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A Note on Isogenous Fibers in Families of Abelian Varieties over Algebraic Closures of Finite Fields

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Abstract. — In the present paper, we discuss isogenous fibers in families of abelian varieties over smooth varieties. In particular, in the present paper, we prove, roughly speaking, that, for a prime number l and a smooth variety over an algebraic closure of a finite field, there are many closed points x of the variety that admit closed points y of the variety such that, $x \neq y$, and, moreover, for an arbitrary abelian scheme over the variety that is able to have a level-l structure, the fiber of the abelian scheme at x is isogenous to the fiber of the abelian scheme at y.

Contents

Introduction

In the present paper, we discuss isogenous fibers in families of abelian varieties over smooth varieties. In the present Introduction, let K be a perfect field, $\overline{K}$ an algebraic closure of K, V^+ a smooth projective variety of positive dimension over K [i.e., a scheme of positive dimension that is smooth, projective, and geometrically connected over K], and $V \subseteq V^+$ a nonempty open subscheme of V^+. Write $V_{\overline{K}} \stackrel{\text{def}}{=} V \times_K \overline{K}$.

For two [necessarily $\overline{K}$-rational] closed points x, $y \in V_{\overline{K}}$ of $V_{\overline{K}}$, we shall say that $x \in V_{\overline{K}}$ is *isogenuous-equivalent* to $y \in V_{\overline{K}}$ [cf. Definition 2.1, (i)] if, for every abelian scheme $\overline{A} \to V_{\overline{K}}$ over $V_{\overline{K}}$, the abelian variety over $\overline{K}$ obtained by forming the fiber of $\overline{A} \to V_{\overline{K}}$ at $x \in V_{\overline{K}}$ is isogenous to the abelian variety over $\overline{K}$ obtained by forming the fiber of $\overline{A} \to V_{\overline{K}}$ at $y \in V_{\overline{K}}$. Write

$$\mathrm{IE}(V_{\overline{K}})$$

for the subset of the underlying set of $V_{\overline{K}}$ consisting of closed points $a \in V_{\overline{K}}$ that satisfy the following condition: There exists a closed point $b \in V_{\overline{K}}$ of $V_{\overline{K}}$ such that $a \neq b$, and, moreover, the closed point $a \in V_{\overline{K}}$ is isogenuous-equivalent to the closed point $b \in V_{\overline{K}}$ [cf. Definition 2.1, (iii)]. In particular, for a closed point $a \in V_{\overline{K}}$ of $V_{\overline{K}}$,

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this closed point $a \in V_{\overline{K}}$ is contained in the set $\mathrm{IE}(V_{\overline{K}})$ if and only if $V_{\overline{K}}$ has a closed point $b \in V_{\overline{K}}$ such that $a \neq b$, and, moreover, for every abelian scheme $\overline{A} \to V_{\overline{K}}$ over $V_{\overline{K}}$, the fiber of $\overline{A} \to V_{\overline{K}}$ at $a \in V_{\overline{K}}$ is isogenous over $\overline{K}$ to the fiber of $\overline{A} \to V_{\overline{K}}$ at $b \in V_{\overline{K}}$.

Therefore, roughly speaking, the larger this subset $\mathrm{IE}(V_{\overline{K}})$ is, the poorer the geometry of abelian schemes over $V_{\overline{K}}$ is. One main motivation of the research of the present paper is the study of this subset $\mathrm{IE}(V_{\overline{K}})$. In particular, in the present paper, we prove that if the field K is finite, and the maximal pro-l quotient of every open subgroup of the étale fundamental group $\pi_1^{\text{ét}}(V_{\overline{K}})$ of $V_{\overline{K}}$ is finite for some prime number l, then every closed point of $V_{\overline{K}}$ is contained in the set $\mathrm{IE}(V_{\overline{K}})$ [cf. Proposition 2.2].

The main result of the present paper is a result with respect to a variant $\mathrm{wIE}(V_{\overline{K}}, l)$ of the subset $\mathrm{IE}(V_{\overline{K}})$ whose definition is given as follows. Let l be a prime number invertible in K. For two [necessarily $\overline{K}$-rational] closed points x, $y \in V_{\overline{K}}$ of $V_{\overline{K}}$, we shall say that $x \in V_{\overline{K}}$ is *l-weakly isogenuous-equivalent* to $y \in V_{\overline{K}}$ [cf. Definition 2.1, (ii)] if, for every abelian scheme $\overline{A} \to V_{\overline{K}}$ over $V_{\overline{K}}$ such that the natural action of the étale fundamental group $\pi_1^{\text{ét}}(V_{\overline{K}})$ of $V_{\overline{K}}$ on the module of l-torsion points of $\overline{A}$ factors through a pro-l quotient of $\pi_1^{\text{ét}}(V_{\overline{K}})$, the abelian variety over $\overline{K}$ obtained by forming the fiber of $\overline{A} \to V_{\overline{K}}$ at $x \in V_{\overline{K}}$ is isogenous to the abelian variety over $\overline{K}$ obtained by forming the fiber of $\overline{A} \to V_{\overline{K}}$ at $y \in V_{\overline{K}}$. Now observe that it is immediate that if a given abelian scheme $\overline{A} \to V_{\overline{K}}$ over $V_{\overline{K}}$ is able to admit a level-l structure, then the natural action of $\pi_1^{\text{ét}}(V_{\overline{K}})$ on the module of l-torsion points of $\overline{A}$ is trivial, hence also factors through a pro-l quotient of $\pi_1^{\text{ét}}(V_{\overline{K}})$. Write

$$\mathrm{wIE}(V_{\overline{K}}, l)$$

for the subset of the underlying set of $V_{\overline{K}}$ consisting of closed points $a \in V_{\overline{K}}$ that satisfy the following condition: There exists a closed point $b \in V_{\overline{K}}$ of $V_{\overline{K}}$ such that $a \neq b$, and, moreover, the closed point $a \in V_{\overline{K}}$ is l-weakly isogenuous-equivalent to the closed point $b \in V_{\overline{K}}$ [cf. Definition 2.1, (iii)]. In particular, for a closed point $a \in V_{\overline{K}}$ of $V_{\overline{K}}$,

this closed point $a \in V_{\overline{K}}$ is contained in the set $\mathrm{wIE}(V_{\overline{K}}, l)$ if and only if $V_{\overline{K}}$ has a closed point $b \in V_{\overline{K}}$ such that $a \neq b$, and, moreover, for every abelian scheme $\overline{A} \to V_{\overline{K}}$ over $V_{\overline{K}}$ such that the natural action of $\pi_1^{\text{ét}}(V_{\overline{K}})$ on the module of l-torsion points of $\overline{A}$ factors through a pro-l quotient of $\pi_1^{\text{ét}}(V_{\overline{K}})$, the fiber of $\overline{A} \to V_{\overline{K}}$ at $a \in V_{\overline{K}}$ is isogenous over $\overline{K}$ to the fiber of $\overline{A} \to V_{\overline{K}}$ at $b \in V_{\overline{K}}$.

Therefore, roughly speaking, the larger this subset $\mathrm{wIE}(V_{\overline{K}}, l)$ is, the poorer the geometry of abelian schemes over $V_{\overline{K}}$ that are able to admit level-l structures is. Moreover, one verifies easily that the inclusion

$$\mathrm{IE}(V_{\overline{K}}) \subseteq \mathrm{wIE}(V_{\overline{K}}, l)$$

holds [cf. Remark 2.1.1]. The main result of the present paper is as follows [cf. Corollary 2.8]:

Theorem A. — *Suppose that the field K is finite. Then the set* $\mathrm{wIE}(V_{\overline{K}}, l)$ *is dense in* $V_{\overline{K}}$.

Note that this main result may essentially be derived from a refinement of the argument given in [2] concerning pro-l Galois sections of hyperbolic curves over finite fields [cf. Proposition 2.5, Remark 2.5.1].

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1. Mod l Galois representations associated to varieties over finite fields

In the present §1, we discuss the mod l Galois representations associated to varieties over finite fields. In the present §1, let

- p, l be distinct prime numbers,
- $\mathbb{F}$ a finite field of characteristic p,
- $\overline{\mathbb{F}}$ an algebraic closure of $\mathbb{F}$,
- X^+ a smooth projective variety of positive dimension over $\mathbb{F}$ [i.e., a scheme of positive dimension that is smooth, projective, and geometrically connected over $\mathbb{F}$], and
- $X \subseteq X^+$ a nonempty open subscheme of X^+.

For each extension field $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ of $\mathbb{F}$ in $\overline{\mathbb{F}}$, write

- $G_{\mathbb{F}'} \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}')$ for the absolute Galois group of $\mathbb{F}'$ determined by the algebraic closure $\overline{\mathbb{F}}$ and
- $X_{\mathbb{F}'} \stackrel{\text{def}}{=} X \times_{\mathbb{F}} \mathbb{F}' \hookrightarrow X^+_{\mathbb{F}'} \stackrel{\text{def}}{=} X^+ \times_{\mathbb{F}} \mathbb{F}'$ for the natural open immersion.

Definition 1.1. — Let G be a topological group. Then we shall write

- G^{ab} for the topological abelianization of G [i.e., the quotient of G by the normal closed subgroup obtained by forming the closure of the commutator subgroup of G] and
- $G^{\mathrm{ab}/l}$ for the quotient of G^{ab} by the [necessarily normal closed] subgroup obtained by forming the closure of $l \cdot G^{\mathrm{ab}} \subseteq G^{\mathrm{ab}}$.

Definition 1.2. — For each extension field $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ of $\mathbb{F}$ in $\overline{\mathbb{F}}$, write

- $\pi_1^{\text{ét}}(X^+_{\mathbb{F}'}) \subseteq \pi_1^{\text{ét}}(X^+)$, $\pi_1^{\text{ét}}(X_{\mathbb{F}'}) \subseteq \pi_1^{\text{ét}}(X)$ for the respective étale fundamental groups of $X^+_{\mathbb{F}'}$, X^+, $X_{\mathbb{F}'}$, X, relative to suitable choices of basepoints,
- $\pi_1^{\text{gm-}l}(X^+_{\mathbb{F}'}) \subseteq \pi_1^{\text{gm-}l}(X^+)$, $\pi_1^{\text{gm-}l}(X_{\mathbb{F}'}) \subseteq \pi_1^{\text{gm-}l}(X)$ for the respective quotients of $\pi_1^{\text{ét}}(X^+_{\mathbb{F}'})$, $\pi_1^{\text{ét}}(X^+)$, $\pi_1^{\text{ét}}(X_{\mathbb{F}'})$, $\pi_1^{\text{ét}}(X)$ by the kernels of the natural continuous surjective homomorphisms from $\pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})$, $\pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})$, $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$, $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$ onto the maximal pro-l quotients of $\pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})$, $\pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})$, $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$, $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$, and
- $\pi_1^{\text{gm-ab}/l}(X^+_{\mathbb{F}'}) \subseteq \pi_1^{\text{gm-ab}/l}(X^+)$, $\pi_1^{\text{gm-ab}/l}(X_{\mathbb{F}'}) \subseteq \pi_1^{\text{gm-ab}/l}(X)$ for the respective quotients of $\pi_1^{\text{ét}}(X^+_{\mathbb{F}'})$, $\pi_1^{\text{ét}}(X^+)$, $\pi_1^{\text{ét}}(X_{\mathbb{F}'})$, $\pi_1^{\text{ét}}(X)$ by the kernels of the natural continuous surjective

homomorphisms $\pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}}) \twoheadrightarrow \pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})^{\text{ab}/l}$, $\pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}}) \twoheadrightarrow \pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})^{\text{ab}/l}$, $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}}) \twoheadrightarrow \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})^{\text{ab}/l}$, $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}}) \twoheadrightarrow \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})^{\text{ab}/l}$.

Thus, we have a natural commutative diagram of profinite groups

$$\begin{array}{ccccccc}
\pi_1^{\text{ét}}(X) & \twoheadrightarrow & \pi_1^{\text{gm-}l}(X) & \twoheadrightarrow & \pi_1^{\text{gm-ab}/l}(X) & \twoheadrightarrow & G_{\mathbb{F}} \\
\downarrow & & \downarrow & & \downarrow & & \| \\
\pi_1^{\text{ét}}(X^+) & \twoheadrightarrow & \pi_1^{\text{gm-}l}(X^+) & \twoheadrightarrow & \pi_1^{\text{gm-ab}/l}(X^+) & \twoheadrightarrow & G_{\mathbb{F}}
\end{array}$$

— where every arrows are surjective.

Lemma 1.3. — *Let V be an irreducible component of $X^+_{\overline{\mathbb{F}}} \setminus X_{\overline{\mathbb{F}}}$. Write $I_V \subseteq \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})^{\text{ab}/l}$ for the inertia subgroup of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})^{\text{ab}/l}$ ($\subseteq \pi_1^{\text{gm-ab}/l}(X)$) associated to V and $G_{\mathbb{F}}(V) \subseteq G_{\mathbb{F}}$ for the image of $N_{\pi_1^{\text{gm-ab}/l}(X)}(I_V)$ in $G_{\mathbb{F}}$. Then the following assertions hold:*

(i) *The subgroup $G_{\mathbb{F}}(V) \subseteq G_{\mathbb{F}}$ of $G_{\mathbb{F}}$ is open.*

(ii) *If V is of codimension ≥ 2 in $X^+_{\overline{\mathbb{F}}}$, then the subgroup I_V is trivial.*

(iii) *There exists a $G_{\mathbb{F}}(V)$-equivariant continuous surjective homomorphism $\mu_l(\overline{\mathbb{F}}) \twoheadrightarrow I_V$, where we write $\mu_l(\overline{\mathbb{F}}) \subseteq \overline{\mathbb{F}}^\times$ for the group of l-th roots of unity in $\overline{\mathbb{F}}$.*

(iv) *The various surjective homomorphisms $\mu_l(\overline{\mathbb{F}}) \twoheadrightarrow I_W$ of* (iii) *— where W ranges over the irreducible components of $X^+_{\overline{\mathbb{F}}} \setminus X^+$ of codimension one in $X^+_{\overline{\mathbb{F}}}$ — and the continuous homomorphism $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})^{\text{ab}/l} \to \pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})^{\text{ab}/l}$ induced by the natural open immersion $X_{\overline{\mathbb{F}}} \hookrightarrow X^+_{\overline{\mathbb{F}}}$ determine an exact sequence of profinite abelian groups*

$$\bigoplus_W \mu_l(\overline{\mathbb{F}}) \longrightarrow \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})^{\text{ab}/l} \longrightarrow \pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})^{\text{ab}/l} \longrightarrow 1.$$

(v) *The maximal pro-l quotient of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$ is topologically finitely generated.*

Proof. — Assertion (i) is immediate. Assertions (ii), (iii), (iv) are formal consequences of the Zariski-Nagata purity theorem [cf., e.g., [8, Exposé X, Théorèm 3.1]] and Abhyankar's lemma [cf., e.g., [8, Exposé XIII, Corollaire 5.3]].

Finally, we verify assertion (v). Observe that it follows from [6, Proposition 2.8.10] that, to verify assertion (v), it suffices to verify that the group $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})^{\text{ab}/l}$ is finite. Moreover, it follows from assertion (iv) that, to verify this assertion, it suffices to verify that the group $\pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})^{\text{ab}/l}$ is finite. On the other hand, it follows immediately from [8, Exposé X, Lemma 2.10] and [8, Exposé X, Corollaire 3.10], together with the well-known theorem of Bertini, that the group $\pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})^{\text{ab}/l}$ is finite, as desired. This completes the proof of assertion (v), hence also of Lemma 1.3. □

Proposition 1.4. — *There exists a finite extension field $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ of $\mathbb{F}$ in $\overline{\mathbb{F}}$ such that the restriction to the open subgroup $G_{\mathbb{F}'} \subseteq G_{\mathbb{F}}$ of the natural action of $G_{\mathbb{F}}$ on $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})^{\text{ab}/l}$ is trivial, hence also factors through a pro-l quotient of $G_{\mathbb{F}'}$.*

Proof. — This assertion follows from Lemma 1.3, (v). □

PROPOSITION 1.5. — *Suppose that the following four conditions are satisfied:*

(1) *The profinite abelian group $\pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})^{\text{ab}}$ is torsion-free.* [*For example, this will be the case if the smooth projective variety X^+ is either a curve or an abelian variety — cf., e.g.,* [7, Corollary 1.2], [8, Exposé XI, Théorème 2.1].]

(2) *The natural action of $G_{\mathbb{F}}$ on the module of l-torsion points of the Albanese variety associated to X^+* [*cf., e.g.,* [4, Proposition A.6, (i)]] *factors through a pro-l quotient of $G_{\mathbb{F}}$.*

(3) *Let V be an irreducible component of $X^+ \setminus X$ of codimension one in X^+. Then the reduced closed subscheme of X^+ whose underlying closed subset is given by V is geometrically connected over $\mathbb{F}$.*

(4) *The integer $\#\mathbb{F} - 1$ is divisible by l.*

Then the natural action of $G_{\mathbb{F}}$ on $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})^{\text{ab}/l}$ factors through a pro-l quotient of $G_{\mathbb{F}}$.

PROOF. — Observe that it follows immediately from [4, Proposition A.6, (iv)], together with condition (1), that there exists a $G_{\mathbb{F}}$-equivariant isomorphism of $\pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})^{\text{ab}/l}$ with the module of l-torsion points of the Albanese variety associated to X^+. Thus, it follows from condition (2) that the natural action of $G_{\mathbb{F}}$ on $\pi_1^{\text{ét}}(X^+_{\overline{\mathbb{F}}})^{\text{ab}/l}$ factors through a pro-l quotient of $G_{\mathbb{F}}$. In particular, it follows immediately from Lemma 1.3, (iv), together with condition (3), that, to verify Proposition 1.5, it suffices to verify that the natural action of $G_{\mathbb{F}}$ on $\mu_l(\overline{\mathbb{F}})$ is trivial. On the other hand, this assertion follows from condition (4). This completes the proof of Proposition 1.5. □

LEMMA 1.6. — *Let Π be a profinite group, and let $\Delta \subseteq \Pi$ be a normal closed subgroup of Π. Write $\Gamma \stackrel{\text{def}}{=} \Pi/\Delta$. Suppose that the following three conditions are satisfied:*

(1) *The profinite group Γ is pronilpotent.*

(2) *The profinite group Δ is pro-l and topologically finitely generated.*

(3) *The natural action of Γ on $\Delta^{\text{ab}/l}$ factors through a pro-l quotient of Γ.*

Then it follows from conditions (1), (2) *that Π is prosolvable. Let $\Pi^{(l)}$, $\Pi^{(\neq l)} \subseteq \Pi$ be pro-l-Sylow, pro-prime-to-l-Hall subgroups of Π, respectively* [*cf.* [6, Corollary 2.3.6], [6, Corollary 2.3.7]]. *Write $\Gamma^{(l)}$, $\Gamma^{(\neq l)} \subseteq \Gamma$ for the pro-l-Sylow, pro-prime-to-l-Hall subgroups of Γ obtained by forming the images of $\Pi^{(l)}$, $\Pi^{(\neq l)} \subseteq \Pi$ in Γ, respectively. Write, moreover, $J \subseteq Z_{\Pi}(\Delta)$ for the inverse image of $\Gamma^{(\neq l)} \subseteq \Gamma$ by the composite $Z_{\Pi}(\Delta) \hookrightarrow \Pi \twoheadrightarrow \Gamma$. Then the following assertions hold:*

(i) *We have an equality $\Gamma^{(l)} \times \Gamma^{(\neq l)} = \Gamma$.*

(ii) *The natural continuous surjective homomorphism $\Pi^{(\neq l)} \twoheadrightarrow \Gamma^{(\neq l)}$ is an isomorphism.*

(iii) *The quotient $\Pi/Z_{\Pi}(\Delta)$ is pro-l.*

(iv) *The inclusion $\Pi^{(\neq l)} \subseteq J$ holds.*

(v) *The composite $J \hookrightarrow \Pi \twoheadrightarrow \Gamma$ determines an exact sequence of profinite groups*

$$1 \longrightarrow Z(\Delta) \longrightarrow J \longrightarrow \Gamma^{(\neq l)} \longrightarrow 1.$$

(vi) *The equality $Z(\Delta) \times \Pi^{(\neq l)} = J$ holds.*

(vii) *The closed subgroup* $\Pi^{(\neq l)} \subseteq J$ *of* J [*cf.* (iv)] *is characteristic.*

(viii) *The closed subgroup* $\Pi^{(\neq l)} \subseteq \Pi$ *of* Π *is normal.*

(ix) *We have an equality* $\Pi^{(l)} \times \Pi^{(\neq l)} = \Pi$.

Proof. — Assertion (i) follows from condition (1). Assertion (ii) follows from condition (2). Next, we verify assertion (iii). Observe that it follows from [1, Corollary 7], together with condition (2), that the kernel of the natural homomorphism $\mathrm{Aut}(\Delta) \to \mathrm{Aut}(\Delta^{\mathrm{ab}/l})$ has a natural structure of pro-l group. Thus, assertion (iii) follows immediately from condition (3) [i.e., by considering the conjugation action of Π on Δ]. This completes the proof of assertion (iii).

Assertion (iv) follows immediately from assertion (iii). Assertion (v) follows from assertions (ii), (iv). Next, we verify assertion (vi). Since the inclusions $\Pi^{(\neq l)} \subseteq J \subseteq Z_\Pi(\Delta)$ hold [cf. assertion (iv)], the conjugation action of $\Pi^{(\neq l)}$ on $Z(\Delta)$ is trivial. In particular, it follows from assertions (ii), (v) that the equality $Z(\Delta) \times \Pi^{(\neq l)} = J$ holds, as desired. This completes the proof of assertion (vi). Assertion (vii) follows from assertion (vi), together with condition (2). Assertion (viii) follows from assertion (vii), together with the [easily verified — cf. assertion (i)] normality of J in Π.

Next, we verify assertion (ix). Observe that it follows immediately from assertion (i) and condition (2) that $\Pi^{(l)} \cdot \Pi^{(\neq l)} = \Pi$. Thus, one verifies easily that, to verify assertion (ix), it suffices to verify that the element γ of Π obtained by forming the commutator of an element of $\Pi^{(l)}$ and an element of $\Pi^{(\neq l)}$ is trivial. To this end, observe that it follows from assertion (i) that $\gamma \in \Delta$. Moreover, it follows from assertion (viii) that $\gamma \in \Pi^{(\neq l)}$. Thus, since $\Delta \cap \Pi^{(\neq l)}$ is trivial [cf. condition (2)], one concludes that γ is trivial, as desired. This completes the proof of assertion (ix), hence also of Lemma 1.6. □

Lemma 1.7. — *Let* Π *be a profinite group, and let* $\Delta \subseteq \Pi$ *be a normal closed subgroup of* Π. *Write* $\Gamma \stackrel{\mathrm{def}}{=} \Pi/\Delta$. *Suppose that the following two conditions are satisfied:*

(1) *The profinite group* Γ *is pronilpotent.*

(2) *The profinite group* Δ *is pro-*l *and topologically finitely generated.*

Then the following two conditions are equivalent:

(a) *The profinite group* Π *is pronilpotent.*

(b) *The natural action of* Γ *on* $\Delta^{\mathrm{ab}/l}$ *factors through a pro-*l *quotient of* Γ.

Proof. — The implication (a) ⇒ (b) is immediate. The implication (b) ⇒ (a) follows from Lemma 1.6, (ii), (ix), together with condition (1). This completes the proof of Lemma 1.7. □

Lemma 1.8. — *Suppose that the natural action of* $G_{\mathbb{F}}$ *on* $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})^{\mathrm{ab}/l}$ *factors through a pro-*l *quotient of* $G_{\mathbb{F}}$. *Write* $G_{\mathbb{F}}^{(\neq l)} \subseteq G_{\mathbb{F}}$ *for the unique pro-prime-to-*l*-Hall subgroup of the profinite abelian group* $G_{\mathbb{F}}$. *Then there exists a unique lifting* $G_{\mathbb{F}}^{(\neq l)} \hookrightarrow \pi_1^{\text{gm-}l}(X)$ *of the natural inclusion* $G_{\mathbb{F}}^{(\neq l)} \hookrightarrow G_{\mathbb{F}}$. *Moreover, the image of this lifting* $G_{\mathbb{F}}^{(\neq l)} \hookrightarrow \pi_1^{\text{gm-}l}(X)$ *is contained in the center of* $\pi_1^{\text{gm-}l}(X)$.

Proof. — Since $G_{\mathbb{F}}$ is abelian, this assertion follows immediately from Lemma 1.6, (ii), (ix), together with Lemma 1.3, (v). □

Lemma 1.9. — *Let Π be a profinite group, and let $\Delta \subseteq \Pi$ be a normal closed subgroup of Π. Write $\Gamma \stackrel{\text{def}}{=} \Pi/\Delta$. Let s_1, $s_2 \colon \Gamma \to \Pi$ be continuous splittings of the natural continuous surjective homomorphism $\Pi \twoheadrightarrow \Gamma$. Suppose that Δ is finite. Then there exists an open subgroup $\Gamma^\circ \subseteq \Gamma$ of Γ such that the equality $s_1(\Gamma^\circ) = s_2(\Gamma^\circ)$ holds.*

Proof. — Observe that since [we have assumed that] the subgroup Δ is finite, to verify Lemma 1.9, we may assume without loss of generality, by replacing Γ by a suitable open subgroup of Γ, that the inclusion $s_1(\Gamma) \subseteq Z_\Pi(\Delta)$ holds. In particular, one verifies easily that the map $\delta \colon \Gamma \to \Delta$ that maps $\gamma \in \Gamma$ to $s_1(\gamma) \cdot s_2(\gamma)^{-1} \in \Delta$ is a continuous homomorphism. Thus, since [we have assumed that] the subgroup Δ is finite, to verify Lemma 1.9, we may assume without loss of generality, by replacing Γ by a suitable open subgroup of Γ, that the image of this homomorphism $\delta \colon \Gamma \to \Delta$ is trivial. Then one concludes that $s_1 = s_2$, as desired. This completes the proof of Lemma 1.9. $\square$

2. Weakly isogenuous-equivalent geometric points

In the present §2, we give a proof of the main theorem of the present paper [cf. Corollary 2.8 below]. In the present §2, we maintain the notational conventions introduced at the beginning of the preceding §1. In particular, we are given the algebraic closure $\overline{\mathbb{F}}$ of the finite field $\mathbb{F}$ and the smooth variety $X_{\overline{\mathbb{F}}}$ of positive dimension over $\overline{\mathbb{F}}$ [i.e., a scheme of positive dimension that is smooth, of finite type, separated, and geometrically connected over $\overline{\mathbb{F}}$]. Let x, $y \in X_{\overline{\mathbb{F}}}$ be two [necessarily $\overline{\mathbb{F}}$-rational] closed points of $X_{\overline{\mathbb{F}}}$. By abuse of notation, write $x \to X_{\overline{\mathbb{F}}}$, $y \to X_{\overline{\mathbb{F}}}$ for the respective $\overline{\mathbb{F}}$-rational geometric points of $X_{\overline{\mathbb{F}}}$ determined by the closed points x, $y \in X_{\overline{\mathbb{F}}}$.

Definition 2.1.

(i) We shall say that $x \in X_{\overline{\mathbb{F}}}$ is *isogenuous-equivalent* to $y \in X_{\overline{\mathbb{F}}}$ if, for every abelian scheme $\overline{A} \to X_{\overline{\mathbb{F}}}$ over $X_{\overline{\mathbb{F}}}$, the abelian variety $\overline{A} \times_{X_{\overline{\mathbb{F}}}} x$ over $\overline{\mathbb{F}}$ is isogenous to the abelian variety $\overline{A} \times_{X_{\overline{\mathbb{F}}}} y$ over $\overline{\mathbb{F}}$.

(ii) We shall say that $x \in X_{\overline{\mathbb{F}}}$ is *l-weakly isogenuous-equivalent* to $y \in X_{\overline{\mathbb{F}}}$ if, for every abelian scheme $\overline{A} \to X_{\overline{\mathbb{F}}}$ over $X_{\overline{\mathbb{F}}}$ such that the natural action of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$ on the module of l-torsion points of $\overline{A}$ factors through a pro-l quotient of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$, the abelian variety $\overline{A} \times_{X_{\overline{\mathbb{F}}}} x$ over $\overline{\mathbb{F}}$ is isogenous to the abelian variety $\overline{A} \times_{X_{\overline{\mathbb{F}}}} y$ over $\overline{\mathbb{F}}$.

(iii) We shall write $\mathrm{IE}(X_{\overline{\mathbb{F}}})$ (respectively, $\mathrm{wIE}(X_{\overline{\mathbb{F}}}, l)$) for the subset of the underlying set of $X_{\overline{\mathbb{F}}}$ consisting of closed points $a \in X_{\overline{\mathbb{F}}}$ that satisfy the following condition: There exists a closed point $b \in X_{\overline{\mathbb{F}}}$ of $X_{\overline{\mathbb{F}}}$ such that $a \neq b$, and, moreover, the closed point $a \in X_{\overline{\mathbb{F}}}$ is isogenuous-equivalent (respectively, l-weakly isogenuous-equivalent) to the closed point $b \in X_{\overline{\mathbb{F}}}$.

Remark 2.1.1. — It is immediate that if $x \in X_{\overline{\mathbb{F}}}$ is isogenuous-equivalent to $y \in X_{\overline{\mathbb{F}}}$, then $x \in X_{\overline{\mathbb{F}}}$ is l-weakly isogenuous-equivalent to $y \in X_{\overline{\mathbb{F}}}$. In particular, the inclusion $\mathrm{IE}(X_{\overline{\mathbb{F}}}) \subseteq \mathrm{wIE}(X_{\overline{\mathbb{F}}}, l)$ holds.

Proposition 2.2. — *Suppose that the maximal pro-l quotient of every open subgroup of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$ is finite. Then $x \in X_{\overline{\mathbb{F}}}$ is isogenuous-equivalent to $y \in X_{\overline{\mathbb{F}}}$. In particular, the set $\mathrm{IE}(X_{\overline{\mathbb{F}}})$ coincides with the set of closed points of $X_{\overline{\mathbb{F}}}$.*

Proof. — Let $\overline{A} \to X_{\overline{\mathbb{F}}}$ be an abelian scheme over $X_{\overline{\mathbb{F}}}$. Then one verifies easily that, to verify Proposition 2.2, we may assume without loss of generality, by replacing $\mathbb{F}$ by a suitable finite extension field of $\mathbb{F}$ in $\overline{\mathbb{F}}$, that there exist an abelian scheme $A \to X$ over X and an isomorphism $A \times_X X_{\overline{\mathbb{F}}} \xrightarrow{\sim} \overline{A}$ over $X_{\overline{\mathbb{F}}}$ of abelian schemes. Write $x_{\mathbb{F}}, y_{\mathbb{F}} \in X$ for the respective images of the composites $x \to X_{\overline{\mathbb{F}}} \to X$, $y \to X_{\overline{\mathbb{F}}} \to X$. Now one verifies easily that, to verify Proposition 2.2, we may assume without loss of generality, by replacing $\mathbb{F}$ by a suitable finite extension field of $\mathbb{F}$ in $\overline{\mathbb{F}}$, that each of $x_{\mathbb{F}}, y_{\mathbb{F}} \in X$ is $\mathbb{F}$-rational. Then since [we have assumed that] the maximal pro-l quotient of every open subgroup of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$ is finite, one verifies immediately from [1, Corollary 7] that the l-adic representation of $\pi_1^{\text{ét}}(X)$ associated to the abelian scheme $A \to X$ over X factors through the quotient of $\pi_1^{\text{ét}}(X)$ by a normal closed subgroup $N \subseteq \pi_1^{\text{ét}}(X)$ of $\pi_1^{\text{ét}}(X)$ such that the intersection $N \cap \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$ is open in $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$. In particular, it follows immediately from Lemma 1.9 that, to verify Proposition 2.2, we may assume without loss of generality, by replacing $\mathbb{F}$ by a suitable finite extension field of $\mathbb{F}$ in $\overline{\mathbb{F}}$, that the characteristic polynomial of the $\#\mathbb{F}$-power Frobenius element with respect to the l-adic representation associated to the fiber of $A \to X$ at $x_{\mathbb{F}} \in X$ coincides with the characteristic polynomial of the $\#\mathbb{F}$-power Frobenius element with respect to the l-adic representation associated to the fiber of $A \to X$ at $y_{\mathbb{F}} \in X$. Then it follows from [5, Appendix I, Theorem 2, (c)] that the abelian variety $\overline{A} \times_{X_{\overline{\mathbb{F}}}} x$ over $\overline{\mathbb{F}}$ is isogenous to the abelian variety $\overline{A} \times_{X_{\overline{\mathbb{F}}}} y$ over $\overline{\mathbb{F}}$, as desired. This completes the proof of Proposition 2.2. □

Definition 2.3.

(i) Let $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ be a finite extension field of $\mathbb{F}$ in $\overline{\mathbb{F}}$. Then we shall say that $x \in X_{\overline{\mathbb{F}}}$ is a *prime-to-l-$\mathbb{F}'$-conjugate* of $y \in X_{\overline{\mathbb{F}}}$ if the following two conditions are satisfied:

(1) The image of the composite $x \to X_{\overline{\mathbb{F}}} \to X_{\mathbb{F}'}$ coincides with the image of the composite $y \to X_{\overline{\mathbb{F}}} \to X_{\mathbb{F}'}$

(2) The extension degree over $\mathbb{F}'$ of the residue field of $X_{\mathbb{F}'}$ at the image of the composite $x \to X_{\overline{\mathbb{F}}} \to X_{\mathbb{F}'}$ [hence also of the composite $y \to X_{\overline{\mathbb{F}}} \to X_{\mathbb{F}'}$ — cf. (1)] is prime to l.

(ii) We shall say that $x \in X_{\overline{\mathbb{F}}}$ is a *prime-to-l-conjugate* of $y \in X_{\overline{\mathbb{F}}}$ if there exists a finite extension field $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ of $\mathbb{F}$ in $\overline{\mathbb{F}}$ such that $x \in X_{\overline{\mathbb{F}}}$ is a prime-to-l-$\mathbb{F}'$-conjugate of $y \in X_{\overline{\mathbb{F}}}$.

Remark 2.3.1.

(i) An example of a prime-to-l-$\mathbb{F}$-conjugate of a closed point of a smooth variety is given as follows: Let d be a positive integer. Suppose that there exists an open immersion $X \hookrightarrow \mathbb{A}^d_{\mathbb{F}}$ over $\mathbb{F}$, by means of which we shall regard X as an open subscheme of $\mathbb{A}^d_{\mathbb{F}}$, hence also of $\mathbb{P}^d_{\mathbb{F}}$. Let r be an integer, and let $c_1, \ldots, c_d$ be elements of $\overline{\mathbb{F}}$. Write $q \stackrel{\text{def}}{=} \#\mathbb{F}$. Suppose, moreover, that

- the inclusion $\{c_1, \ldots, c_{d-1}\} \subseteq \mathbb{F}$ holds, that
- the finite field extension $\mathbb{F}(c_d)/\mathbb{F}$, hence also the finite field extension $\mathbb{F}(c_d^{q^r})/\mathbb{F}$, is of degree prime to l, and that

- the $\overline{\mathbb{F}}$-rational point of $\mathbb{A}^d_{\overline{\mathbb{F}}}$ determined by $(c_1, \ldots, c_d) \in \overline{\mathbb{F}}^d$, hence also the $\overline{\mathbb{F}}$-rational point of $\mathbb{A}^d_{\overline{\mathbb{F}}}$ determined by $(c_1, \ldots, c_{d-1}, c_d^{q^r}) \in \overline{\mathbb{F}}^d$, lies on the open subscheme $X_{\overline{\mathbb{F}}} \subseteq \mathbb{A}^d_{\overline{\mathbb{F}}}$.

Then one verifies easily that the $\overline{\mathbb{F}}$-rational point of $X_{\overline{\mathbb{F}}}$ determined by $(c_1, \ldots, c_d)$ is a prime-to-l-$\mathbb{F}$-conjugate of the $\overline{\mathbb{F}}$-rational point of $X_{\overline{\mathbb{F}}}$ determined by $(c_1, \ldots, c_{d-1}, c_d^{q^r})$.

(ii) One verifies easily that, for the fixed closed point $x \in X_{\overline{\mathbb{F}}}$, the subset of the underlying set of $X_{\overline{\mathbb{F}}}$ consisting of closed points $a \in X_{\overline{\mathbb{F}}}$ that satisfy the following condition is finite: The closed point $x \in X_{\overline{\mathbb{F}}}$ is a prime-to-l-conjugate of the closed point $a \in X_{\overline{\mathbb{F}}}$.

Lemma 2.4. — *The subset of the underlying set of $X_{\overline{\mathbb{F}}}$ consisting of closed points $a \in X_{\overline{\mathbb{F}}}$ that satisfy the following condition is dense in $X_{\overline{\mathbb{F}}}$: There exists a closed point $b \in X_{\overline{\mathbb{F}}}$ of $X_{\overline{\mathbb{F}}}$ such that $a \neq b$, and, moreover, the closed point $a \in X_{\overline{\mathbb{F}}}$ is a prime-to-l-$\mathbb{F}$-conjugate of the closed point $b \in X_{\overline{\mathbb{F}}}$.*

Proof. — Observe that it is immediate that, to verify Lemma 2.4, it suffices to verify that the subset under consideration is nonempty. Next, observe that it follows immediately from the well-known theorem of Bertini that, to verify Lemma 2.4, we may assume without loss of generality, by replacing X^+ by a suitable closed subscheme of X^+, that X is of dimension one. Then it follows from the Weil conjecture for curves over finite fields that there exist a finite extension field $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ of $\mathbb{F}$ in $\overline{\mathbb{F}}$ and an $\mathbb{F}'$-valued point $c \in X(\mathbb{F}')$ of X such that the extension degree $[\mathbb{F}' : \mathbb{F}]$ is prime to l, and, moreover, the closed point of X obtained by forming the image of $c \in X(\mathbb{F}')$ is not $\mathbb{F}$-rational. Then since $G_{\mathbb{F}}$ is abelian, one verifies easily that every $\overline{\mathbb{F}}$-rational geometric point of $X_{\overline{\mathbb{F}}}$ that lies over $c \in X(\mathbb{F}')$ determines an element of the subset under consideration. This completes the proof of Lemma 2.4. □

Proposition 2.5. — *Suppose that the following two conditions are satisfied:*

(1) *The natural action of $G_{\mathbb{F}}$ on $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})^{\text{ab}/l}$ factors through a pro-l quotient of $G_{\mathbb{F}}$.*

(2) *The closed point $x \in X_{\overline{\mathbb{F}}}$ is a prime-to-l-$\mathbb{F}$-conjugate of the closed point $y \in X_{\overline{\mathbb{F}}}$.*

Write $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ for the finite extension field of $\mathbb{F}$ in $\overline{\mathbb{F}}$ obtained by forming the residue field of X at the image of the composite $x \to X_{\overline{\mathbb{F}}} \to X$ [hence also of the composite $y \to X_{\overline{\mathbb{F}}} \to X$ — cf. condition (1) *of Definition* 2.3, (i)] *and $x_{\mathbb{F}'}, y_{\mathbb{F}'} \in X_{\mathbb{F}'}$ for the respective images of the composites $x \to X_{\overline{\mathbb{F}}} \to X_{\mathbb{F}'}$, $y \to X_{\overline{\mathbb{F}}} \to X_{\mathbb{F}'}$. [In particular, each of $x_{\mathbb{F}'}, y_{\mathbb{F}'} \in X_{\mathbb{F}'}$ is $\mathbb{F}'$-rational.] Then the conjugacy class [cf.* [3, Definition 1.1, (i)]] *of pro-l Galois sections of $X_{\mathbb{F}'}$ [cf.* [3, Definition 1.1, (i)]] *which arises from $x_{\mathbb{F}'} \in X_{\mathbb{F}'}(\mathbb{F}')$ [cf.* [3, Definition 1.1, (ii)]] *coincides with the conjugacy class of pro-l Galois sections of $X_{\mathbb{F}'}$ which arises from $y_{\mathbb{F}'} \in X_{\mathbb{F}'}(\mathbb{F}')$.*

Proof. — This assertion follows immediately from Lemma 1.8, together with the various definitions involved. □

Remark 2.5.1. — Proposition 2.5 may be regarded as a refinement of the argument given in [2, Remark 10, (i)].

Theorem 2.6. — *Suppose that the natural action of* $G_{\mathbb{F}}$ *on* $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})^{\text{ab}/l}$ *factors through a pro-*l *quotient of* $G_{\mathbb{F}}$ *[cf., e.g., Proposition* 1.4, *Proposition* 1.5*]. Then if* $x \in X_{\overline{\mathbb{F}}}$ *is a prime-to-*l*-*$\mathbb{F}$*-conjugate of* $y \in X_{\overline{\mathbb{F}}}$*, then* $x \in X_{\overline{\mathbb{F}}}$ *is* l*-weakly isogenuous-equivalent to* $y \in X_{\overline{\mathbb{F}}}$*.*

Proof. — Let $\overline{A} \to X_{\overline{\mathbb{F}}}$ be an abelian scheme over $X_{\overline{\mathbb{F}}}$ such that the natural action of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$ on the module of l-torsion points of $\overline{A}$ factors through a pro-l quotient of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$. Write $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ for the finite extension field of $\mathbb{F}$ in $\overline{\mathbb{F}}$ obtained by forming the residue field of X at the image of the composite $x \to X_{\overline{\mathbb{F}}} \to X$ [hence also of the composite $y \to X_{\overline{\mathbb{F}}} \to X$ — cf. condition (1) of Definition 2.3, (i)]. Then observe that one verifies immediately that there exist

- a finite extension field $\mathbb{F}'' \subseteq \overline{\mathbb{F}}$ of $\mathbb{F}'$ in $\overline{\mathbb{F}}$,
- an abelian scheme $A \to X_{\mathbb{F}''}$ over $X_{\mathbb{F}''}$, and
- an isomorphism $A \times_{X_{\mathbb{F}''}} X_{\overline{\mathbb{F}}} \xrightarrow{\sim} \overline{A}$ over $X_{\overline{\mathbb{F}}}$ of abelian schemes.

Write $x_{\mathbb{F}''}$, $y_{\mathbb{F}''} \in X_{\mathbb{F}''}$ for the respective images of the composites $x \to X_{\overline{\mathbb{F}}} \to X_{\mathbb{F}''}$, $y \to X_{\overline{\mathbb{F}}} \to X_{\mathbb{F}''}$. [In particular, each of $x_{\mathbb{F}''}$, $y_{\mathbb{F}''} \in X_{\mathbb{F}''}$ is $\mathbb{F}''$-rational.] Then since [we have assumed that] the natural action of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$ on the module of l-torsion points of $\overline{A}$ factors through a pro-l quotient of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}})$, one verifies immediately from [1, Corollary 7] that the l-adic representation of $\pi_1^{\text{ét}}(X_{\mathbb{F}''})$ associated to the abelian scheme $A \to X_{\mathbb{F}''}$ over $X_{\mathbb{F}''}$ factors through the natural continuous surjective homomorphism $\pi_1^{\text{ét}}(X_{\mathbb{F}''}) \twoheadrightarrow \pi_1^{\text{gm-}l}(X_{\mathbb{F}''})$. In particular, it follows immediately from Proposition 2.5 that the characteristic polynomial of the $\#\mathbb{F}''$-power Frobenius element with respect to the l-adic representation associated to the fiber of $A \to X_{\mathbb{F}''}$ at $x_{\mathbb{F}''} \in X_{\mathbb{F}''}$ coincides with the characteristic polynomial of the $\#\mathbb{F}''$-power Frobenius element with respect to the l-adic representation associated to the fiber of $A \to X_{\mathbb{F}''}$ at $y_{\mathbb{F}''} \in X_{\mathbb{F}''}$. Then it follows from [5, Appendix I, Theorem 2, (c)] that the abelian variety $\overline{A} \times_{X_{\overline{\mathbb{F}}}} x$ over $\overline{\mathbb{F}}$ is isogenous to the abelian variety $\overline{A} \times_{X_{\overline{\mathbb{F}}}} y$ over $\overline{\mathbb{F}}$, as desired. This completes the proof of Theorem 2.6. □

Corollary 2.7. — *There exists a finite extension field* $\mathbb{F}_0 \subseteq \overline{\mathbb{F}}$ *of* $\mathbb{F}$ *in* $\overline{\mathbb{F}}$ *such that, for an arbitrary finite extension field* $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ *of* $\mathbb{F}_0$ *in* $\overline{\mathbb{F}}$*, if* $x \in X_{\overline{\mathbb{F}}}$ *is a prime-to-*l*-*$\mathbb{F}'$*-conjugate of* $y \in X_{\overline{\mathbb{F}}}$*, then* $x \in X_{\overline{\mathbb{F}}}$ *is* l*-weakly isogenuous-equivalent to* $y \in X_{\overline{\mathbb{F}}}$*.*

Proof. — This assertion follows from Theorem 2.6, together with Proposition 1.4. □

Corollary 2.8. — *The set* $\text{wIE}(X_{\overline{\mathbb{F}}}, l)$ *is dense in* $X_{\overline{\mathbb{F}}}$*.*

Proof. — This assertion follows from Theorem 2.6, together with Proposition 1.4 and Lemma 2.4. □

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