# Random Fields and Processes on Graphs and Fractals

10–12 June 2024

Research Institute for Mathematical Sciences, Kyoto University

*Organisers: David Croydon (Kyoto University), Naotaka Kajino (Kyoto University), Yoshihiro Abe (Tohoku University)*

## 10 June 2024

**14:00–14:40: Ohad FELDHEIM (Hebrew University of Jerusalem)**
. . . . . *The power of two choices in graphical allocation*

**15:00–15:40: Yichao HUANG (Beijing Institute of Technology)**
. . . . . *Gaussian multiplicative cascades and Gaussian multiplicative chaos*

**16:00–16:40: Naomi FELDHEIM (Bar-Ilan University)**
. . . . . *Universality of entropic repulsion for stationary Gaussian fields with spectral singularity*

## 11 June 2024

**09:30–10:10: Yoshinori KAMIJIMA (Toyo University)**
. . *Derivation of a lace expansion for the Ising model by a random current representation on space-time*

**10:25–10:50: Noe KAWAMOTO (NCTS, Taiwan)**
. . . . . *Gaussian deconvolution and the lace expansion for long-range models*

**11:00–11:40: Pierre NOLIN (City University of Hong Kong)**
. . . . . *Monochromatic exponents for two-dimensional percolation*

**13:30–14:10: Naoki KUBOTA (Nihon University)**
. . . . . *Monotonicity of the time constant for the frog model with Bernoulli initial configurations*

**14:25–14:50: Ryoichiro NODA (Kyoto University)**
. . . . . *Convergence of local times of stochastic processes associated with resistance forms*

**15:10–15:50: Masato TAKEI (Yokohama National University)**
. . . . . *Probabilistic analysis of the Takagi function and its generalizations*

**16:00–16:40: Chenlin GU (Tsinghua University)**
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# Abstracts

## 10 June 2024

### 14:00–14:40: Ohad FELDHEIM (Hebrew University of Jerusalem)

*The power of two choices in graphical allocation*

The graphical balls-into-bins process is an overseer process in which balls are assigned one by one to the vertices of a $d$-regular graph $G$. At each time step an edge of $G$ is chosen uniformly at random and the overseer must assign a ball to either of the two endpoints of this edge. The purpose of the overseer is that the allocation will be as well balanced as possible, namely that the load difference between the most loaded and the least loaded bins will be minimal, known as the allocation "gap". The classical 2-choice process corresponds to the case of $G = K_n$. A greedy strategy of allocating to the least loaded bin among the two has been extensively researched, and relations to models in statistical mechanics are conjectured.

Previously it was thought that the greedy strategy is asymptotically optimal up to a constant, and that for fixed $d$ there exists $d$-edge-connected graphs for which the gap has to be polynomially high. We show that this is not the case. For any $k(n)$-edge-connected, $d(n)$-regular graph on $n$ vertices, and any number of balls, we give a non-greedy, efficiently computable allocation strategy that, with high probability, ensures a gap of $O((d/k)\log^4 n \log\log n)$ between the load of any two bins. A not-too-difficult lower bound of $\Omega((d/k) + \log n)$ on the gap achievable by any allocation strategy implies that our strategy achieves the optimal gap up to polylogarithmic factors, for every graph $G$.

A key idea is to relate the problem of designing a good allocation strategy to that of finding suitable multi-commodity flows on $G$. To this end, we consider Räcke's cut-based decomposition tree, a modern algorithmic tool, defining certain orthogonal flows on it.

Based on joint work with Nikhil Bansal from the University of Michigan.

### 15:00–15:40: Yichao HUANG (Beijing Institute of Technology)

*Gaussian multiplicative cascades and Gaussian multiplicative chaos*

We give a short review on some aspects of Kahane's theory of Gaussian multiplicative chaos, focusing on the connection to its ancestor, the Gaussian multiplicative cascades model of Mandelbrot. The main focus of this talk will be a probabilistic toolbox allowing to compare estimates between these two models. Several applications to mathematical physics, e.g. to Polyakov's theory of Liouville conformal field theory, will be discussed. Several extensions and recent results will also be mentioned if time permits.

### 16:00–16:40: Naomi FELDHEIM (Bar-Ilan University)

*Universality of entropic repulsion for stationary Gaussian fields with spectral singularity*

In a seminal 1995 paper, Bolthausen, Deuschel and Zeitouni showed that the probability that the Gaussian free field (GFF) on $\mathbb{Z}^d$, with $d \geq 3$, is positive on a square of side length $T$ is asymptotically $CT^{d-2}\log T$. Moreover, the field exhibits entropic repulsion, namely, when conditioned to be positive on a square of side length $T$, it is lifted to a new height of order $\sqrt{\log T}$ and apart from this, maintains its original distribution. This phenomenon was believed for a long time to be universal for strongly correlated Gaussian fields, though, until recently, rigorous proofs were available only for a handful of models (e.g. the membrane model).

Recently we were able to relate this phenomenon to the spectrum of the process, namely to the Fourier transform of its covariance kernel. As a result we obtain entropic repulsion results for a wide class of stationary $d$-dimensional Gaussian fields on $\mathbb{R}^d$ and $\mathbb{Z}^d$ with spectral singularity at the origin, i.e. whose spectral density at the origin behaves like $|\lambda|^{-\alpha}$ for some $\alpha \in (0, d)$. Our main result is that, under certain mild regularity conditions, the probability of the field being positive on a large ball $B(T)$ decays at a universal log-asymptotic rate of $m(d-\alpha)C(T)\log T$, where $C(T)$ is the capacity of $B(T)$ with respect to the field, and $m$ is the mass of the absolutely continuous component of its spectrum. Using this estimate, we show that the field conditioned on persistence is tending, in a proper integral sense, to a universal limiting shape, shifted by a constant level of $\sqrt{m(d-\alpha)(\log T)}$.

In the talk I will introduce this spectral theory and the notions of capacity and equilibrium potential, and explain the essence of this spectral perspective on entropic repulsion.

Based on joint work with Ohad Feldheim from the Hebrew University of Jerusalem and Stephen Muirhead from the University of Melbourne.

## 11 June 2024

### 09:30–10:10: Yoshinori KAMIJIMA (Toyo University)

*Derivation of a lace expansion for the Ising model by a random current representation on space-time*

The lace expansion is one of the powerful tools to investigate critical phenomena. It has succeeded in getting an asymptotic expansion for the critical point for several models, e.g., the self-avoiding walk, ordinary/oriented percolation, the contact process, etc. Our purpose is to obtain such an asymptotic expansion for the quantum Ising model, in which the different type of phase transition from the classical Ising model is caused by a transverse field, by use of a lace expansion. The lace expansion for the classical Ising model was derived in [Sakai (2007) *Commun. Math. Phys.*], whereas it has not been derived for the quantum Ising model yet.

In this talk, I show a new derivation of a lace expansion for the classical Ising model, which is the special case of the quantum Ising model without the transverse field. The Hamiltonian in the quantum Ising model is expressed by operators. Thanks to the Lie-Trotter product formula, it is enough to consider the $(d+1)$-dimensional space-time Ising model, which is not given by operator language, instead of the $d$-dimensional Ising model. So far, we have derived a new type of the lace expansion without the transverse field based on the random current representation [Björnberg and Grimmett (2009) *J. Stat. Phys.*] [Crawford and Ioffe (2010) *Commun. Math. Phys.*] on the space-time. We also expect that this approach helps us to derive a lace expansion in the case that the transverse field is finite.

This talk is based on joint work with Akira Sakai (Hokkaido University, Japan).

### 10:25–10:50: Noe KAWAMOTO (NCTS, Taiwan)

*Gaussian deconvolution and the lace expansion for long-range models*

We consider long-range statistical mechanical models characterized by power-law decaying pair potentials of the form $D(x) \asymp |x|^{-d-\alpha}$, where $\alpha \in (0, 2)$. It has been established by L-C. Chen and A. Sakai (2015) that the critical two-point function for each long-range model above their upper critical dimension exhibits $|x|^{-(d-\alpha)}$ decay for large $|x|$. Their proof depend on intricate Fourier analysis and convolution estimate. In this talk , we offer a simpler and more transparent approach to $|x|^{-(d-\alpha)}$ decay of crirtical two point function, in which we apply the Gaussian deconvolution theorem introduced by Y. Liu and G. Slade (2023). This talk is based on ongoing joint work with Lung-Chi Chen from NCCU.

### 11:00–11:40: Pierre NOLIN (City University of Hong Kong)

*Monochromatic exponents for two-dimensional percolation*

We consider Bernoulli percolation in dimension two, obtained by coloring independently, black or white, the vertices of an infinite lattice. In order to describe its phase transition, arm events (at or near criticality) play an instrumental role. Roughly speaking, in any annulus on the lattice, such events require the existence of a given number of crossing paths, whose colors are specified by some sequence (in cyclic order). The asymptotic behavior of "polychromatic" arm events, when arms of both colors are present in the sequence, is now very well understood. We discuss the monochromatic case, i.e. the case when all arms have the same color.

In particular, we derive an exact expression for the celebrated backbone exponent, corresponding to two disjoint black arms. This exponent was first considered in statistical physics in the 1970's, and determining its value had remained an open question since then. It turns out to be a root of an elementary function, and contrary to previously-known arm exponents for 2D percolation, which are all rational, it has a transcendental value. More specifically, we use techniques which have been developed recently to compute the conformal radii of random domains defined by SLE curves, based on the coupling between SLE and Liouville quantum gravity (LQG), and using crucially input from Liouville conformal field theory (LCFT).

This talk is based on a joint work with Wei Qian, Xin Sun and Zijie Zhuang.

**13:30–14:10: Naoki KUBOTA (Nihon University)**

*Monotonicity of the time constant for the frog model with Bernoulli initial configurations*

We consider the frog model with Bernoulli initial configuration on the multi-dimensional lattice. This model consists of two types of particles: active and sleeping. Active particles perform independent simple random walks on the lattice. On the other hand, although sleeping particles do not move at first, they become active and can move around when touched by active particles. Initially, only the origin has one active particle, and the other sites have sleeping particles according to a Bernoulli distribution. Then, starting from the original active particle, active ones are gradually generated and propagate across the lattice, with time.

In this talk, we focus on the so-called time constant, which describes the speed of the spread of active particles in the aforementioned model. Notably, the time constant depends on the parameter of the Bernoulli distribution. To understand this dependence in more detail, we discuss how the time constant increases as the parameter changes.

**14:25–14:50: Ryoichiro NODA (Kyoto University)**

*Convergence of local times of stochastic processes associated with resistance forms*

A resistance metric on a space characterizes, as the electrical energy, a corresponding bilinear form (a resistance form) and, combined with a measure on the space (and under certain technical conditions), determines uniquely a Dirichlet form and a stochastic process on the space. Croydon-Hambly-Kumagai (2017) showed that if a sequence of spaces equipped with resistance metrics and measures converge with respect to the Gromov-Hausdorff-vague topology and a uniform volume doubling (UVD) condition is satisfied, then the associated stochastic processes and local times also converge. However, the UVD condition is too strong for many sequences of random graphs. In the subsequent work of Croydon (2018), the UVD condition was relaxed and the convergence of the processes was established under a weaker non-explosion condition. However, the convergence of local times was left open. In this talk, we show that if the spaces additionally satisfy a certain metric-entropy condition, which is weaker than the UVD condition, then the local times of the processes also converge. The metric-entropy condition can be checked using volume estimates of balls in the spaces, and we also present some example applications.

**15:10–15:50: Masato TAKEI (Yokohama National University)**

*Probabilistic analysis of the Takagi function and its generalizations*

The Takagi function is one of the most famous example of nowhere differentiable continuous functions. In the first part, based on a joint work with Shoto Osaka (JJIAM, 2020), we give a set of conditions to describe the rate of convergence of Takagi class functions from the probabilistic point of view. In the second part, based on an ongoing work with Yuzaburo Nakano, we study some properties of Takagi-van der Waerden class functions via variations of elephant random walks.

**16:00–16:40: Chenlin GU (Tsinghua University)**

*Random walk and homogenization on the infinite percolation cluster*

The random walk on percolation cluster is an important example of diffusion in disordered media, which is of interest in the study of both probability and PDEs: from the viewpoint of probability theory, this model illustrates the universality of Brownian motion; in PDEs theory, this is understood as the homogenization phenomenon. In this talk, I will review the classical results from both directions, and present my research results on the local central limit theorem and efficient numerical algorithm on percolation. If time allows, I will also introduce some other related recent progresses based on these results.

## 12 June 2024

**09:30–10:10: Shota OSADA (Kagoshima University)**

*On the ergodicity of unlabeled dynamics associated with random point fields*

We consider a configuration-valued Markov process defined by a Dirichlet form with the standard carré du champ.We say that a Markov process is ergodic if the probability of each event invariant under the time shift is 0 or 1. A typical example is a Poisson random point field, whose corresponding Markov process is an infinite number of Brownian motions. The $\mathrm{Sine}_2$ random point field is number-rigid and tail-trivial, whose corresponding Markov process is Dyson's model in infinite dimensions. There are fewer results for the ergodicity of infinite particle systems. Some results are proved for random point fields close to the Poisson case. As a more general result, for each tail-trivial and number-rigid random point field, the associated Markov processes are ergodic. However, given that Poisson random point fields are the typical example, it seems natural not to assume the number-rigidity.

In our talk, we show that the number-rigidity of random point fields is not needed for the ergodicity of the associated processes. We introduce a $\sigma$-field $\mathcal{G}_\infty$ and decomposition of random point field associated with $\mathcal{G}_\infty$ and show that the Markov process corresponding to each component is ergodic. Especially for a point process that is $\mathcal{G}_\infty$-trivial, the associated Markov process is ergodic. Our decomposition can be applied to number-rigid and tail-trivial random point fields, Coulomb random point fields, and canonical Gibbs measures with Ruelle's class potentials, for example. Remark that combining tail-triviality and number-rigidity, $\mathcal{G}_\infty$-triviality follows. Hence, our result contains the number-rigid case.

**10:25–10:50: Kohei NODA (Kyushu University)**

*Scaling limits of non-Hermitian Wishart ensembles*

Non-Hermitian Wishart matrices were introduced in the context of quantum chromodynamics with a baryon chemical potential. These are non-Hermitian extensions of the classical Wishart/Laguerre ensembles. In this talk, I will discuss correlations for eigenvalues of non-Hermitian Wishart matrices in the symmetry classes of complex and symplectic Ginibre ensembles. After introducing a generalized Christoffel-Darboux formula in the form of a certain second-order differential equation, I will show bulk and edge scaling limits for eigenvalue correlations at both strong and weak non-Hermiticity and some applications. This is based on joint work (arXiv:2402.18257) with Sung-Soo Byun (Seoul National University).

**11:00–11:40: Makoto KATORI (Chuo University)**

*Eigenvalues and pseudospectra in non-normal matrix-valued processes*

Motivated by the recent study of the non-Hermitian matrix-valued Brownian motion, we propose a family of matrix-valued dynamical systems generated by nonnormal Toeplitz matrices with additive perturbations. We first show complicated structures and motions of "eigenvalues" found in numerical calculations. Then we derive the specific equations determining the exact-eigenvalue processes. Coexistence of exact eigenvalues and pseudospectra is clarified by comparing the solutions of these equations with the numerical results. We use the theory of symbol curves of the corresponding nonnormal Toeplitz operators represented by infinite matrices and characterize the numerical results. We report a new phenomenon such that at each time the outermost closed simple curve in a symbol curve is realized as the exact eigenvalues, but the inner part of a symbol curve is reduced in size and embedded in the complicated structure of the pseudospectrum in our process. The matrices in our processes are defective and the Jordan decompositions are studied in connection with eigenvector-overlaps. The asymptotics of the processes in the infinite-matrix limit are also reported. This talk is based on the joint work with Saori Morimoto (Chuo) and Tomoyuki Shirai (Kyushu). A preprint is available at `https://arxiv.org/abs/2401.08129.`

**13:30–14:10: Izumi OKADA (Chiba University)**

*Deviation of capacity of the range of random walk*

We study the capacity of the range of a simple random walk in three and higher dimensions. It is known that the order of the capacity of the random walk range in $n$ dimensions is similar to that of the volume of the random walk range in $n-2$ dimensions. We find the specific difference between the law of the iterated logarithm and large deviation for the capacity of the random walk range and the volume. This is joint work with Arka Adhikari (Stanford), and Amir Dembo (Stanford).

**14:20–15:00: Xin CHEN (Shanghai Jiao Tong University)**

*Heat kernel estimates for Schrödinger operators*

We will introduce some probabilistic methods to study heat kernel estimates for Schrödinger operators (or non-local Schrödinger operators), based on which the interaction between the behaviors of Brownian motions (or stable processes) and potential functions will be applied. Our results include the case that potential functions are unbounded or decaying to 0 at infinity.

**15:20–16:00: Takashi KUMAGAI (Waseda University)**

*Quenched local limit theorem for random conductance models with long-range jumps*

We establish the quenched local limit theorem for reversible random walk on $\mathbb{Z}^d$, $d \geq 2$ among stationary ergodic random conductances that permit jumps of arbitrary length. Our proof is based on the weak parabolic Harnack inequalities and on-diagonal heat kernel estimates for long-range random walks on general ergodic environments.

As a byproduct, we prove the maximal inequality with an extra tail term for long-range reversible random walks, which in turn yields the everywhere sublinear property for the associated corrector. This is a joint work with Xin Chen (Shanghai) and Jian Wang (Fuzhou).