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**Cubic fourfolds with eleven cusps and a related moduli space**

By

Shigeru MUKAI

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**京都大学 数理解析研究所**

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

# CUBIC FOURFOLDS WITH ELEVEN CUSPS AND A RELATED MODULI SPACE

SHIGERU MUKAI

*To Manfred Lehn*

ABSTRACT. First we construct a cubic 4-fold whose singularities are 11 cusps and which has an action of the Mathieu group  $M_{11}$ , all over the ternary field  $\mathbb{F}_3$ . We next consider a certain moduli space of bundles on a supersingular K3 surface of Artin invariant one in characteristic 3. We show that it has 275  $(-2)$  Mukai vectors which form the McLaughlin graph, and ask questions on it and on its relation with our  $M_{11}$ -cubic 4-fold.

We work over an algebraically closed field in characteristic 3, but varieties are mostly defined over  $\mathbb{F}_3$  or  $\mathbb{F}_9$ . A general inseparable triple covering

$$(1) \quad V \rightarrow \mathbb{P}_{(vwxyz)}^4, \quad \tau^3 = G(v, w, x, y, z), \quad \deg G = 3.$$

of the projective 4-space is a cubic 4-fold in  $\mathbb{P}_{(\tau vwxyz)}^5$  with 11 cusps, i.e., simple singularities of type  $A_2$  (since  $c_4(\Omega_{\mathbb{P}^4}(3)) = 11$ ). An example with high symmetry is obtained from the Segre cubic 3-fold

$$Seg^3 : \sum_{i=1}^6 x_i = \sum_{1 \leq i < j < k \leq 6} x_i x_j x_k = 0,$$

which has the maximal number (=10) of nodes (e.g., [6]).

**Example 1.** The inseparable triple covering

$$(2) \quad Seg^4 \rightarrow \mathbb{P}^4, \quad \tau^3 = \sum_{1 \leq i < j < k \leq 6} x_i x_j x_k, \quad \text{with } \mathbb{P}^4 : \sum_{i=1}^6 x_i = 0 \subset \mathbb{P}_{(x)}^5,$$

with formal branch  $Seg^3$  has 10 cusps over its nodes, and one more at  $(x : \tau) = (111111 : -1)$ .

The automorphism group  $\mathfrak{S}_6$  of  $Seg^4$  (and also of  $Seg^3$ ) acts 11 cusps with two orbits of length 10 and 1. A little bit surprisingly there is a *more symmetric* cubic 4-fold with 11 cusps in the sense that the automorphism group acts transitively on the cusps.

The following is our main result of this note, and is regarded as a characteristic 3 analogue of the fact that the Fermat cubic 4-fold has an action of the Mathieu group  $M_{22}$  over  $\mathbb{F}_4$  and the  $M_{22}$ -action on a set of 22 planes in it is (triply) transitive ([7], [4, p. 39]):

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**Theorem 2.** *The cubic 4-fold*

$$(3) \quad V : z^3 = \sum_{i \in \mathbb{Z}/5\mathbb{Z}} (x_{i-1}x_i x_{i+1} - x_{i-2}x_i x_{i+2}) \quad \text{in } \mathbb{P}_{(xz)}^5$$

has an action of the Mathieu group  $M_{11}$  over  $\mathbb{F}_3$  (via the extended ternary Golay code [12, 6, 6]). Moreover,  $V$  has cusps at 11  $\mathbb{F}_3$ -points<sup>1</sup>, on which the  $M_{11}$  acts (quadruply) transitively.  $V$  is smooth elsewhere.

Two cubic 4-folds are closely related with a supersingular K3 surface of Artin invariant one, whose standard projective model is the Fermat quartic surface. Though it does not have an action of  $M_{11}$ , there is a chance for a suitable moduli space of bundles over it to have a birational action of  $M_{11}$ . In §3, we give an 8-dimensional candidate and ask two questions.

### 1. PRELIMINARY

The Mathieu group has a presentation

$$\langle a, b, c, \mid a^{11} = b^5 = c^4 = (ac)^3 = 1, a^b = a^4, b^c = b^2 \rangle$$

with 3 generators ([4, p.18]). The following is our starting point:

**Lemma 3.** ([1, Lemma 3.1]) *The Klein's cubic form  $\sum_{i \in \mathbb{Z}/5\mathbb{Z}} y_i^2 y_{i+1}$  is preserved by the linear*

$$\text{transformations } A' = \text{diag}[\zeta, \zeta^9, \zeta^4, \zeta^3, \zeta^5] \text{ and } B' = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \text{ It is not preserved}$$

$$\text{by } C' = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_3 & a_4 & a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 & a_4 & a_0 \\ a_4 & a_0 & a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 & a_0 & a_1 \end{pmatrix} \text{ but transformed under } C' \text{ to } \sum_{i \in \mathbb{Z}/5\mathbb{Z}} (y_i^2 y_{i+1} + y_i^3), \text{ where}$$

$a_1 = -(\zeta + 1)^2$ ,  $a_3 = a_1^3$ ,  $a_0 = a_3^3$ ,  $a_2 = a_0^3$  and  $a_4 = a_2^3$ . In particular,  $\sum_{i \in \mathbb{Z}/5\mathbb{Z}} y_i^2 y_{i+1}$  is invariant under the action of  $M_{11} = \langle A', B', C' \rangle$  modulo cubes of linear forms.

**Remark 4.** The cubic 4-fold  $\tau^3 - \sum_{i \in \mathbb{Z}/5\mathbb{Z}} y_i^2 y_{i+1} = 0 \subset \mathbb{P}_{(\tau y)}^5$  is interesting over the complex number field  $\mathbb{C}$  in the sense that its automorphism group  $PSL(2, 11)$  is maximal among all finite groups with a symplectic action on a smooth cubic 4-fold ([9]). Similar holds for Klein's cubic 3-fold  $\sum_{i \in \mathbb{Z}/5\mathbb{Z}} y_i^2 y_{i+1} = 0 \subset \mathbb{P}_{(y)}^4$  ([13]).

<sup>1</sup>More precisely, the 11 cusps locate at

$$(x_0 : \dots : x_4 : z) = (10000; 0), (01000; 0), (00100; 0), (00010; 0), (00001; 0), (-1 - 1 - 1 - 1 - 1; 0), \\ (01 - 1 - 11; 1), (101 - 1 - 1; 1), (-1101 - 1; 1), (-1 - 1101; 1), (1 - 1 - 110; 1),$$

which are the 11 points 1, 2, 3, 4, 5, 6 and  $a, b, c, d, e$  in the notation of Coxeter-Todd ([12]).

We make the statement in the lemma into that over  $\mathbb{F}_3$  by the following change of variables:

$$(4) \quad \begin{aligned} y_0 &= \zeta x_0 + \zeta^9 x_1 + \zeta^4 x_2 + \zeta^3 x_3 + \zeta^5 x_4 \\ y_1 &= \zeta^9 x_0 + \zeta^4 x_1 + \zeta^3 x_2 + \zeta^5 x_3 + \zeta x_4 \\ y_2 &= \zeta^4 x_0 + \zeta^3 x_1 + \zeta^5 x_2 + \zeta x_3 + \zeta^9 x_4 \\ y_3 &= \zeta^3 x_0 + \zeta^5 x_1 + \zeta x_2 + \zeta^9 x_3 + \zeta^4 x_4 \\ y_4 &= \zeta^5 x_0 + \zeta x_1 + \zeta^9 x_2 + \zeta^4 x_3 + \zeta^3 x_4 \end{aligned}$$

In fact, Klein's cubic form and the generators  $A', B', C'$  of  $M_{11}$  are transformed to

$$(5) \quad \sum_{i \in \mathbb{Z}/5\mathbb{Z}} (-x_i^3 + x_{i-1}x_i x_{i+1} - x_{i-2}x_i x_{i+2}),$$

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we have the following reformulation over  $\mathbb{F}_3$  of Adler's Lemma 3:

**Lemma 5.** *The cubic form (5) of 5 variables is preserved by the linear transformations  $A$  and  $B$ . It is not preserved by  $C$  but transformed under  $C$  to*

$$\sum_{i \in \mathbb{Z}/5\mathbb{Z}} (x_i^3 + x_{i-1}x_i x_{i+1} - x_{i-2}x_i x_{i+2}).$$

The original Lemma 3 is nothing but the expression by taking the eigenvectors of  $C$  as basis.

## 2. CUBIC 4-FOLDS WITH ACTION OF $M_{11}$ OVER $\mathbb{F}_3$

**2.1. Golay cocodes.** Here, following [4] and [5], we understand the Mathieu group  $M_{11}$  is the group of linear transformations of the ternary Golay code [11, 6, 5], which is a certain 6-dimensional vector subspace  $\mathcal{C}_{11} \subset \bigoplus_{i \in \mathbb{F}_{11}} \mathbb{F}_3 \cdot e_i$  of the 11-dimensional vector space with basis  $\{e_i \mid i \in \mathbb{F}_{11}\}$ .  $M_{11}$  acts on the basis as signed permutations, and is generated by the following three transformations  $A, B$  and  $C$ :

$$(6) \quad \begin{aligned} A &: e_i \mapsto e_{i+1}, & B &: e_i \mapsto e_{3i}, (i \in \mathbb{F}_{11}) \\ C &: e_1 \mapsto -e_5 \mapsto e_9 \mapsto -e_4 \mapsto e_1; & e_3 &\mapsto -e_3; \\ & & e_6 &\mapsto e_{10} \mapsto e_8 \mapsto e_7 \mapsto e_6; & e_2 &\mapsto e_2. \end{aligned}$$

The representation space  $V_5$  of Lemma 5 is the space of cocodes, namely, the quotient space  $(\bigoplus_{i \in \mathbb{F}_{11}} \mathbb{F}_3 \cdot e_i) / \mathcal{C}_{11}$ . Since  $\mathcal{C}_{11}$  is a cyclic code with generating polynomial  $X^5 + X^4 - X^3 + X^2 - 1 = 0$ , the space of cocodes is naturally identified with the finite field  $\mathbb{F}_{243}$  as  $\mathbb{F}_3$ -vector space. Though a standard choice of basis is  $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$  as taken in [1], we take here  $\{\zeta, \zeta^9, \zeta^4, \zeta^3, \zeta^5\}$  as basis, instead.

**2.2. From 5 variables to 6.** We extend the cubic form in the previous section into that of 6 variables which is truly invariant under  $M_{11}$ . For that purpose, we must extend the (linear) representation  $V_5$  of  $M_{11}$  in the previous section to a 6-dimensional one  $V_6$  so that it contains an  $M_{11}$ -invariant vector  $\nu_\infty$  and we have an exact sequence

$$(7) \quad 0 \rightarrow \mathbb{F}_3 \cdot \nu_\infty \rightarrow V_6 \rightarrow V_5 \rightarrow 0$$

of  $M_{11}$ -modules (which does not split).

Our construction of the  $V_6$  is again a space of cocodes. Now we consider the extended Golay code  $[12, 6, 6]$ , which is the extension of (perfect) Golay code  $[11, 6, 5]$  by zero-sum condition. We consider it as a 6-dimensional subspace

$$\mathcal{C}_{12} \subset \bigoplus_{i \in \mathbb{P}^1(\mathbb{F}_{11})} \mathbb{F}_3 \cdot e_i$$

and take  $V_6$  as the quotient vector space  $\left( \bigoplus_{i \in \mathbb{P}^1(\mathbb{F}_{11})} \mathbb{F}_3 \cdot e_i \right) / \mathcal{C}_{12}$ . The code  $\mathcal{C}_{12}$  is self-dual and hence the space of cocodes and  $\mathcal{C}_{12}$  are dual to each other. The set  $\mathbb{P}^1(\mathbb{F}_{11})$  of indices are divided into two parts:

$$Q = \{0, 1, 3, 4, 5, 9\} \quad \text{and} \quad N = \{\infty, 2, 6, 7, 8, X\}, \quad X := 10,$$

according as quadratic residue or non-residue. We denote the image of  $e_i$  in the space of cocodes by  $\nu_i$ . We take  $\nu_i, i \in Q \setminus \{0\}$  and  $\nu_\infty$  as its basis.  $M_{11}$  coincides with the stabilizer group of  $M_{12}$  at  $\nu_\infty$ .

The ternary Golay codes  $\mathcal{C}_{12}$  contains 12 elements  $w_j \in \mathbb{P}^1(\mathbb{F}_{11})$ , called *total words*, including

$$(8) \quad w_\infty = - \sum_{i \in \mathbb{P}^1(\mathbb{F}_{11})} e_i \quad \text{and} \quad w_0 = \sum_{i \in Q} e_i - \sum_{i \in N} e_i.$$

(Other total words  $w_j$ 's are just the translations of  $w_0$ .) Then  $M_{11}$  acts transitively on the set  $w_j \in \mathbb{P}^1(\mathbb{F}_{11})$  of 12 total words. Hence we have the exact sequence (7) of  $M_{11}$ -modules by setting

$$V_6 = \left( \bigoplus_{i \in \mathbb{P}^1(\mathbb{F}_{11})} \mathbb{F}_3 \cdot e_i \right) / \mathcal{C}_{12}, \quad V_5 = \left( \bigoplus_{i \in \mathbb{F}_{11}} \mathbb{F}_3 \cdot e_i \right) / \mathcal{C}_{11}$$

The extended transformation  $\tilde{A}, \tilde{B}$  is defined by putting  $e_\infty \mapsto e_\infty$ , that is, just the trivial extension. Only the extension  $\tilde{C}$  of  $C$  is non-trivial, and it is induced by the permutation of the 12 total words:

$$(0\infty)(12345)(6789X).$$

*Proof of Theorem 2.* The invariance under  $A$  and  $B$  are clear from that under  $A'$  and  $B'$ . The linear transformation  $C$  interchanges two total words  $w_0$  and  $w_\infty$  in (8). It acts on  $x_0, x_1, x_2, x_3, x_4$  by permutation composed with negative by (6). Hence, by Lemma 5, it transforms

$$(\tau - x_0 - x_1 - x_2 - x_3 - x_4)^3 + \sum_{i \in \mathbb{Z}/5\mathbb{Z}} (x_{i-1}x_i x_{i+1} - x_{i-2}x_i x_{i+2})$$

into

$$(\tau + x_0 + x_1 + x_2 + x_3 + x_4)^3 + (x_0 + x_1 + x_2 + x_3 + x_4)^3 + \sum_{i \in \mathbb{Z}/5\mathbb{Z}} (x_{i-1}x_i x_{i+1} - x_{i-2}x_i x_{i+2}),$$

which means the desired invariance.  $\square$

### 3. CONJECTURAL SYMPLECTIC 8-FOLD AS MODULI OF BUNDLES ON FERMAT QUARTIC

**3.1. Two questions.** The Fermat quartic surface  $Fer_4 : \sum_1^4 x_i^4 = 0 \subset \mathbb{P}_{(x)}^3$ , has an action of the finite unitary group  $PGU_4(3)$ . The action of a subgroup of index 4, namely, of  $U_4(3) := PSU_4(3)$  is symplectic. Though  $U_4(3)$  does not contain  $M_{11}$  as a subgroup, the moduli space  $\overline{M}_{Fer}(v)$  of (semi-)stable sheaves on the Fermat quartic  $Fer_4$  might have a birational action of  $M_{11}$ , or even a much larger finite simple group, for suitable Mukai vector  $v = (r, *, s) \in \mathbb{Z} \oplus \text{Pic} \oplus \mathbb{Z}$ . A hopeful candidate, in view of symmetry of the Leech lattice, is 8-dimensional, i.e.,  $\langle v^2 \rangle = 6$ , and the group containing  $M_{11}$  should be the McLaughlin group  $McL$ .

*Question 6.* Does the moduli space  $\overline{M}_{Fer}(3, \alpha, -3)$  have a birational action of  $McL$ , where  $\alpha$  is a  $(-12)$ -divisor class attached to Segre's hemisystem (see §3.2)?

$McL$  contains simple groups  $U_4(3)$  and  $M_{11}$  as maximal subgroups, and hence is generated by these two subgroups. The action of the former on the moduli is not surprising since its  $\mathbb{Q}$ -twisted expression is  $\overline{M}_{Fer}(3, 0, -1)$  (Proposition 9). Seeking after an action of the latter, we pose the following

*Question 7.* Is  $\overline{M}_{Fer}(3, \alpha, -3)$  birational to the conjectural LLSvS 8-fold (see [10] but only over  $\mathbb{C}$  under some condition) associated with the  $M_{11}$ -cubic 4-fold  $V$ ?

**3.2. Segre's hemisystem and the McLaughlin graph in a Picard lattice.** The Fermat quartic surface  $Fer_4$  has 280  $\mathbb{F}_9$ -(rational) points, with weight distribution 2: 24, 3: 64 and 4: 192. For each  $\mathbb{F}_9$ -point  $p$ , the tangent plane  $T_p$  cuts out the union of 4 lines passing through  $p$  from  $Fer_4$ . Since every line has 10  $\mathbb{F}_9$ -points, the number of lines in  $Fer_4$  is  $280 \times 4/10 = 112$ . The Picard lattice is generated by these line classes. Its discriminant group  $\text{Disc}(Fer_4)$  is isomorphic to  $\mathbb{Z}/3 \oplus \mathbb{Z}/3$  (see e.g. [8]).

Segre's hemisystem is a set  $H$  of 56 lines, among the 112, which covers  $Fer_4(\mathbb{F}_9)$  doubly, that is, every  $\mathbb{F}_9$ -point is contained in exactly two members of  $H$ . There are 648 hemisystems and they are divided into 4 orbits of length 162 by the action of  $U_4(3)$ . These 4 orbits corresponds to the four elements of norm  $2/3$  modulo  $2\mathbb{Z}$  in the discriminant group  $\text{Disc}(Fer_4)$  as we will see below. We chose one of them. Then the intersection size  $|H \cap H'|$  of two among our 162 hemisystems are either 20 or 32 ([3, §10.34]).

**Proposition 8.** ([3, §10.61]) *The graph with the following three types of vertices and a suitable adjacency is a strongly regular graph  $srg(275, 112, 30, 56)$ , isomorphic to the McLaughlin graph: (i)  $\infty$ , (ii) the 112 lines in  $Fer_4$  and (iii) the 162 hemisystems.*

We realize this graph inside the extended Picard lattice  $U(-1) \oplus \text{Pic } Fer_4$  of the Fermat quartic surface, or more precisely, in the sublattice  $(3, \alpha, -3)^\perp$ , which is expected to be the Picard lattice of the conjectural moduli symplectic 8-fold ([11], [14], [15] but only over  $\mathbb{C}$ ). Here  $U$  denotes the

standard hyperbolic lattice of rank 2. The intersection pairing  $(D.D')$  on the Picard lattice extends to the orthogonal sum  $\mathbb{Z} \oplus \text{Pic} \oplus \mathbb{Z}$  obviously but with changing the sign of  $U$ , namely,

$$(9) \quad \langle (r, D, s), (r', D', s') \rangle = -rs' + (D.D') - sr', \quad (r, s), (r', s') \in U(-1).$$

Now we define a divisor class for a hemisystem  $H$ . Consider the sum  $\sum_{m \in H} m$  of its all members in the Picard group  $\text{Pic } Fer_4$ . Then we have

$$(10) \quad \left( \sum_{m \in H} m.l \right) = \begin{cases} 8 & \text{if } l \in H, \\ 20 & \text{otherwise.} \end{cases}$$

In particular,  $\sum_{m \in H} m$  is divisible by 4 in the Picard group. So we define

$$\alpha_H := 2h - \frac{1}{4} \sum_{m \in H} m \in \text{Pic } Fer_4,$$

where  $h$  is the hyperplane section class of  $Fer_4$ . Since  $(\alpha_H.l)$  is divisible by 3 for all lines  $l$ ,  $\alpha_H/3$  defines an element in the discriminant group, whose norm is  $2/3$  since  $(\alpha_H^2) = -12$ . The following is equivalent to the preceding proposition:

**Proposition 9.** *The graph on the following three types of  $(-2)$ -vectors in  $(3, 0, -1)^\perp \otimes \mathbb{Q}$ , adjacent when non-orthogonal, is isomorphic to the McLaughlin graph:*

- $(3, h, 1)$ ,
- $(0, l, 0)$  for the 112 lines  $l$  in  $Fer_4$  and
- $(1, -\frac{\alpha_H}{3}, \frac{1}{3})$  for the 162 hemisystems  $H$  chosen as above.

Geometrically, these are the Mukai vectors of the rank 3 bundle  $T_{\mathbb{P}^3}(-1)$  restricted to  $Fer_4$ , torsion sheaves supported on lines and  $\mathbb{Q}$ -line bundles on  $Fer_4$ , respectively.

Now we fix a hemisystem  $F$  among our 162, put  $\alpha = \alpha_F$  and take twist by tensor product of the  $\mathbb{Q}$ -line bundle  $\mathcal{O}_{Fer}(\frac{\alpha}{3})$ . Then all the vertices in the proposition become integral. Since the tensor of a line bundle preserves the inner product (9), we have

**Corollary 10.** *The graph on the following three types of  $(-2)$  Mukai vectors in  $(3, \alpha, -3)^\perp$ , adjacent when non-orthogonal, is isomorphic to the McLaughlin graph:*

- $(3, h + \alpha, -3)$ ,
- $(0, l, *)$  for the 112 lines  $l$  in  $Fer_4$  and
- $(1, \frac{\alpha - \alpha_H}{3}, **)$  for the 162 hemisystems  $H$ ,

where  $*$  is equal to 0 if  $l \in F$  and 1 otherwise, and  $**$  is equal to 1,  $-1$ ,  $-2$  according as  $H = F$ ,  $|H \cap F| = 20$  and  $|H \cap F| = 32$ .

*Proof.*  $\alpha - \alpha_H$  is divisible by 3 since both  $\alpha/3$  and  $\alpha_H/3$  defines the same element in  $\text{Disc}(Fer_4)$ . Hence the vertices are Mukai vectors of a rank 3 bundles, torsion sheaves and the 162 line bundles  $\mathcal{O}_{Fer}(\frac{\alpha - \alpha_H}{3})$ .  $\square$

*Remark 11.* The McLaughlin graph is realized in the Leech lattice  $\Lambda$  using a triangle of type 223 ([4, p.100]). Hence the corollary can be proved using the construction of  $\Lambda$  from the Niemeier lattice of type  $12A_2$  in the way of Borcherds [2].

*Remark 12.* Two more strongly regular graphs are similarly realized by taking  $(-2)$  Mukai vectors as their vertices in characteristic 2 and 5, which will be discussed elsewhere.

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

*Email address:* mukai@kurims.kyoto-u.ac.jp

*Current address:* Morningside Center of Mathematics, Chinese Academy of Sciences, Beijing 100190, China