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A Note on Ordinally Concave Functions

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Abstract

The notion of ordinal concavity of utility functions has recently been considered by Hafalir, Kojima, Yenmez, and Yokote in economics while there exist earlier related works in discrete optimization and operations research. In the present note we consider functions satisfying ordinal concavity and introduce a weaker notion of ordinal weak-concavity as well. We also investigate useful behaviors of ordinally (weak-)concave functions and related choice correspondences, show a characterization of ordinally weak-concave functions, and give an efficient algorithm for maximizing ordinally concave functions. We further examine a duality in ordinally (weak-)concave functions and introduce the lexicographic composition of ordinally weak-concave functions.

Keywords: Discrete optimization, discrete convexity, ordinally concave functions, ordinally weak-concavity, choice functions, lexicographic composition

1. Introduction

I. E. Hafalir, F. Kojima, M. B. Yenmez, and K. Yokote [9] have recently considered a notion of *ordinal concavity* for utility functions defined on the set $\mathbb{Z}^E_{\geq 0}$ of nonnegative integer vectors with a finite nonempty set E. They have shown economic implications of ordinal concavity such as the path-independence property of choice functions associated with ordinally concave utility functions and the rationalizability of path-independent choice rules by ordinally concave utility functions [21]. The notion of ordinal concavity is equivalent to the one that was introduced by the name of *semi-strict quasi M*$^\natural$*-concavity* in [4] (also see [17, 18] and [3]), where the domain of the functions is the integer lattice $\mathbb{Z}^E$.

In the present note we consider functions satisfying ordinal concavity and introduce a weaker notion of ordinal weak-concavity as well. We almost follow the notation in [9, 21]. Note in particular that $\emptyset$ denotes the empty set as usual while it also means a symbol that does not belong to the underlying set E. For any $X \in 2^E$ let $X + x = X \cup \{x\}$ for $x \in E \setminus X$ and $X - x = X \setminus \{x\}$ for $x \in X$. Also for $x = \emptyset$ let $X \pm x = X$.

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Let $u : 2^E \to \mathbb{R}$ be a function on the set of all subsets of a finite nonempty set E. The notion of *ordinal concavity* is defined as follows (see [9, 21] and [4, 17, 18]).

Definition 1.1 (**Ordinal Concavity):** *A function* $u : 2^E \to \mathbb{R}$ *satisfies* ordinal concavity *if for every* $X, X' \in 2^E$ *the following statement holds*:
For every $x \in X \setminus X'$ *there exists* $x' \in (X' \setminus X) \cup \{\emptyset\}$ *such that*

(i) $u(X) < u(X - x + x')$, *or*

(ii) $u(X') < u(X' - x' + x)$, *or*

(iii) $u(X) = u(X - x + x')$ *and* $u(X') = u(X' - x' + x)$.

Let us also consider a weaker version what we call *ordinal weak-concavity* (or *ordinal w-concavity* for short) as follows. (See Appendix A.1 for an example of a function that is ordinally weak-concave but is not ordinally concave.)

Definition 1.2 (**Ordinal Weak-Concavity):** *A function* $u : 2^E \to \mathbb{R}$ *satisfies* ordinal weak-concavity *if for every* $X, X' \in 2^E$ *with* $X \neq X'$ *the following statement holds*: *There exist distinct* $x \in (X \setminus X') \cup \{\emptyset\}$ *and* $x' \in (X' \setminus X) \cup \{\emptyset\}$ *such that*

(i) $u(X) < u(X - x + x')$, *or*

(ii) $u(X') < u(X' - x' + x)$, *or*

(iii) $u(X) = u(X - x + x')$ *and* $u(X') = u(X' - x' + x)$.

Remark 1: A notion of *weak semi-strict quasi M-concavity* (denoted by (SSQM$_w$)) is considered in [17]. Just as we can obtain an M$^\natural$-convex function from an M-convex function by a projection of the domain $\mathbb{R}^n$ along an axis into a one-dimension-lower coordinate space $\mathbb{R}^{n-1}$ (see [8, 14, 16, 19]), we can define a notion of *weak semi-strict quasi M$^\natural$-concavity* from weak semi-strict quasi M-concavity, which has not been explicitly considered in the literature. The notion of ordinal weak-concavity given above is a set-theoretical version of 'weak semi-strict quasi M$^\natural$-concavity.'
□

The present note is organized as follows. In Section 2 we examine some useful behaviors of ordinally w-concave functions and show a characterization of ordinally w-concave functions. We also propose an efficient algorithm for maximizing ordinally concave functions. Sections 3.1 and 3.2 reveal how the results obtained in Section 2 lead us to path-independent choice functions for ordinally concave functions [9, Theorem 2] in view of choice correspondences. We examine behaviors of choice functions associated with ordinally w-concave functions and related choice correspondences in Section 3.3. In Section 4 we examine a duality property in ordinal concavity and introduce the lexicographic composition of two ordinally w-concave functions. Section 5 gives concluding remarks.

2. Ordinally Weak-Concave Functions

For any $X, Y \in 2^E$ define $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$ (the symmetric difference of X and Y). Also for any finite set X denote by $|X|$ the number of elements of X.

2.1. Fundamental operations on functions on 2^E

Consider any function $u : 2^E \to \mathbb{R}$. For any nonempty $X \subseteq E$ define $u^X : 2^X \to \mathbb{R}$ by

$$u^X(Z) = u(Z) \qquad (\forall Z \in 2^X). \tag{2.1}$$

We call u^X the *reduction* of u by X (or the *restriction* of u on X). Also for any $X \subset E$ define $u_X : 2^{E \setminus X} \to \mathbb{R}$ by

$$u_X(Z) = u(Z \cup X) - u(X) \qquad (\forall Z \in 2^{E \setminus X}). \tag{2.2}$$

We call u_X the *contraction* of u by X. Moreover, for any $X, Y \in 2^E$ with $X \subset Y$ define $u_X^Y : 2^{Y \setminus X} \to \mathbb{R}$ by

$$u_X^Y(Z) = u(Z \cup X) - u(X) \qquad (\forall Z \in 2^{Y \setminus X}). \tag{2.3}$$

We call u_X^Y a *minor* of u obtained by the reduction by Y and then by the contraction by X. It should be noted that

- the operations of reduction and contraction keep ordinal (w-)concavity and hence every minor of an ordinally (w-)concave u is ordinally (w-)concave.

2.2. Fundamental properties of ordinally weak-concave functions

Consider a function $u : 2^E \to \mathbb{R}$. Define

$$\mathbf{D}_u^* = \mathrm{Arg}\max\{u(X) \mid X \subseteq E\}. \tag{2.4}$$

Then we see the following fact (this is a projected version of [17, Theorem 3.11(iii)]). For the concept of generalized (poly-)matroid or M$^\natural$-convex set see, e.g., [6, 8, 10, 14, 16].

Lemma 2.1: *For every ordinally w-concave function u the set $\mathbf{D}_u^*$ given by* (2.4) *forms an M$^\natural$-convex set* (*or a generalized matroid*) *on E.*

(Proof) Suppose that u is an ordinally w-concave function. Then for any $X, X' \in \mathbf{D}_u^*$ with $X \neq X'$ there exist distinct $x \in (X \setminus X') \cup \{\emptyset\}$ and $x' \in (X' \setminus X) \cup \{\emptyset\}$ such that (i), (ii), or (iii) in the definition of ordinal w-concavity holds. Since $X, X' \in \mathbf{D}_u^*$, only (iii) holds. This completes the proof of the present lemma, due to a self-dual exchange axiom for M$^\natural$-convex sets or generalized matroids (see [16] (or [14, Theorem 4.3] and [8, Theorem 3.58])). $\square$

Now we show a fundamental fact as a theorem. For every $X \in 2^E$ denote by X^* a set $Z \in \mathbf{D}_u^*$ that minimizes $|X \Delta Z|$ among $\mathbf{D}_u^*$. (In this note X' is used as a variable set that is independent of X, while for a given $X \in 2^E$, X^* is a member of $\mathrm{Arg}\min\{|X \Delta Z| \mid Z \in \mathbf{D}_u^*\}$.)

Theorem 2.2: *Let u be any ordinally w-concave function. Consider any $X, Y \in 2^E \setminus \mathbf{D}_u^*$ satisfying*

$$X \cap X^* \subseteq Y \subseteq X \cup X^*. \tag{2.5}$$

Then there exist distinct $x \in (X^* \setminus Y) \cup \{\emptyset\}$ and $y \in (Y \setminus X^*) \cup \{\emptyset\}$ such that $u(Y) < u(Y - y + x)$.

(Proof) Choose any $X, Y \in 2^E \setminus \mathbf{D}_u^*$ that satisfy (2.5). Note that $Y \neq X^*$ since $Y \notin \mathbf{D}_u^*$. It follows from ordinal w-concavity of u and the definition of X^* that for $X \leftarrow Y$ and $X' \leftarrow X^*$ only (i) in the definition of ordinal w-concavity holds. □

Corollary 2.3: *Let $u : 2^E \to \mathbb{R}$ be any ordinally w-concave function. Consider any $Y \in 2^E \setminus \mathbf{D}_u^*$ and $Z \in \mathbf{D}_u^*$. Then there exist distinct $x \in (Z \setminus Y) \cup \{\emptyset\}$ and $y \in (Y \setminus Z) \cup \{\emptyset\}$ such that $u(Y) < u(Y - y + x)$.*

(Proof) For any $Y \in 2^E \setminus \mathbf{D}_u^*$ and $Z \in \mathbf{D}_u^*$ consider the minor $\bar{u} \equiv u_{Y \cap Z}^{Y \cup Z}$ of u, which is ordinally w-concave. Note that $(Y \cup Z) \setminus (Y \cap Z) \neq \emptyset$. Let us use the unary operator $(\cdot)^*$ (defined for u) for $\bar{u}$ as well. Note that since $Y \notin \mathbf{D}_u^*$ and $Z \in \mathbf{D}_u^*$, we have $(Y \setminus Z)^* \neq Y \setminus Z$ for $\bar{u}$. Then, considering $u \leftarrow \bar{u}$, $X \leftarrow Y \setminus Z$ and $X^* \leftarrow (Y \setminus Z)^*$ (for $\bar{u}$) in Theorem 2.2 with $X = Y$, we have

- there exist distinct $x \in ((Y \setminus Z)^* \setminus Y) \cup \{\emptyset\}$ and $y \in (Y \setminus (Y \setminus Z)^*) \cup \{\emptyset\}$ such that $\bar{u}(Y \setminus (Y \cap Z)) < \bar{u}((Y - y + x) \setminus (Y \cap Z))$.

Since $(Y \setminus Z)^* \setminus Y \subseteq Z \setminus Y$ and $Y \setminus (Y \setminus Z)^* \subseteq Y \setminus Z$ and since $\bar{u}((Y - y + x) \setminus (Y \cap Z)) = u(Y - y + x) - u(Y \cap Z)$ and $\bar{u}(Y \setminus (Y \cap Z)) = u(Y) - u(Y \cap Z)$, the pair of x and y is a desired one for the present corollary. □

From Theorem 2.2 we see the following corollaries, where we suppose that u satisfies ordinal w-concavity.

Corollary 2.4: *For any $X \in 2^E \setminus \mathbf{D}_u^*$ there exists a sequence of distinct subsets $Y_0 (= X), Y_1, \cdots, Y_k$ for a positive integer $k \leq |E|$ such that*

1. $Y_i \in 2^E \setminus \mathbf{D}_u^*$ *for each* $i \in \{0, 1, \cdots, k-1\}$,
2. $u(Y_0) < u(Y_1) < \cdots < u(Y_k)$ *with* $Y_k \in \mathbf{D}_u^*$,
3. *for each* $i \in \{0, 1, \cdots, k-1\}$ *we have* $Y_{i+1} = Y_i - y_i + x_i$ *for distinct* $y_i \in (Y_i \setminus X^*) \cup \{\emptyset\}$ *and* $x_i \in (X^* \setminus Y_i) \cup \{\emptyset\}$.

(Proof) It follows from Theorem 2.2 that repeating the transformation $Y \leftarrow Y - y + x$ as far as $Y \notin \mathbf{D}_u^*$, we obtain $Y \in \mathbf{D}_u^*$ after at most $|X \Delta X^*|$ such transformations, each increasing the value of $u(Y)$. □

A simple consequence of Theorem 2.2 is also given as follows. For any linear ordering $L = (e_1, e_2, \cdots, e_k)$ of distinct k elements of E with $k \leq |E|$ define $L_i = \{e_1, \cdots, e_i\}$ (the set of the initial i elements of L) for each $i \in \{1, \cdots, k\}$.

Corollary 2.5: *For any $Z \in \mathbf{D}_u^*$ of minimum cardinality $|Z|$ there exists a linear ordering $L = (e_1, \cdots, e_k)$ of elements of Z such that $u(\emptyset) < u(L_1) < \cdots < u(L_k)$ with $L_k = Z$.*

(Proof) Starting from $X = \emptyset$, we can reach any maximizer Z of u having the minimum cardinality by repeating the transformation of Y only in a form of $Y+x$ for some $x \in Z \setminus Y$ as in Corollary 2.4. Consider $X \leftarrow Y$ and $X' \leftarrow Z$ in the definition of ordinal w-concavity and note that $X \subset X'$ and $X' \in \mathbf{D}_u^*$ having the minimum cardinality. □

For any $X \in 2^E$ define the *neighborhood* $\mathbf{N}(X)$ of X by

$$\mathbf{N}(X) = \{X - x + x' \mid x \in X \cup \{\emptyset\}, x' \in (E \setminus X) \cup \{\emptyset\}\}. \tag{2.6}$$

Note that $|\mathbf{N}(X)|$ is $\mathrm{O}(|E|^2)$ for any $X \in 2^E$. The following corollary is a projected version of [17, Theorem 4.2(ii)].

Corollary 2.6: *A set $X \in 2^E$ is a maximizer of u if and only if X attains the maximum of $u(Z)$ among all $Z \in \mathbf{N}(X)$.*

(Proof) It suffices to show the if part. Suppose that X is not a maximizer of u. Then it follows from Corollary 2.3 that there exists $Z \in \mathbf{N}(X)$ such that $u(Z) > u(X)$. This completes the proof of the present corollary. □

Corollary 2.6 leads us to a simple hill-climbing algorithm to maximize u satisfying ordinal w-concavity as follows (cf. [17, Sec. 4.2]).

Algorithm 1

Step 0: Choose any $X \in 2^E$ and put $Y \leftarrow X$;
Step 1: While there exists $Z \in \mathbf{N}(Y)$ such that $u(Z) > u(Y)$, do the following:
Choose any $Z \in \mathbf{N}(Y)$ such that $u(Z) > u(Y)$;
Put $Y \leftarrow Z$;
Step 2: Return Y ;

Trivially, Algorithm 1 terminates after updating Y in **Step 1** at most ν times, where ν is the number of distinct function values $u(X)$ for all $X \in 2^E$.

Hafalir et al. [9, Theorem 1] show that when u satisfies ordinal concavity, we can find a maximizer of u after $\mathrm{O}(|E|^3)$ updates of Y in **Step 1** by appropriately choosing Z (also see [17, 18]). It is an interesting problem to find an algorithm faster than Algorithm 1 (if any) for ordinally w-concave functions. In Section 2.4.1 we show an algorithm faster than the one given by Hafalir et al. [9] for ordinally concave functions.

2.3. A characterization of ordinal weak-concavity

For any $X, Y \in 2^E$ with $X \subseteq Y$ define $[X, Y] \equiv \{Z \in 2^E \mid X \subseteq Z \subseteq Y\}$ (an *interval* in 2^E) and

$$\mathbf{C}_u(X, Y) = \mathrm{Arg\,max}\{u(Z) \mid Z \in [X, Y]\}. \tag{2.7}$$

Lemma 2.7: *Let $u : 2^E \to \mathbb{R}$ be any function satisfying ordinal w-concavity. Then for every $X, Y \in 2^E$ and every $Z \in [X \cap Y, X \cup Y]$, if Z maximizes u over*

$$\mathbf{N}(Z) \cap [X \cap Y, X \cup Y],$$

then Z maximizes u over

$$[X \cap Y, X \cup Y],$$

i.e.*,* $Z \in \mathbf{C}_u(X \cap Y, X \cup Y)$.

(Proof) Suppose that $u : 2^E \to \mathbb{R}$ is a function satisfying ordinal w-concavity. Consider any $X, Y \in 2^E$. We can suppose that $X \neq Y$. Then the present lemma is equivalent to the statement of Corollary 2.6 using the minor $u^{X \cup Y}_{X \cap Y}$ in place of u. Recall that $u^{X \cup Y}_{X \cap Y}$ is ordinally w-concave. □

Also, similarly as Lemma 2.1 we can show the following lemma.

Lemma 2.8: *Let $u : 2^E \to \mathbb{R}$ be any function satisfying ordinal w-concavity. For any $X, Y \in 2^E$ with $X \subset Y$ the set $\mathbf{C}_u(X, Y)$ forms an M$^\natural$-convex set.*

(Proof) For any $X, Y \in 2^E$ with $X \subset Y$ consider the minor u^Y_X of u and apply Lemma 2.1 for u^Y_X in place of u. □

Now, we show the following theorem, which means that the properties of u shown in Lemmas 2.7 and 2.8 actually characterize ordinal w-concavity.

Theorem 2.9: *A function $u : 2^E \to \mathbb{R}$ satisfies ordinal w-concavity if and only if the following two statements hold*:

- **(M)** *For any $X, Y \in 2^E$ with $X \subset Y$ the set $\mathbf{C}_u(X, Y)$ forms an M$^\natural$-convex set.*
- **(N)** *For every $X, Y \in 2^E$ and every $Z \in [X \cap Y, X \cup Y]$, if Z maximizes u over $\mathbf{N}(Z) \cap [X \cap Y, X \cup Y]$, then Z maximizes u over $[X \cap Y, X \cup Y]$.*

(Proof) The only-if part follows from Lemmas 2.7 and 2.8. We show the if part.

Suppose that **(M)** and **(N)** hold. Consider any $X, Y \in 2^E$ with $X \neq Y$. Then we have the following three cases:

1. $X \notin \mathbf{C}_u(X \cap Y, X \cup Y)$, or
2. $Y \notin \mathbf{C}_u(X \cap Y, X \cup Y)$, or
3. $X \in \mathbf{C}_u(X \cap Y, X \cup Y)$ and $Y \in \mathbf{C}_u(X \cap Y, X \cup Y)$.

It follows from **(N)** that in Case 1 or Case 2, for some distinct $x \in (X \setminus Y) \cup \{\emptyset\}$ and $y \in (Y \setminus X) \cup \{\emptyset\}$ we have

(i) $u(X - x + y) > u(X)$, or

(ii) $u(Y + x - y) > u(Y)$.

Moreover, it follows from **(M)** that in Case 3, for some distinct $x \in (X \setminus Y) \cup \{\emptyset\}$ and $y \in (Y \setminus X) \cup \{\emptyset\}$ we have

(iii) $u(X - x + y) = u(X)$ and $u(Y + x - y) = u(Y)$.

This completes the proof of the ordinal w-concavity of u. □

2.4. Ordinal concavity vs. ordinal weak-concavity

The lemmas and theorems shown in Section 2.2 can be strengthened if we consider functions satisfying ordinal concavity instead of ordinal weak-concavity. For example, Corollary 2.4 is strengthened as follows. Suppose that $u : 2^E \to \mathbb{R}$ is a function satisfying ordinal concavity.

Corollary 2.10: *For any $X \in 2^E \backslash \mathbf{D}_u^*$ there exists a sequence of distinct subsets $Y_0 (= X), Y_1, \cdots, Y_k$ for a positive integer $k \leq |E|$ such that*

1. $Y_i \in 2^E \setminus \mathbf{D}_u^*$ *for each* $i \in \{0, 1, \cdots, k-1\}$,
2. $u(Y_0) < u(Y_1) < \cdots < u(Y_k)$ *with* $Y_k \in \mathbf{D}_u^*$,
3. *for some integer ℓ with $0 \leq \ell \leq k$ we have*
   (a) *for each $i \in \{0, 1, \cdots, \ell - 1\}$, $Y_{i+1} = Y_i - x_i + x_i'$ such that $x_i \in Y_i \setminus Y_{i+1}$ and $x_i' \in (Y_{i+1} \setminus Y_i) \cup \{\emptyset\}$ and*
   (b) *for each $i \in \{\ell, \cdots, k-1\}$, $Y_{i+1} = Y_i + x_i'$ such that $x_i' \in Y_{i+1} \setminus Y_i$.*

Also, Corollary 2.5 is strengthened as follows.

Corollary 2.11: *Consider any $X \in \mathbf{D}_u^*$ of minimum cardinality. Then for every linear ordering $L = (e_1, \cdots, e_k)$ of elements of X we have $u(\emptyset) < u(L_1) < \cdots < u(L_k)$ with $L_k = X$.*

Corollaries 2.5, 2.10, and 2.11 reflect combinatorial structures like antimatroids or convex geometries behind functions satisfying ordinal (w-)concavity (cf. [7, 12, 13]).

2.4.1. Maximizing ordinally concave functions

Now let us consider an algorithm for maximizing an ordinally concave function $u : 2^E \to \mathbb{R}$. Recall that a set $X \in 2^E$ is called a local maximizer of u if X maximizes u in the neighborhood $\mathbf{N}(X)$ defined by (2.6). Here note that we consider ordinally concave functions but not ordinally w-concave functions in general.

Algorithm 2

Input: An ordinally concave function $u : 2^E \to \mathbb{R}$;
Step 0: Put $W \leftarrow \emptyset$ and let $u' = u$;
Step 1: While $\emptyset$ is not a local maximizer of u', do the following:
  Choose any $x^* \in \operatorname{Arg\,max}\{u(\{x\}) \mid x \in E \setminus W\}$; $W \leftarrow W \cup \{x^*\}$;
  Let u' be the contraction u_W of the original u by the updated W;
Step 2: Return W;

Theorem 2.12: *For any ordinally concave function $u : 2^E \to \mathbb{R}$* Algorithm 2 *finds a maximizer of u and requires $\mathrm{O}(|E|^2)$ function calls for u in total.*

(Proof) If the empty set $\emptyset$ is a local maximizer of u, then it is the global maximizer due to Corollary 2.6. Hence suppose that $\emptyset$ is not a local maximizer of u and choose any $x^* \in \mathrm{Arg\,max}\{u(\{x\}) \mid x \in E \setminus W\}$ with $W = \emptyset$ initially.

We show that there exists some global maximizer $U \in \mathbf{C}_u(E)$ such that $x^* \in U$. For a given $U \in \mathbf{C}_u(E)$ suppose that $x^* \notin U$. Then, putting $X \leftarrow \{x^*\}$ and $X' \leftarrow U$, for these X and X' we have that for any $x \in X \setminus X' = \{x^*\}$ there exists $x' \in (X' \setminus X) \cup \{\emptyset\}$ such that one of the conditions (i), (ii), and (iii) for ordinal concavity holds. That is,

(i) $u(X) < u(X - x + x')$, or

(ii) $u(X') < u(X' - x' + x)$, or

(iii) $u(X) = u(X - x + x')$ and $u(X') = u(X' - x' + x)$.

Note that $x = x^*$ and $X - x^* + x' = x' \in U \cup \{\emptyset\}$. Hence condition (i) does not hold because of the definition of x^* and since $u(\{x^*\}) > u(\emptyset)$. Condition (ii) is null because of the definition of U. It follows that Condition (iii) holds. Then $X' - x' + x^*$ is a global maximizer and satisfies $x^* \in X' - x' + x^*$.

Consequently, updating $W \leftarrow W \cup \{x^*\}$ and considering the contraction $u' = u_W$ of u by $W(= \{x^*\}$ currently), the updated u' is also ordinally concave, and moreover, for any maximizer U' of $u' = u_W$ we have a maximizer $U' \cup W$ of u.

Repeating this process until $\emptyset$ becomes a local maximizer of updated $u' = u_W$, the finally obtained W is a global maximizer of u.

We see that each x^* in **Step 1** is computed by at most $|E|$ function calls for u and the While loop of **Step 1** is repeated at most $|E|$ times. Hence the algorithm terminates after $\mathrm{O}(|E|^2)$ function calls for u. □

Remark 2: Algorithm 2 improves over the algorithms considered in [9] and [17, 18], which require $\mathrm{O}(|E|^3)$ function calls for u. A crucial point is that we start from the special set $\emptyset$ and employ the operation of contraction to execute the algorithm in a sort of recursive way. It can also be understood as an effective use of 'maximizer-cut property' for ordinally concave functions shown in [9, 17, 18]. If we apply Algorithm 2 to an $\mathrm{M}^\natural$-concave function $g : \mathbf{Q} \to \mathbb{R}$ with $\emptyset \in \mathbf{Q} \subseteq 2^E$, a special ordinally concave function, then it becomes an upward-steepest ascent algorithm for such a function (see [19, Theorem 3.7]). □

3. Choice functions and choice correspondences

Let $u : 2^E \to \mathbb{R}$ be a (utility) function. Any function $C : 2^E \to 2^E$ satisfying $C(X) \subseteq X$ ($\forall X \in 2^E$) is called a *choice function* on 2^E. For each $X \in 2^E$ define $\mathbf{C}_u(X) = \mathrm{Arg\,max}\{u(Y) \mid Y \subseteq X\}$. Note that $\mathbf{C}_u(X) = \mathbf{C}_u(\emptyset, X)$ defined in the previous section. The mapping $\mathbf{C}_u : 2^E \to 2^{2^E}$ is called the *choice correspondence rationalized by* u. If C satisfies $C(X) \in \mathbf{C}_u(X)$ for each $X \in 2^E$, we call such C a *choice function associated with* u.

3.1. A choice function associated with an ordinally concave function

In this section we consider ordinally concave functions.

We have the following theorem. The theorem shown here does not necessarily hold for functions satisfying ordinal weak-concavity but not ordinal concavity (see Appendix A.1).

Theorem 3.1: *Let* $u : 2^E \to \mathbb{R}$ *be an ordinally concave function,* $\mathbf{C}_u$ *be the choice correspondence rationalized by* u*, and* $C : 2^E \to 2^E$ *be a choice function associated with* u*. Then the following two statements* (I) *and* (II) *hold*:

(I) *For every* $X, Y \in 2^E$ *and every* $U \in \mathbf{C}_u(X)$ *there exists* $Z \in \mathbf{C}_u(X \cup Y)$ *such that* $Z \cap X \subseteq U$*. In other words, for every* $X, Y \in 2^E$ *we have*

$$\mathbf{C}_u(X \cup Y) \cap \mathbf{C}_u(C(X) \cup (Y \setminus X)) \neq \emptyset.$$

(II) *For every* $X, Y \in 2^E$ *and every* $Z \in \mathbf{C}_u(X \cup Y)$ *there exists* $U \in \mathbf{C}_u(X)$ *such that* $Z \cap X \subseteq U$*. In other words, for every* $X, Y \in 2^E$ *we have*

$$\mathbf{C}_u(X) \cap \mathbf{C}_u(C(X \cup Y) \cap X, X) \neq \emptyset.$$

(Proof) We first show (I) and then (II) similarly as (I).

(I): Consider any $X, Y \in 2^E$. Put $Z = X \cup Y$, $\bar{X} = C(X)$, and $\bar{Z} = C(X \cup Y)$. If for some $W \in \mathbf{C}_u(X \cup Y)$ we have $W \cap (X \setminus \bar{X}) = \emptyset$, we are done.

Put $W \leftarrow \bar{Z}$ and suppose that $W \cap (X \setminus \bar{X}) \neq \emptyset$. Then because of the ordinal concavity of u, for any $z \in W \cap (X \setminus \bar{X})$ there exists $x \in (\bar{X} \setminus W) \cup \{\emptyset\}$ such that

(i) $u(W - z + x) > u(W)$, or

(ii) $u(\bar{X} + z - x) > u(\bar{X})$, or

(iii) $u(W - z + x) = u(W)$ and $u(\bar{X} + z - x) = u(\bar{X})$.

Here both (i) and (ii) are invalid since $W \in \mathbf{C}_u(Z)$ and $\bar{X} \in \mathbf{C}_u(X)$ and since $W - z + x \subseteq Z(= X \cup Y)$ and $\bar{X} + z - x \subseteq X$. Hence (iii) must hold, so that $W - z + x \in \mathbf{C}_u(Z)$. Put $W \leftarrow W - z + x$.

Repeat this process as far as we have $W \cap (X \setminus \bar{X}) \neq \emptyset$. Then we eventually obtain $W \in \mathbf{C}_u(Z)$ with $W \cap (X \setminus \bar{X}) = \emptyset$ since $|W \cap (X \setminus \bar{X})|$ gets smaller every time W is updated.

(II): Consider any $X, Y \in 2^E$, and $\bar{Z} \in \mathbf{C}_u(X \cup Y)$. If for some $W \in \mathbf{C}_u(X)$ we have $\bar{Z} \cap (X \setminus W) = \emptyset$, we are done.

Consider any $\bar{X} \in \mathbf{C}_u(X)$ and put $W \leftarrow \bar{X}$. Suppose that $\bar{Z} \cap (X \setminus W) \neq \emptyset$. Then because of the ordinal concavity of u, for any $z \in \bar{Z} \cap (X \setminus W)$, there exists $x \in (W \setminus \bar{Z}) \cup \{\emptyset\}$ such that

(i) $u(\bar{Z} - z + x) > u(\bar{Z})$, or

(ii) $u(W + z - x) > u(W)$, or

(iii) $u(\bar{Z} - z + x) = u(\bar{Z})$ and $u(W + z - x) = u(W)$.

Here both (i) and (ii) are invalid since $\bar{Z} \in \mathbf{C}_u(X \cup Y)$ and $W \in \mathbf{C}_u(X)$ and since $\bar{Z} - z + x \subseteq X \cup Y$ and $W + z - x \subseteq X$. Hence, (iii) must hold, so that $W + z - x \in \mathbf{C}_u(X)$. Put $W \leftarrow W + z - x$.

Repeat this process as far as we have $\bar{Z} \cap (X \setminus W) \neq \emptyset$. Then we eventually obtain $W \in \mathbf{C}_u(X)$ with $\bar{Z} \cap (X \setminus W) = \emptyset$ since $|\bar{Z} \cap (X \setminus W)|$ gets smaller every time W is updated.

This completes the proof of the present theorem. □

Remark 3: Under a stronger assumption that u is an M$^\natural$-concave function, Murota [15, Theorem 3.8] showed (I) and (II) in the above theorem. The properties of (I) and (II) are known as the substitutability of the choice correspondence $\mathbf{C}_u$ (see [20]). The substitutability plays a crucial role in the two-sided matching setting (see, e.g., [5, 11, 20]) in that a stable matching exists under substitutability and Sen's α: $\forall X \in 2^E, \forall Y \in \mathbf{C}_u(X) : Y \subset X' \subset X \Longrightarrow Y \in \mathbf{C}_u(X')$ (see [2]). Note that Sen's α holds in our problem setting because the choice correspondence is rationalized by u. □

3.2. The unique-maximizer condition

Let us assume that u satisfies the following *unique-maximizer* condition **(UM)** in addition to ordinal concavity.

(UM) For every $X \subseteq E$ there uniquely exists a maximizer of $\max\{u(Y) \mid Y \subseteq X\}$, i.e., $|\mathbf{C}_u(X)| = 1$.

Then the choice function $C : 2^E \to 2^E$ associated with u is uniquely determined, i.e., $\mathbf{C}_u(X) = \{C(X)\}$ for all $X \in 2^E$. Hence we can identify $\mathbf{C}_u(X)$ with $C(X)$ for all $X \in 2^E$.

The following theorem was shown in [9] for functions satisfying ordinal concavity.

Theorem 3.2 ([9, Theorem 2]): *For any function $u : 2^E \to \mathbb{R}$ satisfying ordinal concavity and the unique-maximizer condition* **(UM)***, the choice function $C : 2^E \to 2^E$ associated with u is path-independent, i.e., for every $X, Y \in 2^E$ we have $C(X \cup Y) = C(C(X) \cup Y)$.*

Theorem 3.1 actually leads us to the following theorem for functions satisfying ordinal concavity and Condition **(UM)**.

Theorem 3.3: *For any function $u : 2^E \to \mathbb{R}$ satisfying ordinal concavity and the unique-maximizer condition* **(UM)***, the choice function $C : 2^E \to 2^E$ associated with u satisfies $C(X \cup Y) = C(C(X) \cup (Y \setminus X))$ for all $X, Y \in 2^E$.*

(Proof) From Theorem 3.1(I) we have $\mathbf{C}_u(X \cup Y) \cap \mathbf{C}_u(C(X) \cup (Y \setminus X)) \neq \emptyset$. Then under the present assumption we have $|\mathbf{C}_u(X \cup Y) \cap \mathbf{C}_u(C(X) \cup (Y \setminus X))| = 1$, which implies $C(X \cup Y) = C(C(X) \cup (Y \setminus X))$. □

Note that $C(X \cup Y) = C(C(X) \cup (Y \setminus X))$ for all $X, Y \in 2^E$ if and only if $C(X \cup Y) = C(C(X) \cup Y)$ for all $X, Y \in 2^E$. (This fact seems to be a folklore, but we give its proof in Appendix A.2.) Hence Theorem 3.3 is equivalent to Theorem 3.2.

3.3. A choice function associated with an ordinally weak-concave function

For any function $u : 2^E \to \mathbb{R}$ and $U \in 2^E$ define

$$\mathbf{C}_u^{-1}(U) = \left\{X \in 2^E \mid U \in \mathbf{C}_u(X)\right\}.$$

Here it should be noted that $\mathbf{C}_u^{-1}$ is not the inverse of the mapping (choice correspondence) $\mathbf{C}_u : 2^E \to 2^{2^E}$ in a mathematical sense.

Lemma 3.4: *Let $u : 2^E \to \mathbb{R}$ be an ordinally w-concave function. For any $U \in 2^E$ with $\mathbf{C}_u^{-1}(U) \neq \emptyset$, if $X, Y \in \mathbf{C}_u^{-1}(U)$, then we have $X \cup Y \in \mathbf{C}_u^{-1}(U)$.*

(Proof) Consider $U \in 2^E$ with $\mathbf{C}_u^{-1}(U) \neq \emptyset$ and $X, Y \in \mathbf{C}_u^{-1}(U)$. We show $X \cup Y \in \mathbf{C}_u^{-1}(U)$. We can suppose that $X \neq Y$.

Now suppose to the contrary that $X \cup Y \notin \mathbf{C}_u^{-1}(U)$. Choose any $V \in \mathbf{C}_u(X \cup Y)$ in such a way that the following $(*)$ holds:

$(*)$ V attains the minimum of $|V \Delta U|$.

Since $U \notin \mathbf{C}_u(X \cup Y)$, we have $V \neq U$. Because of the ordinal w-concavity of u, there exist distinct $x \in (U \backslash V) \cup \{\emptyset\}$ and $y \in (V \backslash U) \cup \{\emptyset\}$ such that

(i) $u(U) < u(U - x + y)$, or

(ii) $u(V) < u(V - y + x)$, or

(iii) $u(U) = u(U - x + y)$ and $u(V) = u(V - y + x)$.

If (iii) holds, then because of the definition of V and since $V - y + x \subseteq X \cup Y$, we have $V - y + x \in \mathbf{C}_u(X \cup Y)$. This contradicts the assumption $(*)$ for the choice of V. Also Condition (ii) contradicts $V \in \mathbf{C}_u(X \cup Y)$. Moreover, if (i) holds, then we must have $y \in Y \backslash X$ due to the definition of U for X, which then contradicts the definition of U for Y. This completes the proof of the present lemma. □

Lemma 3.4 implies the following theorem.

Theorem 3.5: *Let $u : 2^E \to \mathbb{R}$ be an ordinally w-concave function. Then, for any $U \in 2^E$ with $\mathbf{C}_u^{-1}(U) \neq \emptyset$, there uniquely exists a set $U^+ \in 2^E$ such that $\mathbf{C}_u^{-1}(U) = [U, U^+]$ being an interval of 2^E.*

For an ordinally w-concave function $u : 2^E \to \mathbb{R}$ that satisfies the unique-maximizer condition **(UM)**, let $C : 2^E \to 2^E$ be a choice function associated with u. Let us call every $U \in \mathrm{Im}(C) \equiv \{C(X) \mid X \in 2^E\}$ a *choice-set* and U^+ the *enclosure* of U. Also we call $X \in 2^E$ a *proper set* with respect to u if for every $x \in E \setminus X$, $C(X + x) \neq C(X)$. The enclosure U^+ of a choice-set $U \in \mathrm{Im}(C)$ is the unique maximal proper set X that contains U and satisfies $C(X) = U$.

Remark 4: Alva and Doğan [1] have shown that for any path-independent choice function $C : 2^E \to 2^E$ and $U \in 2^E$ there uniquely exists a set $U^+ \in 2^E$ such that for every $X \in 2^E$, $C(X) = U$ if and only if $U \subseteq X \subseteq U^+$. This fact also follows from Theorem 3.5 since any path-independent choice function C is associated with an ordinally concave function u that satisfies the unique-maximizer condition **(UM)** (due to Yokote et al. [21]). □

4. Discussions

4.1. Duality in ordinal concavity

Consider any ordinally concave function $u : 2^E \to \mathbb{R}$. In the definition of ordinal concavity, Definition 1.1, the choice of X and X' is for an unordered pair, while the choice of x and x' and the associated conditions are given for an ordered pair (x, x'). If we change the roles of X and X' we have an equivalent definition of ordinal concavity as follows.

Definition 4.1 (Ordinal Concavity*): *A function $u : 2^E \to \mathbb{R}$ satisfies* ordinal concavity *if for every $X, X' \in 2^E$ the following statement holds:*
For every $x' \in X' \setminus X$ there exists $x \in (X \setminus X') \cup \{\emptyset\}$ such that

(i) $u(X) < u(X + x' - x)$, *or*

(ii) $u(X') < u(X' - x' + x)$, *or*

(iii) $u(X) = u(X + x' - x)$ *and* $u(X') = u(X' - x' + x)$.

We discuss some implications of this fact. For any given function $u : 2^E \to \mathbb{R}$ let us define $u^\bullet : 2^E \to \mathbb{R}$ by

$$u^\bullet(X) = u(E \setminus X) \qquad (\forall X \in 2^E). \tag{4.1}$$

We call such $u^\bullet$ the *dual* of u. We may consider that $u^\bullet$ is defined on the dual Boolean lattice of 2^E. Note that $(u^\bullet)^\bullet = u$.

Lemma 4.2: *A function $u : 2^E \to \mathbb{R}$ is ordinally concave if and only if its dual $u^\bullet : 2^E \to \mathbb{R}$ is ordinally concave.*

(Proof) We can see that Definition 4.1 for u gives exactly Definition 1.1 for $u^\bullet$ by considering the complements of X and X'. □

Moreover, we also have the following lemma for ordinal w-concavity.

Lemma 4.3: *A function $u : 2^E \to \mathbb{R}$ is ordinally w-concave if and only if its dual $u^\bullet : 2^E \to \mathbb{R}$ is ordinally w-concave.*

(Proof) The definition of ordinal w-concavity, Definition 1.2, is self-dual, so that the present lemma holds. □

Hence, as a metatheory, if we have a valid statement for an ordinally (w-)concave function u, then the statement obtained by dualization by taking complements is also valid.

For example, as a dual of Theorem 3.1(I) we have the following theorem.

Theorem 4.4: *Suppose that $u : 2^E \to \mathbb{R}$ satisfies ordinal concavity. Then, for every $X, Y \in 2^E$ and every $U \in \mathbf{C}_u(X \cup Y, E)$ there exists $Z \in \mathbf{C}_u(X, E)$ such that*

$$U \setminus (X \cup Y) \subseteq Z \setminus (X \cup Y).$$

(Proof) Let $U' = E \setminus U \in \mathbf{C}_{u^\bullet}(E \setminus (X \cup Y))$. Since $u^\bullet$ is ordinally concave due to Lemma 4.2, it follows from Theorem 3.1(I) that putting $X \leftarrow E \setminus (X \cup Y)$ and $X \cup Y \leftarrow E \setminus X$, there exists $Z' \in \mathbf{C}_{u^\bullet}(E \setminus X)$ such that $Z' \cap (E \setminus (X \cup Y)) \subseteq U'$. Note that putting $Z = E \setminus Z'$, we have

$$Z' \cap (E \setminus (X \cup Y)) \subseteq U' \quad \iff \quad U \setminus (X \cup Y) \subseteq Z \setminus (X \cup Y).$$

This completes the proof of the present theorem. □

It may be worth considering another function associated with u as follows.

$$u^{\#}(X) = u(E) - u(E \setminus X) \qquad (\forall X \in 2^E). \tag{4.2}$$

It should be noted that $u^{\#} : 2^E \to \mathbb{R}$ is *ordinally* (*w-*)*convex* (i.e., $-u^{\#}$ is ordinally (w-)concave) when u is ordinally (w-)concave. We have $u^{\#}(X) = u(E) - u^\bullet(X)$ for all $X \in 2^E$ and we may call $u^{\#}$ the *dual ordinally* (*w-*)*convex function* of the ordinally (w-)concave function u. Also note that when $u(\emptyset) = 0$, we have $(u^{\#})^{\#} = u$.

4.2. Domains of functions

All the functions considered above have the unit hypercube or Boolean lattice 2^E as their domains. Let us consider any sets as domains instead and examine how our above arguments work for the new problem setting.

Let $\mathbf{Q}$ be a nonempty subset of 2^E and consider a function $u : \mathbf{Q} \to \mathbb{R}$. (Formally we may also consider $u : 2^E \to \mathbb{R} \cup \{-\infty\}$ by putting $u(X) = -\infty$ for all $X \in 2^E \setminus \mathbf{Q}$ so that $\mathbf{Q}$ is the effective domain $\mathrm{dom}(u) = \{X \in 2^E \mid u(X) > -\infty\}$.) We call u an *ordinally concave function* on $\mathbf{Q}$ if it satisfies Definition 1.1 with 2^E being replaced by $\mathbf{Q}$. Also we call u an *ordinally w-concave function* on $\mathbf{Q}$ if it satisfies Definition 1.2 with 2^E being replaced by $\mathbf{Q}$. Then we see the following facts.

1. In Section 2.1 define a minor of u as follows: for any $X, Y \in \mathbf{Q}$ such that $X \subset Y$ define $\mathbf{Q}_X^Y = \{Z \in 2^{Y \setminus X} \mid X \cup Z \in \mathbf{Q}\}$ and $u_X^Y(Z) = u(Z \cup X) - u(X)$ for each $Z \in \mathbf{Q}_X^Y$.

2. All the statements given in Sections 2.2, 2.3, and 2.4 hold true by replacing 2^E by $\mathbf{Q}$.

3. In Section 3.1 suppose $\emptyset \in \mathbf{Q}$ and define a choice function $C : 2^E \to \mathbf{Q}$ in such a way that $C(X) \in 2^X \cap \mathbf{Q}$ for each $X \in 2^E$. Also, define the choice correspondence rationalized by u in such a way that $\mathbf{C}_u(X) = \mathrm{Arg\,max}\,\{u(Z) \mid Z \in 2^X \cap \mathbf{Q}\}$ for each $X \in 2^E$. If $C(X) \in \mathbf{C}_u(X)$ for each $X \in 2^E$, we call C a choice function associated with u. Then all the statements in Section 3.1 hold true *mutatis mutandis.*

4. If $u : \mathbf{Q} \to \mathbb{R}$ is ordinally w-concave and is a constant function on $\mathbf{Q}$, then $\mathbf{Q}$ is an M$^\natural$-convex set.

4.3. A lexicographic composition of two functions

Let us consider the lexicographical order $\leq_\ell$ on $\mathbb{R}^2$ defined by $(a,b) <_\ell (c,d) \iff$ (i) $a < c$ or (ii) $a = c$ and $b < d$, for all $a,b,c,d \in \mathbb{R}$. Let $(\mathbb{R}^2)_\ell$ be the set $\mathbb{R}^2$ endowed with the lexicographical order $\leq_\ell$.

Consider two functions $u_i : 2^E \to \mathbb{R}$ for $i = 1,2$. For any $X \in 2^E$ let $(\hat{X}_1, \hat{X}_2)$ be the lexicographic maximizer (X_1, X_2) of

$$\mathrm{Lexico}\max\{(u_1(X_1), u_2(X_2)) \in \mathbb{R}^2 \mid X_1, X_2 \subseteq X,\ X_2 = X \setminus X_1\}. \tag{4.3}$$

Then define a function $\hat{u} : 2^E \to (\mathbb{R}^2)_\ell$ as follows.

$$\hat{u}(X) = (u_1(\hat{X}_1), u_2(\hat{X}_2)) \in (\mathbb{R}^2)_\ell \qquad (\forall X \in 2^E). \tag{4.4}$$

We write $\hat{u}(X) = (\hat{u}_1(X), \hat{u}_2(X))$. Note that for each $X \in 2^E$ we have

$$\hat{X}_1 \in \mathrm{Arg}\max\{u_2(X \setminus Z) \mid Z \in \mathbf{C}_{u_1}(X)\} \tag{4.5}$$

and

$$\hat{u}(X) = (\hat{u}_1(X), \hat{u}_2(X)) = (u_1(\hat{X}_1), u_2(X \setminus \hat{X}_1)). \tag{4.6}$$

We call $\hat{u}$ the *lexicographic composition* of the ordered pair (u_1, u_2) of functions u_1 and u_2. Let us denote $\hat{u} = u_1 \diamond u_2$. The lexicographic composition $u_1 \diamond u_2$ of u_1 and u_2 is ordinally w-concave if for every $X, Y \in 2^E$ with $X \neq Y$ the following statement holds:

There exist distinct $x \in (X \setminus Y) \cup \{\emptyset\}$ and $y \in (Y \setminus X) \cup \{\emptyset\}$ such that

(i)$_\ell$ $\hat{u}(X) <_\ell \hat{u}(X - x + y)$, or

(ii)$_\ell$ $\hat{u}(Y) <_\ell \hat{u}(Y - y + x)$, or

(iii)$_\ell$ $\hat{u}(X) = \hat{u}(X - x + y)$ and $\hat{u}(Y) = \hat{u}(Y - y + x)$.

The lexicographic composition $\hat{u} = u_1 \diamond u_2$ can be regarded as an ordinal analogue of convolution in convex analysis. However, the lexicographic composition $\hat{u} = u_1 \diamond u_2$ is not ordinally w-concave even if both u_1 and u_2 are ordinally w-concave, in general. (See an example confirming this claim in Appendix A.3.)

We have the following theorem on the lexicographic composition $u_1 \diamond u_2$ for a special class of ordinally concave functions u_1.

Theorem 4.5: *If $u_1 : 2^E \to \mathbb{R}$ is an ordinally w-concave function that satisfies the unique-maximizer condition* **(UM)** *and $u_2 : 2^E \to \mathbb{R}$ is ordinally w-concave, then the lexicographic composition $u_1 \diamond u_2$ is ordinally w-concave.*

(Proof) For any $X, Y \in 2^E$ with $X \neq Y$ consider $\hat{X} \in 2^X$ and $\hat{Y} \in 2^Y$ satisfying

$$\hat{X} \in \mathbf{C}_{u_1}(X), \qquad \hat{Y} \in \mathbf{C}_{u_1}(Y). \tag{4.7}$$

Case 1: Suppose that $\hat{X} \neq \hat{Y}$. By ordinal w-concavity of u_1 there exist distinct $\hat{x} \in (\hat{X} \setminus \hat{Y}) \cup \{\emptyset\}$ and $\hat{y} \in (\hat{Y} \setminus \hat{X}) \cup \{\emptyset\}$ such that (i)$_1$ $u_1(\hat{X}) < u_1(\hat{X} - \hat{x} + \hat{y})$, or

(ii)$_1$ $u_1(\hat{Y}) < u_1(\hat{Y} - \hat{y} + \hat{x})$, or

(iii)$_1$ $u_1(\hat{X}) = u_1(\hat{X} - \hat{x} + \hat{y})$ and $u_1(\hat{Y}) = u_1(\hat{Y} - \hat{y} + \hat{x})$

Suppose that (i)$_1$ holds. Then, because of the definition of $\hat{X}$ we have $\hat{X} - \hat{x} + \hat{y} \not\subseteq X$, which implies $\hat{y} \in Y \setminus X$ and $\hat{X} - \hat{x} + \hat{y} \subseteq X + \hat{y}$. It follows from (i)$_1$ that

$$\hat{u}_1(X) = u_1(\hat{X}) < u_1(\hat{X} - \hat{x} + \hat{y}) \leq \hat{u}_1(X + \hat{y}), \tag{4.8}$$

where recall the notation of $\hat{u}_1$ (and $\hat{u}_2$) in (4.6). Hence we see that Condition (i)$_\ell$ holds for $x = \emptyset$ and $y = \hat{y} \in Y \setminus X$.

Similarly, if (ii)$_1$ holds, then we can show that Condition (ii)$_\ell$ holds for $x = \hat{x} \in X \setminus Y$ and $y = \emptyset$.

Suppose that Condition (iii)$_1$ holds. Since u_1 satisfies the unique-maximizer condition ($\mathbf{UM}$), we have $\hat{X} - \hat{x} + \hat{y} \not\subseteq X$, which implies $\hat{y} \in Y \setminus X$ and $\hat{X} - \hat{x} + \hat{y} \subseteq X + \hat{y}$. Since $\hat{X}$ and $\hat{X} - \hat{x} + \hat{y}$ are distinct subsets of $X + \hat{y}$ with $u_1(\hat{X}) = u_1(\hat{X} - \hat{x} + \hat{y})$, by ($\mathbf{UM}$) again there uniquely exists $\hat{X}' \in \mathbf{C}_{u_1}(X + \hat{y})$ such that

$$u_1(\hat{X}') > u_1(\hat{X}) = u_1(\hat{X} - \hat{x} + \hat{y}). \tag{4.9}$$

Therefore,

$$\hat{u}_1(X) = u_1(\hat{X}) < u_1(\hat{X}') = \hat{u}_1(X + \hat{y}). \tag{4.10}$$

Hence we see that Condition (i)$_\ell$ holds for $x = \emptyset$ and $y = \hat{y} \in Y \setminus X$.

Case 2: Suppose that $\hat{X} = \hat{Y}$. Put $U = \hat{X} (= \hat{Y})$, $\tilde{X} = X \setminus U$ and $\tilde{Y} = Y \setminus U$. Since $\tilde{X} \neq \tilde{Y}$ and u_2 is ordinally w-concave, there exist distinct $\tilde{x} \in (\tilde{X} \setminus \tilde{Y}) \cup \{\emptyset\}$ and $\tilde{y} \in (\tilde{Y} \setminus \tilde{X}) \cup \{\emptyset\}$ such that

(i)$_2$ $u_2(\tilde{X}) < u_2(\tilde{X} - \tilde{x} + \tilde{y})$, or

(ii)$_2$ $u_2(\tilde{Y}) < u_2(\tilde{Y} - \tilde{y} + \tilde{x})$, or

(iii)$_2$ $u_2(\tilde{X}) = u_2(\tilde{X} - \tilde{x} + \tilde{y})$ and $u_2(\tilde{Y}) = u_2(\tilde{Y} - \tilde{y} + \tilde{x})$.

If $\tilde{x} \neq \emptyset$, we have $\tilde{x} \in \tilde{X} \subseteq X$. Since $\tilde{x} \notin \tilde{Y} = Y \setminus U$, we have $\tilde{x} \notin Y$. Similarly we can show that $\tilde{y} \notin X$ if $\tilde{y} \neq \emptyset$. Consequently, we have $\tilde{x} \in (X \setminus Y) \cup \{\emptyset\}$ and $\tilde{y} \in (Y \setminus X) \cup \{\emptyset\}$. Also note that $X - \tilde{x} + \tilde{y}, Y - \tilde{y} + \tilde{x} \supseteq U = \hat{X} = \hat{Y}$.

Case 2(i): Suppose that (i)$_2$ holds. Then, since $U \subseteq X - \tilde{x} + \tilde{y} \subseteq X \cup Y$, it follows from Lemma 3.4 and (i)$_2$ that we have

$$\hat{u}_1(X) = \hat{u}_1(X - \tilde{x} + \tilde{y}), \quad \hat{u}_2(X) = u_2(\tilde{X}) < u_2(\tilde{X} - \tilde{x} + \tilde{y}) = \hat{u}_2(X - \tilde{x} + \tilde{y}). \tag{4.11}$$

Hence, for $x = \tilde{x}$ and $y = \tilde{y}$ Condition (i)$_\ell$ holds.

Case 2(ii): When (ii)$_2$ holds, we can similarly show that (ii)$_\ell$ holds.

Case 2(iii): Suppose that (iii)$_2$ holds. Then we have

$$\hat{u}_1(X) = \hat{u}_1(X - \tilde{x} + \tilde{y}), \quad \hat{u}_2(X) = u_2(\tilde{X}) = u_2(\tilde{X} - \tilde{x} + \tilde{y}) = \hat{u}_2(X - \tilde{x} + \tilde{y}), \tag{4.12}$$

$$\hat{u}_1(Y) = \hat{u}_1(Y - \tilde{y} + \tilde{x}), \quad \hat{u}_2(Y) = u_2(\tilde{Y}) = u_2(\tilde{Y} - \tilde{y} + \tilde{x}) = \hat{u}_2(X - \tilde{y} + \tilde{x}). \tag{4.13}$$

Hence (iii)$_\ell$ holds.

This completes the proof of the present theorem. □

Remark 5: If $u_1 : 2^E \to \mathbb{R}$ is ordinally (w-)concave and $u_2 : 2^E \to \mathbb{R}$ is M$^\natural$-concave, then in order to compute $\hat{u}(X) = u_1 \diamond u_2(X)$ for $X \in 2^E$ the lexicographic maximization of (4.3) is reduced to

1. Maximization of u_1^X to obtain $\mathbf{C}_{u_1}(X)$ and
2. Maximization of $(u_2^X)^\bullet : \mathbf{C}_{u_1}(X) \to \mathbb{R}$.

The latter is a special case of maximization of the sum of two M$^\natural$-concave functions, which can be solved if $\mathbf{C}_{u_1}(X)$ is appropriately identified for the maximization. In particular, if u_2 is a modular function, i.e., $u_2(X) = \sum_{x \in X} u_2(x)$ for all $X \in 2^E$, then the latter maximization can be solved by a greedy algorithm over $\mathbf{C}_{u_1}(X)$. □

5. Concluding Remarks

We have investigated combinatorial structures of ordinally concave functions and (newly introduced) ordinally w-concave functions. We have revealed their fundamental properties and facts such as

1. The local optimality implies the global optimality.
2. The set of maximizers in any interval of 2^E forms an M$^\natural$-convex set.
3. We have shown that the above two properties characterize ordinal w-concavity.
4. We have given an $\mathrm{O}(|E|^2)$ algorithm for maximizing ordinally concave functions with a function evaluation oracle.
5. We have shown the weak substitutability of choice functions associated with ordinally concave functions and its implications for path independence under the unique maximizer condition.
6. We have shown the duality in ordinal (w-)concavity and its implications.
7. We have proposed the lexicographic composition of two ordinally w-concave functions.

It is worth further investigating the structures of ordinally (w-)concave functions in view of economics and discrete optimization. An algorithmic open problem is to maximize an ordinally w-concave function in polynomial time, using a function evaluation oracle, even for some special class of ordinally w-concave functions. Also, besides Theorem 4.5 it is interesting to investigate any other appropriate conditions for the lexicographic composition $u_1 \diamond u_2$ of functions u_1 and u_2 to become an ordinally w-concave function.

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A. Appendix

A.1. An example related to Theorem 3.1

Let $E = \{a, b, c\}$ and define a function $u : 2^E \to \mathbb{R}$ as follows.

$X =$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$u(X) =$	0	4	3	1	2	5	6	0

We can see that the instance of $X = \{a\}, X' = \{b, c\}$, and $x = a \in X \setminus X'$ violates the definition of ordinal concavity. On the other hand, u satisfies the definition of ordinal w-concavity.

Take $X = \{a, b\}$, $Y = \{c\}$, and $\{a\} \in \mathbf{C}_u(X) = \{\{a\}\}$, $\{b, c\} \in \mathbf{C}_u(X \cup Y) = \{\{b, c\}\}$. It holds that

$$\{b, c\} \cap \{a, b\} = \{b\} \not\subseteq \{a\}.$$

Therefore, neither (I) nor (II) of Theorem 3.1 holds.

A.2. A proof

Proposition A.1: *Let* $u : 2^E \to \mathbb{R}$ *be any ordinally concave function and* $C : 2^E \to 2^E$ *be a choice function associated with* u*. The following statements* (a) *and* (b) *are equivalent*:

(a) $C(X \cup Y) = C(C(X) \cup (Y \setminus X))$ *for all* $X, Y \in 2^E$.

(b) $C(X \cup Y) = C(C(X) \cup Y)$ *for all* $X, Y \in 2^E$.

(Proof) The implication (b) $\Rightarrow$ (a) clearly holds. We show the converse. Suppose that (a) holds. Consider any $X, Y \in 2^E$ and define

$$Z = X \setminus ((X \setminus C(X)) \cap Y).$$

Then it follows from (a) that we have

$$C(X \cup Y) = C(Z \cup Y) = C(C(Z) \cup (Y \setminus C(X))) = C(C(X) \cup (Y \setminus C(X))) = C(C(X) \cup Y),$$

where the second equality follows from (a) and $Y \setminus Z = Y \setminus C(X)$, the third equality follows from the fact that C is associated with u, because $C(X) \subseteq Z \subseteq X$ and hence $C(Z) = C(X)$. $\square$

A.3. A lexicograhic composition of ordinally w-concave functions is not ordinally w-concave in general

Let $E = \{a, b, c, d\}$ and define functions $u_i : 2^E \to \mathbb{R}$ ($i = 1, 2$) as follows.

$X =$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$
$u_1(X) =$	0	2	2	8	8	2	3	4	5	6	7
$u_2(X) =$	0	6	6	5	1	7	1	2	3	4	0

$X =$	$\{a,b,c\}$	$\{a,b,d\}$	$\{a,c,d\}$	$\{b,c,d\}$	$\{a,b,c,d\}$
$u_1(X) =$	1	1	1	1	0
$u_2(X) =$	1	5	0	0	0

We can see that both u_1 and u_2 are ordinally w-concave. Consider $X = \{a, b, c, d\}$ and $Y = \{c, d\}$, where note that $(X \setminus Y) \cup \{\emptyset\} = \{a, b\} \cup \{\emptyset\}$ and $(Y \setminus X) \cup \{\emptyset\} = \{\emptyset\}$. Then, for the present X and Y we have that for any $x \in \{a, b\}$ and $y = \emptyset$ none of (i)$_\ell$, (ii)$_\ell$, and (iii)$_\ell$ hold. Hence the lexicographic composition $u_1 \diamond u_2$ is not ordinally w-concave.