Un fil d'Ariane pour ce workshop [1]

(Main Tools)

- Modularity Lifting Theorems
  - MLT for residually reducible representations [SW1],
  - MLT for potentially Barsotti-Tate deformations [K1],
  - (MLT for crystalline deformations of intermediate weights [K3]),
  - MLT for unitary groups [CHT], etc.
- Potential Modularity Theorems
  - PMT by using Hilbert-Blumenthal modular varieties
    [T2] (ordinary case), [T3] (crystalline of low weight case),
  - PMT by using a family of Calabi-Yau varieties [HSBT] (GSp case).
- existence of Strictly Compatible Systems
  [KW1], [Kh1], and [KW3]
  - crystalline liftings of low weights,
  - weight 2 liftings etc.

(Conjectures)

- MLT for unitary groups & PMT by using Calabi-Yau family
  $\rightsquigarrow$Sato-Tate conjecture (under mild condition),
- MLT's & the existence of several kinds of SCS's
  $\rightsquigarrow$Serre's conjecture.

(Influence, or logical dependence)

- Wiles' $(3,5)$-trick
  $\rightsquigarrow$PMT,
- Kisin's modified TW argument in non-minimal cases
  $\rightsquigarrow$MLT for unitary groups in non-minimal cases,
- PMT & MLT
  $\rightsquigarrow$the existence of SCS's,
- (p-adic local Langlands
  $\rightsquigarrow$MLT for crystalline deformations of intermediate weights),
- (p-adic local Langlands
  $\rightsquigarrow$Breuil-Mézard conjecture
  $\rightsquigarrow$MLT for potentially semistabe deformations of arbitrary weights).

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Taylor-Wiles system ([W1], [TW], [D1])

- $H^1_{\mathcal{L}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}) \cong \mathrm{Hom}_k(\mathfrak{m}_{R_{\mathcal{L}}}/(\lambda, \mathfrak{m}^2_{R_{\mathcal{L}}}), k)$,
  its dim.= (the number of topological generators of the corresponding universal deformation ring).

- $\dim H^1_{\mathcal{L}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}) = \dim H^1_{\mathcal{L}^*}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}(1)) +$ (sum of local terms)
  by global Tate-Poitou duality, and
  - (local term at ∞)$= -1$,
  - (local term at p in the minimal case)≤ 1 by Fontaine-Laffaille theory,
  - (local terms ($\neq p$) at "minimally ramified" deformations)$= 0$,
  - (local terms at TW-type deformations)$= 1$,
  - $\dim H^1_{\mathcal{L}^*_\emptyset}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}(1)) - \dim H^1_{\mathcal{L}^*_{Q_n}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}(1)) = \#Q_n$,
  - $H^1_{\mathcal{L}^*_{Q_n}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}(1)) = 0$ by Cebotarev arguments,

  $\rightsquigarrow$
  (the number of topological generators of n-th level TW-type universal deformation ring)
  $= \dim H^1_{\mathcal{L}_{Q_n}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}) \leq \dim H^1_{\mathcal{L}^*_{Q_n}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}(1)) + \#Q_n$
  $= \#Q_n = \dim H^1_{\mathcal{L}^*_\emptyset}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}(1))$ (independent of n).

- $\mathbb{T}_{Q_n}$ is free over $\mathcal{O}[\Delta_{Q_n}]$ by de Shalit's argument (need of mod p multiplicity one) (resp. H_{Q_n} is free over $\mathcal{O}[\Delta_{Q_n}]$ by the argument in [D1] (no need of mod p multiplicity one)).

- TW system is *not* a compatible system with respect to n. We make a compatible system from TW system by using the argument of "finite isomorphism classes", and take a projective limit. In the limit level, the situation is simple. So, we get $R_\infty \xrightarrow{\sim} \mathbb{T}_\infty$ in the limit level. We deduce $R_\emptyset \xrightarrow{\sim} \mathbb{T}_\emptyset$ & l.c.i. (resp. + freeness of $H_\emptyset$ over $\mathbb{T}_\emptyset$ [D1]) in the finite level from the limit level.

- $\mathbb{T}_\Sigma$ is reduced $\rightsquigarrow \#(\mathcal{O}/\eta_\Sigma) < \infty$. (cf. we do not know *a priori* $\#(\mathfrak{p}_\Sigma/\mathfrak{p}^2_\Sigma) < \infty$).

- Ihara' lemma and its generalization +Gorenstein-ness of $\mathbb{T}_{\Sigma'}$ and $\mathbb{T}_\Sigma$ (resp. no need of Gorenstein-ness [D1])
  $\rightsquigarrow$calculation of $\#(\eta_\Sigma/\eta_{\Sigma'})$ (resp. $\mathrm{length}_{\mathcal{O}}\Omega_{\Sigma'}/\Omega_\Sigma$, where $\Omega_\Sigma := H_\Sigma/(H_\Sigma[\mathfrak{p}_\Sigma] + H_\Sigma[I_\Sigma])$)
  $\rightsquigarrow \#(\mathfrak{p}_{\Sigma'}/\mathfrak{p}^2_{\Sigma'})/(\mathfrak{p}_\Sigma/\mathfrak{p}^2_\Sigma) \leq \#(\eta_\Sigma/\eta_{\Sigma'})$
  (resp. $\mathrm{length}_{\mathcal{O}}(\mathfrak{p}_{\Sigma'}/\mathfrak{p}^2_{\Sigma'})/(\mathfrak{p}_\Sigma/\mathfrak{p}^2_\Sigma) \leq \mathrm{length}_{\mathcal{O}}\Omega_{\Sigma'}/\Omega_\Sigma$)
  $\rightsquigarrow$
  $R_\Sigma \xrightarrow{\sim} \mathbb{T}_\Sigma$ & l.c.i. (resp. + freeness of H_Σ over $\mathbb{T}_\Sigma$) implies
  $R_{\Sigma'} \xrightarrow{\sim} \mathbb{T}_{\Sigma'}$ & l.c.i. (resp. + freeness of $H_{\Sigma'}$ over $\mathbb{T}_{\Sigma'}$ [D1]).

Kisin's modification of TW argument ([K1], [K3], [K7])

- We study a global deformation ring over local deformation rings
  $\rightsquigarrow$
  - we can show $R^{\mathrm{red}} = T$ even if the local deformation rings at the places dividing p have complicated singularity, and
  - we can show $R^{\mathrm{red}} = T$ without level raising in non-minimal cases.

- We consider framed deformations
  $\rightsquigarrow$we can study local framed deformation rings even if $\overline{\rho}|_{G_{\mathbb{Q}_v}}$ is not irreducible.

- dim. of Selmer group + local contributions
  $\rightsquigarrow$the number of topological generators of R over $\widehat{\otimes}_{v\in\Sigma} R_v$.

- We have to study the following things about local framed deformation rings to apply Kisin's modified TW argument:
  (1) calculation of the dimensions of the local deformation rings,
  (2) to show that the local deformation rings are formally smooth after inverting p, and
  (3) to show that the local deformation rings are domains.

- The above (1), (2), and (3) are easy in the case of $v \nmid p$. In the case of $v \mid p$:
  (1) Calculation of the dimension is easy,
  (2) Formally smooth after inverting p:
  Breuil's theorem (crystalline representations of HT weights in {0,1} come from p-divisible groups)
  $\rightsquigarrow D^{\mathrm{fl}}_{V_{\mathbb{F}},(\xi)} \xrightarrow{\sim} D^{\mathrm{crys}}_{V_\xi}$
  $\rightsquigarrow$check explicitly the formally smoothness by constructing a lifting,
  (3) Domain: Consider a moduli of finite flat models $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}}}$,
    (a) Tate's theorem
    $\rightsquigarrow\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}}}$ is isomorphic to $\mathrm{Spec} R^{\mathbf{v}}$ after inverting p,
    (b) comparing $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}}}$ with a complete local ring of a Hilbert modular variety
    $\rightsquigarrow\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}}} \otimes \mathbb{F}$ is normal, in particular, reduced,
    (c) Kisin's theory of $\mathfrak{S}$-modules in the integral p-adic Hodge theory
    $\rightsquigarrow$the special fiber $\mathcal{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}},0}$ is connected by explicit linear algebra calculations (repeat connecting a point to another point by $\mathbb{P}^1$),
    (d) $H_0(\mathrm{Spec} R^{\mathbf{v},\text{non-ord}}[\frac{1}{p}]) \cong H_0(\mathcal{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}}} \otimes \mathbb{Q}_p)$ by (a)
    $\cong H_0(\mathcal{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}}})$ by (b) $\cong H_0(\widehat{\mathcal{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}}}})$ by formal GAGA
    $\cong H_0(\mathcal{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}},0}) \cong \{*\}$ by (c).

Potential Modularity Theorems ([T2], [T3], [HSBT])

GL_2 case ([T2], [T3]):

- We want to find a Hilbert-Blumenthal abelian variety A such that
$$\overline{\rho} \cong A[\lambda] \longleftarrow T_\lambda A \leftrightsquigarrow T_\wp A \longrightarrow A[\wp] \cong \mathrm{Ind}\overline{\psi}.$$
- We consider Hilbert-Blumenthal modular varieties, and try to find such an abelian variety as a rational point of this modular variety.
- (We allow "potentiality")
  Moret-Bailly's theorem
  $\rightsquigarrow$it suffices to find local points (at λ, $\wp$, and ∞) to get such an abelian variety.
- Ordinary case ([T2]):
  - Honda-Tate theory
    $\rightsquigarrow$find an abelian variety over a finite field,
  - Serre-Tate theory
    $\rightsquigarrow$find an abelian variety over a local field.
- Crystalline of low weight case ([T3]):
  - We consider a twist of the modular variety, which is isomorphic over $\mathbb{Q}_\ell$, $\mathbb{Q}_{p_1}$, $\mathbb{Q}_{p_2}$, and $\mathbb{R}$,
  - CM theory
    $\rightsquigarrow$find a $\mathbb{Q}$ rational point on the twisted variety
    $\rightsquigarrow$find local points on the original variety,
  - studying mod ℓ representations of $\mathrm{GL}_2(O_{F_\lambda})$
    $\rightsquigarrow$change of weights.

GSp_n case ([HSBT]):

- We use Calabi-Yau varieties instead of abelian varieties, and a Calabi-Yau family instead of Hilbert-Blumenthal modular variety.
- The condition of the relation with $\overline{\rho}$
  $\rightsquigarrow$we have to consider a covering of the Calabi-Yau family.
- The Calabi-Yau family has big monodromy
  $\rightsquigarrow$the covering is geometrically connected
  $\rightsquigarrow$we can apply Moret-Bailly's theorem.
- trivial reason, or Fontaine-Laffaille theory, or Serre-Tate theory
  $\rightsquigarrow$find local points.

existence of Strict Compatible Systems ([KW1], [Kh1], and [KW3])

- Savitt's study of local deformation rings
  $\rightsquigarrow$local deformation rings we are considering are not zero.

- (Böckle's method) For $\theta^i : H^i(G_{\mathbb{Q},S}, \mathrm{ad}^0\overline{\rho}) \to \oplus_{v\in\Sigma} H^i(\mathbb{Q}_v, \mathrm{ad}^0\overline{\rho})$,
  - calculation of $\dim \ker\theta^1$
    $\rightsquigarrow$the number of topological generators of R over $\widehat{\otimes}_{v\in\Sigma} R_v$, and
  - calculation of $\dim \mathrm{coker}\theta^1 + \dim \ker\theta^2$
    $\rightsquigarrow$the number of relations of R over $\widehat{\otimes}_{v\in\Sigma} R_v$

  $\rightsquigarrow\dim R \leq 1$.

- PMT
  $\rightsquigarrow$global deformation ring R_F of $\overline{\rho}|_{G_F}$ is flat over $\mathcal{O}$ by MLT
  $\rightsquigarrow R/(p)$ is finite by de Jong's argument
  $\rightsquigarrow R$ is flat over $\mathcal{O}$
  $\rightsquigarrow$we get a minimally ramified lifting ρ (with some conditions) to characteristic 0.

- PMT
  $\rightsquigarrow\rho|_{G_F}$ arises from an automorphic representaion π of $\mathrm{GL}_2(\mathbb{A}_F)$
  $\rightsquigarrow$we can make ρ a part of SCS's by Brauer's thoerem:
  $\rho_\lambda := \sum_i n_i \mathrm{Ind}_{G_{F_i}}^{G_\mathbb{Q}} (\chi_i \otimes \rho_{\pi_{F_i},\lambda})$,
  where $1 = \sum_i n_i \mathrm{Ind}_{G_{F_i}}^{G_\mathbb{Q}} \chi_i$ (F/F_i's are elementary, in particular, solvable), and π_{F_i} is an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{F_i})$ such that $\rho_{\pi_{F_i},\wp} \cong \rho|_{G_{F_i}}$ (we can check that ρ_λ is a true representation).