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The Arithmetic Fundamental Groups of Curves over Local Fields

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ABSTRACT. In the present paper, we study the arithmetic fundamental groups of curves over local fields from the point of view of anabelian geometry. In particular, we prove that, under certain technical assumptions, a continuous isomorphism over the absolute Galois group of the basefield between the tame fundamental groups of hyperbolic curves over a local field arises from a unique isomorphism between the given hyperbolic curves over the basefield. This "certain technical assumptions" are satisfied whenever the basefield of the hyperbolic curves under consideration is a mixed-characteristic local field. Thus, one may conclude that an arbitrary continuous isomorphism over the absolute Galois group of the basefield between the étale fundamental groups of hyperbolic curves over a mixed-characteristic local field arises from a unique isomorphism between the given hyperbolic curves over the basefield. Moreover, this conclusion, together with a "formal argument" in anabelian geometry, leads to an alternative proof of a famous anabelian theorem, i.e., for hyperbolic curves over sub-p-adic fields for some prime number p, proved by Shinichi Mochizuki. Let us recall that main ingredients of the proof of this anabelian theorem by Mochizuki are various results in the study of p-adic Hodge theory. In particular, the present paper yields an alternative proof of this famous anabelian theorem by Mochizuki in which we never apply such a result in p-adic Hodge theory.

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Introduction

In the present paper, we study the arithmetic fundamental groups of curves over local fields from the point of view of anabelian geometry. In the present Introduction, let

- K be a field,
- $\overline{K}$ a separable closure of K, and
- Z_1, Z_2 hyperbolic curves over K.

Write

- $G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group of K determined by the separable closure $\overline{K}$,
- Π_{Z_1}, Π_{Z_2} for the respective tame fundamental groups of Z_1, Z_2, relative to suitable choices of basepoints,
- $\mathrm{Isom}_K(Z_1, Z_2)$ for the set of isomorphisms $Z_1 \xrightarrow{\sim} Z_2$ over K,
- $\mathrm{Isom}_{G_K}(\Pi_{Z_1}, \Pi_{Z_2})$ for the set of continuous isomorphisms $\Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ over G_K, and
- $\overline{\mathrm{Isom}}_{G_K}(\Pi_{Z_1}, \Pi_{Z_2})$ for the quotient set of $\mathrm{Isom}_{G_K}(\Pi_{Z_1}, \Pi_{Z_2})$ with respect to the natural conjugation action of the kernel of the natural continuous outer homomorphism $\Pi_{Z_2} \to G_K$.

Thus, the functoriality of the operation of taking tame fundamental groups determines a natural map

$$\mathrm{Isom}_K(Z_1, Z_2) \longrightarrow \overline{\mathrm{Isom}}_{G_K}(\Pi_{Z_1}, \Pi_{Z_2}).$$

The anabelian Grothendieck conjecture for hyperbolic curves may be formulated as the bijectivity of this map under suitable choices of (K, Z_1, Z_2). A (special case of a) famous theorem proved by Shinichi Mochizuki asserts the bijectivity of the map under consideration in the case where K is sub-p-adic for some prime number p (cf. [18, Theorem A]).

Theorem A. *Suppose that K is sub-p-adic for some prime number p, i.e., that K is isomorphic to a subfield of a finitely generated extension field of the p-adic completion of the field of rational numbers for some prime number p (cf.* [18, Definition 15.4]*). Then the above map*

$$\mathrm{Isom}_K(Z_1, Z_2) \longrightarrow \overline{\mathrm{Isom}}_{G_K}(\Pi_{Z_1}, \Pi_{Z_2})$$

is bijective. Put another way, every continuous isomorphism $\Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ over G_K arises from a unique isomorphism $Z_1 \xrightarrow{\sim} Z_2$ over K.

Let us recall that main ingredients of the proof of this theorem by Mochizuki are various results in the study of p-adic Hodge theory. One main purpose of the present paper is to yield an alternative proof of this theorem in which we never apply such a result in p-adic Hodge theory (cf. Corollary 11.3).

Next, let us explain the main theorem of the present paper. To this end, in the remainder of the present Introduction, suppose that

the field K is a complete discrete valuation field.

Write

- R for the valuation ring of K (which is necessarily a complete discrete valuation ring) and
- k for the residue field of R.

In the remainder of the present Introduction, suppose, moreover, that

the field k is perfect and of characteristic $p > 0$.

One may find two properties for continuous isomorphisms between tame fundamental groups — i.e., LSF-compatibility and Prym-compatibility — and one property of fields — i.e., $\times$-Kummer-faithfulness — in the statement of Theorem B below, i.e., the main theorem of the present paper.

- We shall say that a continuous isomorphism $\phi\colon \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ over G_K is *LSF-compatible* (where the "LSF" stands for "Log Special Fiber") if, roughly speaking, for suitable open subgroups $H \subseteq \Pi_{Z_1}$ of Π_{Z_1}, the induced isomorphism $H \xrightarrow{\sim} \phi(H)$ induces an isomorphism between the "log special fibers" of the respective connected finite étale coverings of Z_1, Z_2 that correspond to $H \subseteq \Pi_{Z_1}$, $\phi(H) \subseteq \Pi_{Z_2}$. The precise definition of the notion of LSF-compatibility is given in Definition 7.4.
- We shall say that a continuous isomorphism $\phi\colon \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ over G_K is *Prym-compatible* if, roughly speaking, for suitable open subgroups $H \subseteq \Pi_{Z_1}$ of Π_{Z_1}, the induced isomorphism $H \xrightarrow{\sim} \phi(H)$ is compatible with certain objects that are related

to the homomorphism "$\iota\colon \underline{Y} \to \widetilde{G} \otimes K$" of (6) that appears in the definition of objects of the category $\mathrm{DD}_{\mathrm{pol}}$ defined in [6, Chapter III, §2]. The precise definition of the notion of Prym-compatibility is given in Definition 10.7.
- We shall say that a field F is *$\times$-Kummer-faithful* if, for every finite separable extension field F' of F and every semi-abelian variety A over F', the intersection $\bigcap_n n \cdot A(F')$ — where n ranges over the positive integers invertible in F — is zero (cf. Definition 10.4, (ii)).

The main theorem of the present paper is as follows (cf. Theorem 11.1, Corollary 11.2):

Theorem B. *Let $\phi\colon \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ be a continuous isomorphism over G_K. Suppose that the isomorphism ϕ is LSF-compatible. Suppose, moreover, that one of the following three conditions is satisfied:*

(1) *The field K is of characteristic zero and $\times$-Kummer-faithful.*
(2) *The field K is of characteristic zero, and the isomorphism ϕ is Prym-compatible.*
(3) *The field K is of characteristic p, the hyperbolic curve Z_1 over K is nonisotrivial (cf. Definition 5.8, (i)), the field k is algebraic over a finite field, and the isomorphism ϕ is Prym-compatible.*

Then the isomorphism ϕ arises from a unique isomorphism $Z_1 \xrightarrow{\sim} Z_2$ over K.

Let us observe that if K is a mixed-characteristic local field (i.e., if K is of characteristic zero, and k is finite), then one may verify that the isomorphism ϕ is LSF-compatible (cf. [17, Theorem 7.2]), and the field K satisfies condition (1) of the statement of Theorem B (cf. [15, Theorem 7]). In particular, it follows from Theorem B (i.e., in the case where condition (1) is satisfied) that Theorem A in the case where K is a mixed-characteristic local field holds. Moreover, it follows immediately from this conclusion, together with a "formal argument", i.e., applied in the proof of [18, Corollary 15.5], that Theorem A for an arbitrary "K" holds.

Next, let us discuss the strategy of the proof of Theorem B. One important observation in the proof of Theorem B is the existence of Prym-faithful Galois étale coverings of stable curves. In order to explain the notion of a Prym-faithful Galois étale covering, let us fix

- stable curves X, Y over the residue field k of R and
- a Galois étale covering $Y \to X$ over k of degree a prime number invertible in k.

Now suppose that we are given a stable curve $\mathcal{X}$ over R that is smooth over K and an isomorphism $X \xrightarrow{\sim} \mathcal{X} \times_R k$ over k, by means of which we identify X with $\mathcal{X} \times_R k$. Then it follows immediately from the topological invariance of étale sites that the Galois étale covering $Y \to X$ over k extends uniquely to a Galois étale covering $\mathcal{Y} \to \mathcal{X}$ over R, where $\mathcal{Y}$ is a stable curve over R. In particular,

(a) by considering the restriction to $X \subseteq \mathcal{X}$ of the log structure on $\mathcal{X}$ associated to the divisor $X \subseteq \mathcal{X}$, one obtains a log structure on X, hence also a log scheme $X^{\log}$ (whose underlying scheme is X), and,

(b) by considering the "difference", relative to the resulting covering $\mathcal{Y} \to \mathcal{X}$, between the Jacobian varieties of $\mathcal{X}$, $\mathcal{Y}$, one obtains a polarized abelian scheme $P_{\mathcal{Y}/\mathcal{X}}$ over R, i.e., the generalized Prym scheme $P_{\mathcal{Y}/\mathcal{X}}$ associated to the Galois étale covering $\mathcal{Y} \to \mathcal{X}$ (cf. Definition 2.1) equipped with the Prym semi-polarization $P_{\mathcal{Y}/\mathcal{X}} \to P^t_{\mathcal{Y}/\mathcal{X}}$ — where we write $P^t_{\mathcal{Y}/\mathcal{X}}$ for the dual semi-abelian scheme of $P_{\mathcal{Y}/\mathcal{X}}$ over R (cf., e.g., [22, Chapitre IV, Théorème 7.1, (i)]) — associated to the Galois étale covering $\mathcal{Y} \to \mathcal{X}$ (cf. Definition 2.3, (ii)).

Then we shall say that the Galois étale covering $Y \to X$ is *Prym-faithful* if, roughly speaking, the assignment "$\mathcal{X} \mapsto (X^{\log}, P_{\mathcal{Y}/\mathcal{X}})$" — i.e., the assignment that assigns, to (the isomorphism class of) $\mathcal{X}$ as above, the (isomorphism class of the) pair that consists of the log scheme of (a) and the polarized abelian scheme of (b) — is injective. The precise definition of the notion of a Prym-faithful Galois étale covering is given in Definition 2.5. Then one main technical result of the present paper is as follows (cf. Theorem 4.5, Lemma 6.10):

Theorem C. *There exist*

- *a positive integer* n*,*
- *a finite extension field* k_Y *of* k*,*
- *for each* $i \in \{0, \ldots, n\}$*, a stable curve* Y_i *over* k_Y*,*
- *for each* $i \in \{1, \ldots, n\}$*, a Galois étale covering* $Y_{i-1} \to Y_i$ *over* k_Y*, and*
- *a Galois étale covering* $Y_n \to X$ *over* k

such that the Galois étale covering $Y_0 \to Y_1$ *over* k_Y *is of degree a prime number invertible in* k*, Prym-faithful, and new-ordinary (cf. Definition 2.2), i.e., and satisfies the condition that every abelian variety quotient of the (semi-abelian variety over* k_Y *obtained by forming the) cokernel of the natural homomorphism over* k_Y *from the Jacobian variety of* Y_1 *to the Jacobian variety of* Y_0 *is ordinary.*

Let us recall that Theorem C in the case where the stable curve X is smooth over k was already essentially proved by Akio Tamagawa. Indeed, Theorem C in the case where X is smooth over k may be regarded as a formal consequence of [28, Theorem 0.5], [28, Theorem 0.7], [28, Proposition 0.8], and [28, Corollary 5.3]. Observe that the problem of the existence of Prym-faithful Galois étale coverings of stable curves may be regarded as the logarithmic infinitesimal version of the "Torelli problem-type result" for generalized Prym varieties. In the present paper, in order to prove Theorem C, we apply the theory of the arithmetic compactifications of Shimura varieties of PEL-type discussed in [13].

Finally, let us explain the strategy of the proof of Theorem B especially in the case where condition (1) is satisfied. To this end, suppose that the field K satisfies condition (1) (i.e., is of characteristic zero and $\times$-Kummer-faithful), and that we are given a hyperbolic curve Z over K. Write Π_Z for the étale fundamental group of Z, relative to a suitable choice of basepoint. Then, roughly speaking, the strategy of the proof of Theorem B (i.e., to reconstruct the curve Z from the topological group Π_Z group-theoretically) may be summarized as follows.

For simplicity, suppose that Z is proper over K. Then observe that it follows from Theorem C, together with the Galois descent argument and some well-known facts concerning the geometry of stable curves, that, to reconstruct Z from Π_Z, we may assume without loss of generality, by replacing Z by a suitable connected finite étale covering of Z, that there exists a commutative diagram of schemes over R

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & Z \arrow[r] \arrow[d, hook] & \mathrm{Spec}(K) \arrow[d, hook] \\ \mathcal{X} \arrow[r] & \mathcal{Z} \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

— where X is a (necessarily proper) hyperbolic curve over K, $\mathcal{X}$ and $\mathcal{Z}$ are stable curves over R, the right-hand vertical arrow is the natural open immersions, the two squares are cartesian, the left-hand horizontal arrows are Galois étale coverings over R, and the Galois étale covering

$$\underline{\mathcal{X}} \stackrel{\mathrm{def}}{=} \mathcal{X} \times_R k \longrightarrow \underline{\mathcal{Z}} \stackrel{\mathrm{def}}{=} \mathcal{Z} \times_R k$$

over k determined by the left-hand lower horizontal arrow is of degree a prime number invertible in k, Prym-faithful, and new-ordinary. Write

- Δ_X for the étale fundamental group of $X \times_K \overline{K}$, relative to a suitable choice of basepoint,

- $\mathcal{Z}^{\log}$ for the log scheme obtained by equipping $\mathcal{Z}$ with the log structure associated to the divisor $\underline{\mathcal{Z}} \subseteq \mathcal{Z}$,
- $\underline{\mathcal{Z}}^{\log}$ for the log scheme obtained by equipping $\underline{\mathcal{Z}}$ with the log structure obtained by pulling back the log structure of $\mathcal{Z}^{\log}$ by the natural closed immersion $\underline{\mathcal{Z}} \hookrightarrow \mathcal{Z}$,
- $P_{\mathcal{X}/\mathcal{Z}}$, $P_{\underline{\mathcal{X}}/\underline{\mathcal{Z}}}$ for the respective generalized Prym schemes associated to the Galois étale coverings $\mathcal{X} \to \mathcal{Z}$, $\underline{\mathcal{X}} \to \underline{\mathcal{Z}}$ (cf. Definition 2.1), and
- $\widetilde{P}_{\mathcal{X}/\mathcal{Z}}$ for the Raynaud extension of the semi-abelian scheme $P_{\mathcal{X}/\mathcal{Z}}$ over R (cf., e.g., [6, Chapter II, §1]).

(i) First, observe that it follows from the LSF-compatibility assumption in the statement of Theorem B that one may reconstruct the log special fiber

$$\underline{\mathcal{Z}}^{\log}$$

of Z and the Galois étale covering

$$\underline{\mathcal{X}} \longrightarrow \underline{\mathcal{Z}}$$

over k.

(ii) Next, observe that, by considering the generalized Prym scheme associated to the Galois étale covering $\underline{\mathcal{X}} \to \underline{\mathcal{Z}}$, which was reconstructed in (i), one may reconstruct the special fiber

$$P_{\underline{\mathcal{X}}/\underline{\mathcal{Z}}} = P_{\mathcal{X}/\mathcal{Z}} \times_R k$$

of the generalized Prym scheme $P_{\mathcal{X}/\mathcal{Z}}$. Note that it is well-known (cf., e.g., [6, Chapter II, §1]) that we have a natural identification $P_{\mathcal{X}/\mathcal{Z}} \times_R k = \widetilde{P}_{\mathcal{X}/\mathcal{Z}} \times_R k$.

(iii) Next, consider the natural continuous outer action of G_K on Δ_X. Then one may verify from some techniques in combinatorial anabelian geometry that one may reconstruct, as a suitable G_K-stable subquotient of the maximal pro-p quotient of the topological abelianization of Δ_X, the p-adic Tate module

$$\mathbb{T}_p(\widetilde{P}_{\mathcal{X}/\mathcal{Z}})$$

associated to the semi-abelian scheme $\widetilde{P}_{\mathcal{X}/\mathcal{Z}}$ equipped with the natural continuous action of G_K. Thus, since (we have assumed that) the field K is of characteristic zero, it follows from a classical theorem in the study of p-divisible groups (cf. [30, Theorem 4]) that one may reconstruct the p-divisible group

$$\widetilde{P}_{\mathcal{X}/\mathcal{Z}}[p^\infty]$$

over R associated to $\widetilde{P}_{\mathcal{X}/\mathcal{Z}}$.

(iv) Next, recall that we have assumed that the semi-abelian variety $P_{\underline{\mathcal{X}}/\underline{\mathcal{Z}}} = \widetilde{P}_{\mathcal{X}/\mathcal{Z}} \times_R k$ (cf. (ii)) over k is ordinary, which thus implies that the p-divisible group over k associated to this semi-abelian variety may be written as the direct product of an étale p-divisible group over k and a multiplicative p-divisible group over k. It follows from this ordinariness, together with our assumption that the field K is of characteristic zero, that one may reconstruct a natural identification between
- the p-divisible group $P_{\underline{\mathcal{X}}/\underline{\mathcal{Z}}}[p^\infty]$ over k associated to the semi-abelian variety $P_{\underline{\mathcal{X}}/\underline{\mathcal{Z}}}$, which was reconstructed in (ii), and
- the special fiber $\widetilde{P}_{\mathcal{X}/\mathcal{Z}}[p^\infty] \times_R k$ of the p-divisible group $\widetilde{P}_{\mathcal{X}/\mathcal{Z}}[p^\infty]$, which was reconstructed in (iii).

In particular, it follows from a classical theorem in the study of deformations of ordinary semi-abelian varieties (cf. [12, Theorem 1.2.1]) that one may reconstruct the semi-abelian scheme

$$\widetilde{P}_{\mathcal{X}/\mathcal{Z}}$$

over R. We note that this reconstruction step gives rise to one main reason why one cannot remove the assumption that K is of characteristic zero from condition (1) of Theorem B (cf. Remark 9.5.1).

(v) Next, recall that we have assumed that the field K is of characteristic zero and $\times$-Kummer-faithful. In particular, by applying some techniques in combinatorial anabelian geometry to the natural continuous outer action of G_K on Δ_X, one may reconstruct the object — which consists of six items, and whose first item is given by the Raynaud extension $\widetilde{P}_{\mathcal{X}/\mathcal{Z}}$ reconstructed in (iv) — of the category $\mathrm{DD}_{\mathrm{pol}}$ defined in [6, Chapter III, §2] that corresponds, relative to the equivalence $M_{\mathrm{pol}} \colon \mathrm{DD}_{\mathrm{pol}} \xrightarrow{\sim} \mathrm{DEG}_{\mathrm{pol}}$ of categories of [6, Chapter III, Corollary 7.2], to the generalized Prym scheme $P_{\mathcal{X}/\mathcal{Z}}$ (i.e., strictly speaking, equipped with the Prym semi-polarization associated to the Galois étale covering $\mathcal{Y} \to \mathcal{X}$ — cf. Definition 2.3, (ii)). Thus, by considering the equivalence $M_{\mathrm{pol}} \colon \mathrm{DD}_{\mathrm{pol}} \xrightarrow{\sim} \mathrm{DEG}_{\mathrm{pol}}$ of categories of [6, Chapter III, Corollary 7.2], one may reconstruct the semi-abelian scheme

$$P_{\mathcal{X}/\mathcal{Z}}$$

over R.

(vi) Finally, observe that since (we have assumed that) the Galois étale covering $\underline{\mathcal{X}} \to \underline{\mathcal{Z}}$ over k is Prym-faithful, one may reconstruct, from the log special fiber $\underline{\mathcal{Z}}^{\log}$ of (i) and the generalized Prym scheme $P_{\mathcal{X}/\mathcal{Z}}$ of (v), the stable curve

$$\mathcal{Z}$$

over R, hence also the hyperbolic curve

$$Z$$

over K, as desired.

This completes the rough explanation of the strategy of the proof of Theorem B in the case where condition (1) is satisfied.

We have the following two remarks concerning the proofs of the main results of the present paper.

- The proof of Theorem B given in the present paper may be regarded as a substantial technical refinement of the argument given in the final portion of [17, §9]. In [17, Theorem 9.7], Mochizuki proved that, roughly speaking, if the field K is p-adic local for some prime number p, and the Jacobian variety of Z_1 has ordinary semistable reduction, then every continuous isomorphism $\Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ over G_K determines functorially an isomorphism $Z_1 \xrightarrow{\sim} Z_2$ over K.
- According to Tamagawa, he has already established (but has not written), more than two decades ago, a special case of Theorem B by similar techniques to the techniques applied in the proof of Theorem B. As pointed out in the discussion following Theorem C, Tamagawa essentially gave the proof of Theorem C in the case where the given stable curve is smooth, which leads to a similar result to Theorem B in the case where the given hyperbolic curves have good reduction.

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1. Stable curves

In the present section, we introduce some notational conventions related to the notion of a stable curve. In the present section, let

- S be a scheme and
- X a stable curve over S (cf. [4, Definition 1.1]).

Definition 1.1. We shall write

- ω_X for the dualizing sheaf of X over S (cf. the discussion following [4, Definition 1.1]) and
- $J_X \stackrel{\text{def}}{=} \mathrm{Pic}^0_{X/S}$ for the Jacobian variety of X over S, which is a semi-abelian scheme over S (cf. [3, §9.4, Theorem 1]).

Definition 1.2. Suppose that S is the spectrum of a separably closed field. Then we shall write

- Γ_X for the dual graph of the stable curve X over S,
- $v(\Gamma_X)$ for the set of vertices of the graph Γ_X (i.e., the set of irreducible components of X), and
- $e(\Gamma_X)$ for the set of edges of the graph Γ_X (i.e., the set of singular points of X).

By abuse of notation, we shall regard $e(\Gamma_X)$ as a closed subset of X in the evident way.

Definition 1.3. Suppose that S is the spectrum of a field k. Let $\overline{k}$ be a separable closure of k and $v \in v(\Gamma_{X \times_k \overline{k}})$ a vertex of $\Gamma_{X \times_k \overline{k}}$. Then we shall write

- $I_v^X = I_v$ for the irreducible component of X obtained by forming the image in X of the irreducible component of $X \times_k \overline{k}$ that corresponds to v and
- $D_v^X = D_v$ for the closed subset of I_v obtained by forming the intersection of I_v with the set of singular points of X.

By abuse of notation, write

- $I_v^X = I_v$ for the reduced closed subscheme of X whose underlying closed subset is given by $I_v \subseteq X$.

Moreover, we shall write

- $U_v^X = U_v \stackrel{\text{def}}{=} I_v \setminus D_v$ for the open subscheme of I_v obtained by forming the complement of D_v in I_v,
- X_v for the smooth proper curve over k obtained by forming the smooth compactification of U_v,
- $g_v^X = g_v$ for the genus of the smooth proper curve X_v over k,
- $\Omega_{X_v} \stackrel{\text{def}}{=} \Omega^1_{X_v/k}$ for the sheaf of relative differentials of X_v over k, and

- $J_v^X = J_v \stackrel{\mathrm{def}}{=} \mathrm{Pic}^0_{X_v/k}$ for the Jacobian variety of X_v over k, which is an abelian variety over k (cf. [3, §9.2, Proposition 3]).

Observe that one verifies easily that if k is perfect, then U_v is a hyperbolic curve over k. If k is perfect, then we shall refer to U_v as the *hyperbolic curve over k associated to v.*

Definition 1.4. Suppose that S is the spectrum of a separably closed field k. Let $v \in v(\Gamma_X)$ be a vertex of Γ_X such that I_v is smooth over k. Then observe that one verifies easily that the natural open immersions $U_v \hookrightarrow I_v$ and $U_v \hookrightarrow X_v$ determine an identification between I_v with X_v, i.e., $I_v = X_v$. By abuse of notation, write D_v for the reduced divisor on $I_v = X_v$ whose support is given by the closed subset $D_v \subseteq I_v = X_v$.

Remark 1.4.1. Let us recall from [14, Chapter 10, Lemma 3.12, (b)] that, in the situation of Definition 1.4, the natural closed immersion $X_v \hookrightarrow X$ determines an isomorphism of $\mathcal{O}_{X_v}$-modules

$$\omega_X|_{X_v} \cong \Omega_{X_v}(D_v).$$

Definition 1.5. Suppose that S is the spectrum of a field k. Let $\overline{k}$ be a separable closure of k.

(i) We shall say that the stable curve X over k is *sturdy* if the inequality $g_v^{X\times_k\overline{k}} > 1$ holds for every $v \in v(\Gamma_{X\times_k\overline{k}})$.

(ii) We shall say that the stable curve X over k is *untangled* if $I_v^{X\times_k\overline{k}}$ is smooth over $\overline{k}$ for every $v \in v(\Gamma_{X\times_k\overline{k}})$.

(iii) We shall say that the stable curve X over k is *split* if the natural action of $\mathrm{Gal}(\overline{k}/k)$ on the graph $\Gamma_{X\times_k\overline{k}}$ is trivial.

Observe that one verifies easily that each of these conditions does not depend on the choice of $\overline{k}$.

Remark 1.5.1. Suppose that S is the spectrum of a field k, and that the stable curve X over k is split. Then one verifies easily that the natural morphism $X \times_k \overline{k} \to X$ determines respective bijections from $v(\Gamma_{X\times_k\overline{k}})$, $e(\Gamma_{X\times_k\overline{k}})$ to the set of irreducible components of X, the set of singular points of X.

Lemma 1.6. *The natural homomorphism* $\mathrm{Aut}_S(X) \to \mathrm{Aut}_S(J_X)$ *is injective.*

Proof. This assertion follows from [4, Theorem 1.13]. □

2. Prym-faithful Galois étale coverings

In the present section, we give the definition of the notion of a Prym-faithful Galois étale covering of a stable curve (cf. Definition 2.5 below), which will play a central role in the proof of the main result of the present paper. In the present section, let

- S be a scheme,
- $g \geq 2$ an integer,
- X a stable curve of genus g over S,
- Y a stable curve over S,
- l a prime number invertible on S, and
- $f\colon Y \to X$ a Galois étale covering of degree l over S (which thus implies that Y is of genus $g_Y \stackrel{\text{def}}{=} l(g-1)+1$), whose Galois group we denote by G.

By considering the natural (necessarily faithful — cf. Lemma 1.6) action of G on J_Y (cf. Definition 1.1) (i.e., induced by the action of G on Y), we shall regard G as a subgroup of the automorphism group of J_Y over S.

Definition 2.1. We shall write

$$h_{\text{new}} \stackrel{\text{def}}{=} l - \sum_{\tau \in G} \tau.$$

Thus, it follows from [28, Proposition-Definition 4.1], together with the discussion following [28, Proposition-Definition 4.1] in the case where we take the "N" of the discussion to be l, that there exists a unique group subscheme of J_Y

$$P_{Y/X} \subseteq J_Y$$

such that

(1) the group scheme $P_{Y/X}$ is geometrically connected over S and is an open group subscheme of the closed group subscheme $\mathrm{Ker}(l - h_{\text{new}}\colon J_Y \to J_Y) \subseteq J_Y$ (which thus implies that $P_{Y/X}$ is a semi-abelian scheme over S), and, moreover,
(2) the endomorphism h_{new} of J_Y factors as the composite of a surjective smooth homomorphism $J_Y \twoheadrightarrow P_{Y/X}$ over S with the natural inclusion $P_{Y/X} \hookrightarrow J_Y$.

We shall refer to this semi-abelian scheme $P_{Y/X}$ over S as the *generalized Prym scheme* associated to the Galois étale covering $f\colon Y \to X$.

Definition 2.2. Suppose that S is the spectrum of a field k of positive characteristic. Then we shall say that the Galois étale covering $f\colon Y \to X$ is *new-ordinary* if the generalized Prym scheme associated to the Galois étale covering $f\colon Y \to X$ (cf. Definition 2.1) is an ordinary semi-abelian variety over k (i.e., satisfies the condition that every abelian variety quotient is ordinary).

Definition 2.3. Suppose that

- the scheme S is noetherian, normal, and integral, that
- the structure morphism $Y \to S$ of Y has a splitting (i.e., a morphism $S \to Y$ over S), and that
- there exists a nonempty open subscheme $U \subseteq S$ of S such that the stable curve $X_U \stackrel{\text{def}}{=} X \times_S U$ over U, hence also the stable curve $Y_U \stackrel{\text{def}}{=} Y \times_S U$ over U, is smooth.

Write $P_{Y/X}$ for the generalized Prym scheme associated to the Galois étale covering $f\colon Y \to X$. Observe that it follows from [3, §9.2, Proposition 3] that the semi-abelian scheme $J_{Y_U} = J_Y \times_S U$ over U, hence also the semi-abelian scheme $P_{Y_U/X_U} = P_{Y/X} \times_S U$ over U, is an abelian scheme over S. Write $P^t_{Y/X}$ for the dual semi-abelian scheme of $P_{Y/X}$ over S (cf., e.g., [22, Chapitre IV, Théorème 7.1, (i)]).

(i) Suppose that the stable curve X over S, hence also the stable curve Y over S, is smooth. Then we shall refer to the polarization $P_{Y/X} \to P^t_{Y/X}$ on $P_{Y/X}$ that arises from the restriction to $P_{Y/X} \subseteq J_Y$ of the (necessarily ample) invertible sheaf on J_Y determined by the theta divisor on $\mathrm{Pic}^{g_Y-1}_{Y/S}$ and some splitting of the structure morphism $Y \to S$ as the *Prym polarization* associated to the Galois étale covering $f\colon Y \to X$. Observe that one verifies easily that this polarization does not depend on the choice of such a splitting of the structure morphism $Y \to S$.

(ii) Observe that it follows from [6, Chapter I, Proposition 2.7] that the Prym polarization $P_{Y_U/X_U} = P_{X/Y} \times_S U \to P^t_{Y_U/X_U} = P^t_{X/Y} \times_S U$ associated to the Galois étale covering $Y_U \to X_U$ determined by f extends uniquely to a homomorphism $P_{Y/X} \to P^t_{Y/X}$ over S. Moreover, observe that one also verifies easily that this extension $P_{Y/X} \to P^t_{Y/X}$ does not depend on the choice of U. We shall refer to this extension $P_{Y/X} \to P^t_{Y/X}$ as the *Prym semi-polarization* associated to the Galois étale covering $f\colon Y \to X$.

Remark 2.3.1. Suppose that we are in the situation of Definition 2.3, (i). Suppose, moreover, that S is the spectrum of a field. Then observe that one verifies easily that the equality $h^2_{\text{new}} = lh_{\text{new}}$ holds. Thus, it follows immediately from the discussion preceding [2, Theorem 5.3.2], together with condition (2) of Definition 2.1 and the primitivity of the norm-endomorphisms associated to nontrivial abelian subvarieties proved in [2, Norm-endomorphism Criterion 5.3.4], that the exponent (cf. the discussion preceding [2, Lemma 5.3.1]) in J_Y of the generalized Prym scheme $P_{Y/X}$ associated to the Galois étale covering $f\colon Y \to X$ divides l (which thus implies that the Prym polarization $P_{Y/X} \to P^t_{Y/X}$ associated to the Galois étale covering $f\colon Y \to X$ is of degree a power of l). In particular, the kernel of the Prym polarization $P_{Y/X} \to P^t_{Y/X}$ is a finite étale commutative group scheme of rank a power of l over S.

Definition 2.4. Suppose that S is the spectrum of a field k. Let R be a noetherian complete local domain whose residue field is given by k and $\mathcal{X}$ a stable curve (necessarily of genus g) over R whose special fiber $\mathcal{X} \times_R k$ is given by X. Then it follows immediately from [7, Exposé X, Théorème 2.1] that there exist

- a unique, up to isomorphism over $\mathcal{X}$, stable curve $\mathcal{Y}$ (necessarily of genus g_Y) over R and
- a unique Galois étale covering $\Phi\colon \mathcal{Y} \to \mathcal{X}$ of degree l over R

that fit into a commutative diagram of schemes over R

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "f"'] & \mathcal{Y} \arrow[d, "\Phi"] \\ X \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

— where the upper horizontal arrow is a morphism that determines an isomorphism $Y \xrightarrow{\sim} \mathcal{Y} \times_R k$ over k, and the lower horizontal arrow is the natural closed immersion. Then we shall say that the Galois étale covering $\Phi\colon \mathcal{Y} \to \mathcal{X}$ over R is the *deformation* of the Galois étale covering $f\colon Y \to X$ associated to the stable curve $\mathcal{X}$ over R.

Definition 2.5. Suppose that S is the spectrum of a separably closed field k. Then we shall say that the Galois étale covering $f\colon Y \to X$ is *Prym-faithful* if the following condition is satisfied: Let R be a complete discrete valuation ring whose residue field is given by k. For each $i \in \{1, 2\}$, let $\mathcal{X}_i$ be a stable curve (necessarily of genus g) over R that is generically smooth over R and $\iota_i\colon \mathcal{X}_i \times_R k \xrightarrow{\sim} X$ an isomorphism over S. Then if the following two conditions are satisfied, then the composite $\iota_2^{-1} \circ \iota_1\colon \mathcal{X}_1 \times_R k \xrightarrow{\sim} \mathcal{X}_2 \times_R k$ lifts to an isomorphism $\mathcal{X}_1 \xrightarrow{\sim} \mathcal{X}_2$ over R:

(1) Write
- $\mathrm{Spec}(R)^{\log}$ for the log scheme obtained by equipping $\mathrm{Spec}(R)$ with the log structure associated to the divisor with normal crossings determined by the closed point of $\mathrm{Spec}(R)$ and
- $S^{\log}$ for the log scheme obtained by equipping S with the log structure obtained by pulling back the log structure of

$\mathrm{Spec}(R)^{\log}$ by the natural surjective homomorphism $R \twoheadrightarrow k$.

Moreover, for each $i \in \{1, 2\}$, write

- $\mathcal{X}_i^{\log}$ for the log scheme over $\mathrm{Spec}(R)^{\log}$ obtained by equipping $\mathcal{X}_i$ with the log structure associated to the divisor $\mathcal{X}_i \times_R k \subseteq \mathcal{X}_i$ and
- $(\mathcal{X}_i \times_R k)^{\log}$ for the log scheme over $S^{\log}$ obtained by equipping $\mathcal{X}_i \times_R k$ with the log structure obtained by pulling back the log structure of $\mathcal{X}_i^{\log}$ by the natural closed immersion $\mathcal{X}_i \times_R k \hookrightarrow \mathcal{X}_i$.

Then the composite $\iota_2^{-1} \circ \iota_1 \colon \mathcal{X}_1 \times_R k \xrightarrow{\sim} \mathcal{X}_2 \times_R k$ induces an isomorphism $(\mathcal{X}_1 \times_R k)^{\log} \xrightarrow{\sim} (\mathcal{X}_2 \times_R k)^{\log}$ of log schemes over $S^{\log}$.

(2) For each $i \in \{1, 2\}$, write

- f_i for the Galois étale covering (necessarily of degree l) of $\mathcal{X}_i \times_R k$ obtained by pulling back $f \colon Y \to X$ by the isomorphism $\iota_i \colon \mathcal{X}_i \times_R k \xrightarrow{\sim} X$ over S and
- $\mathcal{Y}_i \to \mathcal{X}_i$ for the deformation of the Galois étale covering f_i associated to the stable curve $\mathcal{X}_i$ over R (cf. Definition 2.4).

Then the isomorphism $P_{\mathcal{Y}_1/\mathcal{X}_1} \times_R k \xrightarrow{\sim} P_{\mathcal{Y}_2/\mathcal{X}_2} \times_R k$ (cf. Definition 2.1) of semi-abelian schemes over S determined by the composite $\iota_2^{-1} \circ \iota_1 \colon \mathcal{X}_1 \times_R k \xrightarrow{\sim} \mathcal{X}_2 \times_R k$ lifts to an isomorphism $P_{\mathcal{Y}_1/\mathcal{X}_1} \xrightarrow{\sim} P_{\mathcal{Y}_2/\mathcal{X}_2}$ of semi-abelian schemes over R that is compatible with the respective Prym semi-polarizations associated to the Galois étale coverings $\mathcal{Y}_1 \to \mathcal{X}_1$, $\mathcal{Y}_2 \to \mathcal{X}_2$ (cf. Definition 2.3, (ii)).

3. A SUFFICIENT CONDITION TO BE PRYM-FAITHFUL

In the present section, we establish a sufficient condition for a Galois étale covering of a stable curve to be Prym-faithful (cf. Theorem 3.4 below). In the present section, suppose that we are in the situation at the beginning of the preceding §2. Suppose, moreover, that the scheme S is given by the spectrum of a separably closed field k of characteristic $p \geq 0$. Write

- $\Box$ for the set of prime numbers invertible in k,
- $\widehat{\mathbb{Z}}^{\Box}$ for the pro-prime-to-$\Box$ completion of $\mathbb{Z}$,
- $\mathbb{Z}_{(\Box)} \subseteq \mathbb{Q}$ for the localization of $\mathbb{Z}$ by the multiplicatively closed subset of $\mathbb{Z}$ generated by the elements of $\Box$, and

- $\mathcal{O}$ for the $\mathbb{Z}_{(\square)}$-algebra obtained by forming

$$\begin{cases} k & \text{if } p = 0, \\ W(k) & \text{if } p > 0 \end{cases}$$

— where we write $W(k)$ for the ring of Witt vectors with coefficients in k.

Definition 3.1. We shall write

- $\overline{\mathsf{M}}_g$ for the moduli stack over $\mathcal{O}$ that parametrizes stable curves of genus g over $\mathcal{O}$ (cf. [4, §5]),
- M_g for the open substack of $\overline{\mathsf{M}}_g$ that parametrizes smooth stable curves of genus g over $\mathcal{O}$,
- R_X for the completion of a strict henselization of $\overline{\mathsf{M}}_g$ at the geometric point obtained by forming the classifying morphism of the stable curve X of genus g over k,
- K_X for the field of fractions (cf. Remark 3.1.1, (i), below) of R_X (necessarily of characteristic zero),
- $\mathfrak{M} \stackrel{\text{def}}{=} \mathrm{Spec}(R_X) \to \overline{\mathsf{M}}_g$ for the natural morphism over $\mathcal{O}$,
- $\mathfrak{M}_\circ \stackrel{\text{def}}{=} \mathfrak{M} \times_{\overline{\mathsf{M}}_g} \mathsf{M}_g \to \mathsf{M}_g$ for the base-change of the natural morphism $\mathfrak{M} \to \overline{\mathsf{M}}_g$ by the natural open immersion $\mathsf{M}_g \hookrightarrow \overline{\mathsf{M}}_g$,
- $\mathfrak{M}^{\log}$ for the log scheme obtained by equipping $\mathfrak{M}$ with the log structure associated to the divisor $\mathfrak{M} \setminus \mathfrak{M}_\circ$ with normal crossings (cf. [4, Theorem 5.2]),
- $\mathfrak{X}$ for the stable curve of genus g over $\mathfrak{M}$ that corresponds to the natural morphism $\mathfrak{M} \to \overline{\mathsf{M}}_g$,
- $f_{\mathfrak{X}} \colon \mathfrak{X} \to \mathfrak{M}$ for the structure morphism of the stable curve $\mathfrak{X}$ over $\mathfrak{M}$,
- $\mathfrak{X}_\circ$ for the (necessarily smooth) stable curve of genus g over $\mathfrak{M}_\circ$ that corresponds to the morphism $\mathfrak{M}_\circ \to \mathsf{M}_g$,
- $\mathfrak{f} \colon \mathfrak{Y} \to \mathfrak{X}$ for the Galois étale covering (necessarily of degree l) obtained by forming the deformation (cf. Definition 2.4) of the Galois étale covering $f \colon Y \to X$ associated to the stable curve $\mathfrak{X}$ over $\mathfrak{M}$ (cf. Remark 3.1.1, (i), below),
- $f_{\mathfrak{Y}} \stackrel{\text{def}}{=} f_{\mathfrak{X}} \circ \mathfrak{f} \colon \mathfrak{Y} \to \mathfrak{X} \to \mathfrak{M}$ for the structure morphism of the stable curve $\mathfrak{Y}$ over $\mathfrak{M}$,
- $\mathfrak{Y}_\circ$ for the (necessarily smooth) stable curve over $\mathfrak{M}_\circ$ obtained by forming the base-change of $f_{\mathfrak{Y}} \colon \mathfrak{Y} \to \mathfrak{M}$ by the natural open immersion $\mathfrak{M}_\circ \hookrightarrow \mathfrak{M}$,
- $\mathfrak{P} \stackrel{\text{def}}{=} P_{\mathfrak{Y}/\mathfrak{X}}$, $\mathfrak{P}_\circ \stackrel{\text{def}}{=} P_{\mathfrak{Y}_\circ/\mathfrak{X}_\circ}$ (cf. Definition 2.1),
- $\mathfrak{P}^t$, $\mathfrak{P}^t_\circ$ for the respective dual semi-abelian schemes of $\mathfrak{P}$, $\mathfrak{P}_\circ$ over $\mathfrak{M}$, $\mathfrak{M}_\circ$ (cf., e.g., [22, Chapitre IV, Théorème 7.1, (i)]),

- $\mathrm{Lie}(J_{\mathfrak{X}})$, $\mathrm{Lie}(J_{\mathfrak{Y}})$, $\mathrm{Lie}(\mathfrak{P})$, $\mathrm{Lie}(\mathfrak{P}^t)$ for the R_X-modules obtained by forming the tangent spaces of the semi-abelian schemes $J_{\mathfrak{X}}$, $J_{\mathfrak{Y}}$ (cf. Definition 1.1), $\mathfrak{P}$, $\mathfrak{P}^t$ over $\mathfrak{M}$, respectively,
- $\mathrm{Lie}^{\vee}(J_{\mathfrak{X}})$, $\mathrm{Lie}^{\vee}(J_{\mathfrak{Y}})$, $\mathrm{Lie}^{\vee}(\mathfrak{P})$, $\mathrm{Lie}^{\vee}(\mathfrak{P}^t)$ for the R_X-modules obtained by forming the R_X-duals of the R_X-modules $\mathrm{Lie}(J_{\mathfrak{X}})$, $\mathrm{Lie}(J_{\mathfrak{Y}})$, $\mathrm{Lie}(\mathfrak{P})$, $\mathrm{Lie}(\mathfrak{P}^t)$, respectively,
- $\lambda_{\circ}\colon \mathfrak{P}_{\circ} \to \mathfrak{P}_{\circ}^t$ for the Prym polarization associated to the Galois étale covering $\mathfrak{Y}_{\circ} \to \mathfrak{X}_{\circ}$ determined by $\mathfrak{f}$ (cf. Definition 2.3, (i)), and
- $\lambda\colon \mathfrak{P} \to \mathfrak{P}^t$ for the Prym semi-polarization associated to the Galois étale covering $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ (cf. Definition 2.3, (ii)).

Remark 3.1.1.

(i) Since the moduli stack $\overline{\mathsf{M}}_g$ is smooth over $\mathcal{O}$ (cf. [4, Theorem 5.2]), the completion R_X is a noetherian complete local regular domain.

(ii) It follows from [3, §9.2, Proposition 3] that the semi-abelian scheme $J_{\mathfrak{Y}_{\circ}}$ over $\mathfrak{M}_{\circ}$, hence also the semi-abelian scheme $\mathfrak{P}_{\circ}$ over $\mathfrak{M}_{\circ}$, is an abelian scheme over $\mathfrak{M}_{\circ}$.

Definition 3.2. Observe that one verifies easily that there exist a subfield F of K_X, an field embedding $F \hookrightarrow \mathbb{C}$, and a polarized abelian variety (A_F, λ_F) over F such that the polarized abelian variety $(\mathfrak{P}_{\circ}, \lambda_{\circ}) \times_{R_X} K_X$ over K_X determined by $(\mathfrak{P}_{\circ}, \lambda_{\circ})$ and the natural inclusion $R_X \hookrightarrow K_X$ is isomorphic to the polarized abelian variety $(A_F, \lambda_F) \times_F K_X$ over K_X determined by (A_F, λ_F) and the natural inclusion $F \hookrightarrow K_X$. In the remainder of present Definition 3.2, we shall identify $(\mathfrak{P}_{\circ}, \lambda_{\circ}) \times_{R_X} K_X$ with $(A_F, \lambda_F) \times_F K_X$ by means of some fixed isomorphism. Then we shall write

- $(\mathfrak{P}_{\mathbb{C}}, \lambda_{\mathbb{C}})$ for the polarized abelian variety over $\mathbb{C}$ obtained by forming the base-change of (A_F, λ_F) by the natural inclusion $F \hookrightarrow \mathbb{C}$,
- $\mathsf{L} = (L, \langle \cdot, \cdot \rangle, h)$ for the PEL-type $\mathbb{Z}$-lattice (cf. [13, Definition 1.2.1.3]) — i.e., in the case where we take the "$(B, \star, \mathcal{O})$" of [13, §1.2.1] to be $(\mathbb{Q}, \mathrm{id}_{\mathbb{Q}}, \mathbb{Z})$ — that arises from the polarized Hodge structure of weight -1 associated to the polarized abelian variety $(\mathfrak{P}_{\mathbb{C}}, \lambda_{\mathbb{C}})$ over $\mathbb{C}$, and
- $\mathcal{G}$ for the affine group scheme defined in [13, Definition 1.2.1.6] — i.e., in the case where we take the "$(L, \langle \cdot, \cdot \rangle, h)$" of [13, Definition 1.2.1.6] to be L.

Observe that one verifies easily that the reflex field (cf. [13, Definition 1.2.5.4]) of $(L \otimes_{\mathbb{Z}} \mathbb{R}, \langle \cdot, \cdot \rangle, h)$ is given by $\mathbb{Q}$. Moreover, it follows from

Remark 2.3.1 that the triple $(\mathfrak{P}_\circ, \lambda_\circ, \mathbb{Z} \hookrightarrow \mathrm{End}_{\mathfrak{M}_\circ}(\mathfrak{P}_\circ))$ forms a triple as in [13, Definition 1.3.6.1] — i.e., in the case where we take the "$((L, \langle\cdot,\cdot\rangle, h), \Box)$" of [13, §1.3.6] to be $(\mathsf{L}, \Box)$. Moreover, we shall write

- $\alpha = \{\alpha_1\}$ for the level-$\mathcal{G}(\widehat{\mathbb{Z}}^{\Box})$ structure of $(\mathfrak{P}_\circ, \lambda_\circ, \mathbb{Z} \hookrightarrow \mathrm{End}_{\mathfrak{M}_\circ}(\mathfrak{P}_\circ))$ of type $(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Box}, \langle\cdot,\cdot\rangle)$ (cf. [13, Definition 1.3.7.6]) — i.e., in the case where we take the "$\mathcal{H}$" of [13, Definition 1.3.7.6] to be $\mathcal{G}(\widehat{\mathbb{Z}}^{\Box})$ — that consists of the evident identification α_1 between $L/L = \{0\}$ and $\mathfrak{P}_{\mathbb{C}}(\mathbb{C})[1] = \{0\}$.

Note that one verifies easily that, by considering the natural identifications $L/nL = \mathfrak{P}_{\mathbb{C}}(\mathbb{C})[n]$, where n ranges over the positive integers prime to p, one may conclude that the evident identification α_1 between $L/L = \{0\}$ and $\mathfrak{P}_{\mathbb{C}}(\mathbb{C})[1] = \{0\}$ forms a principal level-1 structure of $(\mathfrak{P}_\circ, \lambda_\circ, \mathbb{Z} \hookrightarrow \mathrm{End}_{\mathfrak{M}_\circ}(\mathfrak{P}_\circ))$ of type $(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Box}, \langle\cdot,\cdot\rangle)$ (cf. [13, Definition 1.3.6.2]). Thus, one verifies easily that the tuple $(\mathfrak{P}_\circ, \lambda_\circ, \mathbb{Z} \hookrightarrow \mathrm{End}_{\mathfrak{M}_\circ}(\mathfrak{P}_\circ), \alpha)$ forms an object of the category $\mathsf{M}_{\mathcal{G}(\widehat{\mathbb{Z}}^{\Box})}(\mathfrak{M}_\circ)$ defined in [13, Definition 1.4.1.4], which thus implies that the tuple $(\mathfrak{P}, \lambda, \mathbb{Z} \hookrightarrow \mathrm{End}_{\mathfrak{M}}(\mathfrak{P}), \alpha)$ forms a degenerating family of type $\mathsf{M}_{\mathcal{G}(\widehat{\mathbb{Z}}^{\Box})}$ over $\mathfrak{M}$ (cf. [13, Definition 5.3.2.1]). Let Σ be a compatible choice of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{G}(\widehat{\mathbb{Z}}^{\Box})}$ (cf. [13, Definition 6.3.3.4], [13, Proposition 6.3.3.5]). Then we shall write

- $\mathsf{M}^{\mathrm{tor}}_{\mathcal{G}(\widehat{\mathbb{Z}}^{\Box}),\Sigma}$ for the proper smooth algebraic stack over $\mathbb{Z}_{(\Box)}$ discussed in [13, Theorem 6.4.1.1].

Now observe that one verifies immediately (cf. also Remark 3.1.1, (i); [1, Chapter II, Corollary 4.9, (i)]) that one may replace Σ by a suitable compatible choice of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{G}(\widehat{\mathbb{Z}}^{\Box})}$ so that the degenerating family $(\mathfrak{P}, \lambda, \mathbb{Z} \hookrightarrow \mathrm{End}_{\mathfrak{M}}(\mathfrak{P}), \alpha)$ of type $\mathsf{M}_{\mathcal{G}(\widehat{\mathbb{Z}}^{\Box})}$ over $\mathfrak{M}$ satisfies the condition discussed in [13, Theorem 6.4.1.1.6]), which thus implies that we have a classifying morphism $\mathfrak{M} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{G}(\widehat{\mathbb{Z}}^{\Box}),\Sigma} \times_{\mathbb{Z}_{(\Box)}} \mathcal{O}$ of the degenerating family $(\mathfrak{P}, \lambda, \mathbb{Z} \hookrightarrow \mathrm{End}_{\mathfrak{M}}(\mathfrak{P}), \alpha)$ of type $\mathsf{M}_{\mathcal{G}(\widehat{\mathbb{Z}}^{\Box})}$ over $\mathfrak{M}$. Then we shall write

- R_P for the completion of a strict henselization of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{G}(\widehat{\mathbb{Z}}^{\Box}),\Sigma} \times_{\mathbb{Z}_{(\Box)}} \mathcal{O}$ at the geometric point obtained by forming the composite of the natural closed immersion $\mathrm{Spec}(k) \hookrightarrow \mathfrak{M}$ and the classifying morphism $\mathfrak{M} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{G}(\widehat{\mathbb{Z}}^{\Box}),\Sigma} \times_{\mathbb{Z}_{(\Box)}} \mathcal{O}$ of the degenerating family $(\mathfrak{P}, \lambda, \mathbb{Z} \hookrightarrow \mathrm{End}_{\mathfrak{M}}(\mathfrak{P}), \alpha)$ of type $\mathsf{M}_{\mathcal{G}(\widehat{\mathbb{Z}}^{\Box})}$ over $\mathfrak{M}$,
- $\mathfrak{N} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(R_P) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{G}(\widehat{\mathbb{Z}}^{\Box}),\Sigma} \times_{\mathbb{Z}_{(\Box)}} \mathcal{O}$ for the natural morphism over $\mathcal{O}$,

- $\mathfrak{N}_\circ \stackrel{\text{def}}{=} \mathfrak{N} \times_{(\mathsf{M}^{\text{tor}}_{\mathcal{G}(\widehat{\mathbb{Z}}^\square),\Sigma} \times_{\mathbb{Z}_{(\square)}} \mathcal{O})} (\mathsf{M}_{\mathcal{G}(\widehat{\mathbb{Z}}^\square)} \times_{\mathbb{Z}_{(\square)}} \mathcal{O})$ for the base-change of the natural morphism $\mathfrak{N} \to \mathsf{M}^{\text{tor}}_{\mathcal{G}(\widehat{\mathbb{Z}}^\square),\Sigma} \times_{\mathbb{Z}_{(\square)}} \mathcal{O}$ by the natural open immersion $\mathsf{M}_{\mathcal{G}(\widehat{\mathbb{Z}}^\square)} \times_{\mathbb{Z}_{(\square)}} \mathcal{O} \hookrightarrow \mathsf{M}^{\text{tor}}_{\mathcal{G}(\widehat{\mathbb{Z}}^\square),\Sigma} \times_{\mathbb{Z}_{(\square)}} \mathcal{O}$ (cf. the statement of [13, Theorem 6.4.1.1]),
- $\mathfrak{N}^{\log}$ for the log scheme obtained by equipping $\mathfrak{N}$ with the log structure associated to the divisor $\mathfrak{N} \setminus \mathfrak{N}_\circ$ with normal crossings relative to $\mathcal{O}$ (cf. [13, Theorem 6.4.1.1.3]),
- $\mathfrak{t}\colon \mathfrak{M} \to \mathfrak{N}$ for the morphism over $\mathcal{O}$ induced by the classifying morphism $\mathfrak{M} \to \mathsf{M}^{\text{tor}}_{\mathcal{G}(\widehat{\mathbb{Z}}^\square),\Sigma} \times_{\mathbb{Z}_{(\square)}} \mathcal{O}$ of the degenerating family $(\mathfrak{P}, \lambda, \mathbb{Z} \hookrightarrow \mathrm{End}_{\mathfrak{M}}(\mathfrak{P}), \alpha)$ of type $\mathsf{M}_{\mathcal{G}(\widehat{\mathbb{Z}}^\square)}$ over $\mathfrak{M}$, and
- $\mathfrak{t}^{\log}\colon \mathfrak{M}^{\log} \to \mathfrak{N}^{\log}$ for the morphism of log schemes determined by the morphism $\mathfrak{t}\colon \mathfrak{M} \to \mathfrak{N}$ (cf. Remark 3.1.1, (ii)).

Lemma 3.3. *Let $\mathcal{L}$ be an invertible sheaf on $\mathfrak{X}$ that is of order l and trivialized by the Galois étale covering $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$. By considering the natural (necessarily faithful — cf. Lemma 1.6) action of G on* $\mathrm{Lie}(J_{\mathfrak{Y}})$ *(i.e., induced by the action of G on $\mathfrak{Y}$), we shall regard G as a subgroup of the automorphism group of the R_X-module* $\mathrm{Lie}(J_{\mathfrak{Y}})$*. Write*

$$h_{\text{new}} \stackrel{\text{def}}{=} l - \sum_{\tau \in G} \tau, \qquad h_0 \stackrel{\text{def}}{=} l^{-1} \cdot \sum_{\tau \in G} \tau$$

(cf. Definition 2.1). In the remainder of present Lemma 3.3, we shall identify $\mathrm{Lie}(J_{\mathfrak{Y}})$ *with $\bigoplus_{i=0}^{l-1} H^1(\mathfrak{X}, \mathcal{L}^{\otimes i})$ by means of the composite*

$$\mathrm{Lie}(J_{\mathfrak{Y}}) \xrightarrow{\sim} H^1(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \xrightarrow{\sim} H^1(\mathfrak{X}, \mathfrak{f}_*\mathcal{O}_{\mathfrak{Y}})$$

$$\xrightarrow{\sim} H^1\Big(\mathfrak{X}, \bigoplus_{i=0}^{l-1} \mathcal{L}^{\otimes i}\Big) \xrightarrow{\sim} \bigoplus_{i=0}^{l-1} H^1(\mathfrak{X}, \mathcal{L}^{\otimes i})$$

— where the first arrow is the isomorphism of [3, §8.4, Theorem 1, (a)]*, the second arrow is the isomorphism determined by the Galois étale covering $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$, the third arrow is the isomorphism induced by a trivialization of $\mathfrak{f}^*\mathcal{L}$, and the fourth arrow is the natural isomorphism. Then the following assertions hold:*

(i) *The diagram of R_X-modules*

$$\begin{array}{ccc} \mathrm{Lie}(J_{\mathfrak{Y}}) & \xrightarrow{h_0} & \mathrm{Lie}(J_{\mathfrak{Y}}) \\ \downarrow & & \uparrow \\ \mathrm{Lie}(J_{\mathfrak{X}}) & \xrightarrow[\sim]{} & H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \end{array}$$

— *where the lower horizontal arrow is the isomorphism of* [3, §8.4, Theorem 1, (a)]*, the left-hand vertical arrow is the homomorphism induced by the Galois étale covering* $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$*, and the right-hand vertical arrow is the natural inclusion — commutes.*

(ii) *The factorization of* h_{new}

$$\mathrm{Lie}(J_{\mathfrak{Y}}) \twoheadrightarrow \mathrm{Lie}(\mathfrak{P}) \hookrightarrow \mathrm{Lie}(J_{\mathfrak{Y}})$$

(cf. condition (2) of Definition 2.1) determines an isomorphism of R_X*-modules*

$$\mathrm{Lie}(\mathfrak{P}) \xrightarrow{\sim} \bigoplus_{i=1}^{l-1} H^1(\mathfrak{X}, \mathcal{L}^{\otimes i}).$$

(iii) *The diagram of* R_X*-modules*

$$\begin{tikzcd}
\mathrm{Lie}^{\vee}(\mathfrak{P}) \otimes_{R_X} \mathrm{Lie}^{\vee}(\mathfrak{P}) \arrow[r, two heads] \arrow[d, "\wr"] & \Gamma(\mathfrak{M}, \mathfrak{t}^*\Omega^1_{\mathfrak{N}^{\log}/\mathcal{O}}) \arrow[ddd] \\
\bigoplus_{i,j=1}^{l-1} \Big(\Gamma(\mathfrak{X}, \omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}^{\otimes i}) \otimes_{R_X} \Gamma(\mathfrak{X}, \omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}^{\otimes j})\Big) \arrow[d, two heads] & \\
\bigoplus_{i=1}^{l-1} \Big(\Gamma(\mathfrak{X}, \omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}^{\otimes i}) \otimes_{R_X} \Gamma(\mathfrak{X}, \omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}^{\otimes (l-i)})\Big) \arrow[d] & \\
\Gamma(\mathfrak{X}, \omega_{\mathfrak{X}}^{\otimes 2}) \arrow[r, "\sim"'] & \Gamma(\mathfrak{M}, \Omega^1_{\mathfrak{M}^{\log}/\mathcal{O}})
\end{tikzcd}$$

— *where*

- *the upper horizontal arrow is the homomorphism determined by the extended Kodaira-Spencer map discussed in* [13, Theorem 6.4.1.1.4] *(cf. also* [13, Definition 6.3.1]*) and the isomorphism* $\mathrm{Lie}(\mathfrak{P}) \xrightarrow{\sim} \mathrm{Lie}(\mathfrak{P}^t)$ *induced by the homomorphism* $\lambda\colon \mathfrak{P} \to \mathfrak{P}^t$ *(cf. Remark 2.3.1),*
- *the lower horizontal arrow is the isomorphism that arises from the Kodaira-Spencer homomorphism with respect to the stable curve* $\mathfrak{X} \to \mathfrak{M}$*,*
- *the left-hand upper vertical arrow is the isomorphism determined by the isomorphism of (ii),*

- *the left-hand middle vertical arrow is the natural projection homomorphism,*
- *the left-hand lower vertical arrow is the natural homomorphism determined by a trivialization of $\mathcal{L}^{\otimes l}$, and*
- *the right-hand vertical arrow is the homomorphism induced by the morphism $\mathfrak{t}^{\log}\colon \mathfrak{M}^{\log} \to \mathfrak{N}^{\log}$*

— commutes up to multiplication by an element of $R_X^\times$.

(iv) *Suppose that the natural homomorphism determined by a trivialization of $\mathcal{L}|_X^{\otimes l}$*

$$\Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}|_X) \otimes_k \Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}|_X^{\otimes(l-1)}) \longrightarrow \Gamma(X, \omega_X^{\otimes 2})$$

is surjective. Then the morphism $\mathfrak{t}^{\log}\colon \mathfrak{M}^{\log} \to \mathfrak{N}^{\log}$ is log formally unramified (cf. [23, Chapter IV, Definition 3.1.1]*).*

Proof. Assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows from assertion (i), together with the equality $1 = h_0 + l^{-1} \cdot h_{\mathrm{new}}$. Next, we verify assertion (iii). Let us first observe that one verifies immediately that every R_X-modules that appear in the diagram of assertion (iii) are free R_X-modules of finite rank. In particular, to verify assertion (iii), it suffices to verify that the diagram of K_X-vector spaces obtained by applying "$(-) \otimes_{R_X} K_X$" to the diagram of assertion (iii) commutes up to multiplication by an element of $R_X^\times$. On the other hand, since the stable curve $\mathfrak{X} \times_{R_X} K_X$ over K_X is smooth, this commutativity follows immediately from a similar argument to the argument that was applied in the proofs of [24, Theorem 2.6] and [28, Theorem 4.6]. This completes the proof of assertion (iii).

Finally, we verify assertion (iv). Let us first observe that it follows from [23, Chapter IV, Proposition 2.3.1] and [23, Chapter IV, Proposition 3.1.3] that, to verify assertion (iv), it suffices to verify that the right-hand vertical arrow of the diagram of assertion (iii) is surjective. In particular, since (one verifies immediately that) every R_X-modules that appear in the diagram of assertion (iii) are free R_X-modules of finite rank, it follows from assertion (iii) that, to verify assertion (iv), it suffices to verify that the homomorphism of k-vector spaces obtained by applying "$(-) \otimes_{R_X} k$" to the left-hand lower vertical arrow of the diagram of assertion (iii) is surjective. On the other hand, this surjectivity follows from our assumption (i.e., that appears in the statement of assertion (iv)). This completes the proof of assertion (iv), hence also of Lemma 3.3. ☐

The main result of the present section is as follows.

Theorem 3.4. *Let*

- *k be a separably closed field,*
- *X, Y stable curves over k,*
- *l a prime number invertible in k,*
- *$f\colon Y \to X$ a Galois étale covering of degree l over k, and*
- *$\mathcal{L}$ an invertible sheaf on X that is of order l and trivialized by the Galois étale covering $f\colon Y \to X$.*

Suppose that the natural homomorphism determined by a trivialization of $\mathcal{L}^{\otimes l}$

$$\Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \otimes_k \Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes(l-1)}) \longrightarrow \Gamma(X, \omega_X^{\otimes 2})$$

is surjective. Then the Galois étale covering $f\colon Y \to X$ over k is Prym-faithful (cf. Definition 2.5).

Proof. Let n be a positive integer and R a complete discrete valuation ring whose residue field is given by k. Write $\mathfrak{m} \subseteq R$ for the maximal ideal of R and ${}_nR \stackrel{\mathrm{def}}{=} R/\mathfrak{m}^n$. For each $i \in \{1, 2\}$, let $\mathcal{X}_i$ be a stable curve over R that is generically smooth over R and $\iota_i\colon \mathcal{X}_i \times_R k \xrightarrow{\sim} X$ an isomorphism over k. Suppose that conditions (1), (2) of Definition 2.5 are satisfied. For each $i \in \{1, 2\}$, write, moreover,

- $\mathrm{Spec}({}_nR)^{\log}$ for the log scheme obtained by equipping $\mathrm{Spec}({}_nR)$ with the log structure obtained by pulling back the log structure of $\mathrm{Spec}(R)^{\log}$ (cf. condition (1) of Definition 2.5) by the natural surjective homomorphism $R \twoheadrightarrow {}_nR$,
- $s_i\colon \mathrm{Spec}(R) \to \mathfrak{M}$ for the classifying morphism of the stable curve $\mathcal{X}_i$ over R,
- ${}_ns_i\colon \mathrm{Spec}({}_nR) \to \mathfrak{M}$ for the composite of the natural closed immersion $\mathrm{Spec}({}_nR) \hookrightarrow \mathrm{Spec}(R)$ with s_i,
- $t_i \stackrel{\mathrm{def}}{=} \mathfrak{t} \circ s_i\colon \mathrm{Spec}(R) \to \mathfrak{M} \to \mathfrak{N}$ for the composite of s_i and $\mathfrak{t}$, and
- ${}_nt_i\colon \mathrm{Spec}({}_nR) \to \mathfrak{N}$ for the composite of the natural closed immersion $\mathrm{Spec}({}_nR) \hookrightarrow \mathrm{Spec}(R)$ with t_i.

Now observe that since (we have assumed that) $\mathcal{X}_1$, $\mathcal{X}_2$ are generically smooth over R, the morphisms s_1, $s_2\colon \mathrm{Spec}(R) \to \mathfrak{M}$ uniquely determine morphisms $s_1^{\log}$, $s_2^{\log}\colon \mathrm{Spec}(R)^{\log} \to \mathfrak{M}^{\log}$ of log schemes, respectively. Write $t_1^{\log} \stackrel{\mathrm{def}}{=} \mathfrak{t}^{\log} \circ s_1^{\log}$, $t_2^{\log} \stackrel{\mathrm{def}}{=} \mathfrak{t}^{\log} \circ s_2^{\log}\colon \mathrm{Spec}(R)^{\log} \to \mathfrak{M}^{\log} \to \mathfrak{N}^{\log}$. Write, moreover,

$${}_ns_1^{\log},\ {}_ns_2^{\log}\colon \mathrm{Spec}({}_nR)^{\log} \longrightarrow \mathfrak{M}^{\log},\ {}_nt_1^{\log},\ {}_nt_2^{\log}\colon \mathrm{Spec}({}_nR)^{\log} \longrightarrow \mathfrak{N}^{\log}$$

for the respective composites of the natural strict closed immersion $\mathrm{Spec}({}_nR)^{\log} \hookrightarrow \mathrm{Spec}(R)^{\log}$ with $s_1^{\log}$, $s_2^{\log}$, $t_1^{\log}$, $t_2^{\log}$. Then observe that it follows immediately from condition (1) of Definition 2.5, together with [11, Theorem 4.1], that the equality

$$ {}_1s_1^{\log} = {}_1s_2^{\log} $$

holds. Also, observe that it follows immediately from condition (2) of Definition 2.5 (cf. also the uniqueness discussed in [13, Theorem 6.4.1.1.6]) that the equality $t_1 = t_2$, hence also the equality $t_1^{\log} = t_2^{\log}$, holds, which thus implies the equality

$$ {}_nt_1^{\log} = {}_nt_2^{\log}. $$

In particular, by considering the commutative diagram of log schemes over $\mathrm{Spec}(R)^{\log}$

$$\begin{tikzcd}
\mathrm{Spec}(R/\mathfrak{m})^{\log} \arrow[r, hook] \arrow[d, "{}_1s_1^{\log}={}_1s_2^{\log}"'] & \mathrm{Spec}(R/\mathfrak{m}^n)^{\log} \arrow[dl, shift right, "{}_ns_1^{\log}"'] \arrow[dl, shift left, "{}_ns_2^{\log}"] \arrow[d, "{}_nt_1^{\log}={}_nt_2^{\log}"] \\
\mathfrak{M}^{\log} \arrow[r, "\mathfrak{t}^{\log}"'] & \mathfrak{N}^{\log},
\end{tikzcd}$$

we conclude from Lemma 3.3, (iv), that the equality ${}_ns_1^{\log} = {}_ns_2^{\log}$ holds. In particular, it follows formally that the equality $s_1^{\log} = s_2^{\log}$, hence also the equality $s_1 = s_2$, holds, as desired. This completes the proof of Theorem 3.4. $\square$

4. Existence of Prym-faithful new-ordinary coverings

In the present section, we prove the existence of a Prym-faithful new-ordinary Galois étale covering of a suitable stable curve over a separably closed field of positive characteristic (cf. Theorem 4.5 below). In the present section, let

- k be a separably closed field,
- X a stable curve over k that is sturdy (cf. Definition 1.5, (i)) and untangled (cf. Definition 1.5, (ii)), and
- $\mathcal{L}$ an invertible sheaf on X.

Write

$$ \mathcal{L}^\vee \overset{\mathrm{def}}{=} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X). $$

Also, for each vertex $v \in v(\Gamma_X)$ of Γ_X, write

$$\mathcal{L}_v \stackrel{\text{def}}{=} \mathcal{L}|_{X_v}, \qquad \mathcal{L}_v^\vee \stackrel{\text{def}}{=} \mathcal{H}om_{\mathcal{O}_{X_v}}(\mathcal{L}_v, \mathcal{O}_{X_v}).$$

Definition 4.1. Let d be a positive integer. Then we shall write

$$\Gamma^{\geq d}(X, \mathcal{L}) \subseteq \Gamma(X, \mathcal{L})$$

for the subspace obtained by forming the pull-back of the subspace

$$\bigoplus_{v \in v(\Gamma_X)} \Gamma\big(X_v, \mathcal{L}_v(-dD_v)\big) \subseteq \bigoplus_{v \in v(\Gamma_X)} \Gamma(X_v, \mathcal{L}_v)$$

by the natural homomorphism

$$\Gamma(X, \mathcal{L}) \longrightarrow \bigoplus_{v \in v(\Gamma_X)} \Gamma(X_v, \mathcal{L}_v).$$

Lemma 4.2. *Let d be a positive integer. Then the natural homomorphism*

$$\Gamma(X, \mathcal{L}) \longrightarrow \bigoplus_{v \in v(\Gamma_X)} \Gamma(X_v, \mathcal{L}_v)$$

restricts to an isomorphism of subspaces

$$\Gamma^{\geq d}(X, \mathcal{L}) \xrightarrow{\sim} \bigoplus_{v \in v(\Gamma_X)} \Gamma\big(X_v, \mathcal{L}_v(-dD_v)\big).$$

Proof. This assertion is immediate. ☐

Lemma 4.3. *Suppose that the invertible sheaf $\mathcal{L}_v$ on X_v is nontrivial and of nonnegative degree for each $v \in v(\Gamma_X)$. Then the following assertions hold:*

(i) *The natural homomorphism*

$$\Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \longrightarrow \bigoplus_{x \in e(\Gamma_X)} \omega_X \otimes_k \mathcal{L} \otimes_k k(x)$$

is surjective.

(ii) *The k-vector space $\Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})/\Gamma^{\geq 1}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})$ is of dimension $\# e(\Gamma_X)$.*

(iii) *Suppose that $\Gamma(X_v, \mathcal{L}_v^\vee(D_v)) = \{0\}$ for each $v \in v(\Gamma_X)$. (For example, this will be the case if the inequality $\deg(\mathcal{L}_v) > \deg(D_v)$ holds for each $v \in v(\Gamma_X)$.) Then the k-vector space $\Gamma^{\geq 1}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})/\Gamma^{\geq 2}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})$ is of dimension $2 \cdot \# e(\Gamma_X)$.*

Proof. First, we verify assertion (i). Write $\pi_X \colon \widetilde{X} \to X$ for the normalization of X and $\Omega_{\widetilde{X}}$ for the sheaf of relative differentials of $\widetilde{X}$ over k. Then it follows immediately from the existence of the exact sequence of invertible sheaves on X (cf. [14, Chapter 10, Lemma 3.12, (a)])

$$0 \longrightarrow (\pi_X)_*\Omega_{\widetilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L} \longrightarrow \omega_X \otimes_{\mathcal{O}_X} \mathcal{L} \longrightarrow \bigoplus_{x \in e(\Gamma_X)} \omega_X \otimes_k \mathcal{L} \otimes_k k(x) \longrightarrow 0$$

that, to verify assertion (i), it suffices to verify that $\Gamma(X_v, \mathcal{L}_v^\vee) = \{0\}$ for each $v \in v(\Gamma_X)$. On the other hand, this assertion follows from our assumption that the invertible sheaf $\mathcal{L}_v$ on X_v is nontrivial and of nonnegative degree for each $v \in v(\Gamma_X)$. This completes the proof of assertion (i). Assertion (ii) is a formal consequence of assertion (i).

Finally, we verify assertion (iii). Observe that it follows from Lemma 4.2 (cf. also Remark 1.4.1) that the k-vector space $\Gamma^{\geq 1}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})/\Gamma^{\geq 2}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})$ is isomorphic to the k-vector space

$$\Big(\bigoplus_{v \in v(\Gamma_X)} \Gamma(X_v, \Omega_{X_v} \otimes_{\mathcal{O}_{X_v}} \mathcal{L}_v)\Big) / \Big(\bigoplus_{v \in v(\Gamma_X)} \Gamma\big(X_v, \Omega_{X_v} \otimes_{\mathcal{O}_{X_v}} \mathcal{L}_v(-D_v)\big)\Big).$$

On the other hand, for each $v \in v(\Gamma_X)$, since (we have assumed that) the invertible sheaf $\mathcal{L}_v$ on X_v is nontrivial and of nonnegative degree, it follows that $\dim_k(H^1(X_v, \Omega_{X_v} \otimes_{\mathcal{O}_{X_v}} \mathcal{L}_v)) = \dim_k(\Gamma(X_v, \mathcal{L}_v^\vee)) = 0$, which thus implies that

$$\dim_k\big(\Gamma(X_v, \Omega_{X_v} \otimes_{\mathcal{O}_{X_v}} \mathcal{L}_v)\big) = g_v - 1 + \deg(\mathcal{L}_v).$$

Moreover, for each $v \in v(\Gamma_X)$, since (we have assumed that) $0 = \dim_k(\Gamma(X_v, \mathcal{L}_v^\vee(D_v))) = \dim_k(H^1(X_v, \Omega_{X_v} \otimes_{\mathcal{O}_{X_v}} \mathcal{L}_v(-D_v)))$, it follows that

$$\dim_k\Big(\Gamma\big(X_v, \Omega_{X_v} \otimes_{\mathcal{O}_{X_v}} \mathcal{L}_v(-D_v)\big)\Big) = g_v - 1 + \deg(\mathcal{L}_v) - \deg(D_v).$$

Thus, assertion (iii) follows from the easily verified equality

$$2 \cdot \#e(\Gamma_X) = \sum_{v \in v(\Gamma_X)} \deg(D_v).$$

This completes the proof of assertion (iii), hence also Lemma 4.3. ☐

Lemma 4.4. *Suppose that, for each vertex $v \in v(\Gamma_X)$ of Γ_X, the following three conditions are satisfied:*

(1) *The natural map*

$$\mu_{X_v, \mathcal{L}_v} \colon \Gamma(X_v, \Omega_{X_v} \otimes_{\mathcal{O}_{X_v}} \mathcal{L}_v) \otimes_k \Gamma(X_v, \Omega_{X_v} \otimes_{\mathcal{O}_{X_v}} \mathcal{L}_v^\vee) \longrightarrow \Gamma(X_v, \Omega_{X_v}^{\otimes 2})$$

is surjective.

(2) *The invertible sheaf* $\mathcal{L}_v$ *on* X_v *is nontrivial and of degree zero.*

(3) *The equality* $\Gamma(X_v, \mathcal{L}_v^{\vee}(D_v)) = \{0\}$ *holds.*

Then the natural map

$$\mu_{X,\mathcal{L}} \colon \Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \otimes_k \Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\vee}) \longrightarrow \Gamma(X, \omega_X^{\otimes 2})$$

is surjective.

Proof. Let us first observe that, for each $v \in v(\Gamma_X)$, we have a commutative diagram of k-vector spaces

$$\begin{tikzcd}
\Gamma^{\geq 1}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \otimes_k \Gamma^{\geq 1}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\vee}) \arrow[r, "\mu_{X,\mathcal{L}}"] \arrow[d] & \Gamma^{\geq 2}(X, \omega_X^{\otimes 2}) \arrow[d] \\
\Gamma(X_v, \Omega_{X_v} \otimes_{\mathcal{O}_{X_v}} \mathcal{L}_v) \otimes_k \Gamma(X_v, \Omega_{X_v} \otimes_{\mathcal{O}_{X_v}} \mathcal{L}_v) \arrow[r, "\mu_{X_v,\mathcal{L}_v}"'] & \Gamma(X_v, \Omega_{X_v}^{\otimes 2})
\end{tikzcd}$$

— where the vertical arrows are the homomorphism induced by the natural closed immersion $X_v \hookrightarrow X$ (cf. also Remark 1.4.1 and Lemma 4.2). Thus, it follows immediately from Lemma 4.2, together with condition (1), that the homomorphism

$$\Gamma^{\geq 1}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \otimes_k \Gamma^{\geq 1}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\vee}) \longrightarrow \Gamma^{\geq 2}(X, \omega_X^{\otimes 2})$$

determined by the homomorphism $\mu_{X,\mathcal{L}}$ is surjective. In particular, to verify Lemma 4.4, it suffices to verify that the composite

$$\Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \otimes_k \Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\vee}) \xrightarrow{\mu_{X,\mathcal{L}}} \Gamma(X, \omega_X^{\otimes 2})$$

$$\twoheadrightarrow \Gamma(X, \omega_X^{\otimes 2}) / \Gamma^{\geq 2}(X, \omega_X^{\otimes 2})$$

— where the second arrow is the natural surjective homomorphism — is surjective.

Next, observe that it follows from Lemma 4.3, (i), together with condition (2), that there exists an element $a \in \Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\vee})$ whose image in $\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\vee} \otimes_k k(x)$ is nonzero for each $x \in e(\Gamma_X)$. Write ϕ_a for the composite

$$\Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \longrightarrow \Gamma(X, \omega_X^{\otimes 2}) \twoheadrightarrow \Gamma(X, \omega_X^{\otimes 2}) / \Gamma^{\geq 2}(X, \omega_X^{\otimes 2})$$

— where the first arrow is the homomorphism given by mapping $s \mapsto \mu_{X,\mathcal{L}}(s \otimes a)$, and the second arrows is the natural surjective homomorphism. Observe that it follows immediately from our choice of $a \in \Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\vee})$ that the equality $\mathrm{Ker}(\phi_a) = \Gamma^{\geq 2}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})$

holds. In particular, the homomorphism ϕ_a determines an injective homomorphism

$$\Gamma(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})/\Gamma^{\geq 2}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \hookrightarrow \Gamma(X, \omega_X^{\otimes 2})/\Gamma^{\geq 2}(X, \omega_X^{\otimes 2}).$$

On the other hand, it follows from Lemma 4.3, (ii), (iii), together with conditions (2), (3) (cf. also Remark 1.4.1), that both the domain and the codomain of this injective homomorphism are of dimension $3 \cdot \# e(\Gamma_X)$. In particular, this injective homomorphism is an isomorphism, which thus implies that the composite under consideration is surjective, as desired. This completes the proof of Lemma 4.4. ☐

The main result of the present section is as follows.

Theorem 4.5. *Let*

- *k be a separably closed field and*
- *X a stable curve over k that is untangled (cf. Definition 1.5, (ii)).*

Suppose that, for each vertex $v \in v(\Gamma_X)$ of Γ_X, the following two conditions are satisfied:

(a) *The inequality* $\deg(D_v) < g_v$ *holds, which thus implies that the stable curve X over k is sturdy (cf. Definition 1.5, (i)).*
(b) *The smooth proper curve X_v over k is of gonality* ≥ 5, *i.e., every finite morphism from X_v onto the projective line over k is of degree* ≥ 5.

Then there exist

- *a stable curve Y over k,*
- *a prime number l invertible in k, and*
- *a Galois étale covering $f \colon Y \to X$ of degree l over k*

such that the following three conditions are satisfied:

(1) *For each vertex $v \in v(\Gamma_X)$ of Γ_X, the inverse image $f^{-1}X_v \subseteq Y$ is irreducible.*
(2) *The Galois étale covering $f \colon Y \to X$ is Prym-faithful (cf. Definition 2.5).*
(3) *If, moreover, the field k is of positive characteristic, then the Galois étale covering $f \colon Y \to X$ is new-ordinary (cf. Definition 2.2).*

Proof. Let $v \in v(\Gamma_X)$ be a vertex of Γ_X. Let us first observe that it follows immediately from condition (a) (cf., e.g., [28, Lemma 1.2, (ii)] and [28, Remark 1.7]) that

($\dagger_1$) there exists a nonempty open subscheme $U \subseteq J_v$ of J_v such that, for each $j \in U(k)$, the invertible sheaf $\mathcal{E}_v$ on X_v of degree zero that corresponds to $j \in U(k)$ satisfies the following condition: The equality $\Gamma(X_v, \mathcal{E}_v^\vee(D_v)) = \{0\}$ holds.

Thus, it follows from ($\dagger_1$), together with [28, Lemma 3.9], that

($\dagger_2$) there exists a positive integer c such that, for each prime number l invertible in k, the set that consists of isomorphism classes of invertible sheaves $\mathcal{E}_v$ on X_v of order l that satisfy the following condition is of cardinality $> l^{2g_v-2}(l^2-c)$: The equality $\Gamma(X_v, \mathcal{E}_v^\vee(D_v)) = \{0\}$ holds.

In particular, we conclude immediately from ($\dagger_2$), [28, Corollary 3.10] (cf. also condition (b)), and [28, Corollary 5.3] (cf. also [25, Théorème 4.3.1]) that

($\dagger_3$) there exists a prime number l_v such that, for each prime number $l \geq l_v$, there exists an invertible sheaf $\mathcal{E}_v$ on X_v of order l that satisfies the following three conditions:
- The natural map
$$\Gamma(X_v, \Omega_{X_v} \otimes_{\mathcal{O}_{X_v}} \mathcal{E}_v) \otimes_k \Gamma(X_v, \Omega_{X_v} \otimes_{\mathcal{O}_{X_v}} \mathcal{E}_v^\vee) \longrightarrow \Gamma(X_v, \Omega_{X_v}^{\otimes 2})$$
is surjective.
- The equality $\Gamma(X_v, \mathcal{E}_v^\vee(D_v)) = \{0\}$ holds.
- If the field k is of positive characteristic, then the Galois étale covering of X_v of degree l that trivializes $\mathcal{E}_v$ is new-ordinary.

Thus, it follows from ($\dagger_3$) (cf. also [3, §9.2, Example 8]) that there exist a prime number l and an invertible sheaf $\mathcal{E}$ on X of degree l such that, for each vertex $v \in v(\Gamma_X)$ of Γ_X, the restriction of $\mathcal{E}$ to $X_v \subseteq X$ satisfies the three conditions that appear in ($\dagger_3$). Then it follows from Theorem 3.4 and Lemma 4.4 (cf. also [3, §9.2, Example 8]) that the Galois étale covering of X of degree l that trivializes $\mathcal{E}$ satisfies conditions (1), (2), (3) in the statement of Theorem 4.5. This completes the proof of Theorem 4.5. □

5. Cyclotomes associated to hyperbolic curves

In the present section, we introduce some notational conventions related to the notion of a hyperbolic curve and the notion of a cyclotome. Moreover, we prove some basic facts concerning these notions. In the present section, let

- R be a complete discrete valuation ring whose field of fractions we denote by K, and whose residue field we denote by k,

- $\overline{K}$ a separable closure of K, and
- Z a hyperbolic curve over K.

Suppose that

- the field k is perfect and of characteristic $p > 0$.

Definition 5.1. We shall write

- $K^{\mathrm{ur}} \subseteq K^{\mathrm{tm}} \subseteq \overline{K}$ for the maximal unramified, tamely ramified extension fields of K in $\overline{K}$, respectively,
- $\overline{k}$ for the algebraic closure of k obtained by forming the residue field of the normalization of R in K^{ur},
- $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K) \twoheadrightarrow G_K^{\mathrm{tm}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(K^{\mathrm{tm}}/K) \twoheadrightarrow G_K^{\mathrm{ur}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(K^{\mathrm{ur}}/K) \stackrel{\sim}{\to} \mathrm{Gal}(\overline{k}/k)$, and
- $P_K \stackrel{\mathrm{def}}{=} \mathrm{Ker}(G_K \twoheadrightarrow G_K^{\mathrm{tm}}) \subseteq I_K \stackrel{\mathrm{def}}{=} \mathrm{Ker}(G_K \twoheadrightarrow G_K^{\mathrm{ur}})$ for the wild inertia, inertia subgroups of G_K, respectively.

Definition 5.2. We shall write

- $\widehat{\mathbb{Z}}_\times$ for the profinite (respectively, pro-prime-to-char(K)) completion of the module $\mathbb{Z}$ whenever $\mathrm{char}(K) = 0$ (respectively, $\mathrm{char}(K) \neq 0$) and
- $\Lambda_{\overline{K}}$ for the Tate module of the multiplicative group scheme $\mathbb{G}_{m,K}$ over K.

For a topological module M, we shall write

- $M^D \stackrel{\mathrm{def}}{=} \mathrm{Hom}(M, \Lambda_{\overline{K}})$ for the topological module of continuous homomorphisms $M \to \Lambda_{\overline{K}}$.

Remark 5.2.1. One verifies easily that the module $\Lambda_{\overline{K}}$ has a natural structure of free $\widehat{\mathbb{Z}}_\times$-module of rank one. In particular, if M is a free $\widehat{\mathbb{Z}}_\times$-module of finite rank, then we have a natural identification $M = (M^D)^D$.

Definition 5.3. We shall write

- Z^+ for the smooth compactification of the hyperbolic curve Z over K,
- $\overline{Z} \stackrel{\mathrm{def}}{=} Z \times_K \overline{K} \subseteq \overline{Z}^+ \stackrel{\mathrm{def}}{=} Z^+ \times_K \overline{K}$,
- Π_Z, Δ_Z for the respective tame fundamental groups of $(Z^+, Z^+ \setminus Z)$, $(\overline{Z}^+, \overline{Z}^+ \setminus \overline{Z})$ (i.e., the respective fundamental groups associated to the Galois categories of finite flat coverings of Z^+, $\overline{Z}^+$ that are at most tamely ramified along $Z^+ \setminus Z$, $\overline{Z}^+ \setminus \overline{Z}$), relative to suitable choices of basepoints, and
- Π_{Z^+}, Δ_{Z^+} for the respective étale fundamental groups of Z^+, $\overline{Z}^+$, relative to suitable choices of basepoints.

Thus, the natural morphisms $Z \hookrightarrow Z^+ \to \mathrm{Spec}(K)$ determine a commutative diagram of topological groups

$$\begin{tikzcd}
1 \arrow[r] & \Delta_Z \arrow[r] \arrow[d] & \Pi_Z \arrow[r] \arrow[d] & G_K \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Delta_{Z^+} \arrow[r] & \Pi_{Z^+} \arrow[r] & G_K \arrow[r] & 1
\end{tikzcd}$$

— where the horizontal sequences are exact, and the vertical arrows are surjective.

Remark 5.3.1. Let $W^+ \to Z^+$ be a connected finite flat covering of Z^+ that is at most tamely ramified along $Z^+ \setminus Z$. Write $W \subseteq W^+$ for the open subscheme of W^+ obtained by forming the inverse image of the open subscheme $Z \subseteq Z^+$. Then one verifies easily from the various definitions involved that the scheme W is a hyperbolic curve over the algebraic closure of K in the function field of W.

Definition 5.4. If the smooth proper curve Z^+ over K is of genus ≥ 1, then we shall write $\Lambda_Z \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\widehat{\mathbb{Z}}}(H^2(\Delta_{Z^+}, \widehat{\mathbb{Z}}_\times), \widehat{\mathbb{Z}}_\times)$.

Remark 5.4.1. If the smooth proper curve Z^+ over K is of genus ≥ 1, then it is well-known (cf., e.g., [16, Chapter VI, Theorem 11.1, (a)]) that

(i) the module Λ_Z has a natural structure of free $\widehat{\mathbb{Z}}_\times$-module of rank one, and that
(ii) there exists a natural G_K-equivariant isomorphism $\Lambda_Z \xrightarrow{\sim} \Lambda_{\overline{K}}$.

Definition 5.5. Let Z_1, Z_2 be hyperbolic curves over K and $\phi_{\Pi_Z} : \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) a continuous isomorphism over G_K. Then we shall say that the isomorphism ϕ_{Π_Z} is *compactification-compatible* if the following condition is satisfied: Let $H_1 \subseteq \Pi_{Z_1}$ be an open subgroup of Π_{Z_1}. Write $H_2 \stackrel{\mathrm{def}}{=} \phi_{\Pi_Z}(H_1) \subseteq \Pi_{Z_2}$. For each $i \in \{1, 2\}$, write $W_i^+ \to Z_i^+$ for the finite flat covering of Z_i^+ that corresponds to the open subgroup $H_i \subseteq \Pi_{Z_i}$, which thus implies that we have an identification $H_i = \Pi_{W_i}$ (cf. Definition 5.3), where we write W_i for the open subscheme of W_i^+ obtained by forming the inverse image of $Z_i \subseteq Z_i^+$ (cf. Remark 5.3.1). Then the continuous isomorphism $\Pi_{W_1} \xrightarrow{\sim} \Pi_{W_2}$ determined by the isomorphism ϕ_{Π_Z} fits into a commutative diagram of topological groups

$$\begin{tikzcd}
\Pi_{W_1} \arrow[r, two heads] \arrow[d, "\wr"'] & \Pi_{W_1^+} \arrow[d, "\wr"] \\
\Pi_{W_2} \arrow[r, two heads] & \Pi_{W_2^+}
\end{tikzcd}$$

(cf. Definition 5.3) — where the horizontal arrows are the continuous outer surjective homomorphisms that arise from the open immersions $W_1 \hookrightarrow W_1^+$, $W_2 \hookrightarrow W_2^+$, respectively, and the right-hand vertical arrow is a continuous isomorphism.

Lemma 5.6. *Let Z_1, Z_2 be hyperbolic curves over K and $\phi_{\Pi_Z} : \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) a continuous isomorphism over G_K. Suppose that the l-adic cyclotomic character $G_K \to \mathbb{Z}_l^\times$ on G_K for some prime number l invertible in K is an open homomorphism. (For example, this will be the case if either the field k is finite, or the field K is of characteristic zero.) Then the isomorphism ϕ_{Π_Z} is compactification-compatible.*

Proof. This assertion follows from [21, Corollary 2.7, (i)]. □

Definition 5.7. Let Z_1, Z_2 be hyperbolic curves over K and $\phi_{\Pi_Z} : \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) a continuous isomorphism over G_K. Then we shall say that the isomorphism ϕ_{Π_Z} is *cyclotomically compatible* if the isomorphism ϕ_{Π_Z} is compactification-compatible, and, moreover, the following condition is satisfied: Let $H_1 \subseteq \Pi_{Z_1}$ be an open subgroup of Π_{Z_1}. Write $H_2 \stackrel{\text{def}}{=} \phi_{\Pi_Z}(H_1) \subseteq \Pi_{Z_2}$. For each $i \in \{1, 2\}$, write $W_i^+ \to Z_i^+$ for the finite flat covering of Z_i^+ that corresponds to the open subgroup $H_i \subseteq \Pi_{Z_i}$, which thus implies that we have an identification $H_i = \Pi_{W_i}$ (cf. Definition 5.3), where we write W_i for the open subscheme of W_i^+ obtained by forming the inverse image of $Z_i \subseteq Z_i^+$ (cf. Remark 5.3.1). Suppose that the smooth proper curve W_1^+ is of genus ≥ 1. Then the smooth proper curve W_2^+ is of genus ≥ 1. Moreover, the diagram of topological modules

$$\begin{tikzcd} \Lambda_{W_1} \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \Lambda_{W_2} \arrow[dl, "\sim"] \\ & \Lambda_{\overline{K}} & \end{tikzcd}$$

(cf. Definition 5.4) — where the horizontal arrow is the isomorphism induced by the right-hand vertical arrow of the diagram of Definition 5.5, and the two diagonal arrows are the respective natural isomorphisms of Remark 5.4.1, (ii) — commutes.

Definition 5.8. Let C be a smooth curve over K.

(i) We shall say that the smooth curve C over K is *isotrivial* if there exist a smooth curve C_0 over the separable closure $\overline{K}_0$ in $\overline{K}$ of the minimal subfield of K and an isomorphism $C \times_K \overline{K} \xrightarrow{\sim} C_0 \times_{\overline{K}_0} \overline{K}$ over $\overline{K}$.

(ii) We shall say that the smooth curve C over K has a *good model* (respectively, *nonsmooth stable model*; *nonsmooth sturdy stable model*) if there exist a stable curve $\mathcal{C}$ over R and an isomorphism $\mathcal{C} \times_R K \xrightarrow{\sim} C$ over K such that the stable curve $\mathcal{C} \times_R k$ over k is smooth (respectively, not smooth; sturdy and not smooth) over k.

Remark 5.8.1. Let C be a smooth curve over K. Suppose that C has either a good model or a nonsmooth stable model. Then it is immediate that the smooth curve C is a proper hyperbolic curve over K.

Lemma 5.9. *The following assertions hold:*

(i) *For each integer g_0, there exists a connected finite flat covering $W^+ \to Z^+$ of Z^+ that is at most tamely ramified along $Z^+ \setminus Z$ such that the smooth proper curve W^+ is of genus $\geq g_0$.*

(ii) *Suppose that the smooth curve Z (respectively, the smooth proper curve Z^+) over K is nonisotrivial. Let $W^+ \to Z^+$ be a connected finite flat covering of Z^+ that is at most tamely ramified along $Z^+ \setminus Z$. Write W for the open subscheme of W^+ obtained by forming the inverse image of $Z \subseteq Z^+$. Then the smooth curve W (respectively, the smooth proper curve W^+) is nonisotrivial.*

(iii) *Suppose that the smooth proper curve Z^+ over K has a nonsmooth stable model. Let $W^+ \to Z^+$ be a connected finite flat covering of Z^+ that is at most tamely ramified along $Z^+ \setminus Z$. Then the smooth proper curve W^+ does not have any good model.*

(iv) *Suppose that the smooth curve Z over K is nonisotrivial. Then there exists a connected finite flat covering $W^+ \to Z^+$ of Z^+ that is at most tamely ramified along $Z^+ \setminus Z$ such that the smooth proper curve W^+ is nonisotrivial.*

Proof. Assertion (i) follows immediately from the well-known Riemann-Hurwitz formula, together with the well-known structure of the maximal pro-l quotient of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic $\neq l$ (cf., e.g., [26, Proposition 1.1, (i), (ii)]). Assertion (ii) follows immediately from [27, Lemma 1.32]. Assertion (iii) follows immediately from [19, Corollary 7.4].

Next, we verify assertion (iv). Let us first observe that it follows from assertions (i), (ii) that, to verify assertion (iv), we may assume without loss of generality, by replacing Z by the inverse image of $Z \subseteq Z^+$ by a suitable finite flat covering of Z^+, that the smooth proper curve Z^+ is of genus ≥ 2. Let $W^+ \to Z^+$ be a Galois finite flat covering of Z^+ that is at most tamely ramified along $Z^+ \setminus Z$ whose branch locus is given by $Z^+ \setminus Z$. (Observe that it follow immediately from the well-known structure of the maximal pro-l quotient of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic $\neq l$ — cf., e.g., [26, Proposition 1.1, (i), (ii)] — that such a covering always exists.) Write $W \subseteq W^+$ for the open subscheme of W^+ obtained by forming the inverse image of $Z \subseteq Z^+$ and $L \subseteq \overline{K}$ for the algebraic closure of K in the function field of W.

Assume that the smooth proper curve W^+ over L is isotrivial, i.e., that there exist a smooth curve W_0^+ over the separable closure $\overline{L}_0$ in $\overline{K}$ of the minimal subfield of L and an isomorphism $W^+ \times_L \overline{K} \xrightarrow{\sim} W_0^+ \times_{\overline{L}_0} \overline{K}$ over $\overline{K}$. Then observe that it follows immediately from [4, Theorem 1.11] that the natural action of $\mathrm{Gal}(W^+ \times_L \overline{K}/Z^+ \times_K \overline{K})$ on $W^+ \times_L \overline{K}$ over $\overline{K}$ descends uniquely, relative to a fixed isomorphism $W^+ \times_L \overline{K} \xrightarrow{\sim} W_0^+ \times_{\overline{L}_0} \overline{K}$ as above, to an action on W_0^+ over $\overline{L}_0$. Write Z_0^+ for the quotient of the resulting action of $\mathrm{Gal}(W^+ \times_L \overline{K}/Z^+ \times_K \overline{K})$ on W_0^+ and $Z_0 \subseteq Z_0^+$ for the étale locus of the natural (necessarily finite flat) morphism $W_0^+ \to Z_0^+$. Then one verifies immediately from the various definitions involved that there exists an isomorphism $Z \times_K \overline{K} \xrightarrow{\sim} Z_0 \times_{\overline{L}_0} \overline{K}$ over $\overline{K}$. In particular, the smooth curve Z over K is isotrivial, in contradiction to our assumption that the smooth curve Z over K is nonisotrivial. In particular, the smooth proper curve W^+ over L is nonisotrivial. This completes the proof of assertion (iv), hence also of Lemma 5.9. □

6. The log fundamental groups of log special fibers

In the present section, we discuss some fundamental facts concerning the log fundamental groups of the log special fibers of stable curves over complete discrete valuation rings. In the present section, suppose that we are in the situation at the beginning of the preceding §5. Moreover, let

- $\mathcal{X}$ be a stable curve over R.

Suppose that

- the structure morphism $\mathcal{X} \to \mathrm{Spec}(R)$ is generically smooth, i.e., that the generic fiber $X \stackrel{\mathrm{def}}{=} \mathcal{X} \times_R K$ is smooth over K.

Definition 6.1. We shall write

- $\overline{X} \stackrel{\mathrm{def}}{=} \mathcal{X} \times_R \overline{K}$, $\underline{\mathcal{X}} \stackrel{\mathrm{def}}{=} \mathcal{X} \times_R k$, $\overline{\underline{\mathcal{X}}} \stackrel{\mathrm{def}}{=} \mathcal{X} \times_R \overline{k}$,

- $\mathcal{X}^{\log}$ for the log scheme obtained by equipping $\mathcal{X}$ with the log structure associated to the divisor $\underline{\mathcal{X}} \subseteq \mathcal{X}$,
- $\underline{\mathcal{X}}^{\log}$ for the log scheme obtained by equipping $\underline{\mathcal{X}}$ with the log structure obtained by pulling back the log structure of $\mathcal{X}^{\log}$ by the natural closed immersion $\underline{\mathcal{X}} \hookrightarrow \mathcal{X}$,
- $\mathrm{Spec}(R)^{\log}$ for the log scheme obtained by equipping $\mathrm{Spec}(R)$ with the log structure associated to the divisor with normal crossings determined by the closed point of $\mathrm{Spec}(R)$, and
- $\mathrm{Spec}(k)^{\log}$ for the log scheme obtained by equipping $\mathrm{Spec}(k)$ with the log structure obtained by pulling back the log structure of $\mathrm{Spec}(R)^{\log}$ by the natural surjective homomorphism $R \twoheadrightarrow k$.

Lemma 6.2. *The natural immersions $X \hookrightarrow \mathcal{X} \hookleftarrow \underline{\mathcal{X}}$ determine a sequence of finite (cf.* [4, Theorem 1.11]*) groups*

$$\mathrm{Aut}_K(X) \xleftarrow{\sim} \mathrm{Aut}_R(\mathcal{X}) \hookrightarrow \mathrm{Aut}_k(\underline{\mathcal{X}})$$

— where the first arrow is an isomorphism, and the second arrow is injective.

Proof. The bijectivity of the first arrow is immediate. The injectivity of the second arrow follows from [4, Theorem 1.11]. □

Definition 6.3. We shall write

- Π_X, Δ_X for the respective étale fundamental groups of X, $\overline{X}$, relative to suitable choices of basepoints,
- $\Pi_{\underline{\mathcal{X}}}^{\log}$ for the log fundamental group of $\underline{\mathcal{X}}^{\log}$, relative to a suitable choice of basepoint, and
- $\Delta_{\underline{\mathcal{X}}}^{\mathrm{adm}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{\underline{\mathcal{X}}}^{\log} \twoheadrightarrow G_K^{\mathrm{tm}})$.

Thus, the natural commutative diagram of schemes

$$\begin{tikzcd}
\overline{X} \arrow[r] & X \arrow[r] \arrow[d, hook] & \mathrm{Spec}(K) \arrow[d, hook] \\
& \mathcal{X} \arrow[r] & \mathrm{Spec}(R) \\
\overline{\underline{\mathcal{X}}} \arrow[r] & \underline{\mathcal{X}} \arrow[r] \arrow[u, hook] & \mathrm{Spec}(k) \arrow[u, hook]
\end{tikzcd}$$

determines (cf. also Remark 6.3.1 below) a commutative diagram of topological groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & G_K & \longrightarrow & 1 \\ & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\ 1 & \longrightarrow & \Delta^{\mathrm{adm}}_{\underline{\mathcal{X}}} & \longrightarrow & \Pi^{\log}_{\underline{\mathcal{X}}} & \longrightarrow & G^{\mathrm{tm}}_K & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are exact, and the vertical arrows are surjective. We shall write

- $\mathrm{sp}_{\mathcal{X}} \colon \Pi_X \twoheadrightarrow \Pi^{\log}_{\underline{\mathcal{X}}}$ for the middle vertical continuous surjective homomorphism of this diagram.

We shall refer to this homomorphism $\mathrm{sp}_{\mathcal{X}} \colon \Pi_X \twoheadrightarrow \Pi^{\log}_{\underline{\mathcal{X}}}$ as the *specialization homomorphism* associated to the stable curve $\mathcal{X}$ over R.

Remark 6.3.1. Let us recall from [8, Corollary 1] and [8, Proposition B.7] that the natural open immersion $X \hookrightarrow \mathcal{X}$ induces a continuous outer isomorphism of the tame fundamental group of $(\mathcal{X}, \underline{\mathcal{X}})$ (i.e., the fundamental group associated to the Galois category of finite flat coverings of $\mathcal{X}$ that are at most tamely ramified along $\underline{\mathcal{X}}$) with $\Pi^{\log}_{\underline{\mathcal{X}}}$.

Lemma 6.4. *Suppose that one of the following two conditions is satisfied:*

(1) *The field K is of characteristic zero.*
(2) *The smooth proper curve X over K is nonisotrivial, and the field k is algebraic over a finite field.*

Then there exists a Galois étale covering $Y \to X$ of X such that if one writes $L \subseteq \overline{K}$ for the algebraic closure of K in the function field of Y, then the smooth proper curve Y over L has a nonsmooth stable model.

Proof. Let us first observe that it is immediate that, to verify Lemma 6.4, we may assume without loss of generality that $\underline{\mathcal{X}}$ is smooth over k. Next, observe that it follows immediately from [26, Lemma 5.5] (cf. also Remark 6.3.1 and [4, Corollary 2.7]) that, to verify Lemma 6.4, it suffices to verify that the continuous surjective homomorphism $\Delta_X \twoheadrightarrow \Delta^{\mathrm{adm}}_{\underline{\mathcal{X}}}$ determined by the specialization homomorphism $\mathrm{sp}_{\mathcal{X}} \colon \Pi_X \twoheadrightarrow \Pi^{\log}_{\underline{\mathcal{X}}}$ associated to the stable curve $\mathcal{X}$ over R is not an isomorphism. If condition (1) is satisfied, then this assertion follows from [26, Proposition 1.1, (i), (ii)]. If condition (2) is satisfied, then this assertion follows from [28, Theorem 0.3]. This completes the proof of Lemma 6.4. $\square$

Lemma 6.5. *Let $H \subseteq \Pi_X$ be an open subgroup of Π_X. Write $X_H \to X$ for the finite étale covering of X that corresponds to H, $K_H \subseteq \overline{K}$ for the finite extension field of K that corresponds to the image of H in G_K,* *and* $R_H \subseteq K_H$ *for the normalization of* R *in* K_H*. Then the following assertions hold:*

(i) *Suppose that the kernel of the specialization homomorphism* $\mathrm{sp}_{\mathcal{X}}\colon \Pi_X \twoheadrightarrow \Pi^{\log}_{\underline{\mathcal{X}}}$ *associated to the stable curve* $\mathcal{X}$ *over* R *is contained in* $H \subseteq \Pi_X$*. Then the following four conditions are satisfied:*

  (1) *The inclusion* $K_H \subseteq K^{\mathrm{tm}}$ *holds.*

  (2) *There exist a stable curve* $\mathcal{X}_H$ *over* R_H *and an isomorphism* $\mathcal{X}_H \times_{R_H} K_H \xrightarrow{\sim} X_H$ *over* K_H*. In the remainder of the present (i), we shall identify* $\mathcal{X}_H \times_{R_H} K_H$ *with* X_H *by means of such a fixed isomorphism* $\mathcal{X}_H \times_{R_H} K_H \xrightarrow{\sim} X_H$*. In particular, it follows from* [19, Corollary 7.4] *that the finite étale covering* $X_H \to X$ *extends uniquely to a proper surjective morphism* $\mathcal{X}_H \to \mathcal{X}$*.*

  (3) *The resulting morphism* $\mathcal{X}_H \to \mathcal{X}$ *of (2) is finite.*

  (4) *Let* $v \in v(\Gamma_{\overline{\underline{\mathcal{X}}}})$ *(cf. Definition 1.2) be a vertex of* $\Gamma_{\overline{\underline{\mathcal{X}}}}$ *and* $w \in v(\Gamma_{\overline{\underline{\mathcal{X}}}_H})$ *(cf. Definition 1.2, Definition 6.1) a vertex of* $\Gamma_{\overline{\underline{\mathcal{X}}}_H}$ *that lies over* $v \in v(\Gamma_{\overline{\underline{\mathcal{X}}}})$*. Then the induced morphism* $(\overline{\underline{\mathcal{X}}}_H)_w \to \overline{\underline{\mathcal{X}}}_v$ *(cf. Definition 1.3) restricts to a finite étale covering* $U^{\overline{\underline{\mathcal{X}}}_H}_w \to U^{\overline{\underline{\mathcal{X}}}}_v$ *(cf. Definition 1.3) and is at most tamely ramified along* $D^{\overline{\underline{\mathcal{X}}}}_v \subseteq \overline{\underline{\mathcal{X}}}_v$ *(cf. Definition 1.3).*

(ii) *Suppose, moreover, that the open subgroup* $H \subseteq \Pi_X$ *of* Π_X *is normal. Then the kernel of the specialization homomorphism* $\mathrm{sp}_{\mathcal{X}}\colon \Pi_X \twoheadrightarrow \Pi^{\log}_{\underline{\mathcal{X}}}$ *associated to the stable curve* $\mathcal{X}$ *over* R *is contained in* $H \subseteq \Pi_X$ *if and only if (1), (2), (3) of (i) are satisfied, and, moreover, the following condition is satisfied:*

  (5) *In the situation of (2) of (i), for each vertex* $v \in v(\Gamma_{\overline{\underline{\mathcal{X}}}_H})$ *of* $\Gamma_{\overline{\underline{\mathcal{X}}}_H}$*, the subgroup of the Galois group* $\mathrm{Gal}(\overline{X}_H/\overline{X})$ *(cf. Definition 6.1) that consists of elements that stabilize the closed subscheme* $I^{\overline{\underline{\mathcal{X}}}_H}_v \subseteq \overline{\underline{\mathcal{X}}}_H$ *(cf. Definition 1.3) and induce the identity automorphism of the function field of* $I^{\overline{\underline{\mathcal{X}}}_H}_v$ *is of order prime to* p*.*

Proof. First, we verify assertion (i). The assertion that conditions (1), (2), (3) are satisfied is a formal consequence of Remark 6.3.1. Moreover, the assertion that condition (4) is satisfied is well-known (cf., e.g., [29, Proposition 2.2, (ii)]). This completes the proof of assertion (i). Assertion (ii) follows immediately from Abhyankar's lemma (cf., e.g., [7, Exposé XIII, Proposition 5.5]) and the Zariski-Nagata purity theorem (cf., e.g., [7, Exposé X, Théorème 3.1]), together with Remark 6.3.1. This completes the proof of Lemma 6.5. $\square$

Definition 6.6. Let l be a prime number. Then we shall write

- $\Delta_X^{l\text{-ab}}$ for the topological abelianization of the maximal pro-l quotient of Δ_X.

For each vertex $v \in v(\Gamma_{\underline{\overline{\mathcal{X}}}})$ of $\Gamma_{\underline{\overline{\mathcal{X}}}}$, we shall write

- $\Delta_{U_v^{\underline{\overline{\mathcal{X}}}}}^{l\text{-ab}}$, $\Delta_{\underline{\overline{\mathcal{X}}}_v}^{l\text{-ab}}$ for the respective topological abelianizations of the maximal pro-l quotients of the étale fundamental groups of the smooth curves $U_v^{\underline{\overline{\mathcal{X}}}}$, $\underline{\overline{\mathcal{X}}}_v$ (cf. Definition 1.3) over $\overline{k}$, relative to suitable choices of basepoints, and
- $D_v^{\Delta} \subseteq \Delta_X^{l\text{-ab}}$ for the decomposition subgroup of $\Delta_X^{l\text{-ab}}$ associated to $v \in v(\Gamma_{\underline{\overline{\mathcal{X}}}})$.

For each edge $e \in e(\Gamma_{\underline{\overline{\mathcal{X}}}})$ (cf. Definition 1.2) of $\Gamma_{\underline{\overline{\mathcal{X}}}}$, we shall write

- $D_e^{\Delta} \subseteq \Delta_X^{l\text{-ab}}$ for the decomposition subgroup of $\Delta_X^{l\text{-ab}}$ associated to $e \in e(\Gamma_{\underline{\overline{\mathcal{X}}}})$.

Lemma 6.7. *Let l be a prime number invertible in R. Then the following assertions hold:*

(i) *Let $v \in v(\Gamma_{\underline{\overline{\mathcal{X}}}})$ be a vertex of $\Gamma_{\underline{\overline{\mathcal{X}}}}$. Then the kernel of the natural continuous surjective homomorphism $\Delta_{U_v^{\underline{\overline{\mathcal{X}}}}}^{l\text{-ab}} \twoheadrightarrow D_v^{\Delta}$ induced by the natural open immersion $U_v^{\underline{\overline{\mathcal{X}}}} \hookrightarrow \underline{\overline{\mathcal{X}}}$ (cf. condition (4) of Lemma 6.5, (i)) is contained in the kernel of the continuous (necessarily surjective) homomorphism $\Delta_{U_v^{\underline{\overline{\mathcal{X}}}}}^{l\text{-ab}} \twoheadrightarrow \Delta_{\underline{\overline{\mathcal{X}}}_v}^{l\text{-ab}}$ induced by the natural open immersion $U_v^{\underline{\overline{\mathcal{X}}}} \hookrightarrow \underline{\overline{\mathcal{X}}}_v$, i.e.,*

$$\mathrm{Ker}(\Delta_{U_v^{\underline{\overline{\mathcal{X}}}}}^{l\text{-ab}} \twoheadrightarrow D_v^{\Delta}) \subseteq \mathrm{Ker}(\Delta_{U_v^{\underline{\overline{\mathcal{X}}}}}^{l\text{-ab}} \twoheadrightarrow \Delta_{\underline{\overline{\mathcal{X}}}_v}^{l\text{-ab}}).$$

(ii) *Suppose that the stable curve $\underline{\overline{\mathcal{X}}}$ over $\overline{k}$ is sturdy. Let v, $w \in v(\Gamma_{\underline{\overline{\mathcal{X}}}})$ be vertices of $\Gamma_{\underline{\overline{\mathcal{X}}}}$. Then the equality $v = w$ holds if and only if the equality $D_v^{\Delta} = D_w^{\Delta}$ holds.*

(iii) *Suppose that the graph $\Gamma_{\underline{\overline{\mathcal{X}}}}$ is 2-connected (i.e., that, for each vertex of $\Gamma_{\underline{\overline{\mathcal{X}}}}$, the subgraph of $\Gamma_{\underline{\overline{\mathcal{X}}}}$ obtained by removing the vertex from $\Gamma_{\underline{\overline{\mathcal{X}}}}$ is connected). Let $v \in v(\Gamma_{\underline{\overline{\mathcal{X}}}})$ be a vertex of $\Gamma_{\underline{\overline{\mathcal{X}}}}$. Then the natural continuous surjective homomorphism $\Delta_{U_v^{\underline{\overline{\mathcal{X}}}}}^{l\text{-ab}} \otimes_{\mathbb{Z}_l} \mathbb{F}_l \twoheadrightarrow D_v^{\Delta} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$ induced by the natural open immersion $U_v^{\underline{\overline{\mathcal{X}}}} \hookrightarrow \underline{\overline{\mathcal{X}}}$ (cf. condition (4) of Lemma 6.5, (i)) is an isomorphism.*

Proof. Assertions (i), (ii) follow immediately from [9, Lemma 1.4]. Assertion (iii) follows from [31, Corollary 3.5] (cf. also [29, Proposition 3.4]). $\square$

Lemma 6.8. *Let $Y \to X$ be a finite étale covering of X. Write $K_Y \subseteq \overline{K}$ for the algebraic closure of K in the function field of Y and $R_Y \subseteq K_Y$ for the normalization of R in K_Y. Suppose that the open subgroup $\Pi_Y \subseteq \Pi_X$ of Π_X that corresponds to the finite étale covering $Y \to X$ of X contains the kernel of the specialization homomorphism $\mathrm{sp}_{\mathcal{X}} \colon \Pi_X \twoheadrightarrow \Pi^{\log}_{\underline{\mathcal{X}}}$ associated to the stable curve $\mathcal{X}$ over R. In particular, it follows from condition (2) of Lemma 6.5, (i), that there exist a stable curve $\mathcal{Y}$ over R_Y and an isomorphism $\mathcal{Y} \times_{R_Y} K_Y \xrightarrow{\sim} Y$ over K_Y. In the remainder of the present Lemma 6.8, we shall identify $\mathcal{Y} \times_{R_Y} K_Y$ with Y by means of such a fixed isomorphism $\mathcal{Y} \times_{R_Y} K_Y \xrightarrow{\sim} Y$. Then the following assertions hold:*

(i) *Suppose that the stable curve $\underline{\mathcal{X}}$ over k is sturdy. Then the stable curve $\underline{\mathcal{Y}}$ (cf. Definition 6.1) over k is sturdy.*
(ii) *Suppose that the stable curve $\underline{\mathcal{X}}$ over k untangled. Then the stable curve $\underline{\mathcal{Y}}$ over k is untangled.*
(iii) *Let $v \in v(\Gamma_{\overline{\mathcal{X}}})$ be a vertex of $\Gamma_{\overline{\mathcal{X}}}$ and $w \in v(\Gamma_{\overline{\mathcal{Y}}})$ (cf. Definition 1.2, Definition 6.1) a vertex of $\Gamma_{\overline{\mathcal{Y}}}$ that lies over $v \in v(\Gamma_{\overline{\mathcal{X}}})$. Suppose that the inequality $\deg(D^{\overline{\mathcal{X}}}_v) < g^{\overline{\mathcal{X}}}_v$ (cf. Definition 1.3) holds. Then the inequality $\deg(D^{\overline{\mathcal{Y}}}_w) < g^{\overline{\mathcal{Y}}}_w$ (cf. Definition 1.3) holds.*

Proof. Assertions (i), (ii) are immediate. Assertion (iii) follows immediately from the well-known Riemann-Hurwitz formula, together with condition (4) of Lemma 6.5, (i). □

Definition 6.9.

(i) We shall say that a subgroup H of a group G is *subnormal* if there exist a positive integer n and a sequence $H = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_{n-1} \subseteq H_n = G$ of subgroups of G such that, for each $i \in \{1, \ldots, n\}$, the subgroup H_{i-1} is normal in H_i.
(ii) We shall define a *sub-Galois étale covering* of a scheme to be a (necessarily connected) finite étale covering of the scheme obtained by forming the composite of finitely many Galois étale coverings.

Lemma 6.10. *There exists a sub-Galois étale covering $Y \to X$ of X such that if one writes $K_Y \subseteq \overline{K}$ for the algebraic closure of K in the function field of Y and $R_Y \subseteq K_Y$ for the normalization of R in K_Y, then the following four conditions are satisfied:*

(1) *The open subgroup $\Pi_Y \subseteq \Pi_X$ of Π_X that corresponds to the finite étale covering $Y \to X$ of X contains the kernel of the specialization homomorphism $\mathrm{sp}_{\mathcal{X}} \colon \Pi_X \twoheadrightarrow \Pi^{\log}_{\underline{\mathcal{X}}}$ associated to*

the stable curve $\mathcal{X}$ over R. In particular, it follows from condition (2) of Lemma 6.5, (i), that there exist a stable curve $\mathcal{Y}$ over R_Y and an isomorphism $\mathcal{Y} \times_{R_Y} K_Y \xrightarrow{\sim} Y$ over K_Y. In the remainder of the present Lemma 6.10, we shall identify $\mathcal{Y} \times_{R_Y} K_Y$ with Y by means of such a fixed isomorphism $\mathcal{Y} \times_{R_Y} K_Y \xrightarrow{\sim} Y$.

(2) *The stable curve $\underline{\mathcal{Y}}$ (cf. Definition 6.1) over k is untangled.*

(3) *For each vertex $v \in v(\Gamma_{\overline{\mathcal{Y}}})$ (cf. Definition 1.2, Definition 6.1) of $\Gamma_{\overline{\mathcal{Y}}}$, the inequality $\deg(D_v^{\overline{\mathcal{Y}}}) < g_v^{\overline{\mathcal{Y}}}$ (cf. Definition 1.3) holds (which thus implies that the stable curve $\underline{\mathcal{Y}}$ over k is sturdy).*

(4) *For each vertex $v \in v(\Gamma_{\overline{\mathcal{Y}}})$ of $\Gamma_{\overline{\mathcal{Y}}}$, the smooth proper curve $\overline{\underline{\mathcal{Y}}}_v$ (cf. Definition 1.3) over $\overline{k}$ is of gonality ≥ 5, i.e., every finite morphism from $\overline{\underline{\mathcal{Y}}}_v$ onto the projective line over $\overline{k}$ is of degree ≥ 5.*

Proof. Let us first observe that it follows immediately from [21, Remark 1.1.5] and [32, Lemma 3.2], together with Lemma 6.8, (i), (ii), that, to verify Lemma 6.10, we may assume without loss of generality, by replacing X by a suitable sub-Galois étale covering of X, that

(a) the stable curve $\underline{\mathcal{X}}$ over k is sturdy and untangled, that,

(b) for each vertex of $\Gamma_{\overline{\mathcal{X}}}$, there exist at least two edges of $\Gamma_{\overline{\mathcal{X}}}$ that abut to the vertex, and that

(c) the graph $\Gamma_{\overline{\mathcal{X}}}$ is 2-connected.

Let us fix a vertex $v_0 \in v(\Gamma_{\overline{\mathcal{X}}})$ of $\Gamma_{\overline{\mathcal{X}}}$. Let l_0 be an odd prime number invertible in R. Then it follows immediately from (a), (b), (c), and Lemma 6.7, (iii), together with the well-known structure of the maximal pro-l quotient of the admissible fundamental group of a stable curve over an algebraically closed field of characteristic $\neq l$ (cf., e.g., [26, Proposition 1.1, (i), (ii)], [9, Lemma 1.4]), that there exists a continuous surjective homomorphism $\Delta_X^{l_0\text{-ab}} \twoheadrightarrow \mathbb{F}_{l_0}$ such that,

- for each $v \in v(\Gamma_{\overline{\mathcal{X}}})$, the image by this continuous surjective homomorphism of $D_v^{\Delta} \subseteq \Delta_X^{l_0\text{-ab}}$ is nontrivial, and, moreover,
- for each $e \in e(\Gamma_{\overline{\mathcal{X}}})$ that abuts to v_0, the image by this continuous surjective homomorphism of $D_e^{\Delta} \subseteq \Delta_X^{l_0\text{-ab}}$ is nontrivial.

In particular, one concludes from Lemma 6.8, (ii), and the well-known Riemann-Hurwitz formula, together with condition (4) of Lemma 6.5, (i) (cf. also our assumption that $l_0 > 2$, and that the stable curve $\underline{\mathcal{X}}$ over k is sturdy), that there exists a Galois étale covering ${}^{v_0}Y \to X$ of X such that conditions (1), (2) of the statement of the present

Lemma 6.10 are satisfied, and, moreover, a similar inequality to the inequality "$\deg(D_v^{\overline{\mathcal{Y}}}) < g_v^{\overline{\mathcal{Y}}}$" for each vertex of $\Gamma_{\underline{\overline{\mathcal{Y}}}}$ that lies over $v_0 \in v(\Gamma_{\underline{\overline{\mathcal{X}}}})$ is satisfied. Thus, by considering a connected component of the fiber product of these "${}^{v_0}Y$" over X — where v_0 ranges over the vertices of $\Gamma_{\underline{\overline{\mathcal{X}}}}$ — one verifies immediately from Lemma 6.8, (ii), (iii), that there exists a sub-Galois étale covering of X such that conditions (1), (2), (3) of the statement of the present Lemma 6.10 are satisfied. In particular, it follows immediately from [28, Theorem 0.7] and [28, Proposition 0.8, (i), (ii)], together with Lemma 6.8, (ii), (iii) (cf. also [9, Lemma 1.4]), that there exists a sub-Galois étale covering of X such that conditions (1), (2), (3), (4) of the statement of the present Lemma 6.10 are satisfied, as desired. This completes the proof of Lemma 6.10. $\square$

7. Reconstruction of log special fibers

In the present section, we show how the log special fiber of a stable curve can be recovered from the étale fundamental group (cf. Lemma 7.3, (iv), below). In the present section, suppose that we are in the situation at the beginning of the preceding §6.

Lemma 7.1. *Suppose that the equality $k = \overline{k}$ holds, and that the stable curve $\underline{\mathcal{X}}$ over k is sturdy. Let l be a prime number invertible in R and $H \subseteq \Pi_X$ a normal open subgroup of Π_X. Write $X_H \to X$ for the Galois étale covering that corresponds to H, $K_H \subseteq \overline{K}$ for the finite Galois extension field of K that corresponds to the image of H in G_K, and $R_H \subseteq K_H$ for the normalization of R in K_H. Then the following assertions hold:*

- (i) *Condition (1) of Lemma 6.5, (i), is satisfied if and only if the image of H in G_K contains the subgroup $P_K \subseteq G_K$.*
- (ii) *Condition (2) of Lemma 6.5, (i), is satisfied if and only if there exists a sub-$\mathbb{Z}_l$-module $M \subseteq \Delta_{X_H}^{l\text{-ab}}$ (cf. Definition 6.6) of $\Delta_{X_H}^{l\text{-ab}}$ such that the conjugation action of H on $\Delta_{X_H}^{l\text{-ab}}$ determines the respective trivial actions of H on M and $\Delta_{X_H}^{l\text{-ab}}/M$.*
- (iii) *Suppose that condition (2) of Lemma 6.5, (i), is satisfied. Thus, there exist a stable curve $\mathcal{X}_H$ over R_H and an isomorphism $\mathcal{X}_H \times_{R_H} K_H \xrightarrow{\sim} X_H$ over K_H. Here, we shall identify $\mathcal{X}_H \times_{R_H} K_H$ with X_H by means of such a fixed isomorphism $\mathcal{X}_H \times_{R_H} K_H \xrightarrow{\sim} X_H$. Then condition (3) of Lemma 6.5, (i), is satisfied if and only if, for each vertex $v \in v(\Gamma_{\underline{\overline{\mathcal{X}}}_H})$ (cf. Definition 1.2, Definition 6.1) of $\Gamma_{\underline{\overline{\mathcal{X}}}_H}$, the image of $D_v^{\Delta} \subseteq \Delta_{X_H}^{l\text{-ab}}$ (cf. Definition 6.6) by the natural continuous homomorphism*

$\Delta^{l\text{-ab}}_{X_H} \to \Delta^{l\text{-ab}}_X$ *has a natural structure of free* $\mathbb{Z}_l$*-module of rank* ≥ 4.

(iv) *Suppose that condition (2) of Lemma 6.5, (i), is satisfied. Thus, there exist a stable curve* $\mathcal{X}_H$ *over* R_H *and an isomorphism* $\mathcal{X}_H \times_{R_H} K_H \xrightarrow{\sim} X_H$ *over* K_H*. Here, we shall identify* $\mathcal{X}_H \times_{R_H} K_H$ *with* X_H *by means of such a fixed isomorphism* $\mathcal{X}_H \times_{R_H} K_H \xrightarrow{\sim} X_H$*. Then condition (5) of Lemma 6.5, (ii), is satisfied if and only if, for each vertex* $v \in v(\Gamma_{\underline{\mathcal{X}}_H})$ *of* $\Gamma_{\underline{\mathcal{X}}_H}$*, the subgroup of* Δ_X/Δ_{X_H} *(cf. Definition 6.3) that consists of* $\gamma \in \Delta_X/\Delta_{X_H}$ *such that the conjugation action of* γ *on* $\Delta^{l\text{-ab}}_{X_H}$ *stabilizes the closed subgroup* $D^{\Delta}_v \subseteq \Delta^{l\text{-ab}}_{X_H}$ *and induces the identity automorphism of* D^{Δ}_v *is of order prime to* p.

Proof. Assertion (i) is immediate. Assertion (ii) follows from the well-known stable reduction criterion (cf., e.g., [3, §7.4, Theorem 6]), together with [4, Theorem 2.4]. Assertion (iii) follows immediately from Lemma 6.7, (i), together with the well-known structure of the maximal pro-l quotient of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic $\neq l$ (cf., e.g., [26, Proposition 1.1, (i), (ii)]). Assertion (iv) follows immediately from Lemma 1.6 and Lemma 7.2, (i), below, together with Lemma 6.7, (i), (ii). ☐

Lemma 7.2. *Let* A *be an abelian variety over* K *and* l *an odd prime number invertible in* K*. Then the following assertions hold:*

(i) *Write* $A[l] \subseteq A$ *for the group subscheme of* A *obtained by forming the kernel of the endomorphism of* A *given by multiplication by* l*. Then the natural homomorphism* $\mathrm{Aut}_K(A) \to \mathrm{Aut}(A[l](\overline{K}))$ *is injective.*

(ii) *Let* B *be an abelian variety over* K*. Write* $\mathbb{T}_l(A)$*,* $\mathbb{T}_l(B)$ *for the respective* l*-adic Tate modules of* A*,* B*. Then the natural map* $\mathrm{Hom}_K(A, B) \to \mathrm{Hom}_{\widehat{\mathbb{Z}}}(\mathbb{T}_l(A), \mathbb{T}_l(B))$ *is injective.*

Proof. Assertion (i) follows from [5, Lemme 5.17]. Assertion (ii) follows from the (easily verified) fact that the subset of the underlying topological space of A that consists of torsion points of A of l-power order is dense. ☐

Lemma 7.3. *Let* $\mathcal{X}_1$*,* $\mathcal{X}_2$ *be stable curves over* R *such that the generic fibers* $X_1 \stackrel{\text{def}}{=} \mathcal{X}_1 \times_R K$*,* $X_2 \stackrel{\text{def}}{=} \mathcal{X}_2 \times_R K$ *are smooth over* K*, respectively, and* $\phi_{\Pi_X} : \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ *(cf. Definition 6.3) a continuous isomorphism over* G_K*. Suppose that the stable curve* $\underline{\mathcal{X}}_1$ *(cf. Definition 6.1) over* k *is sturdy. Then the following assertions hold:*

(i) *Let l be a prime number. Then, for each $v_1 \in v(\Gamma_{\overline{\underline{\mathcal{X}}}_1})$ (cf. Definition 1.2, Definition 6.1), there exists a unique vertex $v_2 \in v(\Gamma_{\overline{\underline{\mathcal{X}}}_2})$ such that the image of $D^{\Delta}_{v_1} \subseteq \Delta^{l\text{-ab}}_{X_1}$ (cf. Definition 6.6) by the isomorphism $\Delta^{l\text{-ab}}_{X_1} \xrightarrow{\sim} \Delta^{l\text{-ab}}_{X_2}$ induced by the isomorphism ϕ_{Π_X} is given by $D^{\Delta}_{v_2} \subseteq \Delta^{l\text{-ab}}_{X_2}$. Moreover, for each $e_1 \in e(\Gamma_{\overline{\underline{\mathcal{X}}}_1})$ (cf. Definition 1.2), there exists a(n) (not necessarily unique) edge $e_2 \in e(\Gamma_{\overline{\underline{\mathcal{X}}}_2})$ such that the image of $D^{\Delta}_{e_1} \subseteq \Delta^{l\text{-ab}}_{X_1}$ (cf. Definition 6.6) by the isomorphism $\Delta^{l\text{-ab}}_{X_1} \xrightarrow{\sim} \Delta^{l\text{-ab}}_{X_2}$ induced by the isomorphism ϕ_{Π_X} is given by $D^{\Delta}_{e_2} \subseteq \Delta^{l\text{-ab}}_{X_2}$.*

(ii) *The isomorphism ϕ_{Π_X} fits into a commutative diagram of topological groups*

$$\begin{tikzcd} \Pi_{X_1} \arrow[r, two heads, "\mathrm{sp}_{\mathcal{X}_1}"] \arrow[d, "\phi_{\Pi_X}"', "\wr"] & \Pi^{\log}_{\underline{\mathcal{X}}_1} \arrow[d, "\phi_{\Pi^{\log}_{\underline{\mathcal{X}}}}"', "\wr"] \arrow[dr] & \\ & & G^{\mathrm{tm}}_K \\ \Pi_{X_2} \arrow[r, two heads, "\mathrm{sp}_{\mathcal{X}_2}"'] & \Pi^{\log}_{\underline{\mathcal{X}}_2} \arrow[ur] & \end{tikzcd}$$

(cf. Definition 5.1, Definition 6.1, Definition 6.3) — where the diagonal arrows are the natural continuous surjective homomorphisms, and the vertical arrows are continuous isomorphisms.

(iii) *Suppose, moreover, that the field k is finite, and that $\mathcal{X}_1$ is not smooth over R. Then there exists an isomorphism $\phi_{\underline{\mathcal{X}}} \colon \underline{\mathcal{X}}_1 \xrightarrow{\sim} \underline{\mathcal{X}}_2$ of schemes (not necessarily over k). Moreover, the assignment "$\phi_{\Pi_X} \mapsto \phi_{\underline{\mathcal{X}}}$" is functorial.*

(iv) *Suppose, moreover, that the field k is finite, that $\mathcal{X}_1$ is not smooth over R, and that the isomorphism ϕ_{Π_X} is cyclotomically compatible. Then the isomorphism $\phi_{\Pi^{\log}_{\underline{\mathcal{X}}}} \colon \Pi^{\log}_{\underline{\mathcal{X}}_1} \xrightarrow{\sim} \Pi^{\log}_{\underline{\mathcal{X}}_2}$ of (ii) arises from an isomorphism $\phi_{\underline{\mathcal{X}}^{\log}} \colon \underline{\mathcal{X}}^{\log}_1 \xrightarrow{\sim} \underline{\mathcal{X}}^{\log}_2$ (cf. Definition 6.1) over* $\mathrm{Spec}(k)^{\log}$.

Proof. Assertion (i) follows immediately — in light of Lemma 6.7, (ii), Lemma 6.8, (i), and [4, Corollary 2.7] — by applying [21, Corollary 2.7, (iii)] to the various isomorphisms between the respective maximal pro-q quotients of suitable open subgroups of Δ_{X_1}, Δ_{X_2} for some prime number q invertible in R. Assertion (ii) follows from Lemma 6.5, (ii), and Lemma 7.1, together with assertion (i).

Next, we verify assertions (iii), (iv). We begin by observing that, to verify assertions (iii), (iv), by applying Lemma 5.9, (iii), and Lemma 6.8, (i), together with Galois descent, we may pass to a suitable Galois étale covering of $\mathcal{X}_1$. In particular, it follows from Lemma 6.10, together with Lemma 6.8, (ii), that, to verify assertions (iii), (iv), we may assume without loss of generality that each of the stable curves $\underline{\mathcal{X}}_1$, $\underline{\mathcal{X}}_2$ over k is sturdy, untangled, and split. Then assertion (iii) follows immediately from a similar argument to the argument applied in the discussion given in [17, pp.600-602]. Moreover, assertion (iv) follows immediately from a similar argument to the argument applied in the proof of [17, Theorem 7.2] (cf. also the discussion preceding [17, Theorem 7.2]). This completes the proofs of assertions (iii), (iv), hence also of Lemma 7.3. $\square$

Definition 7.4. Let Z_1, Z_2 be hyperbolic curves over K and $\phi_{\Pi_Z} : \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) a continuous isomorphism over G_K. Then we shall say that the isomorphism ϕ_{Π_Z} is *LSF-compatible* (where the "LSF" stands for "Log Special Fiber") if the isomorphism ϕ_{Π_Z} is compactification-compatible (cf. Definition 5.5), and, moreover, the following condition is satisfied: Let $H_1 \subseteq \Pi_{Z_1}$ be an open subgroup of Π_{Z_1}. Write $H_2 \stackrel{\text{def}}{=} \phi_{\Pi_Z}(H_1) \subseteq \Pi_{Z_2}$, $L \subseteq \overline{K}$ for the finite extension field of K that corresponds to the image of H_1 (i.e., of H_2) in G_K, $R_L \subseteq L$ for the normalization of R in L, and k_L for the residue field of R_L. Moreover, for each $i \in \{1, 2\}$, write $W_i^+ \to Z_i^+$ for the finite flat covering of Z_i^+ that corresponds to the open subgroup $H_i \subseteq \Pi_{Z_i}$, which thus implies that we have an identification $H_i = \Pi_{W_i}$ (cf. Definition 5.3), where we write W_i for the open subscheme of W_i^+ obtained by forming the inverse image of $Z_i \subseteq Z_i^+$ (cf. Remark 5.3.1). Suppose that there exist a stable curve $\mathcal{W}_1$ over R_L such that $\underline{\mathcal{W}}_1$ (cf. Definition 6.1) is sturdy and not smooth over k_L and an isomorphism $\mathcal{W}_1 \times_{R_L} L \xrightarrow{\sim} W_1^+$ over L. Then there exist a stable curve $\mathcal{W}_2$ over R_L and an isomorphism $\mathcal{W}_2 \times_{R_L} L \xrightarrow{\sim} W_2^+$ over L. Moreover, if we identify $\mathcal{W}_i \times_{R_L} L$ with W_i^+ by means of such a fixed isomorphism $\mathcal{W}_i \times_{R_L} L \xrightarrow{\sim} W_i^+$ for each $i \in \{1, 2\}$, then the isomorphism $\phi_{\Pi^{\log}_{\underline{\mathcal{Z}}}} : \Pi^{\log}_{\underline{\mathcal{W}}_1} \xrightarrow{\sim} \Pi^{\log}_{\underline{\mathcal{W}}_2}$ (cf. Definition 6.1, Definition 6.3) induced by ϕ_{Π_Z} (cf. Lemma 7.3, (ii); our assumption that the isomorphism ϕ_{Π_Z} is compactification-compatible) arises from an isomorphism $\underline{\mathcal{W}}_1^{\log} \xrightarrow{\sim} \underline{\mathcal{W}}_2^{\log}$ (cf. Definition 6.1) over $\mathrm{Spec}(k_L)^{\log}$ (cf. Definition 6.1).

Lemma 7.5. *Let Z_1, Z_2 be hyperbolic curves over K and $\phi_{\Pi_Z} : \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) a continuous isomorphism over G_K. Suppose that the field k is finite, and that the isomorphism ϕ_{Π_X} is cyclotomically* *compatible (cf. Definition 5.7). Then the isomorphism ϕ_{Π_Z} is LSF-compatible.*

Proof. This assertion follows immediately from Lemma 7.3, (iv) (cf. also Lemma 5.6). ☐

Definition 7.6. Suppose that the field K is of characteristic p. Let n be a nonnegative integer. Then we shall write

- $\mathrm{Fr}_K(n)$ for the absolute p^n-th power Frobenius endomorphism of $\mathrm{Spec}(K)$,
- $Z(n)$ for the hyperbolic curve over K obtained by pulling back the hyperbolic curve Z over K by $\mathrm{Fr}_K(n)$, and
- $\mathrm{Fr}_{Z/K}(n)\colon Z \to Z(n)$ for the relative p^n-th power Frobenius morphism over K (i.e., the morphism determined by the absolute p^n-th power Frobenius endomorphism of Z).

Remark 7.6.1. One verifies easily that, in the situation of Definition 7.6, the continuous outer homomorphism $\Pi_Z \to \Pi_{Z(n)}$ (cf. Definition 5.3) induced by the morphism $\mathrm{Fr}_{Z/K}(n)\colon Z \to Z(n)$ is a continuous outer isomorphism.

Lemma 7.7. *Let Z_1, Z_2 be hyperbolic curves over K and $\phi_{\Pi_Z}\colon \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) a continuous isomorphism over G_K. Suppose that the field k is finite. Then the following assertions hold:*

(i) *Suppose that the field K is of characteristic zero. Then the isomorphism ϕ_{Π_Z} is cyclotomically compatible.*

(ii) *Suppose that the field K is of characteristic p, and that the hyperbolic curve Z_1 over K is nonisotrivial. Then there exists a uniquely determined integer n that satisfies the following condition: If n is nonnegative, then the composite $\Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2(n)}$ (cf. Definition 5.3, Definition 7.6) of the given isomorphism $\phi_{\Pi_Z}\colon \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ with the continuous isomorphism $\Pi_{Z_2} \xrightarrow{\sim} \Pi_{Z_2(n)}$ induced by $\mathrm{Fr}_{Z_2/K}(n)$ (cf. Remark 7.6.1) is cyclotomically compatible. If n is negative, then the composite $\Pi_{Z_1(n)} \xrightarrow{\sim} \Pi_{Z_2}$ of the inverse of the continuous isomorphism $\Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_1(n)}$ induced by $\mathrm{Fr}_{Z_1/K}(n)$ with the given isomorphism $\phi_{\Pi_Z}\colon \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ is cyclotomically compatible.*

Proof. Observe that it follows from [16, Chapter VI, Theorem 11.1, (a)], together with Lemma 5.9, (ii), that, to verify Lemma 7.7, we may pass to a suitable finite étale covering of Z_1. In particular, it follows from Lemma 6.4 and Lemma 6.10, together with Lemma 5.9, (iii), that we may assume without loss of generality that Z_1^+, hence also Z_2^+ (cf. Lemma 7.3, (iv)), has a nonsmooth sturdy stable model. Then, by applying a similar argument to the argument applied in the discussion given in [17, pp.601-603] (or, alternatively, by applying [10, Lemma 5.2, (ii)]), one concludes that the diagram of topological modules

$$\begin{array}{ccc} \Lambda_{Z_1^+} & \xrightarrow{\sim} & \Lambda_{Z_2^+} \\ \wr\downarrow & & \downarrow\wr \\ \Lambda_{\overline{K}} & \xrightarrow[\sim]{} & \Lambda_{\overline{K}} \end{array}$$

— where the vertical arrows are the respective natural isomorphisms of Remark 5.4.1, (ii), the upper horizontal arrow is the isomorphism induced by ϕ_{Π_Z} (cf. Lemma 5.6), and the lower horizontal arrow is the isomorphism obtained by multiplying 1 (respectively, a power of p) if K is of characteristic zero (respectively, of characteristic p) — commutes. Thus, the desired assertion follows immediately from the definition of the relative p^n-th power Frobenius morphism defined in Definition 7.6. This completes the proof of Lemma 7.7. $\square$

8. Tate modules of Raynaud extensions of generalized Prym schemes

In the present section, we introduce some notational conventions related to the notion of the generalized Prym scheme of a finite étale covering of a stable curve and the notion of the Raynaud extension of the generalized Prym scheme. Moreover, we prove some basic facts concerning these notions. In the present section, suppose that we are in the situation at the beginning of §6. Moreover, let

- $\mathcal{Y}$ be a stable curve over R such that the generic fiber $Y \stackrel{\text{def}}{=} \mathcal{Y} \times_R K$ is smooth over K,
- l a prime number invertible in R, and
- $\mathcal{Y} \to \mathcal{X}$ a Galois étale covering of degree l over R, whose Galois group we denote by G.

By considering the natural (necessarily faithful — cf. Lemma 1.6, Lemma 6.2) action of G on the Jacobian variety $J_{\underline{\mathcal{Y}}}$ (cf. Definition 1.1) of the stable curve $\underline{\mathcal{Y}} = \mathcal{Y} \times_R k$ (cf. Definition 6.1) over k (i.e., induced by the action of G on $\mathcal{Y}$ over R), we shall regard G as a subgroup of the automorphism group of $J_{\underline{\mathcal{Y}}}$ over k. Suppose, moreover, that

- the structure morphism $\mathcal{Y} \to \mathrm{Spec}(R)$ of $\mathcal{Y}$ has a splitting, and that
- the stable curve $\underline{\mathcal{Y}}$ over k is split.

Definition 8.1. We shall write

- $J^t_{\mathcal{Y}}$, $P^t_{\mathcal{Y}/\mathcal{X}}$ for the respective dual semi-abelian schemes (cf., e.g., [22, Chapitre IV, Théorème 7.1, (i)]) of the Jacobian variety $J_{\mathcal{Y}}$ of $\mathcal{Y}$ over R, the generalized Prym scheme $P_{\mathcal{Y}/\mathcal{X}}$ associated to the Galois étale covering $\mathcal{Y} \to \mathcal{X}$ (cf. Definition 2.1),
- $0 \to T(\widetilde{J}_{\mathcal{Y}}) \to \widetilde{J}_{\mathcal{Y}} \to A(\widetilde{J}_{\mathcal{Y}}) \to 0$, $0 \to T(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \to \widetilde{P}_{\mathcal{Y}/\mathcal{X}} \to A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \to 0$, $0 \to T(\widetilde{J}^t_{\mathcal{Y}}) \to \widetilde{J}^t_{\mathcal{Y}} \to A(\widetilde{J}^t_{\mathcal{Y}}) \to 0$, $0 \to T(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \to \widetilde{P}^t_{\mathcal{Y}/\mathcal{X}} \to A(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \to 0$ for the respecrtive Raynaud extensions associated to $J_{\mathcal{Y}}$, $P_{\mathcal{Y}/\mathcal{X}}$, $J^t_{\mathcal{Y}}$, $P^t_{\mathcal{Y}/\mathcal{X}}$ (cf., e.g., [6, Chapter II, §1]),
- $\mathbb{T}_\times(J_{\mathcal{Y}})$, $\mathbb{T}_\times(P_{\mathcal{Y}/\mathcal{X}})$, $\mathbb{T}_\times(J^t_{\mathcal{Y}})$, $\mathbb{T}_\times(P^t_{\mathcal{Y}/\mathcal{X}})$, $\mathbb{T}_\times(T(\widetilde{J}_{\mathcal{Y}}))$, $\mathbb{T}_\times(\widetilde{J}_{\mathcal{Y}})$, $\mathbb{T}_\times(A(\widetilde{J}_{\mathcal{Y}}))$, $\mathbb{T}_\times(T(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}))$, $\mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$, $\mathbb{T}_\times(A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}))$, $\mathbb{T}_\times(T(\widetilde{J}^t_{\mathcal{Y}}))$, $\mathbb{T}_\times(\widetilde{J}^t_{\mathcal{Y}})$, $\mathbb{T}_\times(A(\widetilde{J}^t_{\mathcal{Y}}))$, $\mathbb{T}_\times(T(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}))$, $\mathbb{T}_\times(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})$, $\mathbb{T}_\times(A(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}))$ for the respective full profinite (respectively, pro-prime-to-char(K)-adic) Tate modules of (the generic fibers of) $J_{\mathcal{Y}}$, $P_{\mathcal{Y}/\mathcal{X}}$, $J^t_{\mathcal{Y}}$, $P^t_{\mathcal{Y}/\mathcal{X}}$, $T(\widetilde{J}_{\mathcal{Y}})$, $\widetilde{J}_{\mathcal{Y}}$, $A(\widetilde{J}_{\mathcal{Y}})$, $T(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$, $\widetilde{P}_{\mathcal{Y}/\mathcal{X}}$, $A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$, $T(\widetilde{J}^t_{\mathcal{Y}})$, $\widetilde{J}^t_{\mathcal{Y}}$, $A(\widetilde{J}^t_{\mathcal{Y}})$, $T(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})$, $\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}$, $A(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})$ whenever $\mathrm{char}(K) = 0$ (respectively, $\mathrm{char}(K) \neq 0$), and
- $C(\widetilde{J}_{\mathcal{Y}})$, $C(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$, $C(\widetilde{J}^t_{\mathcal{Y}})$, $C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})$ for the respective character groups of the tori $T(\widetilde{J}_{\mathcal{Y}})$, $T(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$, $T(\widetilde{J}^t_{\mathcal{Y}})$, $T(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})$ over R.

Remark 8.1.1.

(i) It is well-known (cf., e.g., [6], Chapter II, §1) that we have a natural identification $J_{\underline{\mathcal{Y}}} = \widetilde{J}_{\mathcal{Y}} \times_R k$, which thus determines a natural identification of the connected component of $\mathrm{Ker}(l - h_{\mathrm{new}} \colon J_{\underline{\mathcal{Y}}} \to J_{\underline{\mathcal{Y}}})$ with $\widetilde{P}_{\mathcal{Y}/\mathcal{X}} \times_R k$ (cf. Definition 2.1).

(ii) It is well-known (cf., e.g., [6], Chapter I, §1; [6], Chapter III, Corollary 7.4) that there exist natural commutative diagrams of topological modules

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \arrow[r] \arrow[d, hook] & \mathbb{T}_\times(P_{\mathcal{Y}/\mathcal{X}}) \arrow[r] \arrow[d, hook] & C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_\times \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{T}_\times(\widetilde{J}_{\mathcal{Y}}) \arrow[r] & \mathbb{T}_\times(J_{\mathcal{Y}}) \arrow[r] & C(\widetilde{J}^t_{\mathcal{Y}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_\times \arrow[r] & 0,
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{T}_\times(\widetilde{J}^t_{\mathcal{Y}}) \arrow[r] \arrow[d] & \mathbb{T}_\times(J^t_{\mathcal{Y}}) \arrow[r] \arrow[d] & C(\widetilde{J}_{\mathcal{Y}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_\times \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{T}_\times(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \arrow[r] & \mathbb{T}_\times(P^t_{\mathcal{Y}/\mathcal{X}}) \arrow[r] & C(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_\times \arrow[r] & 0
\end{tikzcd}$$

— where the horizontal sequences are exact, and the vertical arrows of the first diagram are injective.

(iii) It is well-known (cf., e.g., [6], Chapter II, §2) that the abelian scheme $A(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})$ over R is the dual abelian scheme of $A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$.

(iv) It is immediate that we have natural identifications $\mathbb{T}_\times(T(\widetilde{J}_\mathcal{Y})) = C(\widetilde{J}_\mathcal{Y})^D$ (cf. Definition 5.2), $\mathbb{T}_\times(T(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})) = C(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})^D$, $\mathbb{T}_\times(T(\widetilde{J}^t_\mathcal{Y})) = C(\widetilde{J}^t_\mathcal{Y})^D$, $\mathbb{T}_\times(T(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})) = C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})^D$.

(v) It follows from (iii), (iv) (cf. also Remark 5.2.1) that, by applying "$(-)^D$" to the natural exact sequences $0 \to \mathbb{T}_\times(T(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})) \to \mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \to \mathbb{T}_\times(A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})) \to 0$, $0 \to \mathbb{T}_\times(T(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})) \to \mathbb{T}_\times(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \to \mathbb{T}_\times(A(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})) \to 0$, we obtain exact sequences of topological modules

$$0 \longrightarrow \mathbb{T}_\times(A(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})) \longrightarrow \mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})^D \longrightarrow C(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \otimes_\mathbb{Z} \widehat{\mathbb{Z}}_\times \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{T}_\times(A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})) \longrightarrow \mathbb{T}_\times(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})^D \longrightarrow C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \otimes_\mathbb{Z} \widehat{\mathbb{Z}}_\times \longrightarrow 0.$$

Definition 8.2. We shall write

- $\mathbb{M}_Y$ for the topological abelianization of (respectively, the topological abelianization of the maximal pro-prime-to-char(K) quotient of) Δ_Y whenever char$(K) = 0$ (respectively, char$(K) \neq 0$) and
- $\mathbb{M}_Y^{\mathrm{Prym}} \subseteq \mathbb{M}_Y$ for the image of the endomorphism of the topological group $\mathbb{M}_Y$ determined by h_{new}.

Remark 8.2.1. It follows from the well-known theory of Jacobian varieties of curves that the morphism $Y \to J_Y$ (cf. Definition 1.1) determined by a splitting of the structure morphism of $\mathcal{Y}$ over R determines a G_K-equivariant continuous isomorphism

$$\mathbb{M}_Y \xrightarrow{\sim} \mathbb{T}_\times(J_\mathcal{Y}).$$

Moreover, this isomorphism does not depend on the choice of a splitting of the structure morphism of $\mathcal{Y}$ over R.

Lemma 8.3. *The isomorphism $\mathbb{M}_Y \xrightarrow{\sim} \mathbb{T}_\times(J_\mathcal{Y})$ of Remark 8.2.1 fits into a commutative diagram of topological modules*

$$\begin{tikzcd}
\mathbb{M}_Y^{\mathrm{Prym}} \arrow[r, hook] \arrow[d, "\wr"] & \mathbb{M}_Y \arrow[d, "\wr"] \\
\mathbb{T}_\times(P_{\mathcal{Y}/\mathcal{X}}) \arrow[r, hook] & \mathbb{T}_\times(J_\mathcal{Y})
\end{tikzcd}$$

— *where the horizontal arrows are the natural inclusions (cf. also Remark 8.1.1, (ii)), and the vertical arrows are continuous isomorphisms.*

Proof. This assertion follows immediately from the various definitions involved. □

Definition 8.4. We shall write

- $\mathbb{M}_Y^{\mathrm{adm}}$ for the topological quotient of $\mathbb{M}_Y$ determined by the quotient $\Delta_{\underline{\mathcal{Y}}}^{\mathrm{adm}}$ (cf. Definition 6.3) of Δ_Y,
- $\mathbb{M}_Y^{/\mathrm{vr}}$ for the topological quotient of $\mathbb{M}_Y^{\mathrm{adm}}$ by the closed subgroup of $\mathbb{M}_Y^{\mathrm{adm}}$ topologically generated by the images of the decomposition subgroups of $\Delta_{\underline{\mathcal{Y}}}^{\mathrm{adm}}$ associated to the elements of $v(\Gamma_{\overline{\underline{\mathcal{Y}}}})$ (cf. Definition 1.2, Definition 6.1),
- $\mathbb{M}_Y^{\mathrm{vr}} \subseteq \mathbb{M}_Y$ for the kernel of the natural continuous surjective homomorphism $\mathbb{M}_Y \twoheadrightarrow \mathbb{M}_Y^{/\mathrm{vr}}$,
- $\mathbb{M}_Y^{\text{Prym-vr}} \stackrel{\mathrm{def}}{=} \mathbb{M}_Y^{\mathrm{Prym}} \cap \mathbb{M}_Y^{\mathrm{vr}} \subseteq \mathbb{M}_Y^{\mathrm{Prym}}$ for the intersection of $\mathbb{M}_Y^{\mathrm{Prym}}$ with $\mathbb{M}_Y^{\mathrm{vr}}$,
- $((\mathbb{M}_Y^{/\mathrm{vr}})^D)_{\text{pre-Prym}} \subseteq (\mathbb{M}_Y^{\mathrm{Prym}})^D$ for the image by the natural homomorphism $\mathbb{M}_Y^D \to (\mathbb{M}_Y^{\mathrm{Prym}})^D$ of the closed subgroup $(\mathbb{M}_Y^{/\mathrm{vr}})^D \subseteq \mathbb{M}_Y^D$,
- $((\mathbb{M}_Y^{/\mathrm{vr}})^D)_{\mathrm{Prym}} \subseteq (\mathbb{M}_Y^{\mathrm{Prym}})^D$ for the kernel of the natural homomorphism from $(\mathbb{M}_Y^{\mathrm{Prym}})^D$ onto
$$\Big((\mathbb{M}_Y^{\mathrm{Prym}})^D / \big((\mathbb{M}_Y^{/\mathrm{vr}})^D\big)_{\text{pre-Prym}}\Big) \otimes_{\mathbb{Z}} \mathbb{Q},$$
- $\mathbb{M}_Y^{\text{Prym-nd}} \subseteq \mathbb{M}_Y^{\mathrm{Prym}}$ for the image of the submodule $\mathbb{T}_\times(T(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})) \subseteq \mathbb{T}_\times(P_{\mathcal{Y}/\mathcal{X}})$ (cf. Remark 8.1.1, (ii)) by the inverse of the left-hand vertical arrow of the diagram of Lemma 8.3,
- $\mathbb{M}_Y^{\text{Prym-(/nd)}} \stackrel{\mathrm{def}}{=} \mathbb{M}_Y^{\mathrm{Prym}}/\mathbb{M}_Y^{\text{Prym-nd}}$ for the quotient of $\mathbb{M}_Y^{\mathrm{Prym}}$ by the submodule $\mathbb{M}_Y^{\text{Prym-nd}} \subseteq \mathbb{M}_Y^{\mathrm{Prym}}$ of $\mathbb{M}_Y^{\mathrm{Prym}}$, and
- $((\mathbb{M}_Y^{/\mathrm{nd}})^D)_{\mathrm{Prym}} \subseteq (\mathbb{M}_Y^{\mathrm{Prym}})^D$ for the kernel of the natural homomorphisms from $(\mathbb{M}_Y^{\mathrm{Prym}})^D$ onto
$$\Big((\mathbb{M}_Y^{\mathrm{Prym}})^D / (\mathbb{M}_Y^{\text{Prym-(/nd)}})^D\Big) \otimes_{\mathbb{Z}} \mathbb{Q}$$
— where we regard $(\mathbb{M}_Y^{\text{Prym-(/nd)}})^D$ as a subgroup of $(\mathbb{M}_Y^{\mathrm{Prym}})^D$ by the natural continuous injective homomorphism $(\mathbb{M}_Y^{\text{Prym-(/nd)}})^D \hookrightarrow (\mathbb{M}_Y^{\mathrm{Prym}})^D$.

Lemma 8.5. *The following assertions hold:*

(i) *The isomorphism* $\mathbb{M}_Y \xrightarrow{\sim} \mathbb{T}_\times(J_{\mathcal{Y}})$ *of Remark 8.2.1 fits into a commutative diagram of topological modules*

$$\begin{tikzcd}
\mathbb{M}_Y^{\text{Prym-nd}} \arrow[r, hook] \arrow[d, "\wr"] & \mathbb{M}_Y^{\text{Prym-vr}} \arrow[r, hook] \arrow[d, "\wr"] & \mathbb{M}_Y^{\text{Prym}} \arrow[r, hook] \arrow[d, "\wr"] & \mathbb{M}_Y \arrow[d, "\wr"] \\
\mathbb{T}_\times\big(T(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})\big) \arrow[r, hook] & \mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \arrow[r, hook] & \mathbb{T}_\times(P_{\mathcal{Y}/\mathcal{X}}) \arrow[r, hook] & \mathbb{T}_\times(J_{\mathcal{Y}})
\end{tikzcd}$$

— where the horizontal arrows are the natural inclusions (cf. also Remark 8.1.1, (ii)), and the vertical arrows are continuous isomorphisms.

(ii) *The diagram of (i) determines a commutative diagram of topological modules*

$$\begin{tikzcd}
\mathbb{M}_Y^{\text{Prym-vr}} \arrow[r, two heads] \arrow[d, "\wr"] & \mathbb{M}_Y^{\text{Prym-vr}}/\mathbb{M}_Y^{\text{Prym-nd}} \arrow[d, "\wr"] \\
\mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \arrow[r, two heads] & \mathbb{T}_\times\big(A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})\big)
\end{tikzcd}$$

— where the horizontal arrows are the natural surjective homomorphisms, and the vertical arrows are continuous isomorphisms.

(iii) *The isomorphism* $\mathbb{T}_\times(J^t_{\mathcal{Y}}) \xrightarrow{\sim} \mathbb{M}_Y^D$ *determined by the isomorphism of Remark 8.2.1 fits into a commutative diagram of topological modules*

$$\begin{tikzcd}
\mathbb{T}_\times\big(T(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})\big) \arrow[r, hook] \arrow[d, "\wr"] & \mathbb{T}_\times(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \arrow[r, hook] \arrow[d, "\wr"] & \mathbb{T}_\times(P^t_{\mathcal{Y}/\mathcal{X}}) \arrow[d, "\wr"] & \mathbb{T}_\times(J^t_{\mathcal{Y}}) \arrow[l] \arrow[d, "\wr"] \\
\big((\mathbb{M}_Y^{/\text{vr}})^D\big)_{\text{Prym}} \arrow[r, hook] & \big((\mathbb{M}_Y^{/\text{nd}})^D\big)_{\text{Prym}} \arrow[r, hook] & (\mathbb{M}_Y^{\text{Prym}})^D & \mathbb{M}_Y^D \arrow[l]
\end{tikzcd}$$

— where the horizontal arrows are the natural homomorphisms (cf. also Remark 8.1.1, (ii)), and the vertical arrows are continuous isomorphisms.

(iv) *The diagram of (iii) determines a commutative diagram of topological modules*

$$\begin{tikzcd}
\mathbb{T}_\times(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \arrow[r, two heads] \arrow[d, "\wr"] & \mathbb{T}_\times\big(A(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})\big) \arrow[d, "\wr"] \\
\big((\mathbb{M}_Y^{/\text{nd}})^D\big)_{\text{Prym}} \arrow[r, two heads] & \big((\mathbb{M}_Y^{/\text{nd}})^D\big)_{\text{Prym}}/\big((\mathbb{M}_Y^{/\text{vr}})^D\big)_{\text{Prym}}
\end{tikzcd}$$

— where the horizontal arrows are the natural surjective homomorphisms, and the vertical arrows are continuous isomorphisms.

(v) *The natural identification* $\mathbb{M}_Y^{\mathrm{Prym}} = ((\mathbb{M}_Y^{\mathrm{Prym}})^D)^D$ *(cf. Remark 5.2.1) determines an identification of the quotients of* $\mathbb{M}_Y^{\mathrm{Prym}}$

$$\mathbb{M}_Y^{\mathrm{Prym}}/\mathbb{M}_Y^{\mathrm{Prym\text{-}vr}} = \Big(\big((\mathbb{M}_Y^{/\mathrm{vr}})^D\big)_{\mathrm{Prym}}\Big)^D,$$

which fits into the decomposition

$$\mathbb{T}_\times(P_{\mathcal{Y}/\mathcal{X}})/\mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \xleftarrow{\sim} \mathbb{M}_Y^{\mathrm{Prym}}/\mathbb{M}_Y^{\mathrm{Prym\text{-}vr}} = \Big(\big((\mathbb{M}_Y^{/\mathrm{vr}})^D\big)_{\mathrm{Prym}}\Big)^D$$

$$\xrightarrow{\sim} \mathbb{T}_\times\big(T(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})\big)^D = C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_\times$$

— where the first arrow is the isomorphism determined by the diagram of (i), the second arrow is the isomorphism determined by the left-hand vertical arrow of the diagram of (iii), and the second equality is the equality determined by the equality $\mathbb{T}_\times(T(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})) = C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})^D$ *that appears in Remark 8.1.1, (iv) — of the natural identification* $\mathbb{T}_\times(P_{\mathcal{Y}/\mathcal{X}})/\mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) = C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_\times$ *that appears in the upper horizontal sequence of the first diagram of Remark 8.1.1, (ii).*

Proof. These assertions follow immediately from [3, §9.2, Example 8] (cf. also [6, Chapter III, Corollary 8.2]). □

Lemma 8.6. *Let*

- $\mathcal{X}_1$, $\mathcal{X}_2$ *be stable curves over* R *such that, for each* $i \in \{1, 2\}$,
  - *the generic fiber* $X_i \stackrel{\mathrm{def}}{=} \mathcal{X}_i \times_R K$ *is smooth over* K*, and that*
  - *the stable curve* $\underline{\mathcal{X}}_i$ *(cf. Definition 6.1) over* k *is sturdy,*
- $\mathcal{Y}_1$, $\mathcal{Y}_2$ *stable curves over* R *such that, for each* $i \in \{1, 2\}$,
  - *the generic fiber* $Y_i \stackrel{\mathrm{def}}{=} \mathcal{Y}_i \times_R K$ *is smooth over* K*, that*
  - *the structure morphism* $\mathcal{Y}_i \to \mathrm{Spec}(R)$ *has a splitting, and that*
  - *the stable curve* $\underline{\mathcal{Y}}_i$ *(cf. Definition 6.1) over* k *is split, and*
- $\mathcal{Y}_1 \to \mathcal{X}_1$, $\mathcal{Y}_2 \to \mathcal{X}_2$ *Galois étale coverings of degree* l *over* R.

Let $\phi_{\Pi_X} \colon \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ *(cf. Definition 6.3) be a continuous isomorphism over* G_K *that restricts to an isomorphism* $\Pi_{Y_1} \xrightarrow{\sim} \Pi_{Y_2}$ *(cf. Definition 6.3) necessarily over* G_K*. Then the following assertions hold:*

(i) *The isomorphism* $\mathbb{M}_{Y_1} \xrightarrow{\sim} \mathbb{M}_{Y_2}$ *(cf. Definition 8.2) determined by the isomorphism* ϕ_{Π_X} *fits into a commutative diagram of topological groups*

$$\begin{tikzcd}
\mathbb{M}_{Y_1}^{\text{Prym-vr}} \arrow[r, hook] \arrow[d, "\wr"] & \mathbb{M}_{Y_1}^{\text{Prym}} \arrow[r, hook] \arrow[d, "\wr"] & \mathbb{M}_{Y_1} \arrow[d, "\wr"] \\
\mathbb{M}_{Y_2}^{\text{Prym-vr}} \arrow[r, hook] & \mathbb{M}_{Y_2}^{\text{Prym}} \arrow[r, hook] & \mathbb{M}_{Y_2}
\end{tikzcd}$$

(cf. Definition 8.4) — where the horizontal arrows are the natural inclusions, and the vertical arrows are continuous isomorphisms.

(ii) *The isomorphism* $\mathbb{M}_{Y_2}^{D} \xrightarrow{\sim} \mathbb{M}_{Y_1}^{D}$ *determined by the isomorphism* ϕ_{Π_X} *fits into a commutative diagram of topological groups*

$$\begin{tikzcd}
\big((\mathbb{M}_{Y_2}^{/\text{vr}})^D\big)_{\text{Prym}} \arrow[r, hook] \arrow[d, "\wr"] & (\mathbb{M}_{Y_2}^{\text{Prym}})^D \arrow[d, "\wr"] & \mathbb{M}_{Y_2}^{D} \arrow[l] \arrow[d, "\wr"] \\
\big((\mathbb{M}_{Y_1}^{/\text{vr}})^D\big)_{\text{Prym}} \arrow[r, hook] & (\mathbb{M}_{Y_1}^{\text{Prym}})^D & \mathbb{M}_{Y_1}^{D} \arrow[l]
\end{tikzcd}$$

(cf. Definition 8.4) — where the horizontal arrows are the natural homomorphisms, and the vertical arrows are continuous isomorphisms.

Proof. These assertions follow from Lemma 7.3, (i), (ii), together with the various definitions involved (cf. also Lemma 6.8, (i)). □

9. Reconstruction of Raynaud extensions of generalized Prym schemes

In the present section, we show how the Raynaud extension of the generalized Prym scheme of a finite étale covering in a certain situation can be recovered from the étale fundamental group (cf. Lemma 9.3 below). In the present section, suppose that we are in the situation at the beginning of the preceding §8.

Definition 9.1. Suppose that the field K is of characteristic zero. Then we shall write

- $\widetilde{P}_{\mathcal{Y}/\mathcal{X}}[p^\infty]$ for the p-divisible group over R determined by the semi-abelian scheme $\widetilde{P}_{\mathcal{Y}/\mathcal{X}}$,

- $\mathcal{E}_{\mathcal{Y}/\mathcal{X}}$ for the p-divisible group over R whose p-adic Tate module is given (cf. [30, Theorem 4]; Lemma 8.5, (i)) by the G_K-module obtained by forming the maximal G_K-stable torsion-free quotient of $\mathbb{M}_Y^{\text{Prym-vr}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_p$ on which the natural action of I_K (cf. Definition 5.1) is trivial, and
- $\mathcal{T}_{\mathcal{Y}/\mathcal{X}}$ for the p-divisible group over R whose p-adic Tate module is given (cf. [30, Theorem 4]; Lemma 8.5, (i)) by the G_K-module $(((\mathbb{M}_Y^{\text{Prym-vr}})^D)^{I_K})^D \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_p$, where we write $((\mathbb{M}_Y^{\text{Prym-vr}})^D)^{I_K} \subseteq (\mathbb{M}_Y^{\text{Prym-vr}})^D$ for the submodule of $(\mathbb{M}_Y^{\text{Prym-vr}})^D$ of I_K-invariants.

Remark 9.1.1. It is immediate that if the field K is of characteristic zero, then the p-divisible group $\widetilde{P}_{\mathcal{Y}/\mathcal{X}}[p^\infty]$ over R is the p-divisible group over R whose p-adic Tate module is given (cf. [30, Theorem 4]) by the G_K-module $\mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_p$.

Lemma 9.2. *Suppose that the field K is of characteristic zero, and that the Galois étale covering $\underline{\mathcal{Y}} \to \underline{\mathcal{X}}$ is new-ordinary (cf. Definition 2.2). Then the isomorphism $\mathbb{M}_Y^{\text{Prym-vr}} \xrightarrow{\sim} \mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$ that appears in Lemma 8.5, (i), determines (cf. also Remark 9.1.1,* [30, Theorem 4]*) an isomorphism of p-divisible groups over k*

$$(\mathcal{E}_{\mathcal{Y}/\mathcal{X}} \times \mathcal{T}_{\mathcal{Y}/\mathcal{X}}) \times_R k \xrightarrow{\sim} \widetilde{P}_{\mathcal{Y}/\mathcal{X}}[p^\infty] \times_R k.$$

Proof. Let us first recall that since (we have assumed that) the field k is perfect, every p-divisible group over k may be decomposed into the product of an étale p-divisible group over k and a connected p-divisible group over k. Moreover, one verifies easily that an arbitrary homomorphism over k from an étale (respectively, a connected) p-divisible group over k to a connected (respectively, an étale) p-divisible group over k is trivial. Now let us recall that we have assumed that the Galois étale covering $\underline{\mathcal{Y}} \to \underline{\mathcal{X}}$ is new-ordinary. Thus, the desired assertion follows immediately from the well-known structure of the p-adic Tate module of the p-divisible group over R that arises from an extension by a torus over R of an abelian scheme over R whose special fiber over k is ordinary. This completes the proof of Lemma 9.2. □

Lemma 9.3. *Let*

- *$\mathcal{X}_1$, $\mathcal{X}_2$ be stable curves over R such that, for each $i \in \{1, 2\}$,*
  - *the generic fiber $X_i \stackrel{\text{def}}{=} \mathcal{X}_i \times_R K$ is smooth over K, and that*
  - *the stable curve $\underline{\mathcal{X}}_i$ (cf. Definition 6.1) over k is sturdy and not smooth over k,*
- *$\mathcal{Y}_1$, $\mathcal{Y}_2$ stable curves over R such that, for each $i \in \{1, 2\}$,*

- *the generic fiber* $Y_i \stackrel{\mathrm{def}}{=} \mathcal{Y}_i \times_R K$ *is smooth over* K, *that*
- *the structure morphism* $\mathcal{Y}_i \to \mathrm{Spec}(R)$ *has a splitting, and that*
- *the stable curve* $\underline{\mathcal{Y}}_i$ *(cf. Definition 6.1) over* k *is split, and*

- $\mathcal{Y}_1 \to \mathcal{X}_1$, $\mathcal{Y}_2 \to \mathcal{X}_2$ *Galois étale coverings of degree* l *over* R.

Let $\phi_{\Pi_X} : \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ *(cf. Definition 6.3) be a continuous isomorphism over* G_K *that restricts to an isomorphism* $\Pi_{Y_1} \xrightarrow{\sim} \Pi_{Y_2}$ *(cf. Definition 6.3) necessarily over* G_K. *Suppose that the following three conditions are satisfied:*

- *The field* K *is of characteristic zero.*
- *The Galois étale covering* $\underline{\mathcal{Y}}_1 \to \underline{\mathcal{X}}_1$ *is new-ordinary.*
- *The isomorphism* ϕ_{Π_X} *is LSF-compatible (cf. Definition 7.4).*

Then the isomorphism $\mathbb{M}_{Y_1}^{\text{Prym-vr}} \xrightarrow{\sim} \mathbb{M}_{Y_2}^{\text{Prym-vr}}$ *(cf. Definition 8.4) determined by* ϕ_{Π_X} *(cf. Lemma 8.6, (i)) arises — relative to the second vertical arrow of the diagram of Lemma 8.5, (i) — from an isomorphism* $\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1} \xrightarrow{\sim} \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}$ *(cf. Definition 8.1) of semi-abelian schemes over* R *(cf. Remark 9.3.1 below).*

Proof. Let us first observe that since (we have assumed that) the isomorphism ϕ_{Π_X} is LSF-compatible, the isomorphism $\Pi_{\underline{\mathcal{Y}}_1}^{\log} \xrightarrow{\sim} \Pi_{\underline{\mathcal{Y}}_2}^{\log}$ (cf. Definition 6.3) induced by ϕ_{Π_X} (cf. Lemma 7.3, (ii)) arises (cf. also Lemma 5.9, (iii)) from an isomorphism $\underline{\mathcal{Y}}_1^{\log} \xrightarrow{\sim} \underline{\mathcal{Y}}_2^{\log}$ (cf. Definition 6.1) over $\mathrm{Spec}(k)^{\log}$ (cf. Definition 6.1), which determines (cf. Remark 8.1.1, (i)) an isomorphism $\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1} \times_R k \xrightarrow{\sim} \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2} \times_R k$ over k. In particular, since (we have assumed that) the field K is of characteristic zero, and the Galois étale covering $\underline{\mathcal{Y}}_1 \to \underline{\mathcal{X}}_1$, hence also the Galois étale covering $\underline{\mathcal{Y}}_2 \to \underline{\mathcal{X}}_2$, is new-ordinary, one concludes immediately from Lemma 9.2 that the isomorphism $\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}[p^\infty] \times_R k \xrightarrow{\sim} \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}[p^\infty] \times_R k$ (cf. Definition 9.1) induced by the above isomorphism $\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1} \times_R k \xrightarrow{\sim} \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2} \times_R k$ coincides with the isomorphism $\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}[p^\infty] \times_R k \xrightarrow{\sim} \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}[p^\infty] \times_R k$ induced by the isomorphism $\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}[p^\infty] \xrightarrow{\sim} \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}[p^\infty]$ determined (cf. Remark 9.1.1, [30, Theorem 4]) by the composite

$$\mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}) \xleftarrow{\sim} \mathbb{M}_{Y_1}^{\text{Prym-vr}} \xrightarrow{\sim} \mathbb{M}_{Y_2}^{\text{Prym-vr}} \xrightarrow{\sim} \mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2})$$

(cf. Definition 8.1) — where the first and third arrows are the second vertical arrow of the diagram of Lemma 8.5, (i), and the second arrow is the isomorphism determined by ϕ_{Π_X}. Thus, it follows immediately from [12, Theorem 1.2.1] that the isomorphism $\mathbb{M}_{Y_1}^{\text{Prym-vr}} \xrightarrow{\sim} \mathbb{M}_{Y_2}^{\text{Prym-vr}}$ determined by ϕ_{Π_X} arises — relative to the second vertical arrow of the diagram of Lemma 8.5, (i) — from an isomorphism $\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1} \xrightarrow{\sim} \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}$ of semi-abelian schemes over R, as desired. This completes the proof of Lemma 9.3. $\square$

Remark 9.3.1. Suppose that, in the situation of Lemma 9.3, we are given an isomorphism $\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1} \xrightarrow{\sim} \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}$ of semi-abelian schemes over R. Then observe that one verifies easily that this isomorphism $\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1} \xrightarrow{\sim} \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}$ of semi-abelian schemes over R determines a commutative diagram of semi-abelian schemes over R

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}) & \longrightarrow & \widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1} & \longrightarrow & A(\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}) & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \longrightarrow & T(\widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}) & \longrightarrow & \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2} & \longrightarrow & A(\widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}) & \longrightarrow & 0 \end{array}$$

(cf. Definition 8.1) — where the horizontal sequences are exact, the vertical arrows are isomorphisms, and the left-hand vertical arrow determines an isomorphism of modules

$$C(\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}) \xrightarrow{\sim} C(\widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2})$$

(cf. Definition 8.1).

Definition 9.4. Let Z_1, Z_2 be hyperbolic curves over K and $\phi_{\Pi_Z} : \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) a continuous isomorphism over G_K. Then we shall say that the isomorphism ϕ_{Π_Z} is *REP-compatible* (where the "REP" stands for "Raynaud Extension of the generalized Prym scheme") if the isomorphism ϕ_{Π_Z} is compactification-compatible (cf. Definition 5.5), and, moreover, the following condition is satisfied: Let $H_1' \subseteq H_1 \subseteq \Pi_{Z_1}$ be open subgroups of Π_{Z_1} such that the image of H_1' in G_K coincides with the image of H_1 in G_K. Write $H_2' \stackrel{\text{def}}{=} \phi_{\Pi_Z}(H_1') \subseteq H_2 \stackrel{\text{def}}{=} \phi_{\Pi_Z}(H_1) \subseteq \Pi_{Z_2}$, $L \subseteq \overline{K}$ for the finite extension field of K that corresponds to the image of H_1 (i.e., of H_2) in G_K, $R_L \subseteq L$ for the normalization of R in L, and k_L for the residue field of R_L. Moreover, for each $i \in \{1, 2\}$, write $V_i^+ \to W_i^+ \to Z_i^+$ (cf. Definition 5.3) for the finite flat coverings of Z_i^+ that correspond to the open subgroups $H_i' \subseteq H_i \subseteq \Pi_{Z_i}$, respectively. Suppose that there exist

- a prime number q invertible in R,
- stable curves $\mathcal{V}_1$, $\mathcal{W}_1$, $\mathcal{V}_2$, $\mathcal{W}_2$ over R_L, and
- isomorphisms $\mathcal{V}_1 \times_{R_L} L \xrightarrow{\sim} V_1^+$, $\mathcal{W}_1 \times_{R_L} L \xrightarrow{\sim} W_1^+$, $\mathcal{V}_2 \times_{R_L} L \xrightarrow{\sim} V_2^+$, $\mathcal{W}_2 \times_{R_L} L \xrightarrow{\sim} W_2^+$ over L, by means of which we identify $\mathcal{V}_1 \times_{R_L} L$, $\mathcal{W}_1 \times_{R_L} L$, $\mathcal{V}_2 \times_{R_L} L$, $\mathcal{W}_2 \times_{R_L} L$ with V_1^+, W_1^+, V_2^+, W_2^+, respectively,

such that

- each of the stable curves $\underline{\mathcal{V}}_1$, $\underline{\mathcal{W}}_1$, $\underline{\mathcal{V}}_2$, $\underline{\mathcal{W}}_2$ (cf. Definition 6.1) over k_L is sturdy, split, and not smooth over k_L, that
- each of the morphisms $\mathcal{V}_1 \to \mathcal{W}_1$, $\mathcal{V}_2 \to \mathcal{W}_2$ induced by the finite flat coverings $V_1^+ \to W_1^+$, $V_2^+ \to W_2^+$ (cf. [19, Corollary 7.4]), respectively, is a Galois étale covering of degree q and new-ordinary, and, moreover, that
- each of the structure morphisms $\mathcal{V}_1 \to \mathrm{Spec}(R_L)$, $\mathcal{V}_2 \to \mathrm{Spec}(R_L)$ has a splitting.

Then the isomorphism $\mathbb{M}_{V_1^+}^{\text{Prym-vr}} \xrightarrow{\sim} \mathbb{M}_{V_2^+}^{\text{Prym-vr}}$ (cf. Definition 8.4) determined by ϕ_{Π_Z} (cf. our assumption that the isomorphism ϕ_{Π_Z} is compactification-compatible; Lemma 8.6, (i)) arises — relative to the second vertical arrow of the diagram of Lemma 8.5, (i) — from an isomorphism $\widetilde{P}_{\mathcal{V}_1/\mathcal{W}_1} \xrightarrow{\sim} \widetilde{P}_{\mathcal{V}_2/\mathcal{W}_2}$ (cf. Definition 8.1) of semi-abelian schemes over R_L.

Lemma 9.5. *Let Z_1, Z_2 be hyperbolic curves over K and $\phi_{\Pi_Z} \colon \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) a continuous isomorphism over G_K. Suppose that the field K is of characteristic zero, and that the isomorphism ϕ_{Π_X} is LSF-compatible. Then the isomorphism ϕ_{Π_Z} is REP-compatible.*

Proof. This assertion follows from Lemma 9.3. □

Remark 9.5.1. One main difficulty to work with basefields of positive characteristic in the present paper is as follows. Suppose that the Galois étale covering $\underline{\mathcal{Y}} \to \underline{\mathcal{X}}$ is new-ordinary. As discussed in Lemma 9.2, in the case where the field K is of characteristic zero, one may relate the p-divisible group $\widetilde{P}_{\mathcal{Y}/\mathcal{X}}[p^\infty]$ associated to the semi-abelian scheme $\widetilde{P}_{\mathcal{Y}/\mathcal{X}}$ with the étale fundamental group Π_X of the generic fiber X of $\mathcal{X}$. On the other hand, in the case where the field K is of positive characteristic, since every nontrivial multiplicative p-divisible group over K is not étale, at the time of writing, the authors of the present paper are not able to relate $\widetilde{P}_{\mathcal{Y}/\mathcal{X}}[p^\infty]$ with Π_X. In particular, at the time of writing, the authors of the present paper are not able to prove Lemma 9.5 without assuming that K is of characteristic zero.

10. Reconstruction of generalized Prym schemes

In the present section, we show how the generalized Prym scheme of a finite étale covering in a certain situation can be recovered from the étale fundamental group (cf. Lemma 10.6, (iii), below). In the present section, suppose that we are in the situation at the beginning of §8.

Definition 10.1. We shall write

- $\lambda_{\mathcal{Y}}\colon J_{\mathcal{Y}} \xrightarrow{\sim} J^t_{\mathcal{Y}}$ for the isomorphism over R obtained by forming the uniquely determined extension of the principal polarization on J_Y determined by the theta divisor on $\mathrm{Pic}^{g_Y-1}_{Y/K}$, where we write g_Y for the genus of the curve Y over K, and some splitting of the structure morphism $\mathcal{Y} \to \mathrm{Spec}(R)$,
- $\lambda^P_{\mathcal{Y}/\mathcal{X}}\colon P_{\mathcal{Y}/\mathcal{X}} \to P^t_{\mathcal{Y}/\mathcal{X}}$ for the Prym semi-polarization associated to the Galois étale covering $\mathcal{Y} \to \mathcal{X}$ (cf. Definition 2.3, (ii)),
- $\lambda^A_{\mathcal{Y}/\mathcal{X}}\colon A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \to A(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})$ for the polarization on $A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$ (cf. Remark 8.1.1, (iii)) induced by $\lambda^P_{\mathcal{Y}/\mathcal{X}}\colon P_{\mathcal{Y}/\mathcal{X}} \to P^t_{\mathcal{Y}/\mathcal{X}}$,
- $\lambda^C_{\mathcal{Y}/\mathcal{X}}\colon C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \to C(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$ for the homomorphism determined by the homomorphism $T(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \to T(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})$ induced by $\lambda^P_{\mathcal{Y}/\mathcal{X}}\colon P_{\mathcal{Y}/\mathcal{X}} \to P^t_{\mathcal{Y}/\mathcal{X}}$, and
- $\mathbb{T}_\times(\lambda_{\mathcal{Y}})\colon \mathbb{T}_\times(J_{\mathcal{Y}}) \xrightarrow{\sim} \mathbb{T}_\times(J^t_{\mathcal{Y}}) = \mathbb{T}_\times(J_{\mathcal{Y}})^D$, $\mathbb{T}_\times(\lambda^P_{\mathcal{Y}/\mathcal{X}})\colon \mathbb{T}_\times(P_{\mathcal{Y}/\mathcal{X}}) \to \mathbb{T}_\times(P^t_{\mathcal{Y}/\mathcal{X}}) = \mathbb{T}_\times(P_{\mathcal{Y}/\mathcal{X}})^D$, $\mathbb{T}_\times(\lambda^A_{\mathcal{Y}/\mathcal{X}})\colon \mathbb{T}_\times(A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})) \to \mathbb{T}_\times(A(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})) = \mathbb{T}_\times(A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}))^D$ (cf. Definition 5.2) for the continuous homomorphisms induced by $\lambda_{\mathcal{Y}}$, $\lambda^P_{\mathcal{Y}/\mathcal{X}}$, $\lambda^A_{\mathcal{Y}/\mathcal{X}}$, respectively.

Lemma 10.2. *The following assertions hold:*

(i) *The cup pairing*

$$\mathrm{Hom}_{\widehat{\mathbb{Z}}_\times}(\mathbb{M}_Y, \mathbb{M}_Y^D) = H^1(\Delta_Y, \widehat{\mathbb{Z}}_\times) \otimes_{\widehat{\mathbb{Z}}_\times} H^1(\Delta_Y, \Lambda_{\overline{K}})$$

$$\longrightarrow H^2(\Delta_Y, \Lambda_{\overline{K}}) = \mathrm{Hom}_{\widehat{\mathbb{Z}}_\times}(\Lambda_Y, \Lambda_{\overline{K}})$$

determines a continuous isomorphism

$$\mathbb{M}_Y^D \xrightarrow{\sim} \mathbb{M}_Y \otimes_{\widehat{\mathbb{Z}}_\times} \mathrm{Hom}_{\widehat{\mathbb{Z}}_\times}(\Lambda_Y, \Lambda_{\overline{K}}).$$

(ii) *The isomorphism $\mathbb{M}_Y \xrightarrow{\sim} \mathbb{M}_Y^D \otimes_{\widehat{\mathbb{Z}}_\times} \mathrm{Hom}_{\widehat{\mathbb{Z}}_\times}(\Lambda_{\overline{K}}, \Lambda_Y)$ determined by the isomorphism of (i) fits into a commutative diagram of topological modules*

$$\begin{tikzcd}
\mathbb{M}_Y \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathbb{M}_Y^D \otimes_{\widehat{\mathbb{Z}}_\times} \mathrm{Hom}_{\widehat{\mathbb{Z}}_\times}(\Lambda_{\overline{K}}, \Lambda_Y) \arrow[r, "\sim"] & \mathbb{M}_Y^D \\
\mathbb{T}_\times(J_{\mathcal{Y}}) \arrow[r, "\sim", "\mathbb{T}_\times(\lambda_{\mathcal{Y}})"'] & \mathbb{T}_\times(J^t_{\mathcal{Y}}) \arrow[r, equal] & \mathbb{T}_\times(J_{\mathcal{Y}})^D \arrow[u, "\wr"]
\end{tikzcd}$$

— where the vertical arrows are the isomorphisms determined by the isomorphism of Remark 8.2.1, and the right-hand upper

horizontal arrow is the isomorphism determined by the isomorphism

$$\widehat{\mathbb{Z}}_\times \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathbb{Z}}_\times}(\Lambda_Y, \Lambda_{\overline{K}})$$

given by the natural isomorphism $\Lambda_Y \xrightarrow{\sim} \Lambda_{\overline{K}}$ *of Remark 5.4.1, (ii) (cf. also Remark 5.2.1; Remark 5.4.1, (i)).*

(iii) *The composite* $\mathbb{M}_Y \xrightarrow{\sim} \mathbb{M}_Y^D$ *of the upper horizontal arrows of the diagram of (ii) fits into a commutative diagram of topological modules*

$$\begin{tikzcd}
\mathbb{M}_Y^{\text{Prym-nd}} \arrow[r, hook] \arrow[d] & \mathbb{M}_Y^{\text{Prym-vr}} \arrow[r, hook] \arrow[d] & \mathbb{M}_Y^{\text{Prym}} \arrow[r, hook] \arrow[d] & \mathbb{M}_Y \arrow[d, "\wr"] \\
\big((\mathbb{M}_Y^{/\text{vr}})^D\big)_{\text{Prym}} \arrow[r, hook] & \big((\mathbb{M}_Y^{/\text{nd}})^D\big)_{\text{Prym}} \arrow[r, hook] & (\mathbb{M}_Y^{\text{Prym}})^D & \mathbb{M}_Y^D \arrow[l]
\end{tikzcd}$$

— where the upper sequence is the upper sequence of the diagram of Lemma 8.5, (i), and the lower sequence is the lower sequence of the diagram of Lemma 8.5, (iii).

(iv) *We have a commutative diagram of topological modules*

$$\begin{tikzcd}
\mathbb{M}_Y^{\text{Prym-vr}}/\mathbb{M}_Y^{\text{Prym-nd}} \arrow[r] \arrow[d, "\wr"'] & \big((\mathbb{M}_Y^{/\text{nd}})^D\big)_{\text{Prym}}/\big((\mathbb{M}_Y^{/\text{vr}})^D\big)_{\text{Prym}} \\
\mathbb{T}_\times\big(A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})\big) \arrow[r, "\mathbb{T}_\times(\lambda^A_{\mathcal{Y}/\mathcal{X}})"'] & \mathbb{T}_\times\big(A(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})\big) \arrow[u, "\wr"']
\end{tikzcd}$$

— where the upper horizontal arrow is the homomorphism determined by the diagram of (iii), and the left-hand, right-hand vertical arrows are the right-hand vertical arrows of the diagrams of Lemma 8.5, (ii), (iv), respectively.

(v) *We have a commutative diagram of topological modules*

$$\begin{tikzcd}
\mathbb{M}_Y^{\text{Prym-nd}} \arrow[rrr] \arrow[d, "\wr"'] & & & \big((\mathbb{M}_Y^{/\text{vr}})^D\big)_{\text{Prym}} \\
\mathbb{T}_\times\big(T(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})\big) \arrow[r, equal] & C(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})^D \arrow[r] & C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})^D \arrow[r, equal] & \mathbb{T}_\times\big(T(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})\big) \arrow[u, "\wr"']
\end{tikzcd}$$

(cf. Remark 8.1.1, (iv)) — where the upper horizontal arrow is the left-hand vertical arrow of the diagram of (iii), the lower horizontal arrow is the homomorphism induced by $\lambda^C_{\mathcal{Y}/\mathcal{X}}$*, and the left-hand, right-hand vertical arrows are the right-hand vertical arrows of the diagrams of Lemma 8.5, (i), (iii), respectively.*

Proof. These assertions follow immediately from the well-known theory of Jacobian varieties of curves, together with the various definitions involved. □

Lemma 10.3. *Consider the diagram*

$$\begin{tikzcd}
(\mathbb{M}_Y^{\text{Prym-nd}})^D & \Big(((\mathbb{M}_Y^{/\text{vr}})^D)_{\text{Prym}} \Big)^D \arrow[l] \arrow[d, "\wr"] \\
C(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_{\times} \arrow[u, "\wr"] & C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_{\times} \arrow[l, "\lambda^C_{\mathcal{Y}/\mathcal{X}}"]
\end{tikzcd}$$

obtained by applying "$(-)^D$" to the diagram of Lemma 10.2, (v) (cf. also Remark 5.2.1). In the remainder of Lemma 10.3, we identify $(\mathbb{M}_Y^{\text{Prym-nd}})^D$, $(((\mathbb{M}_Y^{/\text{vr}})^D)_{\text{Prym}})^D$ with $C(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_{\times}$, $C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_{\times}$ by means of the left-hand, right-hand vertical arrows of this diagram, respectively. Let $C \subsetneq (((\mathbb{M}_Y^{/\text{vr}})^D)_{\text{Prym}})^D$ be a submodule of $(((\mathbb{M}_Y^{/\text{vr}})^D)_{\text{Prym}})^D$ and $f_C \colon C \to C(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$ a homomorphism of modules. Then the following two conditions are equivalent:

1. *The equality $(C, f_C) = (C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}), \lambda^C_{\mathcal{Y}/\mathcal{X}})$ holds.*
2. *The inclusion $C \hookrightarrow (((\mathbb{M}_Y^{/\text{vr}})^D)_{\text{Prym}})^D$ determines an isomorphism $C \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_{\times} \xrightarrow{\sim} (((\mathbb{M}_Y^{/\text{vr}})^D)_{\text{Prym}})^D$, and, moreover, the homomorphism $(((\mathbb{M}_Y^{/\text{vr}})^D)_{\text{Prym}})^D \to (\mathbb{M}_Y^{\text{Prym-nd}})^D$ determined by this resulting isomorphism and the homomorphism $f_C \colon C \to C(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$ under consideration coincides with the upper horizontal arrow $(((\mathbb{M}_Y^{/\text{vr}})^D)_{\text{Prym}})^D \to (\mathbb{M}_Y^{\text{Prym-nd}})^D$ of the above diagram.*

Proof. This assertion follows immediately from the (easily verified) fact that the modules $C(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$, $C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})$ are finitely generated and free, and, moreover, the homomorphism $\lambda^C_{\mathcal{Y}/\mathcal{X}} \colon C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \to C(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$ is an injective homomorphism whose cokernel is (finite and) of order a power of l (cf. Remark 2.3.1), hence also prime to p. □

Definition 10.4. Let F be a field and $\overline{F}$ a separable closure of F.

(i) Let A be a semi-abelian variety over F. Then we shall refer to the homomorphism $A(F) \to H^1(\text{Gal}(\overline{F}/F), \mathbb{T}_{\times}(A))$ — where we write $\mathbb{T}_{\times}(A)$ for the full profinite (respectively, pro-prime-to-char(K)-adic) Tate module of A whenever char$(K) = 0$ (respectively, char$(K) \neq 0$) — induced by the various Kummer exact sequences associated to A as the $\times$-*Kummer homomorphism* associated to the semi-abelian variety A over F.

(ii) We shall say that the field F is $\times$-*Kummer-faithful* if, for every finite extension field F' of F contained in $\overline{F}$ and every semi-abelian variety A over F', the $\times$-Kummer homomorphism $A(F') \to H^1(\mathrm{Gal}(\overline{F}/F'), \mathbb{T}_\times(A))$ associated to A is injective. Observe that one verifies easily that this condition does not depend on the choice of $\overline{F}$.

Definition 10.5. We shall refer to the homomorphisms

$$C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \longrightarrow A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})(K), \qquad C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \longrightarrow \widetilde{P}_{\mathcal{Y}/\mathcal{X}}(K)$$

discussed in [6, Chapter II, Theorem 6.2, (3)], [6, Chapter II, Theorem 6.2, (5)] associated to the semi-abelian scheme $P_{\mathcal{Y}/\mathcal{X}}$ over R, equipped with the Prym semi-polarization associated to the Galois étale covering $\mathcal{Y} \to \mathcal{X}$, as the *dual-extension-homomorphism*, the *Prym-period-homomorphism* associated to the Galois étale covering $\mathcal{Y} \to \mathcal{X}$, respectively.

Remark 10.5.1. Suppose that the field K is $\times$-Kummer-faithful. Then one verifies immediately from the various definitions involved (respectively, from [6, Chapter III, Corollary 7.3]) that the dual-extension-homomorphism $C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \to A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})(K)$ (respectively, the Prym-period-homomorphism $C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \to \widetilde{P}_{\mathcal{Y}/\mathcal{X}}(K)$) associated to the Galois étale covering $\mathcal{Y} \to \mathcal{X}$ is a uniquely determined homomorphism $C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \to A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})(K)$ (respectively, $C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \to \widetilde{P}_{\mathcal{Y}/\mathcal{X}}(K)$) such that the image of $c \in C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}})$ by the composite of the homomorphism $C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \to A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})(K)$ (respectively, $C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \to \widetilde{P}_{\mathcal{Y}/\mathcal{X}}(K)$) under consideration with the $\times$-Kummer homomorphism $A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})(K) \to H^1(G_K, \mathbb{T}_\times(A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})))$ (respectively, $\widetilde{P}_{\mathcal{Y}/\mathcal{X}}(K) \to H^1(G_K, \mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})))$ associated to $A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}) \times_R K$ (respectively, $\widetilde{P}_{\mathcal{Y}/\mathcal{X}} \times_R K$) is given by the G_K-torsor under $\mathbb{T}_\times(A(\widetilde{P}_{\mathcal{Y}/\mathcal{X}}))$ (respectively, $\mathbb{T}_\times(\widetilde{P}_{\mathcal{Y}/\mathcal{X}})$) obtained by forming the fiber of $c \otimes 1 \in C(\widetilde{P}^t_{\mathcal{Y}/\mathcal{X}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_\times$ by the third arrow of the second exact sequence of Remark 8.1.1, (v) (respectively, the third arrow of the upper horizontal sequence of the first diagram of Remark 8.1.1, (ii)).

Lemma 10.6. *Let*

- *$\mathcal{X}_1$, $\mathcal{X}_2$ be stable curves over R such that, for each $i \in \{1, 2\}$,*
  - *the generic fiber $X_i \stackrel{\mathrm{def}}{=} \mathcal{X}_i \times_R K$ is smooth over K, and that*

   – *the stable curve* $\underline{\mathcal{X}}_i$ *(cf. Definition 6.1) over* k *is sturdy,*
- $\mathcal{Y}_1$*,* $\mathcal{Y}_2$ *stable curves over* R *such that, for each* $i \in \{1, 2\}$*,*
   – *the generic fiber* $Y_i \stackrel{\text{def}}{=} \mathcal{Y}_i \times_R K$ *is smooth over* K*, that*
   – *the structure morphism* $\mathcal{Y}_i \to \mathrm{Spec}(R)$ *has a splitting, and that*
   – *the stable curve* $\underline{\mathcal{Y}}_i$ *(cf. Definition 6.1) over* k *is split, and*
- $\mathcal{Y}_1 \to \mathcal{X}_1$*,* $\mathcal{Y}_2 \to \mathcal{X}_2$ *Galois étale coverings of degree* l *over* R*.*

Let $\phi_{\Pi_X} : \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ *(cf. Definition 6.3) be a continuous isomorphism over* G_K *that restricts to an isomorphism* $\Pi_{Y_1} \xrightarrow{\sim} \Pi_{Y_2}$ *(cf. Definition 6.3) necessarily over* G_K*. Suppose that the following two conditions are satisfied:*

(a) *The isomorphism* ϕ_{Π_X} *is cyclotomically compatible (cf. Definition 5.7).*
(b) *The isomorphism* $\mathbb{M}^{\text{Prym-vr}}_{Y_1} \xrightarrow{\sim} \mathbb{M}^{\text{Prym-vr}}_{Y_2}$ *(cf. Definition 8.4) determined by* ϕ_{Π_X} *(cf. Lemma 8.6, (i)) arises — relative to the second vertical arrow of the diagram of Lemma 8.5, (i) — from an isomorphism* $\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1} \xrightarrow{\sim} \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}$ *(cf. Definition 8.1) of semi-abelian schemes over* R*.*

Then the following assertions hold:

(i) *The isomorphism* $(((\mathbb{M}^{/\text{vr}}_{Y_1})^D)_{\text{Prym}})^D \xrightarrow{\sim} (((\mathbb{M}^{/\text{vr}}_{Y_2})^D)_{\text{Prym}})^D$ *(cf. Definition 8.4) determined by* ϕ_{Π_X} *(cf. Lemma 8.6, (ii)) arises — relative to the right-hand vertical arrow of the diagram of Lemma 10.3 — from an isomorphism* $C(\widetilde{P}^t_{\mathcal{Y}_1/\mathcal{X}_1}) \xrightarrow{\sim} C(\widetilde{P}^t_{\mathcal{Y}_2/\mathcal{X}_2})$ *(cf. Definition 8.1) of modules. Moreover, the resulting isomorphism* $C(\widetilde{P}^t_{\mathcal{Y}_1/\mathcal{X}_1}) \xrightarrow{\sim} C(\widetilde{P}^t_{\mathcal{Y}_2/\mathcal{X}_2})$ *fits into a commutative diagram of modules*

$$\begin{array}{ccc} C(\widetilde{P}^t_{\mathcal{Y}_1/\mathcal{X}_1}) & \xrightarrow{\lambda^C_{\mathcal{Y}_1/\mathcal{X}_1}} & C(\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}) \\ \wr\downarrow & & \downarrow\wr \\ C(\widetilde{P}^t_{\mathcal{Y}_2/\mathcal{X}_2}) & \xrightarrow[\lambda^C_{\mathcal{Y}_2/\mathcal{X}_2}]{} & C(\widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}) \end{array}$$

(cf. Definition 8.1) — where the right-hand vertical arrow is the isomorphism induced by the isomorphism of (b) (cf. also Remark 9.3.1).

(ii) *If the field K is $\times$-Kummer-faithful, then the diagrams of groups*

$$\begin{array}{ccc} C(\widetilde{P}^t_{\mathcal{Y}_1/\mathcal{X}_1}) & \longrightarrow & A(\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1})(K) \\ \wr\downarrow & & \downarrow\wr \\ C(\widetilde{P}^t_{\mathcal{Y}_2/\mathcal{X}_2}) & \longrightarrow & A(\widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2})(K), \end{array} \qquad \begin{array}{ccc} C(\widetilde{P}^t_{\mathcal{Y}_1/\mathcal{X}_1}) & \longrightarrow & \widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}(K) \\ \wr\downarrow & & \downarrow\wr \\ C(\widetilde{P}^t_{\mathcal{Y}_2/\mathcal{X}_2}) & \longrightarrow & \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}(K) \end{array}$$

(cf. Definition 8.1) — where the first and third vertical arrows are the isomorphism obtained by (i), the second and fourth vertical arrows are the isomorphisms induced by the isomorphism $\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1} \xrightarrow{\sim} \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}$ of condition (b) (cf. Remark 9.3.1), the left-hand upper, lower horizontal arrows are the dual-extension-homomorphisms associated to the Galois étale coverings $\mathcal{Y}_1 \to \mathcal{X}_1$, $\mathcal{Y}_2 \to \mathcal{X}_2$, respectively, and the right-hand upper, lower horizontal arrows are the Prym-period-homomorphisms associated to the Galois étale coverings $\mathcal{Y}_1 \to \mathcal{X}_1$, $\mathcal{Y}_2 \to \mathcal{X}_2$, respectively — commute.

(iii) *Suppose that the two diagrams of (ii) commute. Then the isomorphism $\mathbb{M}^{\mathrm{Prym}}_{Y_1} \xrightarrow{\sim} \mathbb{M}^{\mathrm{Prym}}_{Y_2}$ (cf. Definition 8.4) determined by ϕ_{Π_X} (cf. Lemma 8.6, (i)) arises — relative to the third vertical arrow of the diagram of Lemma 8.5, (i) — from an isomorphism $P_{\mathcal{Y}_1/\mathcal{X}_1} \xrightarrow{\sim} P_{\mathcal{Y}_2/\mathcal{X}_2}$ (cf. Definition 8.1) of semi-abelian schemes over R that is compatible with the respective Prym semi-polarizations associated to the Galois étale coverings $\mathcal{Y}_1 \to \mathcal{X}_1$, $\mathcal{Y}_2 \to \mathcal{X}_2$ (cf. Definition 2.3, (ii)).*

Proof. First, we verify assertion (i). Observe that it follows from condition (a) that we obtain a commutative diagram of topological modules

$$\begin{array}{ccc} \Big(\big((\mathbb{M}^{/\mathrm{vr}}_{Y_1})^D\big)_{\mathrm{Prym}}\Big)^D & \longrightarrow & (\mathbb{M}^{\text{Prym-nd}}_{Y_1})^D \\ \wr\downarrow & & \downarrow\wr \\ \Big(\big((\mathbb{M}^{/\mathrm{vr}}_{Y_2})^D\big)_{\mathrm{Prym}}\Big)^D & \longrightarrow & (\mathbb{M}^{\text{Prym-nd}}_{Y_2})^D \end{array}$$

— where the horizontal arrows are the upper horizontal arrow of the diagram of Lemma 10.3, and the vertical arrows are continuous isomorphisms determined by the isomorphism ϕ_{Π_X} (cf. Lemma 8.6, (ii); condition (b); Remark 9.3.1). Moreover, observe that it follows from condition (b) (cf. also Remark 9.3.1) that the right-hand vertical arrow of this diagram restricts — relative to the left-hand vertical arrow of the diagram of Lemma 10.3 — to an isomorphism $C(\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}) \xrightarrow{\sim} C(\widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2})$ of submodules. In particular, assertion (i) follows from Lemma 10.3.

This completes the proof of assertion (i). Assertion (ii) follows immediately form Lemma 8.5, (i), (ii), (v), and Lemma 8.6, (i), together with assertion (i) and Remark 10.5.1.

Next, we verify assertion (iii). Let us first observe that it follows from condition (b), together with Remark 9.3.1, that we have

(1) a commutative diagram of semi-abelian schemes over R

$$\begin{tikzcd}
0 \arrow[r] & T(\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}) \arrow[r] \arrow[d, "\wr"] & \widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1} \arrow[r] \arrow[d, "\wr"] & A(\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}) \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & T(\widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}) \arrow[r] & \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2} \arrow[r] & A(\widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}) \arrow[r] & 0
\end{tikzcd}$$

— where the horizontal sequences are exact, the vertical arrows are isomorphisms, and the middle vertical arrow is the isomorphism of condition (b).

Next, recall that it follows from assertion (i) that we have

(2) an isomorphism of modules

$$C(\widetilde{P}^t_{\mathcal{Y}_1/\mathcal{X}_1}) \xrightarrow{\sim} C(\widetilde{P}^t_{\mathcal{Y}_2/\mathcal{X}_2}).$$

Next, recall that it follows from our assumption that we have

(3) a commutative diagram of groups

$$\begin{tikzcd}
C(\widetilde{P}^t_{\mathcal{Y}_1/\mathcal{X}_1}) \arrow[r] \arrow[d, "\wr"'] & A(\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1})(K) \arrow[d, "\wr"] \\
C(\widetilde{P}^t_{\mathcal{Y}_2/\mathcal{X}_2}) \arrow[r] & A(\widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2})(K)
\end{tikzcd}$$

— where the left-hand vertical arrow is the isomorphism of (2), the right-hand vertical arrow is the isomorphism determined by the right-hand vertical arrow of the diagram of (1), and the upper, lower horizontal arrows are the dual-extension-homomorphisms associated to the Galois étale coverings $\mathcal{Y}_1 \to \mathcal{X}_1$, $\mathcal{Y}_2 \to \mathcal{X}_2$, respectively.

Next, observe that it follows from Lemma 8.5, (ii), (iv); Lemma 8.6, (i), (ii); Lemma 10.2, (iv), together with Lemma 7.2, (ii), and condition (a), that we have

(4) a commutative diagram of abelian schemes over R

$$\begin{tikzcd} A(\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}) \arrow[r, "\lambda^A_{\mathcal{Y}_1/\mathcal{X}_1}"] \arrow[d, "\wr"'] & A(\widetilde{P}^t_{\mathcal{Y}_1/\mathcal{X}_1}) \arrow[d, "\wr"] \\ A(\widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}) \arrow[r, "\lambda^A_{\mathcal{Y}_2/\mathcal{X}_2}"'] & A(\widetilde{P}^t_{\mathcal{Y}_2/\mathcal{X}_2}). \end{tikzcd}$$

— where the left-hand vertical arrow is the right-hand vertical arrow of the diagram of (1), and the right-hand vertical arrow is the isomorphism determined by the left-hand vertical arrow (cf. also Remark 8.1.1, (iii)).

Next, recall that it follows from assertion (i) that we have

(5) a commutative diagram of modules

$$\begin{tikzcd} C(\widetilde{P}^t_{\mathcal{Y}_1/\mathcal{X}_1}) \arrow[r, "\lambda^C_{\mathcal{Y}_1/\mathcal{X}_1}"] \arrow[d, "\wr"'] & C(\widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}) \arrow[d, "\wr"] \\ C(\widetilde{P}^t_{\mathcal{Y}_2/\mathcal{X}_2}) \arrow[r, "\lambda^C_{\mathcal{Y}_2/\mathcal{X}_2}"'] & C(\widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}) \end{tikzcd}$$

— where the left-hand vertical arrow is the isomorphism of (2), and the right-hand vertical arrow is the isomorphism induced by the left-hand vertical arrow of the diagram of (1).

Next, recall that it follows from our assumption that we have

(6) a commutative diagram of groups

$$\begin{tikzcd} C(\widetilde{P}^t_{\mathcal{Y}_1/\mathcal{X}_1}) \arrow[r] \arrow[d, "\wr"'] & \widetilde{P}_{\mathcal{Y}_1/\mathcal{X}_1}(K) \arrow[d, "\wr"] \\ C(\widetilde{P}^t_{\mathcal{Y}_2/\mathcal{X}_2}) \arrow[r] & \widetilde{P}_{\mathcal{Y}_2/\mathcal{X}_2}(K) \end{tikzcd}$$

— where the left-hand vertical arrow is the isomorphism of (2), the right-hand vertical arrow is the isomorphism determined by the middle vertical arrow of the diagram of (1), and the upper, lower horizontal arrows are the respective Prym-period-homomorphisms associated to the Galois étale coverings $\mathcal{Y}_1 \to \mathcal{X}_1$, $\mathcal{Y}_2 \to \mathcal{X}_2$.

Thus, one concludes immediately from the equivalence $M_{\mathrm{pol}} \colon \mathrm{DD}_{\mathrm{pol}} \xrightarrow{\sim} \mathrm{DEG}_{\mathrm{pol}}$ of categories of [6, Chapter III, Corollary 7.2] that the isomorphism $\mathbb{M}^{\mathrm{Prym}}_{Y_1} \xrightarrow{\sim} \mathbb{M}^{\mathrm{Prym}}_{Y_2}$ determined by ϕ_{Π_X} arises — relative to the third vertical arrow of the diagram of Lemma 8.5, (i) — from an isomorphism $P_{\mathcal{Y}_1/\mathcal{X}_1} \xrightarrow{\sim} P_{\mathcal{Y}_2/\mathcal{X}_2}$ of semi-abelian schemes over R that is compatible with the respective Prym semi-polarizations associated to the Galois étale coverings $\mathcal{Y}_1 \to \mathcal{X}_1$, $\mathcal{Y}_2 \to \mathcal{X}_2$, as desired. This completes the proof of assertion (iii), hence also of Lemma 10.6. $\square$

Definition 10.7. Let Z_1, Z_2 be hyperbolic curves over K and $\phi_{\Pi_Z} : \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) a continuous isomorphism over G_K. Then we shall say that the isomorphism ϕ_{Π_Z} is *Prym-compatible* if the isomorphism ϕ_{Π_Z} is cyclotomically compatible and REP-compatible, and, moreover, in the situation discussed in Definition 9.4, the diagrams of groups

$$\begin{array}{ccc} C(\widetilde{P}^t_{\mathcal{V}_1/\mathcal{W}_1}) & \longrightarrow & A(\widetilde{P}_{\mathcal{V}_1/\mathcal{W}_1})(L) \\ \wr\downarrow & & \downarrow\wr \\ C(\widetilde{P}^t_{\mathcal{V}_2/\mathcal{W}_2}) & \longrightarrow & A(\widetilde{P}_{\mathcal{V}_2/\mathcal{W}_2})(L), \end{array} \qquad \begin{array}{ccc} C(\widetilde{P}^t_{\mathcal{V}_1/\mathcal{W}_1}) & \longrightarrow & \widetilde{P}_{\mathcal{V}_1/\mathcal{W}_1}(L) \\ \wr\downarrow & & \downarrow\wr \\ C(\widetilde{P}^t_{\mathcal{V}_2/\mathcal{W}_2}) & \longrightarrow & \widetilde{P}_{\mathcal{V}_2/\mathcal{W}_2}(L) \end{array}$$

(cf. Definition 8.1) — where the first and third vertical arrows are the isomorphisms obtained by Lemma 10.6, (i) (cf. our assumption that the isomorphism ϕ_{Π_Z} is cyclotomically compatible and REP-compatible), the second and fourth vertical arrows are the isomorphisms induced by the isomorphism $\widetilde{P}_{\mathcal{V}_1/\mathcal{W}_1} \xrightarrow{\sim} \widetilde{P}_{\mathcal{V}_2/\mathcal{W}_2}$ (cf. Remark 9.3.1), the left-hand upper, lower horizontal arrows are the dual-extension-homomorphisms associated to the Galois étale coverings $\mathcal{V}_1 \to \mathcal{W}_1$, $\mathcal{V}_2 \to \mathcal{W}_2$, respectively, and the right-hand upper, lower horizontal arrows are the Prym-period-homomorphisms associated to the Galois étale coverings $\mathcal{V}_1 \to \mathcal{W}_1$, $\mathcal{V}_2 \to \mathcal{W}_2$, respectively — commute.

Lemma 10.8. *Let Z_1, Z_2 be hyperbolic curves over K and $\phi_{\Pi_Z} : \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) a continuous isomorphism over G_K. Suppose that the field K is of characteristic zero and $\times$-Kummer-faithful, and that the isomorphism ϕ_{Π_X} is LSF-compatible. Then the isomorphism ϕ_{Π_Z} is Prym-compatible.*

Proof. This assertion follows from Lemma 7.7, (i); Lemma 9.5; Lemma 10.6, (ii), (ii). $\square$

11. Anabelian consequences

In the present section, we give proofs of the main results of the present paper (cf. Theorem 11.1, Corollary 11.2, Corollary 11.3 below).

Theorem 11.1. *Let*

- *R be a complete discrete valuation ring whose field of fractions we denote by K, and whose residue field we denote by k,*
- *$\overline{K}$ a separable closure of K, and*
- *Z_1, Z_2 hyperbolic curves over K.*

Let $\phi_{\Pi_Z} : \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) be a continuous isomorphism over G_K (cf. Definition 5.1). Suppose that the following three conditions are satisfied.

(a) *The field k is perfect and of characteristic $p > 0$.*
(b) *The isomorphism ϕ_{Π_Z} is LSF-compatible (cf. Definition 7.4) and Prym-compatible (cf. Definition 10.7).*
(c) *If K is of characteristic p, then the hyperbolic curve Z_1 over K is nonisotrivial (cf. Definition 5.8, (i)), and the field k is algebraic over a finite field.*

Then the isomorphism ϕ_{Π_Z} arises from a unique isomorphism $Z_1 \xrightarrow{\sim} Z_2$ over K.

Proof. The uniqueness portion of Theorem 11.1 follows immediately from a similar argument to the argument applied in the first paragraph of the proof of [18, Theorem 14.1], together wth Lemma 7.7, (ii).

Next, to verify the existence portion of Theorem 11.1, let $H_1 \subseteq \Pi_{Z_1}$ be a subnormal (cf. Definition 6.9, (i)) open subgroup of Π_{Z_1}. Write $H_2 \stackrel{\mathrm{def}}{=} \phi_{\Pi_Z}(H_1) \subseteq \Pi_{Z_2}$, $L \subseteq \overline{K}$ for the finite extension field of K that corresponds to the image of H_1 (i.e., of H_2) in G_K, $R_L \subseteq L$ for the normalization of R in L, and k_L for the residue field of R_L. Moreover, for each $i \in \{1, 2\}$, write $W_i^+ \to Z_i^+$ (cf. Definition 5.3) for the finite flat covering of Z_i^+ that corresponds to the open subgroup $H_i \subseteq \Pi_{Z_i}$, which thus implies that we have an identification $H_i = \Pi_{W_i}$ (cf. Definition 5.3), where we write W_i for the open subscheme of W_i^+ obtained by forming the inverse image of $Z_i \subseteq Z_i^+$ (cf. Remark 5.3.1). Then it follows from [4, Corollary 2.7] (cf. also Lemma 5.9, (i)), together with the well-known structure of the maximal pro-l quotient of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic $\neq l$ (cf., e.g., [26, Proposition 1.1, (i), (ii)]), that we may take "H_1" so that,

(1) for each $i \in \{1, 2\}$, there exist a stable curve $\mathcal{W}_i$ over R_L and an isomorphism $\mathcal{W}_i \times_{R_L} L \xrightarrow{\sim} W_i^+$ over L, by means of which we identify $\mathcal{W}_i \times_{R_L} L$ with W_i^+, and, moreover,
(2) for each $i \in \{1, 2\}$, the branch locus of the finite flat covering $W_i^+ \to Z_i^+$ is given by $Z_i^+ \setminus Z_i$.

Next, observe that it follows from Lemma 6.4, together with Lemma 5.9, (ii), (iv), that we may take "H_1" so that W_1^+ has a nonsmooth stable model. Thus, it follows from Lemma 6.10, together with Lemma 5.9, (iii), that we may take "H_1" so that

(3) the stable curve $\underline{\mathcal{W}}_1$ (cf. Definition 6.1) over k_L is untangled and not smooth over k_L,

(4) for each vertex $w \in v(\Gamma_{\overline{\mathcal{W}}_1})$ (cf. Definition 1.2, Definition 6.1) of $\Gamma_{\overline{\mathcal{W}}_1}$, the inequality $\deg(D_w^{\overline{\mathcal{W}}_1}) < g_w^{\overline{\mathcal{W}}_1}$ (cf. Definition 1.3) holds (which thus implies that the stable curve $\underline{\mathcal{W}}_1$ over k_L is sturdy), and, moreover,

(5) for each vertex $w \in v(\Gamma_{\overline{\mathcal{W}}_1})$ of $\Gamma_{\overline{\mathcal{W}}_1}$, the smooth proper curve $(\overline{\mathcal{W}}_1)_w$ (cf. Definition 1.3) over $\overline{k}$ is of gonality ≥ 5.

In particular, it follows from Theorem 4.5 that we may assume without loss of generality, after possibly replacing L by a suitable finite extension field of L in $\overline{K}$, that there exist

- a stable curve $\mathcal{V}_1$ over R_L,
- a prime number l invertible in R, and
- a Galois étale covering $\mathcal{V}_1 \to \mathcal{W}_1$ of degree l over R_L

such that

(6) the induced Galois étale covering $\overline{\mathcal{V}}_1 \to \overline{\mathcal{W}}_1$ (cf. Definition 6.1) over $\overline{k}$ is Prym-faithful and new-ordinary.

Write $H'_1 \subseteq H_1$ for the open subgroup of H_1 that corresponds to the induced Galois étale covering $V_1^+ \stackrel{\text{def}}{=} \mathcal{V}_1 \times_{R_L} L \to W_1^+$, $H'_2 \stackrel{\text{def}}{=} \phi_{\Pi_Z}(H'_1) \subseteq H_2$, $V_2^+ \to W_2^+$ for the finite flat covering of W_2^+ that corresponds to the open subgroup $H'_2 \subseteq H_2$. Then it follows from condition (2) of Lemma 6.5, (i), and Lemma 7.3, (ii) (cf. also (4); Lemma 5.6), that there exist a stable curve $\mathcal{V}_2$ over R_L and an isomorphism $\mathcal{V}_2 \times_{R_L} L \xrightarrow{\sim} V_2^+$ over L, by means of which we identify $\mathcal{V}_2 \times_{R_L} L$ with V_2^+.

Next, observe that since (we have assumed that) the isomorphism ϕ_{Π_Z} is LSF-compatible (cf. also (3); (4); Lemma 6.8, (i)),

(7) the commutative diagram of topological groups

$$\begin{tikzcd}
\Pi^{\log}_{\underline{\mathcal{V}}_1} \arrow[r] \arrow[d, "\wr"'] & \Pi^{\log}_{\underline{\mathcal{W}}_1} \arrow[d, "\wr"] \\
\Pi^{\log}_{\underline{\mathcal{V}}_2} \arrow[r] & \Pi^{\log}_{\underline{\mathcal{W}}_2}
\end{tikzcd}$$

(cf. Definition 6.1, Definition 6.3) — where the horizontal arrows are the natural homomorphisms, and the vertical arrows are continuous isomorphisms — induced by ϕ_{Π_Z} (cf. Lemma 5.6; Lemma 7.3, (ii)) arises from a commutative diagram of log schemes over $\mathrm{Spec}(k_L)^{\log}$ (cf. Definition 6.1)

$$\begin{tikzcd} \underline{\mathcal{V}}_1^{\log} \arrow[r] \arrow[d, "\wr"'] & \underline{\mathcal{W}}_1^{\log} \arrow[d, "\wr"] \\ \underline{\mathcal{V}}_2^{\log} \arrow[r] & \underline{\mathcal{W}}_2^{\log} \end{tikzcd}$$

(cf. Definition 6.1) — where the horizontal arrows are the natural morphisms, and the vertical arrows are isomorphisms.

In particular, one concludes from (4), (6) that

(8) the stable curve $\underline{\mathcal{W}}_2$ over k_L is sturdy, and, moreover,
(9) the induced Galois étale covering $\overline{\underline{\mathcal{V}}}_2 \to \overline{\underline{\mathcal{W}}}_2$ (cf. Definition 6.1) over $\overline{k}$ is Prym-faithful and new-ordinary.

Moreover, it is immediate that we may assume without loss of generality, after possibly replacing L by a suitable finite extension field of L in $\overline{K}$, that

(10) each of the stable curves $\underline{\mathcal{V}}_1$, $\underline{\mathcal{W}}_1$, $\underline{\mathcal{V}}_2$, $\underline{\mathcal{W}}_2$ over k_L is split, and, moreover,
(11) each of the structure morphisms $\mathcal{V}_1 \to \mathrm{Spec}(R_L)$, $\mathcal{V}_2 \to \mathrm{Spec}(R_L)$ has a splitting.

Next, observe that since (we have assumed that) the isomorphism ϕ_{Π_Z} is Prym-compatible, it follows from Lemma 10.6, (iii) (cf. also (4), (6), (8), (9), (10), (11)), that

(12) the isomorphism $\mathbb{M}^{\mathrm{Prym}}_{V_1^+} \xrightarrow{\sim} \mathbb{M}^{\mathrm{Prym}}_{V_2^+}$ (cf. Definition 8.4) determined by ϕ_{Π_Z} (cf. Lemma 8.6, (i)) arises — relative to the third vertical arrow of the diagram of Lemma 8.5, (i) — from an isomorphism $P_{\mathcal{V}_1/\mathcal{W}_1} \xrightarrow{\sim} P_{\mathcal{V}_2/\mathcal{W}_2}$ (cf. Definition 8.1) of semi-abelian schemes over R_L that is compatible with the respective Prym semi-polarizations associated to the Galois étale coverings $\mathcal{V}_1 \to \mathcal{W}_1$, $\mathcal{V}_2 \to \mathcal{W}_2$ (cf. Definition 2.3, (ii)).

In particular, it follows immediately from (6), (7), (9) that the isomorphism $\underline{\mathcal{W}}_1 \xrightarrow{\sim} \underline{\mathcal{W}}_2$ determined by the isomorphism $\underline{\mathcal{W}}_1^{\log} \xrightarrow{\sim} \underline{\mathcal{W}}_2^{\log}$ that appears in (7) lifts uniquely (cf. Lemma 6.2) to an isomorphism $\mathcal{W}_1 \xrightarrow{\sim} \mathcal{W}_2$ over R_L, which restricts to an isomorphism $W_1^+ \xrightarrow{\sim} W_2^+$ over L.

Next, observe that since (we have assumed that) the open subgroup $H_1 \subseteq \Pi_{Z_1}$ of Π_{Z_1} is subnormal, for each $i \in \{1, 2\}$, the finite flat covering $W_i^+ \to Z_i^+$ may be written as the composite of finitely many Galois finite flat coverings. In particular, by applying Galois descent inductively, one concludes immediately that this isomorphism $W_1^+ \xrightarrow{\sim}$

W_2^+ fits into a commutative diagram of schemes over K

$$\begin{tikzcd} W_1^+ \arrow[r] \arrow[d, "\wr"'] & Z_1^+ \arrow[d, "\wr"] \\ W_2^+ \arrow[r] & Z_2^+ \end{tikzcd}$$

— where the horizontal arrows are the natural finite flat coverings, and the vertical arrows are isomorphisms. In particular, it follows from (2) that the lower horizontal arrow restricts to an isomorphism $f_Z \colon Z_1 \xrightarrow{\sim} Z_2$ over K.

Finally, observe that one verifies immediately from the various definitions involved that the assignment "$\phi_{\Pi_Z} \mapsto f_Z$" is functorial, i.e., with respect to isomorphisms. Thus, one concludes formally, by applying the assignment "$\phi_{\Pi_Z} \mapsto f_Z$" to the restrictions of ϕ_{Π_Z} to the various normal open subgroups of Π_{Z_1}, that the isomorphism ϕ_{Π_Z} arises from the isomorphism f_Z. This completes the proof of Theorem 11.1. □

Corollary 11.2. *Let*

- *R be a complete discrete valuation ring whose field of fractions we denote by K and whose residue field we denote by k,*
- *$\overline{K}$ a separable closure of K, and*
- *Z_1, Z_2 hyperbolic curves over K.*

Suppose that the field k is perfect and of characteristic $p > 0$. Let $\phi_{\Pi_Z} : \Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) be a continuous isomorphism over G_K (cf. Definition 5.1). Suppose, moreover, that the field K is of characteristic zero and $\times$-Kummer-faithful (cf. Definition 10.4, (ii)), and that the isomorphism ϕ_{Π_X} is LSF-compatible (cf. Definition 7.4). Then the isomorphism ϕ_{Π_Z} arises from a unique isomorphism $Z_1 \xrightarrow{\sim} Z_2$ over K.

Proof. Observe that since (we have assumed that) the field K is of characteristic zero and $\times$-Kummer-faithful, and the isomorphism ϕ_{Π_X} is LSF-compatible, it follows from Lemma 10.8 that the isomorphism ϕ_{Π_Z} is Prym-compatible. Thus, Corollary 11.2 follows from Theorem 11.1. This completes the proof of Corollary 11.2. □

Corollary 11.3. *Let p be a prime number, K a sub-p-adic field (cf.* [18, Definition 15.4]*), $\overline{K}$ an algebraic closure of K, and Z_1, Z_2 hyperbolic curves over K. Then every continuous isomorphism $\Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ (cf. Definition 5.3) over G_K (cf. Definition 5.1) arises from a unique isomorphism $Z_1 \xrightarrow{\sim} Z_2$ over K. Put another way, if one writes* $\mathrm{Isom}_K(Z_1, Z_2)$ *for the set of isomorphisms $Z_1 \xrightarrow{\sim} Z_2$ over K,* $\mathrm{Isom}_{G_K}(\Pi_{Z_1}, \Pi_{Z_2})$

for the set of continuous isomorphisms $\Pi_{Z_1} \xrightarrow{\sim} \Pi_{Z_2}$ *over* G_K*, and* $\overline{\mathrm{Isom}}_{G_K}(\Pi_{Z_1}, \Pi_{Z_2})$ *for the quotient set of* $\mathrm{Isom}_{G_K}(\Pi_{Z_1}, \Pi_{Z_2})$ *with respect to the natural conjugation action of the kernel of the natural continuous outer homomorphism* $\Pi_{Z_2} \to G_K$*, then the natural map*

$$\mathrm{Isom}_K(Z_1, Z_2) \longrightarrow \overline{\mathrm{Isom}}_{G_K}(\Pi_{Z_1}, \Pi_{Z_2})$$

is bijective.

Proof. If K is a p-adic local field, then this assertion follows from Corollary 11.2, together with Lemma 7.5; Lemma 7.7, (i); [15, Theorem 7]. The general case follows then immediately from this case, together with a "formal argument", i.e., applied in the proof of [18, Corollary 15.5]. □

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