Character sheaves and modular generalized Springer correspondence
Part 2: The generalized Springer correspondence

Anthony Henderson
(joint with Pramod Achar, Daniel Juteau, Simon Riche)

University of Sydney

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Simplifying the context

To get a 'toy model' of character sheaves on G:

1. Instead of G-equivariant perverse sheaves on G, consider G-equivariant perverse sheaves on the **unipotent variety** $\mathcal{U}_G$. This is simpler because there are only finitely many G-orbits, but still highly relevant e.g. for cuspidal character sheaves.
2. Assume p is large enough so that there is a G-equivariant isomorphism $\mathcal{U}_G \xrightarrow{\sim} \mathcal{N}_G$ where $\mathcal{N}_G$ is the **nilpotent cone** in the Lie algebra $\mathfrak{g}$; then we can use Fourier transform on $\mathfrak{g}$.
3. The behaviour for large p is no different from considering G over $\mathbb{C}$ with the usual topology rather than étale topology.

This setting (first with simplification 1 only, later with 2 also) was studied by Lusztig in the case of $\overline{\mathbb{Q}_\ell}$-sheaves: one of his main results here was the 'generalized Springer correspondence'.
Aim: to prove an analogue in the **modular** case where $\mathrm{char}(k) = \ell$, as a first step towards understanding modular character sheaves.

The new set-up

New notation:

- G is a connected reductive algebraic group over $\mathbb{C}$,
- $\mathfrak{g}$ is its Lie algebra, on which G has the adjoint action,
- $\mathcal{N}_G = \{x \in \mathfrak{g} \,|\, x \text{ nilpotent}\}$ is the nilpotent cone, on which G has finitely many orbits,
- k is a sufficiently large field of characteristic $\ell \geq 0$,
- $\mathcal{D}_G(\mathcal{N}_G, k)$ is the constructible equivariant derived category.

For any $A \in \mathcal{D}_G(\mathcal{N}_G, k)$ and G-orbit $\mathcal{O}$ in $\mathcal{N}_G$, the restrictions $\mathcal{H}^i A|_{\mathcal{O}}$ are G-equivariant local systems (i.e. G-equivariant sheaves of finite-dimensional k-vector spaces) on $\mathcal{O}$, so they correspond to finite-dimensional representations over k of the finite group

$$A_G(x) = G_x/G_x^\circ, \text{ where } G_x \text{ is the stabilizer in } G \text{ of } x \in \mathcal{O}.$$

Let $\mathfrak{N}_{G,k}$ denote the set of pairs $(\mathcal{O}, \mathcal{E})$ where $\mathcal{O}$ is a G-orbit in $\mathcal{N}_G$ and $\mathcal{E}$ is an **irreducible** G-equivariant local system on $\mathcal{O}$.

Example ($G = GL_n$)

When $G = GL_n$, $\mathfrak{g} = \mathrm{Mat}_n$ and $\mathcal{N}_G = \{x \in \mathrm{Mat}_n \,|\, x^n = 0\}$. By the Jordan form theorem, we have a bijection

$$G\backslash\mathcal{N}_G \longleftrightarrow \mathcal{P}_n = \{\text{partitions } \lambda \text{ of } n\},$$

where $x \in \mathcal{O}_\lambda$ means that x has Jordan blocks of sizes $\lambda_1, \lambda_2, \cdots$. In this case $A_G(x) = 1$ for all x, so $\mathfrak{N}_{G,k} \longleftrightarrow \mathcal{P}_n$ for all fields k.

Example (G of type G_2)

The five nilpotent orbits, in order of decreasing dimension, are:

$$G_2 \text{ (regular)},\ G_2(a_1) \text{ (subregular)},\ \widetilde{A_1},\ A_1,\ 0.$$

These Bala–Carter labels record the type of the smallest Levi subalgebra meeting the orbit (where $\widetilde{A_1}$ means the short-root A_1). We have $A_G(x) = 1$ for all x except $A_G(x) = S_3$ for $x \in G_2(a_1)$, so $|\mathfrak{N}_{G,k}| = 7$ usually, $|\mathfrak{N}_{G,k}| = 6$ if $\mathrm{char}(k) \in \{2, 3\}$.

There is an anti-autoequivalence D of $\mathcal{D}_G(\mathcal{N}_G, k)$, *Verdier duality*. We study the abelian subcategory $\mathrm{Perv}_G(\mathcal{N}_G, k)$ of G-equivariant perverse k-sheaves on $\mathcal{N}_G$, where $A \in \mathcal{D}_G(\mathcal{N}_G, k)$ is *perverse* if

$$\mathcal{H}^i A|_{\mathcal{O}} = \mathcal{H}^i(DA)|_{\mathcal{O}} = 0 \text{ whenever } i > -\dim \mathcal{O}.$$

The simple objects in $\mathrm{Perv}_G(\mathcal{N}_G, k)$ are in bijection with $\mathfrak{N}_{G,k}$:

$$\mathrm{IC}(\mathcal{O}, \mathcal{E}) = \begin{array}{l}\text{`intermediate extension' of } \mathcal{E}[\dim \mathcal{O}] \text{ to } \overline{\mathcal{O}}, \\ \text{extended by zero to the whole of } \mathcal{N}_G.\end{array}$$

Example ($G = GL_2$, cf. Juteau–Mautner–Williamson)

The two orbits are $\mathcal{O}_{(1,1)} = \{0\}$ and $\mathcal{O}_{(2)} = \mathcal{N}_G \setminus \{0\}$. We have

$$\mathrm{IC}(\mathcal{O}_{(1,1)}, \underline{k}) = \underline{k}_0 \text{ (skyscraper sheaf)},$$
$$\mathrm{IC}(\mathcal{O}_{(2)}, \underline{k}) = \underline{k}_{\mathcal{N}_G}[2] \textbf{ if } \ell \neq 2.$$

The $\ell = 2$ case is different, because then $H^1(\mathcal{O}_{(2)}, k) \neq 0$.

Cuspidal pairs and induction series

Let P be a parabolic subgroup of G and L a Levi factor of P. We have a geometric parabolic induction functor

$$\mathbf{I}_{L\subset P}^{G} = \mathrm{Ind}_P^G \circ \mathrm{Res}_P^L : \mathcal{D}_L(\mathcal{N}_L, k) \to \mathcal{D}_G(\mathcal{N}_G, k),$$

defined in the same way as for character sheaves:

$$\mathrm{Res}_P^L : \mathcal{D}_L(\mathcal{N}_L, k) \xrightarrow{\sim} \mathcal{D}_P(\mathcal{N}_L, k) \xrightarrow{(\cdot)^*} \mathcal{D}_P(\mathcal{N}_P, k),$$

$$\mathrm{Ind}_P^G : \mathcal{D}_P(\mathcal{N}_P, k) \xrightarrow{\sim} \mathcal{D}_G(G \times_P \mathcal{N}_P, k) \xrightarrow{(\cdot)_!} \mathcal{D}_G(\mathcal{N}_G, k).$$

Lemma (Lusztig when $\ell = 0$, [AHR] when $\ell > 0$)

$\mathbf{I}_{L\subset P}^{G}$ commutes with D and maps $\mathrm{Perv}_L(\mathcal{N}_L, k)$ to $\mathrm{Perv}_G(\mathcal{N}_G, k)$. It has left adjoint $\mathbf{R}_{L\subset P}^{G} = \mathrm{Ind}_P^L \circ \mathrm{Res}_P^G$ and right adjoint $\mathbf{R}_{L\subset P^-}^{G}$ where P^- denotes the opposite parabolic with the same Levi L.

We say that a pair $(\mathcal{O}, \mathcal{E}) \in \mathfrak{N}_{G,k}$, or the corresponding $\mathrm{IC}(\mathcal{O}, \mathcal{E})$, is *cuspidal* if the following equivalent conditions hold:

1. $\mathbf{R}^G_{L\subset P}(\mathrm{IC}(\mathcal{O}, \mathcal{E})) = 0$ for all $L \subset P \subsetneq G$;
2. $\mathrm{IC}(\mathcal{O}, \mathcal{E})$ is not a quotient of $\mathbf{I}^G_{L\subset P}(A)$ for any $L \subset P \subsetneq G$ and any $A \in \mathrm{Perv}_L(\mathcal{N}_L, k)$;
3. $\mathrm{IC}(\mathcal{O}, \mathcal{E})$ is not a subobject of $\mathbf{I}^G_{L\subset P}(A)$ for any $L \subset P \subsetneq G$ and any $A \in \mathrm{Perv}_L(\mathcal{N}_L, k)$.

Remark

When $\ell = 0$, the Decomposition Theorem of [BBD] implies that if $A \in \mathrm{Perv}_L(\mathcal{N}_L, k)$ is simple, then $\mathbf{I}^G_{L\subset P}(A)$ is semisimple, so one can replace 'quotient'/'subobject' with 'summand'. Semisimplicity can fail if $\ell > 0$, and cuspidals **can** occur as constituents of $\mathbf{I}^G_{L\subset P}(A)$. This is analogous to modular representations of $G(\mathbb{F}_q)$ when $\ell \neq p$.

Lemma (Lusztig – same proof works for $\ell > 0$)

If $(\mathcal{O}, \mathcal{E})$ *is cuspidal,* $\mathcal{O}$ *is* distinguished, *i.e. meets no proper Levi.*

Let $\mathfrak{M}_{G,k}$ be the set of *cuspidal data* $(L, \mathcal{O}_L, \mathcal{E}_L)$ where L is a Levi subgroup of G (take only one representative of each G-conjugacy class, allowing $L = G$) and $(\mathcal{O}_L, \mathcal{E}_L)$ is a cuspidal pair for L.

Proposition (Lusztig when $\ell = 0$, [AHJR] when $\ell > 0$)

For any $(L, \mathcal{O}_L, \mathcal{E}_L) \in \mathfrak{M}_{G,k}$, $\mathbf{I}_{L \subset P}^G(\mathrm{IC}(\mathcal{O}_L, \mathcal{E}_L))$ is independent of the parabolic P, and its head and socle are isomorphic.

Remark

The analogue for modular representations is by Geck–Hiss–Malle.

The *induction series* associated to $(L, \mathcal{O}_L, \mathcal{E}_L) \in \mathfrak{M}_{G,k}$ is the set of simple quotients (equivalently, subobjects) of $\mathbf{I}_{L \subset P}^G(\mathrm{IC}(\mathcal{O}_L, \mathcal{E}_L))$.

Lemma (Lusztig – same proof works for $\ell > 0$)

Any simple object $\mathrm{IC}(\mathcal{O}, \mathcal{E})$ in $\mathrm{Perv}_G(\mathcal{N}_G, k)$ belongs to the induction series associated to some $(L, \mathcal{O}_L, \mathcal{E}_L) \in \mathfrak{M}_{G,k}$ as above. (If $\mathrm{IC}(\mathcal{O}, \mathcal{E})$ is cuspidal, then $(L, \mathcal{O}_L, \mathcal{E}_L) = (G, \mathcal{O}, \mathcal{E})$.)

The **(modular) generalized Springer correspondence** is:

Theorem (Lusztig when $\ell = 0$, [AHJR] when $\ell > 0$)

1. *Induction series associated to different cuspidal data are disjoint: in other words, a given* $\mathrm{IC}(\mathcal{O}, \mathcal{E})$ *belongs to the induction series associated to a* **unique** $(L, \mathcal{O}_L, \mathcal{E}_L) \in \mathfrak{M}_{G,k}$.
2. *The induction series associated to* $(L, \mathcal{O}_L, \mathcal{E}_L)$ *is canonically in bijection with the set of irreducible* k*-reps of* $N_G(L)/L$.
3. *Hence we have a bijection*

$$\mathfrak{N}_{G,k} \longleftrightarrow \bigsqcup_{(L,\mathcal{O}_L,\mathcal{E}_L)\in\mathfrak{M}_{G,k}} \mathrm{Irr}(N_G(L)/L, k).$$

The proof will be discussed in the next lecture.

Remark

The analogue of 1 holds for modular representations of $G(\mathbb{F}_q)$ also; for the analogue of 2 one needs a q-deformed group algebra.

Background: the Springer correspondence

- In the mid-1970s, Springer gave a geometric construction of the irreducible $\overline{\mathbb{Q}_\ell}$-reps of the Weyl group $W = N_G(T)/T$.
- As reformulated by Lusztig and Borho–Macpherson, this comes from an action of W on the semisimple perverse sheaf

$$\mathrm{Spr} = \mathbf{I}_{T\subset B}^{G}(\underline{\overline{\mathbb{Q}_\ell}}_0) = \mu_! \underline{\overline{\mathbb{Q}_\ell}}[\dim \mathcal{N}_G] \in \mathrm{Perv}_G(\mathcal{N}_G, \overline{\mathbb{Q}_\ell}),$$

where $\mu : G \times_B \mathcal{N}_B \to \mathcal{N}_G$ is the *Springer resolution* of $\mathcal{N}_G$. The *Springer correspondence* is the resulting bijection

$$\{\text{simple summands of } \mathrm{Spr}\} \longleftrightarrow \mathrm{Irr}(W, \overline{\mathbb{Q}_\ell})$$

$$\mathrm{IC}(\mathcal{O}, \mathcal{E}) \mapsto \mathrm{Hom}_{\mathrm{Perv}_G(\mathcal{N}_G, \overline{\mathbb{Q}_\ell})}(\mathrm{Spr}, \mathrm{IC}(\mathcal{O}, \mathcal{E})).$$

- Lusztig then found that this was the $(L, \mathcal{O}_L, \mathcal{E}_L) = (T, 0, \underline{\overline{\mathbb{Q}_\ell}})$ case of the generalized Springer correspondence, thus accounting for the $\mathrm{IC}(\mathcal{O}, \mathcal{E})$'s that are not summands of Spr.
- Juteau (2007) showed that the Springer correspondence holds with k instead of $\overline{\mathbb{Q}_\ell}$ and 'quotients' instead of 'summands'.

Example ($G = GL_n$, $W = S_n$)

- When $\ell = 0$, $|\mathrm{Irr}(S_n, k)| = |\mathcal{P}_n|$, so every $\mathrm{IC}(\mathcal{O}_\lambda, \underline{k})$ is a summand of Spr, i.e. the Springer correspondence for GL_n is already 'generalized'. In particular, GL_n does not have a cuspidal pair unless $n = 1$.
- When $\ell > 0$, James constructed the irreps D^λ of S_n over k, labelled by λ that are *ℓ-regular* (no part occurs $\geq \ell$ times). Under Juteau's correspondence, D^λ maps to $\mathrm{IC}(\mathcal{O}_{\lambda^{\mathbf{t}}}, \underline{k})$ where $\lambda^{\mathbf{t}}$ is the transpose partition; so these are the simple quotients of Spr. (All simples occur as constituents of Spr.)
  The only distinguished orbit in $\mathcal{N}_G$ is $\mathcal{O}_{(n)}$; we will see that

$$(\mathcal{O}_{(n)}, \underline{\Bbbk}) \text{ is cuspidal } \iff n \text{ is a power of } \ell.$$

  So $\mathfrak{M}_{G,k}$ is essentially the set of Levis of the form $\prod_{i \geq 0} GL_{\ell^i}^{m_i}$, where m_i are nonnegative integers such that $\sum_{i \geq 0} m_i \ell^i = n$.

Example ($G = GL_n$, $\ell > 0$ continued)

For $L = \prod_{i\geq 0} GL_{\ell^i}^{m_i}$ such a Levi subgroup of GL_n, we have

$$N_G(L)/L \cong \prod_{i\geq 0} S_{m_i},$$

$$\mathrm{Irr}(N_G(L)/L, k) \leftrightarrow \prod_{i\geq 0}\{\ell\text{-regular } \lambda^{(i)} \vdash m_i\}.$$

Under our correspondence, the collection $(\lambda^{(i)})$ maps to $\mathrm{IC}(\mathcal{O}_\lambda, \underline{k})$ where $\lambda = \sum_{i\geq 0} \ell^i (\lambda^{(i)})^{\mathbf{t}}$. Note that $\mathrm{IC}(\mathcal{O}_{(n)}, \underline{\Bbbk})$ occurs in the series of $L = \prod_{i\geq 0} GL_{\ell^i}^{b_i}$ where $\sum_{i\geq 0} b_i \ell^i = n$ and all $b_i < \ell$.

Remark

The above combinatorial correspondence is a simplified version of what appears in the analogous theory of induction series for modular representations of $GL_n(\mathbb{F}_q)$ (Dipper–Du).

Example ($G = G_2$, W dihedral of order 12)

- When $\ell = 0$, $|\mathrm{Irr}(W, k)| = 6 < 7 = |\mathfrak{N}_{G,k}|$. The non-Springer pair is $(G_2(a_1), \mathcal{E}_{\mathrm{sign}})$, which must be cuspidal because the other proper Levi subgroups are both isomorphic to GL_2.
- When $\ell = 2$, $|\mathrm{Irr}(W, k)| = 2$, and only $\mathrm{IC}(0, \underline{k})$ and $\mathrm{IC}(\widetilde{A_1}, \underline{k})$ belong to Juteau's correspondence. The other series are:

$$(L \text{ of type } A_1, \mathcal{O}_{(2)}, \underline{k}),\ |N_G(L)/L| = 2 : \quad \mathrm{IC}(A_1, \underline{k}),$$
$$(L \text{ of type } \widetilde{A_1}, \mathcal{O}_{(2)}, \underline{k}),\ |N_G(L)/L| = 2 : \quad \mathrm{IC}(G_2(a_1), \mathcal{E}_{\mathrm{refln}}),$$

leaving 2 cuspidal pairs, $(G_2(a_1), \underline{k})$ and $(G_2, \underline{k})$.

The above GL_n and G_2 examples illustrate:

Theorem ([AHJR])

When $\ell > 0$, $\mathrm{IC}(\mathcal{O}_{\mathrm{reg}}, \underline{k})$ belongs to the induction series associated to $(L, \mathcal{O}_{L,\mathrm{reg}}, \underline{k})$ where L is minimal such that $\ell \nmid |W/W_L|$.