Lecture R.I.M.S. - Kyoto University

HOMOTOPICAL ARITHMETIC GEOMETRY OF MODULI STACKS OF CURVES

2018, Oct.1 - 15

Abstract: Moduli spaces of curves are ideal spaces for studying fundamental abstract theories of arithmetic geometry: they give geometric Galois representations that can be explicitly computed, furnish examples of anabelian spaces, and in genus zero generate the category of mixed Tate motives. They also possess a dual nature, being either considered as schemes or algebraic stacks.
The goal of this series of talks is to provide a basic introduction to these aspects by covering various fundamental geometric and arithmetic properties. It is intended for graduate students in algebraic geometry and non-specialists researchers. Elementary notions will be either recalled or illustrated with pictures or examples.

ALGEBRAIC & DELIGNE-MUMFORD STACKS — LECTURES 1 AND 2

Taking the functor of points for schemes as initial motivation, we introduce the notion of stacks as lax functors in groupoids with descent conditions and show how to recover Laumon-Moret-Bailly's original definition. We present how the Artin and Deligne-Mumford algebraic versions – that admit topological coverings by schemes – allow to "push" algebraic geometry properties in this context.

Keywords: diagrams of groupoids, Grothendieck topologies, examples of global quotient and inertia stacks.

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MODULI PROBLEMS & MODULI SPACES OF CURVES — LECTURE 3

We present how the scheme-stack structures and the geometry of curves lead to two solutions for building classifying spaces. Having introduced the notion of functor of moduli, we present Gieseker and Deligne-Mumford constructions of the moduli space of curves: the former follows Mumford G.I.T-theory and gives a quasi-projective scheme, the latter produces a smooth algebraic Deligne-Mumford global stack with nice stable compactification.

Keywords: Hilbert scheme, explicit examples in low genus, stable compactification, formal neighbourhood.

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FUNDAMENTAL GROUP & ARITHMETIC LECTURE 4

We follow Grothendieck construction of the étale fundamental group that leads to Geometric Galois actions of the absolute Galois group of rational on the geometric fundamental group of moduli stack of curves. We adapt this approach in the case of Deligne-Mumford stacks and show how it leads to a divisorial and a stack arithmetic of the spaces. Following the seminal work of Ihara, Matsumoto and Nakamura, we present explicit results and properties of the former, then recent similar results in the case of cyclic inertia for the latter.

Keywords: étale fundamental group for stacks, explicit computations in low dimensions, tangential Galois representations.

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MOTIVIC THEORY FOR MODULI STACK OF CURVES LECTURE 5

We present recent progress on an ongoing project on the construction of a category of motives for the moduli stacks of curves, whose main property is to reflect the arithmetic properties of the cyclic stack inertia. Having recalled briefly some already available categories of Chow, Grothendieck and Voevodsky (derived) motives, we first present Morel-Voevodsky stable/unstable motivic homotopy categories. We then show how the homotopical-simplicial approach is compatible with Cox-Friedlander étale topological type and is well adapted to our goal.

The various model categories hidden in the previous lectures are revealed.

Keywords: Quillen model category, Artin-Mazur étale topological type, Mixed Tate motives and loop space.

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Benjamin Collas
Institute of Mathematics Bayreuth - DFG
Bayreuth, Germany
`http://collas.perso.math.cnrs.fr/`
`benjamin.collas@uni-bayreuth.de`