Trace decategorification of categorified quantum $sl(2)$

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Trace of a linear category

Let C be a (small) k-linear category, with k a commutative, unital ring.

Definition

The *trace* of C is defined by

$$\mathrm{Tr}(C) = \left(\bigoplus_{x \in \mathrm{Ob}(C)} \mathrm{End}_C(x) \right) / \, \mathrm{Span}_k\{fg - gf\},$$

where f, g run through all $f\colon x \to y$, $g\colon y \to x$ in C.
$\mathrm{Tr}(C)$ is also called the 0th Hochschild–Mitchell homology $HH_0(C)$.

Fact

The trace is functorial:

$$\mathrm{Tr}\colon \{\textit{linear categories}\} \to \mathbf{Mod}_k$$

In fact, for a linear functor $F\colon C \to D$, we set $\mathrm{Tr}(F)([f]) = [F(f)]$.

Trace of the additive closure

Let $C^{\oplus}$ be the additive closure of C.
Then the inclusion functor $i\colon C \to C^{\oplus}$ induces an isomorphism

$$\mathrm{Tr}(i)\colon \mathrm{Tr}(C) \xrightarrow{\cong} \mathrm{Tr}(C^{\oplus}).$$

Indeed, the inverse is given by the "trace"

$$\mathrm{Tr}(C^{\oplus}) \ni [(f_{i,j})_{i,j}] \mapsto \sum_i [f_{i,i}] \in \mathrm{Tr}(C).$$

for $(f_{i,j})_{i,j}\colon \bigoplus_i x_i \to \bigoplus_i x_i$, $f_{i,j}\colon x_i \to x_j$.

To compute the trace $\mathrm{Tr}(D)$ of an additive category D, it suffices to compute $\mathrm{Tr}(C)$ for a full subcategory C of D such that $D \simeq C^{\oplus}$.

Trace of the Karoubi envelope

Let Kar(C) denote the *Karoubi envelope* (or *idempotent completion*) of C, which is the "universal" linear category containing C in which idempotents split, and which can be constructed by

$$\mathrm{Ob}(\mathrm{Kar}(C)) = \{(x,e) \mid x \in \mathrm{Ob}(C),\ e\colon x \to x,\ e^2 = e\},$$
$$\mathrm{Kar}(C)((x,e),(y,e')) = \{f\colon x \to y \mid f = e' f e\}.$$

Then the inclusion functor $i\colon C \to Kar(C)$, $x \mapsto (x, 1_x)$, induces

$$\mathrm{Tr}(i)\colon\ \mathrm{Tr}(C) \xrightarrow{\cong} \mathrm{Tr}(\mathrm{Kar}(C)).$$

Indeed, the inverse is given by

$$\mathrm{Tr}(\mathrm{Kar}(C)) \ni [f\colon (x,e) \to (x,e)] \mapsto [f] \in \mathrm{Tr}(C).$$

Chern character

For an additive category C, let $K_0(C) \in \mathbf{Ab}$ denote the *split Grothendieck group* of C, defined by

$$K_0(C) = \frac{\mathbb{Z}(\mathrm{Ob}(C)/\cong)}{[x \oplus y]_{\cong} = [x]_{\cong} + [y]_{\cong}, \quad x, y \in \mathrm{Ob}(C)}.$$

The *Chern character map* is the $\mathbb{Z}$-linear map

$$\mathrm{ch}\colon K_0(C) \to \mathrm{Tr}(C)$$

defined by

$$\mathrm{ch}([x]_{\cong}) = [1_x].$$

K_0 and Tr: Injectivity of ch

In many cases, the Chern character map ch is injective.
Indeed, we have the following.

Proposition (Beliakova–H–Lauda–Webster)

Let k be a perfect field.
Let C be a k-linear Krull-Schmidt category such that $\dim_k \mathrm{End}_C(x) < \infty$ for each indecomposable object x in C.
Then

$$\mathrm{ch}\colon K_0(C) \otimes k \to Tr(C)$$

is injective.

Trace of a 2-category

Let $\mathbf{C}$ be a linear 2-category.
Then, for $x, y \in \mathrm{Ob}(\mathbf{C})$, the composition functor

$$\circ\colon \ \mathbf{C}(y,z) \times \mathbf{C}(x,y) \to \mathbf{C}(x,z)$$

induces a bilinear map

$$\circ\colon \ \mathrm{Tr}(\mathbf{C}(y,z)) \times \mathrm{Tr}(\mathbf{C}(x,y)) \to \mathrm{Tr}(\mathbf{C}(x,z))$$

Thus, we have a linear category $\mathrm{Tr}(\mathbf{C})$ with

- $\mathrm{Ob}(\mathrm{Tr}(\mathbf{C})) := \mathrm{Ob}(\mathbf{C})$,
- $\mathrm{Tr}(\mathbf{C})(x,y) := \mathrm{Tr}(\mathbf{C}(x,y))$.

This gives a functor

$$\mathrm{Tr}\colon \ \{\text{linear 2-categories}\} \to \{\text{linear categories}\}$$

Symmetric functions and symmetric polynomials

Let $\Lambda = \mathbb{Z}[e_1, e_2, \ldots] = \mathbb{Z}[h_1, h_2, \ldots] = \bigoplus_{\lambda \text{ partitions}} \mathbb{Z}s_\lambda$ denote the ring of symmetric functions, where

$$\begin{aligned} e_k &= \text{the elementary symmetric function of degree } k, \\ h_k &= \text{the complete symmetric function of degree } k, \\ s_\lambda &= \text{the Schur function associated to } \lambda. \end{aligned}$$

For $m \geq 0$, set $\Lambda_m := \mathbb{Z}[x_1, \ldots, x_m]^{S_m}$, the ring of symmetric polynomials.

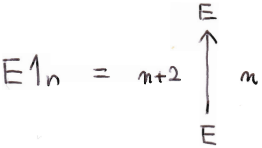
2-category $\mathcal{U}^*$: objects and 1-morphisms

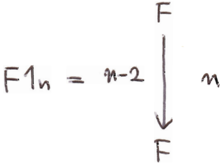
$\mathcal{U}^*$ is the additive 2-category enriched in graded abelian groups such that

- $\mathrm{Ob}(\mathcal{U}^*) = \mathbb{Z} =$ (weight lattice of sl_2),
- 1-morphisms are generated (under $\circ$ and $\oplus$) by

$$E1_n\colon\ n \to n+2, \qquad F1_n\colon\ n \to n-2,$$

depicted by



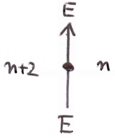


Compositions are abbreviated as

$$(E1_{n+2})(E1_n) = E^2 1_n, \quad (E1_n)(E1_{n-2})(F1_n) = E^2 F1_n, \quad \text{etc.}$$

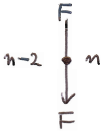
2-category $\mathcal{U}^*$: generating 2-morphisms

The 2-morphisms are generated by



$: E1_n \longrightarrow E1_n$

degree 2



$: F1_n \longrightarrow F1_n$

degree 2

$: E^21_n \longrightarrow E^21_n$

degree −2

$: F^21_n \longrightarrow F^21_n$

degree −2

$: FE1_n \longrightarrow 1_n$

degree $n-1$

$: EF1_n \longrightarrow 1_n$

degree $-n-1$

$: 1_n \longrightarrow FE1_n$

degree $n-1$

$: 1_n \longrightarrow EF1_n$

degree $-n-1$

$e_m1_n = \boxed{e_m}\ ^n \; : 1_n \longrightarrow 1_n \quad (m \geq 1)$ degree $2m$

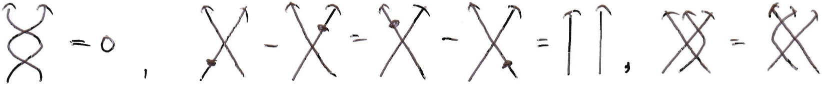
2-category $\mathcal{U}^*$: relations (1)

The 2-morphisms are subject to the following relations.

Isotopy

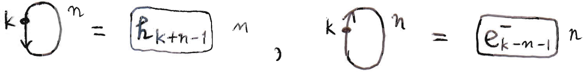
· where we set

NilHecke

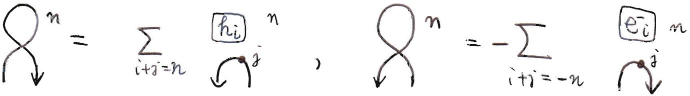


2-category $\mathcal{U}^*$: relations (2)

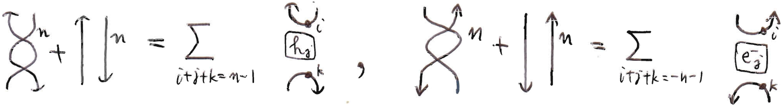
Bubbles



Loops



Bigons



(We set $e_k^- = (-1)^k e_k$.)

2-category $\dot{\mathcal{U}}^*$

Let $\dot{\mathcal{U}}^* = \mathrm{Kar}(\mathcal{U}^*)$, the Karoubi envelope of $\mathcal{U}^*$.
Thus,

- $\mathrm{Ob}(\dot{\mathcal{U}}^*) = \mathrm{Ob}(\mathcal{U}^*) = \mathbb{Z}$,
- $\dot{\mathcal{U}}^*(m, n) = \mathrm{Kar}(\mathcal{U}^*(m, n))$.

In $\dot{\mathcal{U}}^*$, there are 1-morphisms

$$E^{(a)}1_n = (E^a 1_n, u_a)\colon\ n \to n + 2a,$$
$$F^{(a)}1_n = (F^a 1_n, u_a^*)\colon\ n \to n - 2a$$

corresponding to the divided powers

$$E^{(a)} = E^a/[a]!, \quad F^{(a)} = F^a/[a]!.$$

2-categories $\mathcal{U}$ and $\dot{\mathcal{U}}$

Let $\mathcal{U}$ be the additive 2-category enriched over abelian groups such that

- $\mathrm{Ob}(\mathcal{U}) = \mathrm{Ob}(\mathcal{U}^*) = \mathbb{Z}$,
- 1-morphisms are generated under $\oplus$ by "degree shifts" $f\langle j\rangle$, $j \in \mathbb{Z}$, where f is a monomial 1-morphisms in $\mathcal{U}^*$. For example, $E^2 1_n\langle 2\rangle\colon\ n \to n+4$.
- For 1-morphisms $f\langle j\rangle, g\langle j'\rangle\colon\ m \to n$, we set

$$\mathcal{U}(m,n)(f\langle j\rangle, g\langle j'\rangle) := \mathcal{U}^*(m,n)(f,g)_{j'-j},$$

  the degree $j'-j$ part of $\mathcal{U}^*(m,n)(f,g)$.

Let $\dot{\mathcal{U}} = \mathrm{Kar}(\mathcal{U})$, the Karoubi envelope of the 2-category $\mathcal{U}$.
I.e., $\mathrm{Ob}(\dot{\mathcal{U}}) = \mathrm{Ob}(\mathcal{U}) = \mathbb{Z}$ and $\dot{\mathcal{U}}(m,n) = \mathrm{Kar}(\mathcal{U}(m,n))$.

$K_0(\dot{\mathcal{U}})$ and $\mathrm{Tr}(\dot{\mathcal{U}})$

Theorem (Lauda, Khovanov–Lauda–Mackaay–Stošić)

The split Grothendieck group $K_0(\dot{\mathcal{U}})$ of $\dot{\mathcal{U}}$ is isomorphic to the Beilinson-Lusztig-MacPherson idempotented integral form of the quantized enveloping algebra of sl_2:

$$K_0(\dot{\mathcal{U}}) \cong \dot{\mathbf{U}}(sl_2) \qquad (\text{over } \mathbb{Z}).$$

Theorem (Beliakova–H–Lauda–Živković)

The Chern character map ch *for $\dot{\mathcal{U}}$ is an isomorphism*

$$\mathrm{ch}\colon\ K_0(\dot{\mathcal{U}}) \xrightarrow{\cong} \mathrm{Tr}(\dot{\mathcal{U}}) \qquad (\text{over } \mathbb{Z}).$$

Remark: This theorem is generalized to the simply laced case over a field (Beliakova–H–Lauda–Webster).

Remark: We also have $HH_k(\dot{\mathcal{U}}) = 0$ for $k > 0$.

Proof (sketch)

We use results in [KLMS]. Let $m, n \in \mathbb{Z}$, $m - n \in 2\mathbb{Z}$.
Define $B_{m,n} \subset \mathrm{Ob}(\dot{\mathcal{U}}(m,n))$ by

$$B_{m,n} = \begin{cases} \{1_n F^{(b)} E^{(a)} 1_m \langle j \rangle \mid a, b \geq 0,\ 2(a-b) = n-m,\ j \in \mathbb{Z}\} & \text{if } m+n \geq 0, \\ \{1_n E^{(a)} F^{(b)} 1_m \langle j \rangle \mid a, b \geq 0,\ 2(a-b) = n-m,\ j \in \mathbb{Z}\} & \text{if } m+n < 0. \end{cases}$$

Let $\mathcal{B}(m,n) = \dot{\mathcal{U}}(m,n)|_{B_{m,n}}$, the full subcategory with $\mathrm{Ob} = B_{m,n}$. Then

- $\dot{\mathcal{U}}(m,n) = \mathcal{B}(m,n)^{\oplus}$,
- $K_0(\dot{\mathcal{U}}(m,n)) \cong \mathbb{Z} \cdot B_{m,n}$,
- for $x \in B_{m,n}$, we have $\mathrm{End}_{\mathcal{B}(m,n)}(x) = \mathbb{Z} \cdot 1_x$,
- for $x, y \in B_{m,n}$, $x \neq y$, we have either

$$\mathcal{B}(m,n)(x,y) = 0 \quad \text{or} \quad \mathcal{B}(m,n)(y,x) = 0.$$

Therefore

$$\mathrm{Tr}(\dot{\mathcal{U}}(m,n)) \cong \mathrm{Tr}(\mathcal{B}(m,n)) = \bigoplus_{x \in B_{m,n}} \mathbb{Z} \cdot [1_x] \cong \mathbb{Z} \cdot B_{m,n} \cong K_0(\dot{\mathcal{U}}(m,n)).$$

Current algebra $U(sl_2[t])$

The *current algebra* $U(sl_2[t])$ of $sl_2 = \mathbb{C}\{H, E, F\}$ is generated by

$$H_i := H \otimes t^i, \quad E_i := E \otimes t^i, \quad F_i := F \otimes t^i \quad (i \geq 0)$$

with relations

$$[H_i, H_j] = 0, \quad [E_i, E_j] = 0, \quad [F_i, F_j] = 0,$$
$$[H_i, E_j] = 2E_{i+j}, \quad [H_i, F_j] = -2F_{i+j}, \quad [E_i, F_j] = H_{i+j}.$$

$U(sl_2[t])$ has the idempotented form $\dot{U}(sl_2[t])$, which is the linear category with $\mathrm{Ob} = \mathbb{Z}$ such that

$$\dot{U}(sl_2[t])(m, n) = U(sl_2[t])/(U(sl_2[t])(H_0 - m) + (H_0 - n)U(sl_2[t]))$$

Garland's integral form $U_{\mathbb{Z}}(sl_2[t])$ of $U(sl_2[t])$ is the $\mathbb{Z}$-subalgebra generated by $E_i^{(a)} = E_i^a/a!$, $F_i^{(a)} = F_i^a/a!$ for $i \geq 0$, $a > 0$.
We have the idempotented form $\dot{U}_{\mathbb{Z}}(sl_2[t])$ as well.

Current algebra and $\mathrm{Tr}(\mathcal{U}^*)$

Theorem (Beliakova–H–Lauda–Živković)

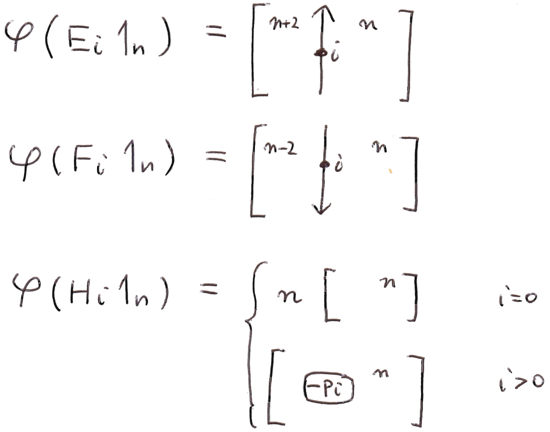
There is an isomorphism of linear categories

$$\mathrm{Tr}(\mathcal{U}^*) \cong \dot{U}_{\mathbb{Z}}(sl_2[t]).$$

Remark

Over a field, the theorem is generalized to simply laced case by Beliakova–H–Lauda–Webster.
The proof uses the isomorphisms between the trace of the cyclotomic quotients of the KLR algebras and the Weyl modules of the current algebra proved by Shan-Varagnolo-Vasserot and B-H-L-W.

We define a map $\varphi\colon\ \dot{U}(sl_2[t]) \to \mathrm{Tr}(\mathcal{U}^*)$ as follows.



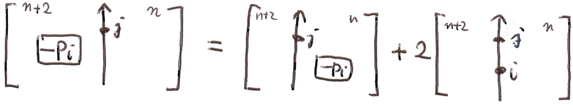
Here $p_i \in \Lambda$ is the power sum symmetric function of degree i.

Sample proofs (1)

$[H_i, E_j]1_n = 2E_{i+j}1_n$:

We have

$$\varphi(H_i E_j 1_n) =$$

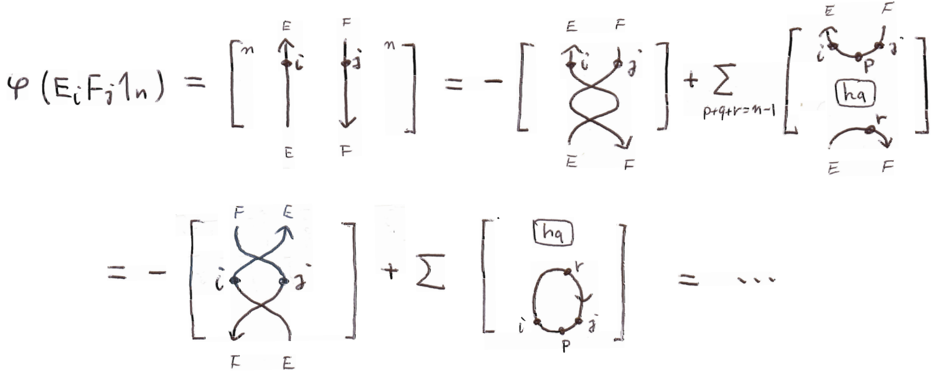


$$= \varphi(E_j H_i 1_n) + 2\varphi(E_{i+j} 1_n)$$

Here we use the bubble slide relation

Sample proofs (2)

$[E_i, F_j]1_n = H_{i+j}1_n$:
We have



Outline of proof of theorem

One can construct a map

$$\varphi\colon\ \dot{U}_{\mathbb{Z}}(sl_2[t])(m,n) \to \mathrm{Tr}(\dot{\mathcal{U}}^*)(m,n).$$

We have a triangular decomposition of $\mathrm{Tr}(\mathcal{U}^*)(m,n)$

$$\begin{aligned}\mathrm{Tr}(\mathcal{U}^*)(m,n) &= \bigoplus_{a,b\geq 0,\ 2(a-b)=n-m}\ \bigoplus_{\lambda,\mu,\nu} \mathbb{Z}\, F_\lambda^{(b)} s_\mu E_\nu^{(a)} 1_m \\ &\cong \bigoplus_{a,b} \Lambda_b \otimes \Lambda \otimes \Lambda_a.\end{aligned}$$

where λ,μ,ν are partitions, and

$$E_\nu^{(a)} 1_m\colon\ m \to m+2a, \quad F_\lambda^{(b)} 1_{m+2a}\colon\ m+2a \to n,$$

are 2-morphisms corresponding to the Schur polynomials $s_\nu \in \Lambda_a$, $s_\lambda \in \Lambda_b$.
One can prove that φ is an isomorphism by comparing the basis of $\mathrm{Tr}(\mathcal{U}^*)(m,n)$ and Garland's basis of $\dot{U}_{\mathbb{Z}}(sl_2[t])(m,n)$.