Comparison of the two specializations of nonsymmetric Macdonald polynomials: at $t = 0$ and at $t = \infty$

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Basic notation

$\mathfrak{g}_{\mathrm{af}}$: untwisted affine Lie algebra over $\mathbb{C}$

$\mathfrak{h}_{\mathrm{af}} \subset \mathfrak{g}_{\mathrm{af}}$: Cartan subalgebra

$\Delta_{\mathrm{af}}^{+} \subset (\mathfrak{h}_{\mathrm{af}})^{*}$: positive affine roots

$c = \sum_{i \in I_{\mathrm{af}}} a_i^{\vee} \alpha_i^{\vee} \in \mathfrak{g}_{\mathrm{af}}$: canonical central element

$\alpha_i^{\vee}$, $i \in I_{\mathrm{af}} = I \cup \{0\}$: simple coroots

$\delta = \sum_{i \in I_{\mathrm{af}}} a_i \alpha_i \in \Delta_{\mathrm{af}}^{+}$: (primitive) null root

α_i, $i \in I_{\mathrm{af}} = I \cup \{0\}$: simple roots

$P = \sum_{i \in I} \mathbb{Z} \varpi_i$: classical weight lattice

E_i, F_i, $i \in I_{\mathrm{af}} = I \cup \{0\}$: Chevalley generators for $\mathfrak{g}_{\mathrm{af}}$

$\varpi_i = \Lambda_i - a_i^{\vee} \Lambda_0$, $i \in I$: level-zero fundamental weights

$\Lambda_i,\ i \in I_{\mathrm{af}} = I \cup \{0\}$: affine fundamental weights

$W = \langle r_i \mid i \in I \rangle$: finite Weyl group

$r_i,\ i \in I$: simple reflections

$W_{\mathrm{af}} = W \ltimes Q^\vee$: affine Weyl group

$Q^\vee = \sum_{i \in I} \mathbb{Z}\alpha_i^\vee$

$Q^{\vee,+} = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee$

$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = \sum_{i \in I} \varpi_i \in P$: Weyl vector

$\Delta^+ \subset \mathfrak{h}^*$: positive roots of the finite-dim. subalgebra $\mathfrak{g}$ ($\subset \mathfrak{g}_{\mathrm{af}}$)

$\mathfrak{h} \subset \mathfrak{g}$: Cartan subalgebra

Semi-infinite Bruhat graph

$\ell^{\frac{\infty}{2}}(x)$, $x \in W_{\mathrm{af}}$: semi-infinite length, defined by

$$\ell^{\frac{\infty}{2}}(wt_\mu) := \ell(w) + 2\langle \rho, \mu \rangle$$

for $w \in W$ and $\mu \in Q^\vee$

Semi-infinite Bruhat graph is a Δ_{af}^+-labeled, directed graph with vertex set W_{af} whose edges are of the form:

$$x \xrightarrow{\beta} r_\beta x, \quad x \in W_{\mathrm{af}},\ \beta \in \Delta_{\mathrm{af}}^+,$$

$$\text{with} \quad \ell^{\frac{\infty}{2}}(r_\beta x) = \ell^{\frac{\infty}{2}}(x) + 1.$$

For $\boldsymbol{x}$, $\boldsymbol{y} \in \boldsymbol{W}_{\mathrm{af}}$,

$$x \le_{\frac{\infty}{2}} y \overset{\text{def}}{\Longleftrightarrow}$$

$^{\exists}$**directed path from $\boldsymbol{x}$ to $\boldsymbol{y}$ in the semi-infinite Bruhat graph;**

$$x <_{\frac{\infty}{2}} y \overset{\text{def}}{\Longleftrightarrow} x \le_{\frac{\infty}{2}} y \quad \text{and} \quad x \neq y$$

Semi-infinite LS paths

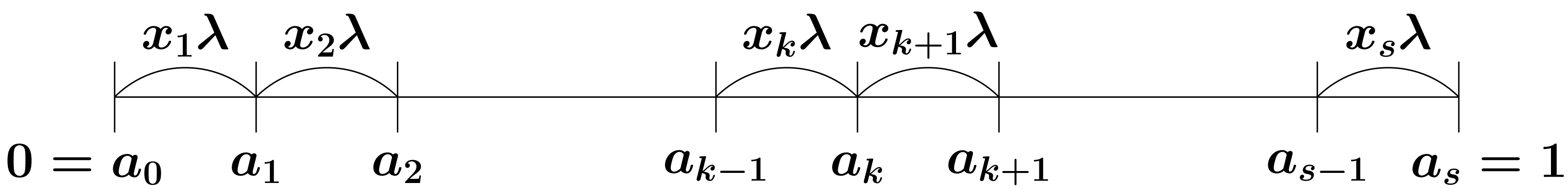
$\lambda \in P_+ = \sum_{i \in I} \mathbb{Z}_{\gneq 0} \varpi_i$: level-zero dominant and regular

$$\eta = (x_1 >_{\frac{\infty}{2}} x_2 >_{\frac{\infty}{2}} \cdots >_{\frac{\infty}{2}} x_s \,; \quad 0 = a_0 < a_1 < \cdots < a_s = 1),$$

where $x_k \in W_{\mathrm{af}}$ and $a_k \in \mathbb{Q}$, is a semi-infinite LS path of shape λ if for all $1 \leq k \leq s-1$, there exists a directed path

$$x_k = y_t \xleftarrow{\beta_t} y_{t-1} \xleftarrow{\beta_{t-1}} \cdots \xleftarrow{\beta_2} y_1 \xleftarrow{\beta_1} y_0 = x_{k+1}$$

with $a_k \langle y_{l-1}\lambda, \beta_l^\vee \rangle \in \mathbb{Z}$ $(1 \leq {}^\forall l \leq t)$.



$\mathbb{B}^{\frac{\infty}{2}}(\boldsymbol{\lambda})$: the set of all semi-infinite LS paths of shape $\boldsymbol{\lambda}$

$\mathbb{B}_0^{\frac{\infty}{2}}(\boldsymbol{\lambda})$: connected component of $\mathbb{B}^{\frac{\infty}{2}}(\boldsymbol{\lambda})$ containing $(e\,;\,0=a_0,\ a_1=1)$

For

$$\eta = (x_1 >_{\frac{\infty}{2}} \cdots >_{\frac{\infty}{2}} x_s\,;\,0 = a_0 < a_1 < \cdots < a_s = 1) \in \mathbb{B}^{\frac{\infty}{2}}(\boldsymbol{\lambda})$$

above, we set

$$\iota(\eta) := x_1 \in W_{\mathrm{af}} : \text{ initial direction of } \eta,$$

$$\kappa(\eta) := x_s \in W_{\mathrm{af}} : \text{ final direction of } \eta.$$

Remark

Here, for simplicity of explanation, we have assumed that

$$\boldsymbol{\lambda} \in \sum_{i \in I} \mathbb{Z}_{\gneqq 0} \varpi_i \qquad \text{(i.e., level-zero dominant and regular).}$$

Extremal weight modules and their crystal bases

$\boldsymbol{\lambda} = \sum_{i \in I} m_i \varpi_i \in P_+,\ m_i \in \mathbb{Z}_{\geq 0}$: level-zero dominant

$U_q(\mathfrak{g}_{\mathrm{af}})$: quantum affine algebra

$V(\boldsymbol{\lambda})$: extremal weight module of extremal weight $\boldsymbol{\lambda}$ over $U_q(\mathfrak{g}_{\mathrm{af}})$;

this is a module generated by a vector $v_{\boldsymbol{\lambda}}$ over $U_q(\mathfrak{g}_{\mathrm{af}})$

with the relation that $v_{\boldsymbol{\lambda}}$ is "extremal of weight $\boldsymbol{\lambda}$" in the sense:

${}^{\exists}\{S_w v_{\boldsymbol{\lambda}}\}_{w \in W_{\mathrm{af}}} \subset V(\boldsymbol{\lambda})$ such that

$S_e v_{\boldsymbol{\lambda}} = v_{\boldsymbol{\lambda}}$, and such that for all $w \in W_{\mathrm{af}}$ and $i \in I_{\mathrm{af}}$,

if $\langle w\boldsymbol{\lambda}, \alpha_i^\vee \rangle \geq 0$, then $E_i S_w v_{\boldsymbol{\lambda}} = 0$ and $F_i^{(\langle w\boldsymbol{\lambda}, \alpha_i^\vee \rangle)} S_w v_{\boldsymbol{\lambda}} = S_{r_i w} v_{\boldsymbol{\lambda}}$,

if $\langle w\boldsymbol{\lambda}, \alpha_i^\vee \rangle \leq 0$, then $F_i S_w v_{\boldsymbol{\lambda}} = 0$ and $E_i^{(-\langle w\boldsymbol{\lambda}, \alpha_i^\vee \rangle)} S_w v_{\boldsymbol{\lambda}} = S_{r_i w} v_{\boldsymbol{\lambda}}$.

$\mathcal{B}(\lambda)$: crystal basis of $V(\lambda)$

$u_\lambda \in \mathcal{B}(\lambda)$: extremal element corresponding to v_λ;

this element is "extremal of weight λ" in the following sense:

$^\exists\{S_w u_\lambda\}_{w \in W_{\mathrm{af}}} \subset \mathcal{B}(\lambda)$ such that

$S_e u_\lambda = u_\lambda$, and such that

if $\langle w\lambda, \alpha_i^\vee \rangle \geq 0$, then $e_i S_w u_\lambda = 0$ and $f_i^{\langle w\lambda, \alpha_i^\vee \rangle} S_w u_\lambda = S_{r_i w} u_\lambda$,

if $\langle w\lambda, \alpha_i^\vee \rangle \leq 0$, then $f_i S_w u_\lambda = 0$ and $e_i^{-\langle w\lambda, \alpha_i^\vee \rangle} S_w u_\lambda = S_{r_i w} u_\lambda$

for all $w \in W_{\mathrm{af}}$ and $i \in I_{\mathrm{af}}$.

Connected components of the crystal basis $\mathcal{B}(\lambda)$

$\lambda = \sum_{i \in I} m_i \varpi_i \in P_+$, $m_i \in \mathbb{Z}_{\geq 0}$: level-zero dominant

$V(\lambda)$: extremal weight module of extremal weight λ over $U_q(\mathfrak{g}_{\mathrm{af}})$

$v_\lambda \in V(\lambda)$: (generating) extremal vector of weight λ

$U_q(\mathfrak{g}_{\mathrm{af}})$: quantum affine algebra

$U_q^+(\mathfrak{g}_{\mathrm{af}})$: positive part of $U_q(\mathfrak{g}_{\mathrm{af}})$

$\mathcal{B}(-\infty) \ni u_{-\infty}$: crystal basis of $U_q^+(\mathfrak{g}_{\mathrm{af}})$

$U_q^-(\mathfrak{g}_{\mathrm{af}})$: negative part of $U_q(\mathfrak{g}_{\mathrm{af}})$

$\mathcal{B}(\infty) \ni u_{\infty}$: crystal basis of $U_q^-(\mathfrak{g}_{\mathrm{af}})$

$\boldsymbol{\mathcal{B}(\lambda)}$: **crystal basis of** $\boldsymbol{V(\lambda)}$

$\boldsymbol{u_\lambda \in \mathcal{B}(\lambda)}$: **extremal element corresponding to** $\boldsymbol{v_\lambda}$

$\mathbf{Par}(\boldsymbol{\lambda})$: **the set of** $\boldsymbol{I}$**-tuples** $\boldsymbol{c_0 = (\rho^{(i)})_{i\in I}}$ **of partitions**

such that the length of the partition $\boldsymbol{\rho^{(i)}}$ **is** $\boldsymbol{\lneqq m_i}$ $(\forall \boldsymbol{i \in I})$;

for $\boldsymbol{c_0 = (\rho^{(i)})_{i\in I} \in \mathrm{Par}(\lambda)}$, **we set** $\boldsymbol{|c_0| := \sum_{i\in I} |\rho^{(i)}|}$,

where $\boldsymbol{|\rho^{(i)}|}$ **is the size of the partition** $\boldsymbol{\rho^{(i)}}$ **for** $\boldsymbol{i \in I}$.

$\overline{\mathbf{Par}}(\boldsymbol{\lambda})$: **the set of** $\boldsymbol{I}$**-tuples** $\boldsymbol{c_0 = (\rho^{(i)})_{i\in I}}$ **of partitions**

such that the length of the partitions $\boldsymbol{\rho^{(i)}}$ **is** $\boldsymbol{\le m_i}$ $(\forall \boldsymbol{i \in I})$

Fact (Kashiwara, Beck-Nakajima)

As crystals,

$$\boldsymbol{\mathcal{B}(\lambda) \subset \mathcal{B}(\infty) \otimes \{\tau_\lambda\} \otimes \mathcal{B}(-\infty).}$$

Moreover, every extremal element in $\boldsymbol{\mathcal{B}(\lambda)}$ is connected to an extremal element of the form:

$$\boldsymbol{S^-_{\mathrm{c}_0} u_\infty \otimes \tau_\lambda \otimes u_{-\infty} \in \mathcal{B}(\infty) \otimes \{\tau_\lambda\} \otimes \mathcal{B}(-\infty),}$$

for some $\boldsymbol{\mathrm{c}_0 = (\rho^{(i)})_{i \in I} \in \overline{\mathrm{Par}}(\lambda)}$ (or, $\boldsymbol{\mathrm{c}_0 = (\rho^{(i)})_{i \in I} \in \mathrm{Par}(\lambda)}$);

also, we have

$$\boldsymbol{S^-_{\mathrm{c}_0} u_\infty \otimes \tau_\lambda \otimes u_{-\infty} \equiv S^-_{\mathrm{c}_0} v_\lambda \pmod{q}.}$$

Remark

The elements S_{c_0}, where $c_0 = (\rho^{(i)})_{i\in I}$ are I-tuples of partitions, are the "purely imaginary" PBW-type basis elements in $U_q^+(\mathfrak{g}_{\mathrm{af}})$, and $S^-_{c_0} := \overline{S^\vee_{c_0}}$, where the $\mathbb{C}(q)$-algebra automorphism $^\vee$ of $U_q(\mathfrak{g}_{\mathrm{af}})$ is given by

$$E_i^\vee := F_i, \quad F_i^\vee := E_i, \quad (q^h)^\vee = q^{-h},$$

and the $\mathbb{C}$-algebra automorphism $^-$ of $U_q(\mathfrak{g}_{\mathrm{af}})$ is given by

$$\overline{E_i} := E_i, \quad \overline{F_i} := F_i, \quad \overline{q^h} := q^{-h}, \quad \overline{q} := q^{-1}.$$

Realization of the crystal $\mathcal{B}(\lambda)$

Assume that

$$\lambda = \sum_{i \in I} m_i \varpi_i \in P_+ \quad \text{is such that } m_i \gneqq 0 \ ({}^\forall i \in I).$$

Theorem

We have an isomorphism

$$\boxed{\Phi_\lambda : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathbb{B}^{\frac{\infty}{2}}(\lambda)}$$

of crystals such that

$$\Phi_\lambda(S_{\mathrm{c}_0}^- u_\infty \otimes \tau_\lambda \otimes u_{-\infty}) = \eta^{\mathrm{c}_0}$$

for all $\mathrm{c}_0 \in \mathrm{Par}(\lambda)$.

Here, for each $\mathrm{c}_0 \in \mathrm{Par}(\lambda)$, the element $\eta^{\mathrm{c}_0} \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is an extremal element of the form:

$$\eta^{\mathrm{c}_0} = (t_{\xi_1}, \ldots, t_{\xi_{s-1}}, t_{\xi_s} = e\,;\, 0 = a_0, \ldots, a_s = 1), \quad s \geq 1,$$

with $\xi_k \in Q^\vee$ $(1 \leq k \leq s-1)$, such that

$$\xi_k - \xi_{k+1} \in \sum_{\substack{i \in I \\ a_k \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}}} \mathbb{Z}_{\geq 0} \alpha_i^\vee$$

for all $1 \leq k \leq s-1$; $\xi_s := 0$ by convention.

Demazure submodules of $V(\lambda)$

$\lambda = \sum_{i \in I} m_i \varpi_i \in P_+$: level-zero dominant and regular

$U_q^-(\mathfrak{g}_{\mathrm{af}})$: negative part of $U_q(\mathfrak{g}_{\mathrm{af}})$

For each $x \in W_{\mathrm{af}}$, we set

$$V_x^-(\lambda) := U_q^-(\mathfrak{g}_{\mathrm{af}}) S_x v_\lambda \subset V(\lambda),$$

where $S_x v_\lambda \in V(\lambda)$ is an extremal vector of weight $x\lambda$.

Remark

$$V_x^-(\lambda) \cong V_e^-(x\lambda) \subset V(x\lambda).$$

Fact (Kashiwara)

For each $x \in W_{\mathrm{af}}$, $V_x^-(\lambda)$ has the crystal basis

$$\mathcal{B}_x^-(\lambda) = (S_x^*)^{-1}\bigl(\mathcal{B}(x\lambda) \cap (\mathcal{B}(\infty) \otimes \tau_{x\lambda} \otimes u_{-\infty})\bigr),$$

where $S_x^* : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathcal{B}(x\lambda)$ is an isomorphism of crystals.

Characterization of $\mathcal{B}_x^-(\lambda)$

Assume that $\lambda = \sum_{i \in I} m_i \varpi_i \in P_+$ is such that $m_i \gneqq 0$ $({}^\forall i \in I)$.

For each $x \in W_{\mathrm{af}}$, we set

$$\mathbb{B}^{\frac{\infty}{2}}_{\geq x}(\lambda) := \left\{ \eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda) \mid \kappa(\eta) \geq_{\frac{\infty}{2}} x \right\}.$$

Theorem

For each $x \in W_{\mathrm{af}}$,

$$\Phi_\lambda(\mathcal{B}_x^-(\lambda)) = \mathbb{B}^{\frac{\infty}{2}}_{\geq x}(\lambda),$$

where $\Phi_\lambda : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is the isomorphism above of crystals.

Relation with symmetric Macdonald polynomials

Assume that $\lambda = \sum_{i \in I} m_i \varpi_i \in P_+$ **is such that** $m_i \gneq 0 \quad ({}^{\forall} i \in I)$.

Write $V_e^-(\lambda)$ **as:**

$$V_e^-(\lambda) = \bigoplus_{\substack{\gamma \in Q \\ k \in \mathbb{Z}_{\geq 0}}} V_e^-(\lambda)_{\lambda+\gamma-k\delta},$$

where $Q = \sum_{i \in I} \mathbb{Z}\alpha_i$ **; we set**

$$\text{gr-ch}(V_e^-(\lambda)) := \sum_{\substack{\gamma \in Q \\ k \in \mathbb{Z}_{\geq 0}}} (\dim_{\mathbb{C}(q)} V_e^-(\lambda)_{\lambda+\gamma-k\delta}) e^{\lambda+\gamma} q^{-k}.$$

Theorem

$$\text{gr-ch}(V_e^-(\lambda)) = \frac{P_\lambda(x\,;\, q^{-1},\, 0)}{\prod_{i\in I}\prod_{r=1}^{m_i}(1-q^{-r})},$$

with $x = e^{\lambda+\gamma}$, where $P_\lambda(x\,;\, q,\, 0)$ denotes the specialization at $t = 0$ of the symmetric Macdonald polynomial $P_\lambda(x; q, t)$.

Remark

Here we have used our previous result that the "graded character" $\text{gr-ch}(W_e(\lambda))$ of the local Weyl module $W_e(\lambda)$ (in the notation below) is identical to the specialization $P_\lambda(x\,;\, q^{-1},\, 0)$ at $t = 0$ of the symmetric Macdonald polynomial $P_\lambda(x\,;\, q^{-1},\, t)$.

Remark

(the Ram-Yip formula)

The nonsymmetric Macdonald polynomial $E_{w_\circ \lambda}(x\,;\,q,\,t)$ is equal to the following:

$$\sum_{p_J} x^{\mathrm{wt}(p_J)} t^{\ell(\mathrm{dir}(p_J))} (t^{-1}-t)^{|J|} \frac{\prod_{j \in J_-} q^{\deg(\beta_j^\vee)} t^{\langle 2\rho, -\,\mathrm{cl}(\beta_j^\vee)\rangle}}{\prod_{j \in J} (1 - q^{\deg(\beta_j^\vee)} t^{\langle 2\rho, -\,\mathrm{cl}(\beta_j^\vee)\rangle})},$$

where p_J runs over specific finite sequences of elements in W_{af} corresponding to certain finite sets J determined by λ;

$\mathrm{wt}(p_J) \in P$, $\mathrm{dir}(p_J) \in W$, $\mathrm{cl}(\beta_j) \in -\Delta^+$ and $\deg(\beta_j^\vee) \in \mathbb{Z}_{\gneq 0}$ for $j \in J$, with $J_- \subset J$.

Also, note that $P_\lambda(x\,;\,q^{-1},\,0) = E_{w_\circ \lambda}(x\,;\,q^{-1},\,0)$.

Relation with level-zero fundamental representations

Assume that $\lambda = \sum_{i \in I} m_i \varpi_i \in P_+$ is such that $m_i \gneqq 0$ $(^\forall i \in I)$.

For the unit element $e \in W$, we set

$$W_e(\lambda) := V_e^-(\lambda) \Big/ \sum_{c_0 \in \overline{\mathrm{Par}}(\lambda) \setminus (\emptyset)_{i \in I}} U_q^-(\mathfrak{g}_{\mathrm{af}}) S_{c_0}^- v_\lambda;$$

recall that $\overline{\mathrm{Par}}(\lambda)$ is the set of I-tuples $c_0 = (\rho^{(i)})_{i \in I}$ of partitions such that the length of the partition $\rho^{(i)}$ is $\leq m_i$ for all $i \in I$, and $S_{c_0}^- v_\lambda \in V(\lambda)$ is an extremal vector of weight $\lambda - |c_0|\delta$.

We denote the quotient map by

$$\mathrm{cl} : V_e^-(\lambda) \twoheadrightarrow W_e(\lambda).$$

Remark $\boldsymbol{W_e(\lambda)}$ **has the crystal basis**

$$\left\{\boldsymbol{\eta} \in \mathbb{B}_0^{\frac{\infty}{2}}(\boldsymbol{\lambda}) \mid \boldsymbol{\kappa}(\boldsymbol{\eta}) \in \boldsymbol{W}\right\}.$$

Now, for $\boldsymbol{\eta} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_s\,;\, \boldsymbol{a}_0, \ldots, \boldsymbol{a}_s) \in \mathbb{B}^{\frac{\infty}{2}}(\boldsymbol{\lambda})$**, we set**

$$\mathbf{cl}(\boldsymbol{\eta}) := (\mathbf{cl}(\boldsymbol{x}_1), \ldots, \mathbf{cl}(\boldsymbol{x}_s)\,;\, \boldsymbol{a}_0, \ldots, \boldsymbol{a}_s),$$

where $\mathbf{cl} : \boldsymbol{W}_{\mathrm{af}} \twoheadrightarrow \boldsymbol{W}$ **is a (surjective) homomorphism given by:**

$$\mathbf{cl}(\boldsymbol{w t_\mu}) = \boldsymbol{w} \quad \textbf{for } \boldsymbol{w} \in \boldsymbol{W},\ \boldsymbol{\mu} \in \boldsymbol{Q}^\vee.$$

Note that

$$\left\{\mathbf{cl}(\boldsymbol{\eta}) \mid \boldsymbol{\eta} \in \mathbb{B}_0^{\frac{\infty}{2}}(\boldsymbol{\lambda}) \textbf{ and } \boldsymbol{\kappa}(\boldsymbol{\eta}) \in \boldsymbol{W}\right\}$$

$$= \left\{\mathbf{cl}(\boldsymbol{\eta}) \mid \boldsymbol{\eta} \in \mathbb{B}^{\frac{\infty}{2}}(\boldsymbol{\lambda})\right\} =: \mathbb{B}(\boldsymbol{\lambda})_{\mathrm{cl}}.$$

Note

For $x = wt_\mu \in W_{\mathrm{af}}$ and $\beta = \alpha + k\delta \in \Delta_{\mathrm{af}}^+$,

$x \xrightarrow{\beta} r_\beta x$ in the semi-infinite Bruhat graph if and only if

(1) $k = 0$, $w^{-1}\alpha \in \Delta^+$, and $\ell(wr_{w^{-1}\alpha}) = \ell(w) + 1$, or

(2) $k = 1$, $w^{-1}\alpha \in \Delta^+$, and $\ell(wr_{w^{-1}\alpha}) = \ell(w) - 2\langle \rho, w^{-1}(\alpha^\vee)\rangle + 1$.

We set

$$\mathbb{B}(\lambda)_{\mathrm{cl}} := \left\{ \mathrm{cl}(\eta) \mid \eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda) \right\},$$

the set of quantum LS paths of shape λ.

Then, as a $U_q(\mathfrak{g})$-crystal, $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is isomorphic to

$$\left\{ \eta \in \mathbb{B}_0^{\frac{\infty}{2}}(\lambda) \mid \kappa(\eta) \in W \right\} \subset \mathbb{B}^{\frac{\infty}{2}}(\lambda).$$

Remark

As $U_q(\mathfrak{g})$-modules,

$$W_e(\lambda) \cong \bigotimes_{i \in I} W(\varpi_i)^{\otimes m_i},$$

where

$W(\varpi_i)$: **i-th level-zero fundamental representation of $U'_q(\mathfrak{g}_{\mathrm{af}})$;**

$U_q(\mathfrak{g}) \subset U'_q(\mathfrak{g}_{\mathrm{af}}) = U_q((\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}) \oplus \mathbb{C}c)$.

Proposition

For the unit element $e \in W$, the graded character gr-ch$(W_e(\lambda))$ of $W_e(\lambda)$ is identical to the specialization $E_{w_\circ\lambda}(x\,;\,q^{-1},\,0)$ at $t=0$ of the nonsymmetric Macdonald polynomial $E_{w_\circ\lambda}(x\,;\,q^{-1},\,t)$, where $w_\circ \in W$ denotes the longest element.

Remark

We have

$$E_{w_\circ\lambda}(x\,;\,q^{-1},\,0) = P_\lambda(x\,;\,q^{-1},\,0),$$

where $w_\circ \in W$ is the longest element.

Quantum Bruhat graph

QBG : quantum Bruhat graph associated with W and Δ^+;

this is a labeled, directed graph with

vertex set W,

edges : $u \xrightarrow{\beta} v$, $u, v \in W$ and $\beta \in \Delta^+$,

where $u \xrightarrow{\beta} v$ means that

(1) $v = ur_\beta$ and $\ell(v) = \ell(u) + 1$ (Bruhat edge),

or

(2) $v = ur_\beta$ and

$$\ell(v) = \ell(u) - 2\langle \rho, \beta^\vee \rangle + 1 \text{ (quantum edge).}$$

Specialization at $t = \infty$ of nonsymmetric Macdonald polynomials

Let $\lambda = \sum_{i \in I} m_i \varpi_i \in P_+$ be level-zero dominant and regular,

i.e., $m_i \gneq 0$ for all $i \in I$.

For an edge $u \xrightarrow{\beta} v$ in the QBG, we set

$$\mathrm{wt}_\lambda(u \to v) := \begin{cases} 0 & \text{(for a Bruhat edge)}, \\ \langle \lambda, \beta^\vee \rangle & \text{(for a quantum edge)}. \end{cases}$$

Also, for $u, v \in W$, we set

$$\mathrm{wt}_\lambda(u \Rightarrow v) := \mathrm{wt}_\lambda(u_0 \to u_1) + \cdots + \mathrm{wt}_\lambda(u_{k-1} \to u_k),$$

by taking a shortest directed path

$$u = u_0 \to u_1 \to \cdots \to u_{k-1} \to u_k = v$$

in the QBG.

For $\boldsymbol{\eta} \in \mathbb{B}(\boldsymbol{\lambda})_{\mathrm{cl}}$ of the form:

$$\boldsymbol{\eta} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_s; 0 = \boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_s = 1),$$

we set $\boldsymbol{\kappa}(\boldsymbol{\eta}) := \boldsymbol{w}_s \in \boldsymbol{W}$, and

$$\mathrm{wt}(\boldsymbol{\eta}) := \sum_{i=0}^{s-1} (\boldsymbol{a}_{i+1} - \boldsymbol{a}_i) \boldsymbol{w}_{i+1} \boldsymbol{\lambda} \in \boldsymbol{P};$$

we also set

$$\deg_{w_\circ}(\boldsymbol{\eta}) := - \sum_{i=1}^{s} \boldsymbol{a}_i \mathrm{wt}_{\boldsymbol{\lambda}}(\boldsymbol{w}_{i+1} \Rightarrow \boldsymbol{w}_i),$$

where $\boldsymbol{w}_{s+1} := \boldsymbol{w}_\circ \in \boldsymbol{W}$ (the longest element in $\boldsymbol{W}$).

Now, we define:

$$\mathbf{gch}_{w_\circ}(\mathbb{B}(\boldsymbol{\lambda})_{\mathrm{cl}}) := \sum_{\eta \in \mathbb{B}(\boldsymbol{\lambda})_{\mathrm{cl}}} \boldsymbol{q}^{\deg_{w_\circ}(\eta)} \boldsymbol{e}^{\mathrm{wt}(\eta)}.$$

Theorem

In the notation and setting above, we have

$$\boldsymbol{E}_{w_\circ \boldsymbol{\lambda}}(\boldsymbol{x}\,;\,\boldsymbol{q},\,\infty) = \mathbf{gch}_{w_\circ}(\mathbb{B}(\boldsymbol{\lambda})_{\mathrm{cl}}).$$

Comparison with the specialization at $t = 0$

For $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ of the form:

$$\eta = (w_1, \cdots, w_s; 0 = a_0, a_1, \cdots, a_s = 1),$$

we set

$$\mathrm{Deg}(\eta) := \sum_{i=1}^{s-1} a_i \mathrm{wt}_\lambda(w_{i+1} \Rightarrow w_i).$$

Remark

We have

$$\deg_{w_\circ}(\eta) = -\mathrm{Deg}(\eta) - \mathrm{wt}_\lambda(w_\circ \Rightarrow \kappa(\eta)).$$

Also, we set

$$\mathrm{gr\text{-}ch}(\mathbb{B}(\boldsymbol{\lambda})_{\mathrm{cl}}) := \sum_{\eta \in \mathbb{B}(\boldsymbol{\lambda})_{\mathrm{cl}}} \boldsymbol{q}^{-\mathrm{Deg}(\eta)} \boldsymbol{e}^{\mathrm{wt}(\eta)}.$$

Theorem

In the notation and setting above, we have

$$\mathrm{gr\text{-}ch}(\mathbb{B}(\boldsymbol{\lambda})_{\mathrm{cl}}) = \mathrm{gr\text{-}ch}(\boldsymbol{W}_e(\boldsymbol{\lambda})) = \boldsymbol{E}_{w_\circ \boldsymbol{\lambda}}(\boldsymbol{x}\,;\, \boldsymbol{q}^{-1},\, 0).$$