime number; F a number field. Write E ($\subseteq \overline{\mathbb{Q}}$) for the field obtained by adjoining all roots of p to F [so E contains all roots of unity, and $F \subseteq E$ is a nonabelian metabelian Galois extension]. Let K be a subfield of a finitely generated extension field of E; M_1, M_2 function fields of one variable over K. Write Δ for the kernel of the natural surjection $G_{M_2} \twoheadrightarrow G_K$. Then the natural map*

$$\mathrm{Isom}_K(M_2, M_1) \longrightarrow \mathrm{Isom}_{G_K}(G_{M_1}, G_{M_2})/\mathrm{Inn}(\Delta)$$

is bijective.

Proof. In light of Theorems 1.8, 4.7, to verify Corollary 4.10, it suffices to prove that there exists a surjective homomorphism $K^\times \twoheadrightarrow \mathbb{Z}$. First, by applying Proposition 4.8, we may assume without loss of generality that K is a subfield of E. Next, suppose that $K^\times$ does not admit any surjective homomorphism onto $\mathbb{Z}$. Then it follows immediately from Proposition 4.9 that $K^\times \subseteq E^{\times\infty}$. This is a contradiction. Thus, we conclude that there exists a surjective homomorphism $K^\times \twoheadrightarrow \mathbb{Z}$. This completes the proof of Corollary 4.10. $\square$

Remark 4.10.1. It appears to the author that the above result may be regarded as the first result concerning [the strong/desired form of] the Grothendieck Conjecture for the function fields of smooth curves over fields whose associated cyclotomic characters totally vanish.

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