A conjecture of Kawashima-Poëls and holomorphic extensions of Bernoullization map

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Notation

- B_n : Bernoulli numbers (with $B_1 = +\frac{1}{2}$), i.e., defined by $\frac{te^t}{e^t-1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}$,
- $B_n(x)$: Bernoulli polynomials, i.e., defined by $\frac{te^{xt}}{e^t-1} = \sum_{n\geq 0} B_n(x) \frac{t^n}{n!}$,
- $B_{n,\chi}$: generalized Bernoulli numbers for a Dirichlet character χ of conductor N, i.e., defined by $\sum_{1\leq a\leq N} \frac{\chi(a)te^{at}}{e^{Nt}-1} = \sum_{n\geq 0} B_{n,\chi} \frac{t^n}{n!}$,
- $B(s_0, s_1)$: beta function,
- $\Gamma(s)$: gamma function,
- $\psi^{(k)}(s) := \frac{d^{k+1}}{ds^{k+1}} \log \Gamma(s)$: polygamma function ($\psi(s) := \psi^{(0)}(s)$: digamma function).

Notation

- $\mathbb{N} := \mathbb{Z}_{\geq 0}$,
- $(x)_n := x(x+1)\cdots(x+n-1)$: Pochhammer symbol,
- For $\mathbf{s} = (s_0, \ldots, s_m), \mathbf{s}' = (s'_0, \ldots, s'_m) \in \mathbb{C}^{m+1}$,
    - $|\mathbf{s}| := s_0 + \cdots + s_m$,
    - $\mathbf{s}! := \Gamma(s_0+1)\cdots\Gamma(s_m+1)$ if $\mathbf{s} \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^{m+1}$,
    - $\mathbf{s} + \mathbf{s}' := (s_0 + s'_0, \ldots, s_m + s'_m)$,
    - $(0, \mathbf{s}) := (0, s_0, \ldots, s_m) \in \mathbb{C}^{m+2}$.

§1, Introduction

Bernoullization map

Let R be a commutative ring containing $\mathbb{Q}$, $R[B]$ the polynomial ring with coefficient in R with a formal variable B, and an R-linear homomorphism

$$\varphi : R[B] \to R.$$

We write the generating function $f_\varphi(t) := \sum_{n\geq 0} \frac{\varphi(B^n)}{n!} t^n \in R[[t]]$ for φ.

Remark (generating function)

If we extend φ to the $R[[t]]$-linear hom. $R[B][[t]] \to R[[t]]$, then $f_\varphi(t) = \varphi(e^{Bt})$.

Remark (R-algebra homomorphism)

φ is an R-*algebra* hom. if and only if $\varphi = \mathrm{ev}_a : B^n \mapsto a^n$ for some $a \in R$.

Bernoullization map

Example (Bernoullization map)

Bernoullization map is the $\mathbb{Q}$-linear homomorphism

$$\mathrm{Ber} : \mathbb{Q}[B] \to \mathbb{Q}; \qquad B^n \mapsto B_n.$$

We have $f_{\mathrm{Ber}}(t) = \mathrm{Ber}(e^{Bt}) = \frac{te^t}{e^t - 1}$.

Remark (Bernoulli polynomial)

$\mathrm{Ber}((B-1+a)^n) = B_n(a)$, since $\mathrm{Ber}(e^{(B-1+a)t}) = e^{(-1+a)t}\mathrm{Ber}(e^{Bt}) = \frac{te^{at}}{e^t-1}$.

Eisensteinization map

Example (Eisensteinization map)

For $R := \mathbb{C}[\mathrm{Eis}_2, \mathrm{Eis}_4, \mathrm{Eis}_6] \subset$ (hol. fct's. on upper half plane), **Eisensteinization map** is the R-linear homomorphism

$$\mathrm{Eis} : R[B] \to R; \qquad B^k \mapsto \mathrm{Eis}_k,$$

where $\mathrm{Eis}_k(\tau) := B_k - 2k \sum_{n \geq 1} \sigma_{k-1}(n) q^n$ for k is even, 0 for k is odd ($q := e^{2\pi i \tau}$). We have

$$f_{\mathrm{Eis}}(z) = \mathrm{Eis}_2(\tau)\frac{z^2}{2} + z\frac{d}{dz}\log\sigma(z/2\pi i, \tau) = z\frac{d}{dz}\log\theta_{11}(z/2\pi i, \tau),$$

where σ is **Weierstrass' $\boldsymbol{\sigma}$-function**, and θ_{11} is one of **Jacobi's theta functions**. Note that $\lim_{\tau \to i\infty} \mathrm{Eis}(B^n)(\tau) = \mathrm{Ber}(B^n)$ for $n \neq 1$.

Kawashima-Poëls determinant

For $n \in \mathbb{N}$ and $\mathbf{n} = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$, we write

$$P_n := \frac{(B)_n}{n!}, \qquad P_{\mathbf{n}} := P_{n_0} \cdots P_{n_m}.$$

We define the **Kawashima-Poëls determinant** to be

$$\Theta_{\mathbf{n}}^{\varphi} := \det\left(\varphi\left(\frac{d^i}{dB^i}(P_{\mathbf{n}}P_j)\right)\right)_{0 \le i,j \le m-1} \in R$$

(the determinant of an m-by-m matrix). We also write

$$\Theta_{\mathbf{n}} := \Theta_{\mathbf{n}}^{\mathrm{Ber}} \in \mathbb{Q}.$$

A conjecture of Kawashima-Poëls

In the studies of irrational number theory on the p-adic Hurwitz zeta function, Kawashima-Poëls proposed the following conjecture:

Conjecture 1 (Kawashima-Poëls)

For any $m \geq 1$ and $\mathbf{n} \in \mathbb{N}^{m+1}$, it holds that $\Theta_{\mathbf{n}} = \frac{m!\mathbf{n}!}{(m+|\mathbf{n}|)!}$.

Remark (a theorem of Kawashima-Poëls)

Kawashima-Poëls proved $\Theta_{\mathbf{n}} \neq 0$, and that it implied, by constructing Padé approximants, some $\mathbb{Q}$-**linear independence results** on special values of p-adic Hurwitz zeta function.

First Main Theorem

Theorem 1 (KP-conjecture)

Conjecture 1 is true.

The following corollary was also conjectured by Kawashima from a point of view of *multivariable beta functions*:

Corollary (Distribution relation)

It holds that $\Theta_{\mathbf{n}} = \Theta_{\mathbf{n}+(1,0,\ldots,0)} + \cdots + \Theta_{\mathbf{n}+(0,\ldots,0,1)}$.

Two holomorphic extensions of Bernoullization map

Next, we extend the Bernoullization map, **in two ways**, to holomorphic functions satisfying certain conditions.

- The first extension ($=:$ series ext'n) is motivated from the fact that **the Bernoulli numbers appear in special values of Riemann's zeta fct**.
- The second extension ($=:$ integral ext'n) is motivated from the fact that **the Bernoulli numbers appear as the solution of the difference problem**, i.e., for a poly. $f(n)$, find a poly. $F(n)$ s.t. $f(n) = F(n+1) - F(n)$.
- They **coincide** on the overlap of the domains of the definitions (Theorem 2).

Remark (philosophical remark)

The series (resp. integral) ext'n "comes from" $\mathrm{Re}(s) < 0$ (resp. $\mathrm{Re}(s) > -1$).

Examples of holomorphic extensions of Bernoullization map

Examples (holomorphic extension of Bernoullization)

- $\mathrm{Ber}(B^s) = -s\zeta(1-s)$,
- $\mathrm{Ber}(e^{Bt}) = \frac{te^t}{e^t-1}$,
- $\mathrm{Ber}(\log(B-a)) = \psi(1-a)$, in particular, $\mathrm{Ber}(-\log B) = \gamma$ (Euler's cst.),
- $\mathrm{Ber}((B-a)(\log(B-a)-1)) = \log\frac{\Gamma(1-a)}{\sqrt{2\pi}} = \zeta'(0,1-a)$.

Remark ($B^i(\log B)^j$ and the j-th derivative at $s=1-i$)

$\mathrm{Ber}((B-a)^i(\log(B-a))^j)$ is related to $\zeta^{(j)}(1-i,1-a)$
(resp. $((s-1)\zeta(s,1-a))^{(j)}|_{s=1}$) for $i>0$ (resp. $i=0$), $j\geq 0$.

Complex KP-conjecture

We extend the polynomial $P_{\mathbf{n}} = \prod_{0 \leq i \leq m} \frac{(B)_{n_i}}{n_i!} \in \mathbb{Q}[B]$ for $\mathbf{n} = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$ to the meromorphic function of $(B, \mathbf{s}) \in \mathbb{C} \times \mathbb{C}^{m+1}$:

$$P_{\mathbf{s}} := \prod_{0 \leq i \leq m} \frac{\Gamma(B + s_i)}{\Gamma(B)\Gamma(s_i + 1)},$$

where $\mathbf{s} = (s_0, \ldots, s_m)$, and we define the **complex KP-determinant**

$$\Theta_{\mathbf{s}} := \det \left(\mathrm{Ber} \left(\frac{d^i}{dB^i} (P_{\mathbf{s}} P_j) \right) \right)_{0 \leq i,j \leq m-1}.$$

Second Main Theorem

Theorem 3 (Complex KP-conjecture)

It holds that $\Theta_{\mathbf{s}} = \frac{m!\mathbf{s}!}{(m+|\mathbf{s}|)!}$ as holomorphic functions on $(\mathbb{C} \setminus \mathbb{Z}_{<0})^{m+1}$.

Remark ($m = 1$ case)

Theorem 3 is proved for the integral extension of Bernoullization, and, by Theorem 2 (Comparison Theorem), Theorem 3 holds for the series extension of Bernoullization, whose $m = 1$ case is equivalent to

$$B(s_0, s_1) = -\sum_{n\geq 0} \frac{(s_0)_n (s_1)_n}{n!^2} \left(\psi(s_0 + n) + \psi(s_1 + n) - 2\psi(n + 1)\right)$$

for $(s_0, s_1) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^2$ with $\mathrm{Re}(s_0 + s_1) < 2$.

§2, Proof of KP-conjecture

Strategy of the proof: $\Theta_{\mathbf{n}} = \frac{m!\mathbf{n}!}{(m+|\mathbf{n}|)!}$

Proposition 1 (Boundary relation)

For $\mathbf{n} \in \mathbb{N}^{m+1}$, it holds that $\Theta_{(0,\mathbf{n})} = \frac{m+1}{m+1+|\mathbf{n}|}\Theta_{\mathbf{n}}$.

Proposition 2 (Increment relation)

For $\mathbf{n} = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$, it holds that $\Theta_{\mathbf{n}+(1,0,\ldots,0)} = \frac{n_0+1}{m+|\mathbf{n}|+1}\Theta_{\mathbf{n}}$.

Remark (two Propositions imply Theorem 1)

Boundary relation, Increment relation, $\Theta_{(0,0)} = 1$, and the symmetry of $\Theta_{(n_0,\ldots,n_m)}$ with respect to $n_0, \ldots, n_m$ imply Theorem 1.

Preparation via Wronskian (valid for all φ)

For $Q, Q_0, \ldots, Q_{m-1} \in R[B]$, we write $\mathbf{v}_m^{\varphi}(Q) := \left(\varphi\left(\frac{d^i Q}{dB^i}\right)\right)_{0\le i\le m-1} \in R^m$,
$\mathbf{v}_m(Q) := \mathbf{v}_m^{\mathrm{Ber}}(Q) \in \mathbb{Q}^m$ (column vectors), and
$W(Q_0, \ldots, Q_{m-1}) := \det\left(\frac{d^i Q_j}{dB^i}\right)_{0\le i,j\le m-1} \in R[B]$.

Lemma 1 (Wronskian relation)

Let $Q_0, \ldots, Q_{m-1} \in R[B]$ with deg $< m$. Then $W(Q_0, \ldots, Q_{m-1}) \in R$, and

$$\det\left(\ \mathbf{v}_m^{\varphi}(P_{\mathbf{n}} Q_0) \ \vdots\ \cdots \ \vdots\ \mathbf{v}_m^{\varphi}(P_{\mathbf{n}} Q_{m-1})\ \right) = W(Q_0, \ldots, Q_{m-1}) \Theta_{\mathbf{n}}^{\varphi}.$$

Proof of Lemma 1

Since $1, \frac{(B)_1}{1!}, \dots, \frac{(B)_{m-1}}{(m-1)!}$ form an R-basis of $R[B]_{\deg<m}$, $\exists A \in M_m(R)$ such that

$$(Q_0, \dots, Q_{m-1}) = \left(1, \frac{(B)_1}{1!}, \dots, \frac{(B)_{m-1}}{(m-1)!}\right) A.$$

Hence (LHS) $= \det A \cdot \Theta^{\varphi}_{\mathbf{n}}$. For similar reasons, we have
$W(Q_0, \dots, Q_{m-1}) = \det A \cdot W\left(1, \frac{(B)_1}{1!}, \dots, \frac{(B)_{m-1}}{(m-1)!}\right) = \det A \in R$. $\square$

Corollaries to Lemma 1

Corollary (Independence of lower terms)

Let $Q_i = \frac{a_i}{i!}B^i + (\text{lower terms}) \in R[B]$ with $\deg = i$ for $0 \leq i \leq m-1$. Then

$$\det\left(\mathbf{v}_m^{\varphi}(P_{\mathbf{n}}Q_0) \,\vdots\, \cdots \,\vdots\, \mathbf{v}_m^{\varphi}(P_{\mathbf{n}}Q_{m-1}) \right) = a_0 \cdots a_{m-1}\Theta_{\mathbf{n}}^{\varphi}.$$

Corollary (KP-determinants for algebra hom's)

For any $m \geq 1$, $\mathbf{n} \in \mathbb{N}^{m+1}$ and $a \in R$, it holds that $\Theta_{\mathbf{n}}^{\mathrm{ev}_a} = P_{\mathbf{n}}(a)^m$.

Proof: By Lemma 1,

$$\Theta_{\mathbf{n}}^{\mathrm{ev}_a} = \det\left(\mathbf{v}_m^{\mathrm{ev}_a}(P_{\mathbf{n}}) \,\vdots\, \mathbf{v}_m^{\mathrm{ev}_a}\left(P_{\mathbf{n}}\frac{B-a}{1!}\right) \,\vdots\, \cdots \,\vdots\, \mathbf{v}_m^{\mathrm{ev}_a}\left(P_{\mathbf{n}}\frac{(B-a)^{m-1}}{(m-1)!}\right) \right) = P_{\mathbf{n}}(a)^m. \qquad \square$$

Preparation via generating function (valid only for Ber)

Put $\mathbf{v}_{\mathbf{n}}^{\varphi} := \mathbf{v}_m^{\varphi}(P_{\mathbf{n}}) \in R^m$, $\mathbf{v}_{\mathbf{n}} := \mathbf{v}_{\mathbf{n}}^{\mathrm{Ber}} \in \mathbb{Q}^m$. Write $\mathbf{t}^{\mathbf{n}} := t_0^{n_0} \cdots t_m^{n_m}$.

Lemma 2 (Generating function)

$$\sum_{\mathbf{n} \in \mathbb{N}^{m+1}} \mathbf{v}_{\mathbf{n}}^{\varphi} \mathbf{t}^{\mathbf{n}} = \left((-\log T)^i f_{\varphi}(-\log T)\right)_{0 \le i \le m-1} \quad (\in R[[t_0, \ldots, t_m]]^m),$$

where $T := (1-t_0)\cdots(1-t_m)$. In particular,

$$(1-T) \sum_{\mathbf{n} \in \mathbb{N}^{m+1}} \mathbf{v}_{\mathbf{n}} \mathbf{t}^{\mathbf{n}} = \left((-\log T)^{i+1}\right)_{0 \le i \le m-1}.$$

Proof of Lemma 2

Proof: Since $\sum_{n_i \geq 0} \frac{(B)_{n_i}}{n_i!} t_i^n = \sum_{n_i \geq 0} \binom{-B}{n_i} (-t_i)^{n_i} = (1 - t_i)^{-B}$, we have $P := \sum_{\mathbf{n} \in \mathbb{N}^{m+1}} P_{\mathbf{n}} \mathbf{t}^{\mathbf{n}} = \frac{1}{T^B}$. The first assertion follows from $\frac{d^i P}{dB^i} = \frac{(-\log T)^i}{T^B}$ and $\varphi(\frac{(-\log T)^i}{T^B}) = (-\log T)^i \varphi(e^{B(-\log T)}) = (-\log T)^i f_\varphi(-\log T)$.

The second assertion follows from $f_{\mathrm{Ber}}(-\log T) = \frac{(-\log T)e^{-\log T}}{e^{-\log T} - 1} = \frac{-\log T}{1 - T}$. $\square$

Remark (p-adic L-function)

Under the well-known correspondence between the formal power series and the p-adic measures, after replacing $\mathrm{Ber}(B) = 1/2$ by $\mathrm{Ber}(B) = 0$, $f_{\mathrm{Ber}}(\log(1 + t) + a) \in \mathbb{Z}_p^{\mathrm{ur}\wedge}[[t]]$ for $a \in \mathbb{Z}_{(p)}$ correspond to the p-adic measures giving rise to Kubota-Leopoldt' p-adic L-function.

Corollary of Lemma 2

Corollary (Alternating relation)

1. For $\mathbf{n} \in \mathbb{N}^{m+1}$, it holds that $\sum_{\mathbf{e}\in\{0,1\}^{m+1},\mathbf{e}\neq(1,\dots,1)}(-1)^{|\mathbf{e}|}\mathbf{v}_{\mathbf{n}+\mathbf{e}} = (0,\dots,0)^t$.
2. For $\mathbf{n} = (n_1,\dots,n_m) \in \mathbb{N}^m$, it holds that $\sum_{\mathbf{e}\in\{0,1\}^{m},\mathbf{e}\neq(1,\dots,1)}(-1)^{|\mathbf{e}|}\mathbf{v}_{(0,\mathbf{n}+\mathbf{e})} = (-1)^{m-1}(0,\dots,\frac{m!}{(n_1+1)\cdots(n_m+1)})^t$.

Proof: Compare the coefficient of $\mathbf{t}^{\mathbf{n}+(1,\dots,1)}$ and $\mathbf{t}^{(0,\mathbf{n}+(1,\dots,1))}$ in the second formula of Lemma 2, and use $(-\log T)^{i+1} = (\sum_{n_0>0}\frac{t_0^{n_0}}{n_0}+\cdots+\sum_{n_m>0}\frac{t_m^{n_m}}{n_m})^{i+1}$.
$\square$

Proof of Boundary relation $\Theta_{(0,\mathbf{n})} = \frac{m+1}{m+1+|\mathbf{n}|}\Theta_{\mathbf{n}}$

We extend the maps $\mathbb{N}^{m+1} \to \mathbb{Q}[B] : \mathbf{n} \mapsto P_{\mathbf{n}}$ and $\mathbb{N}^{m+1} \to \mathbb{Q}^{m+1} : \mathbf{n} \mapsto \mathbf{v}_{\mathbf{n}}$ to $\mathbb{Z}$-linear maps $\mathbb{Z}[\mathbb{N}^{m+1}] \to \mathbb{Q}[B]$ and $\mathbb{Z}[\mathbb{N}^{m+1}] \to \mathbb{Q}^{m+1}$, respectively.

For $\mathbf{n} \in \mathbb{N}^{m+1}$ and $0 \le i \le m$, put

$$\alpha_i := \sum_{\mathbf{e}\in\{0,1\}^{m+1},|\mathbf{e}|=i} [(0, \mathbf{n}+\mathbf{e})] \in \mathbb{Z}[\mathbb{N}^{m+2}], \quad \alpha'_i := \sum_{\mathbf{e}\in\{0,1\}^{m+1},|\mathbf{e}|=i} [\mathbf{n}+\mathbf{e}] \in \mathbb{Z}[\mathbb{N}^{m+1}].$$

Note that $\alpha_0 = (0, \mathbf{n})$, $\alpha'_0 = \mathbf{n}$, and $P_{\alpha_i} = P_{\alpha'_i}$.

Note also that $P_{\alpha_i} = P_{\mathbf{n}} \cdot \left(s_i(\frac{1}{n_0+1}, \ldots, \frac{1}{n_m+1})B^i + (\text{lower terms}) \right)$, where $s_i(x_0, \ldots, x_m)$ denotes the elementary symmetric polynomial of degree i. Hence, by Corollary of Wronskian relation,

$$\det \left(\mathbf{v}_{\alpha_0} \vdots \mathbf{v}_{\alpha_1} \vdots \cdots \vdots \mathbf{v}_{\alpha_m} \right) = \left(\prod_{i=0}^{m} i! s_i(\frac{1}{n_0+1}, \ldots, \frac{1}{n_m+1}) \right) \Theta_{(0,\mathbf{n})}.$$

On the other hand, by alternating relation (2), we have

$$\sum_{i=0}^{m} (-1)^i \mathbf{v}_{\alpha_i} = (-1)^m \left(0, \ldots, 0, \frac{(m+1)!}{(n_0+1)\cdots(n_m+1)} \right)^t.$$

Thus, we have

$$
\begin{aligned}
&\det\left(\begin{array}{c:c:c:c} \mathbf{v}_{\alpha_0} & \mathbf{v}_{\alpha_1} & \cdots & \mathbf{v}_{\alpha_m} \end{array}\right)\\
=&\det\left(\begin{array}{c:c:c:c:c} \mathbf{v}_{\alpha_0} & \mathbf{v}_{\alpha_1} & \cdots & \mathbf{v}_{\alpha_{m-1}} & (-1)^m \sum_{i=0}^m (-1)^i \mathbf{v}_{\alpha_i} \end{array}\right)\\
=&\det\left(\begin{array}{c:c:c:c:c} & & & & 0 \\ \mathbf{v}_{\alpha'_0} & \mathbf{v}_{\alpha'_1} & \cdots & \mathbf{v}_{\alpha'_{m-1}} & \vdots \\ & & & & 0 \\ \hdashline * & * & \cdots & * & \frac{(m+1)!}{(n_0+1)\cdots(n_m+1)} \end{array}\right)\\
=&\frac{(m+1)!}{(n_0+1)\cdots(n_m+1)}\cdot\left(\prod_{i=0}^{m-1} i!\,s_i\left(\frac{1}{n_0+1},\ldots,\frac{1}{n_m+1}\right)\right)\Theta_{\mathbf{n}}.
\end{aligned}
$$

Therefore, we obtain

$$\Theta_{(0,\mathbf{n})} = \frac{1}{m! s_m\left(\frac{1}{n_0+1}, \ldots, \frac{1}{n_m+1}\right)} \cdot \frac{(m+1)!}{(n_0+1)\cdots(n_m+1)} \cdot \Theta_{\mathbf{n}}$$
$$= \frac{m+1}{m+1+n_0+\cdots+n_m}\Theta_{\mathbf{n}}$$

as desired. □

Proof of Increment relation $\Theta_{\mathbf{n}+(1,0,\dots,0)} = \frac{n_0+1}{m+|\mathbf{n}|+1}\Theta_{\mathbf{n}}$

For $m \geq 1$, $\mathbf{n} \in \mathbb{N}^{m+1}$, and $0 \leq i \leq m-1$, put

$$\alpha := \sum_{\mathbf{e}\in\{0,1\}^m} (-1)^{|\mathbf{e}|}[\mathbf{n} + (0, \mathbf{e})] \in \mathbb{Z}[\mathbb{N}^{m+1}],$$

$$\beta_i := \sum_{\substack{\mathbf{e}\in\{0,1\}^m \\ |\mathbf{e}|=i}} [\mathbf{n} + (1, 0, \dots, 0) + (0, \mathbf{e})] \in \mathbb{Z}[\mathbb{N}^{m+1}].$$

By alternating relation (1), we have $\mathbf{v}_\alpha = \sum_{i=0}^{m-1}(-1)^i \mathbf{v}_{\beta_i}$. In particular,

$$\begin{aligned}
&\det\left(\mathbf{v}_\alpha \,\vdots\, \mathbf{v}_{\beta_0} \,\vdots\, \cdots \,\vdots\, \mathbf{v}_{\beta_{m-2}} \right) = \det\left((-1)^{m-1}\mathbf{v}_{\beta_{m-1}} \,\vdots\, \mathbf{v}_{\beta_0} \,\vdots\, \cdots \,\vdots\, \mathbf{v}_{\beta_{m-2}} \right) \\
&= \det\left(\mathbf{v}_{\beta_0} \,\vdots\, \cdots \,\vdots\, \mathbf{v}_{\beta_{m-2}} \,\vdots\, \mathbf{v}_{\beta_{m-1}} \right) \\
&= \left(\prod_{i=0}^{m-1} i!s_i\left(\frac{1}{n_1+1}, \ldots, \frac{1}{n_m+1}\right)\right) \Theta_{\mathbf{n}+(1,0,\ldots,0)}
\end{aligned}$$

by Corollary of Wronskian relation.

On the other hand, $P_\alpha = P_{\mathbf{n}} P$, where

$$P = \prod_{i=1}^{m} \left(1 - \frac{B + n_i}{n_i + 1} \right).$$

Let Q (resp. R) denote the quotient (resp. the remainder) of P divided by $\frac{B+n_0}{n_0+1}$. Then,

$$Q = (-1)^m \frac{n_0 + 1}{(n_1 + 1) \cdots (n_m + 1)} B^{m-1} + (\text{lower terms}),$$

and

$$R = P(-n_0) = \prod_{i=1}^{m} \left(1 - \frac{-n_0 + n_i}{n_i + 1} \right) = \prod_{i=1}^{m} \frac{n_0 + 1}{n_i + 1}.$$

Therefore, we obtain

$$
\begin{aligned}
&\left(\prod_{i=0}^{m-1} i! s_i\left(\frac{1}{n_1+1},\dots,\frac{1}{n_m+1}\right)\right)\Theta_{\mathbf{n}+(1,0,\dots,0)} = \det\left(\ \mathbf{v}_\alpha \ \vdots\ \mathbf{v}_{\beta_0} \ \vdots\ \cdots \ \vdots\ \mathbf{v}_{\beta_{m-2}}\ \right)\\
&= \det\left(\ \mathbf{v}_m(P_{\mathbf{n}+(1,0,\dots,0)}Q) \ \vdots\ \mathbf{v}_{\beta_0} \ \vdots\ \cdots \ \vdots\ \mathbf{v}_{\beta_{m-2}}\ \right) + R\det\left(\ \mathbf{v}_{\mathbf{n}} \ \vdots\ \mathbf{v}_{\beta_0} \ \vdots\ \cdots \ \vdots\ \mathbf{v}_{\beta_{m-2}}\ \right)\\
&=(-1)^{m-1}\det\left(\mathbf{v}_{\beta_0} \ \vdots\ \cdots \ \vdots\ \mathbf{v}_{\beta_{m-2}} \ \vdots\ \mathbf{v}_m(P_{\mathbf{n}+(1,0,\dots,0)}Q)\right) + R\det\left(\mathbf{v}_{\mathbf{n}} \ \vdots\ \mathbf{v}_{\beta_0} \ \vdots\ \cdots \ \vdots\ \mathbf{v}_{\beta_{m-2}}\right)\\
&= -\left(\prod_{i=0}^{m-2} i! s_i\left(\frac{1}{n_1+1},\dots,\frac{1}{n_m+1}\right)\right)\frac{(m-1)!(n_0+1)}{(n_1+1)\cdots(n_m+1)}\Theta_{\mathbf{n}+(1,0,\dots,0)}\\
&\quad + \left(\prod_{i=1}^{m}\frac{n_0+1}{n_i+1}\right)\left(\prod_{i=1}^{m-1} i!\frac{1}{n_0+1}s_{i-1}\left(\frac{1}{n_1+1},\dots,\frac{1}{n_m+1}\right)\right)\Theta_{\mathbf{n}}
\end{aligned}
$$

by Corollary of Wronskian relation.

Hence we have

$$\Theta_{\mathbf{n}} = s_{m-1}\left(\frac{1}{n_1+1},\dots,\frac{1}{n_m+1}\right)\frac{(n_1+1)\cdots(n_m+1)}{n_0+1}\Theta_{\mathbf{n}+(1,0,\dots,0)} + \Theta_{\mathbf{n}+(1,0,\dots,0)}$$
$$= \left(\frac{(n_1+1)+\cdots+(n_m+1)}{n_0+1}+1\right)\Theta_{\mathbf{n}+(1,0,\dots,0)} = \frac{m+1+|\mathbf{n}|}{n_0+1}\Theta_{\mathbf{n}+(1,0,\dots,0)},$$

as desired. □

The proof of Theorem 1 is completed. □

§3, Extensions of Bernoullization map

Series extension of Bernoullization map

Heuristic observation:

$$\mathrm{Ber}: B^k \mapsto B_k = -k\zeta(1-k) \text{ “}{=}\text{” } -k\sum_{n\geq 1} n^{k-1} = -\sum_{n=1}^{\infty} \frac{d}{dB}B^k\Big|_{B=n}.$$

Series extension of Bernoullization map

Definition (Series extension of Bernoullization map)

For a domain $\Omega \subset \mathbb{C}$ satisfying $\mathbb{Z}_{\geq 1} \subset \Omega$, we define

$$\mathcal{F}^{\Sigma}(\Omega) := \left\{ \begin{array}{c} \text{meromorphic fct. } f(B) \text{ on } \Omega \text{ s.t.} \\ f \text{ has no poles on } \mathbb{Z}_{\geq 1}, \text{ and } \sum_{n=1}^{\infty} \frac{df}{dB}(n) \text{ abs. conv.} \end{array} \right\}$$

and **the series extension of Bernoullization map**

$$\mathrm{Ber}^{\Omega,\Sigma} : \mathcal{F}^{\Sigma}(\Omega) \to \mathbb{C}\,;\, f(B) \mapsto -\sum_{n=1}^{\infty} \frac{df}{dB}(n).$$

Remark ($P_{\mathbf{s}}$ case)

For $\mathbf{s} \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^{m+1}$ with $\mathrm{Re}(|\mathbf{s}|) < 0$, we have $P_{\mathbf{s}} \in \mathcal{F}^{\Sigma}(\mathbb{C})$.

Family version of $\mathrm{Ber}^{\Omega,\Sigma}$

For a domain $X \subset \mathbb{C}^N$, we define

$$\mathcal{M}^{\Omega,\Sigma}(X) := \left\{ \begin{array}{c} \text{mero. fct. } f \text{ on } \Omega \times X \\ \forall x_0 \in X - \cup_{n\geq 1}(\text{poles of } f|_{B=n}), \exists U \text{ open nbd. of } x_0 \text{ s.t.} \\ \sum_{n=1}^{\infty} \frac{df}{dB}(n)_{|U} \text{ converges uniformly and absolutely} \end{array} \right\}$$

and extend $\mathrm{Ber}^{\Omega,\Sigma}$ to the $\mathcal{M}(X)$-linear homomorphism

$$\mathrm{Ber}^{\Omega,\Sigma}_X : \mathcal{M}^{\Omega,\Sigma}(X) \to \mathcal{M}(X) := \{\text{mero. fct. on } X\}.$$

Extension by meromorphic continuation

Let $\widetilde{X} \subset \mathbb{C}^N$ be a domain containing X. For $f \in \mathcal{M}(\Omega \times \widetilde{X})$, if there exist $g \in \mathcal{M}(\widetilde{X})$ such that $\mathrm{Ber}_X^{\Omega,\Sigma}(f|_{\Omega \times X}) = g|_X$, then we define $\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}(f) := g$.

Indefinite form "$\infty - \infty$" for Bernoullization

Remark (indefinite form "$\infty - \infty$" for Bernoullization)

There are **indefinite forms** ("$\infty - \infty$") for the Bernoullization. For example, the Bernoullization of $e^{2\pi iBt}$ is originally defined on $\mathrm{Im}(t) > 0$ and is extended by the meromorphic continuation as $\frac{2\pi ite^{2\pi it}}{e^{2\pi it}-1}$ to $t \in \mathbb{C}$. This has a pole at $t \in \mathbb{Z}$, hence the Bernoulization of $e^{2\pi iB}$ is not defined. By the same reason, the Bernoulization of $e^{-2\pi iB}$ is not defined either. Therefore, the Bernoullization of $e^{2\pi iB} - e^{-2\pi iB}$ is so-called an indefinite form ("$\infty - \infty$") and *is not defined*.

Indefinite form “$\infty - \infty$” for Bernoullization

Remark (value of indefinite form depends on how to cancel ∞’s)

As the usual phenomenon of the indefinite forms, if one resolves an indefinite form of the Bernoullization by canceling the poles via meromorphic continuations, then the Bernoullization **depends on how to cancel them**.

For example, on one hand, the Bernoullization of $e^{2\pi iBt} - e^{-2\pi iBt}$, whose value at $t = 1$ is $e^{2\pi iB} - e^{-2\pi iB}$, is $\frac{2\pi ite^{2\pi it}}{e^{2\pi it}-1} - \frac{2\pi it}{e^{2\pi it}-1} = 2\pi it$, and it can be specialized at $t = 1$ to $2\pi i$. On the other hand, the Bernoullization of $e^{-2\pi it}e^{2\pi iBt} - e^{2\pi it}e^{-2\pi iBt}$, whose value at $t = 1$ is $e^{2\pi iB} - e^{-2\pi iB}$ as well, is $\frac{2\pi it}{e^{2\pi it}-1} - \frac{2\pi ite^{2\pi it}}{e^{2\pi it}-1} = -2\pi it$, and it can be specialized at $t = 1$ to $-2\pi i$.

Hurwitz variant ${}^a f$ and Dirichlet variant ${}^\chi f$

- For $a \in \mathbb{C}$, we define the **Hurwitz variant** ${}^a f$ of f to be

$$ {}^a f(B) := f(B - 1 + a), $$

- for a Dirichlet character χ of conductor N, we define the **Dirichlet variant** ${}^\chi f$ of f to be

$$ {}^\chi f(B) := \frac{1}{N} \sum_{1 \leq a \leq N} \chi(a) f(N(B-1) + a). $$

Examples : (i) Powers

For $\Omega = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and $X = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\} \subset \widetilde{X} = \mathbb{C}$,

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left(\tfrac{1}{B^s}\right) = s\zeta(s+1).$$

In particular,

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left(\tfrac{1}{B^s}\right)\Big|_{s=-k} = B_k = \mathrm{Ber}(B^k),$$

as intended, for $k \in \mathbb{N}$. If we write $s! := \Gamma(s+1)$ and $(-)^{[s]} := \frac{(-)^s}{s!}$ (divided powers), then **the functional equation** of Riemann's zeta function can be written as

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left((\pi B^2)^{[s/2]}\right) = \mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left((\pi B^2)^{[(1-s)/2]}\right).$$

Examples : (i) Powers

More generally, for $f(B) = \frac{1}{B^s}$, $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and a Dirichlet character χ of conductor N, we have

$$\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}({}^a f) = s\zeta(s+1,a) = -\frac{d}{da}\zeta(s,a), \qquad \mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}({}^\chi f) = sL(s+1,\chi).$$

Since, for $k \in \mathbb{N}$, $\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}((B-1+a)^k) = \mathrm{Ber}((B-1+a)^k) = B_k(a)$ and $B_{k,\chi} = N^{k-1}\sum_{1\leq a\leq N}\chi(a)B_k(a/N)$, these recover the classical formulae

$$\zeta(1-k,a) = -\frac{B_k(a)}{k}, \qquad L(1-k,\chi) = -\frac{B_{k,\chi}}{k}.$$

Examples : (ii) Exponentials

Since $-\sum_{n\geq 1} \frac{de^{Bt}}{dB}\big|_{B=n} = \frac{te^t}{e^t-1}$ for $\mathrm{Re}(t) < 0$, we have

$$\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}(e^{Bt}) = \frac{te^t}{e^t - 1}$$

for $\Omega = \mathbb{C}$, and $X = \{\mathrm{Re}(t) < 0\} \subset \widetilde{X} = \mathbb{C}$. For $a \in \mathbb{C}$, a Dirichlet character χ of conductor N, and $f(B) = e^{Bt}$,

$$\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}({}^{a}f) = \frac{te^{at}}{e^t - 1}, \qquad \mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}({}^{\chi}f) = \sum_{1\leq a\leq N} \frac{\chi(a)te^{at}}{e^{Nt} - 1}.$$

Examples : (ii) Exponentials

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}(\sin(Bt)) = \mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left(\frac{e^{iBt} - e^{-iBt}}{2i}\right) = \frac{t}{2},$$

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}(\cos(Bt)) = \mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left(\frac{e^{iBt} + e^{-iBt}}{2}\right) = \frac{t}{2}\cot\frac{t}{2}.$$

(Note that $\sin(2\pi nB)\,(n \in \mathbb{Z})$ is an indefinite form "$\infty - \infty$" for the Bernoullization.)

Examples : (iii) Logarithms

By differentiating $\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}(\frac{1}{(B-1+a)^s}) = -\frac{d}{da}\zeta(s,a)$ with respect to s and specializing $s=0$, for $f(B) = -\log B$, we obtain

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}({}^a f) = -\frac{d}{da}\zeta'(0,a) = -\frac{d}{da}\log\frac{\Gamma(a)}{\sqrt{2\pi}} = -\psi(a)$$

for $\Omega = \mathbb{C}\setminus\mathbb{R}_{\leq 0}$ and $X = \{a\in\mathbb{C} \mid |1-a|<1\} \subset \widetilde{X} = \mathbb{C}$, by *Lerch's formula*.

Examples : (iii) Logarithms

By differentiating $\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}(\frac{1}{(B-1+a)^s}) = s\zeta(s+1,a)$ with respect to s and specializing $s=-1$, for $f(B) = B\log B$, we obtain

$$\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}({}^a f) = -\zeta(0,a) + \zeta'(0,a) = B_1(a) + \log\frac{\Gamma(a)}{\sqrt{2\pi}}$$

for $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. It can be written as, for $g(B) = B(\log B - 1)$ (note that $\frac{dg}{dB} = \log B$)

$$\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}({}^a g) = \log\frac{\Gamma(a)}{\sqrt{2\pi}}.$$

Examples : (iii) Logarithms

For the quadratic character $\chi = \chi_K$ of conductor N associated to an imaginary quadratic field K, by using *class number formula and Chowla-Selberg formula*,

$$
\begin{aligned}
\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}({}^{\chi}g) &= \sum_{1\le a\le N} \chi(a)\log\Gamma(a/N) + \frac{\log N}{N}\sum_{1\le a\le N}\chi(a)a \\
&= -\frac{2h_K}{w_K}\log\frac{2\pi}{N} + \frac{1}{6w_K}\sum_{[\mathfrak{a}]\in\mathrm{Cl}(K)}\log((2\pi)^{24}\Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1})) - \frac{2h_K}{w_K}\log N \\
&= \frac{1}{6w_K}\sum_{[\mathfrak{a}]\in\mathrm{Cl}(K)}\log((2\pi)^{12}\Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1})),
\end{aligned}
$$

where $w_K := \#\mathcal{O}_K^{\times}$, h_K is the class number of K, and Δ is Ramanujan's delta fct.

Integral extension of Bernoullization map

For $R > 1$, $0 < \epsilon < 1$, we write

$$\Omega_{R,\epsilon} := \left\{ B \in \mathbb{C} \mid \text{either } \mathrm{Re}(B) > R - \epsilon \text{ or } (\mathrm{Re}(B) > 1 - 2\epsilon \text{ and } |\mathrm{Im}(B)| < 2\epsilon) \right\}.$$

For an unbounded domain $\Omega \subset \mathbb{C}$, we write $\mathcal{F}^{\int}(\Omega)$ for the ring of holomorphic functions on Ω with polynomial growth, i.e., there exists $N \in \mathbb{N}$ such that $|f(B)| < |B|^N$ on Ω for $|B| \gg 0$. Put $\mathcal{F}^{\int}_{R,\epsilon} := \mathcal{F}^{\int}(\Omega_{R,\epsilon})$.

Remark ($\frac{df}{dB}$ and $P_{\mathbf{s}}$ case)

Note that, for $f \in \mathcal{F}^{\int}_{R,\epsilon}$, we have $\frac{df}{dB} \in \mathcal{F}^{\int}_{R,\epsilon'}$ for any $0 < \epsilon' < \epsilon$, and that, for $\mathbf{s} \in (\mathbb{C} \setminus \mathbb{R}_{\leq -1})^{m+1}$, $P_{\mathbf{s}}|_{\Omega_{R,\epsilon}} \in \mathcal{F}^{\int}_{R,\epsilon}$ for suitable $R > 1, 0 < \epsilon < 1$.

Integral extension of Bernoullization map

Definition (Integral extension of Bernoullization map)

For $f \in \mathcal{F}^{\int}_{R,\epsilon}$, we define **the integral extension fo Bernoullization map** by

$$\mathrm{Ber}^{\int}(f) := 2\pi i \left(\int_{R-\infty i}^{R-\epsilon i} + \int_{R-\epsilon i}^{1-\epsilon-\epsilon i} + \int_{1-\epsilon-\epsilon i}^{1-\epsilon+\epsilon i} + \int_{1-\epsilon+\epsilon i}^{R+\epsilon i} + \int_{R+\epsilon i}^{R+\infty i} \right) \frac{f(B)e^{2\pi iB}}{(e^{2\pi iB}-1)^2} dB.$$

Note that $\frac{f(B)e^{2\pi iB}}{(e^{2\pi iB}-1)^2}$ exponentially decays for both $\mathrm{Im}(B) \to \pm\infty$, hence the integral converges.

Remark (weakening the growth condition)

We may weaken the growth condition by the following condition: $|f(B)| < e^{\alpha|\mathrm{Im}(B)|}$ on Ω for $|\mathrm{Im}(B)| \to \infty$ for some $0 \leq \alpha < 2\pi$.

Reflection formula

Proposition 3 (Reflection formula for entire functions)

For an entire function f such that $|f(B)| < e^{\alpha|\mathrm{Im}(B)|}$ on Ω for $|\mathrm{Im}(B)| \to \infty$ for some $0 \leq \alpha < 2\pi$, it holds that $\mathrm{Ber}^{\int}(f(B)) = \mathrm{Ber}^{\int}(f(1-B))$.

Proof: (RHS) is equal to

$$2\pi i \left(\int_{1-R-\infty i}^{1-R-\epsilon i} + \int_{1-R-\epsilon i}^{\epsilon-\epsilon i} + \int_{\epsilon-\epsilon i}^{\epsilon+\epsilon i} + \int_{\epsilon+\epsilon i}^{1-R+\epsilon i} + \int_{1-R+\epsilon i}^{1-R+\infty i} \right) \frac{f(B)e^{2\pi i B}}{\left(e^{2\pi i B}-1\right)^2} dB =$$

$\mathrm{Ber}^{\int}(f(B))$. $\square$

Remark (classical polynomial case)

For $f \in \mathbb{Q}[B]$, it follows from $\mathrm{Ber}(e^{(1-B)t}) = e^t \frac{(-t)e^{-t}}{e^{-t}-1} = \frac{te^t}{e^t-1} = \mathrm{Ber}(e^{Bt})$.

Remark (entireness assumption and growth condition)

In Proposition 3, f needs to be hol. on $\Omega_{R,\epsilon}$ for some $R > 1, 0 < \epsilon < 1$ in order to define $\mathrm{Ber}^{\int}(f(B))$, needs to be hol. on $1 - \Omega_{R,\epsilon}$ in order to define $\mathrm{Ber}^{\int}(f(1-B))$, and needs to be hol. on $\mathbb{C} \setminus \overline{(\Omega_{R+2\epsilon,\epsilon/4} \cup (1 - \Omega_{R+2\epsilon,\epsilon/4}))}$ as well in order to move the path of the integral in the proof of Proposition 3 in the desired manner. Therefore, f must be an entire function. If, moreover, we assume that f is of polynomial order, then, f is a polynomial (if $|f| \leq C|B|^N$ for $|B| \gg 0$, then, for any $T \gg 0$ and $n > N$, $|f^{(n)}(0)| \leq \frac{n!}{2\pi} \oint |\frac{f(B)}{B^{n+1}}| dB \leq \frac{n! C |B|^N}{|T|^n}$ on $|B| = T$ implies $f^{(n)}(0) = 0$).
In conclusion, in Proposition 3, we **cannot remove the assumption of entireness** of f, and if we had assumed that f was of polynomial growth, then Proposition 3 **would be nothing new**.

Bernoullization via solution of difference problem

Proposition 4 (Bernoullization via solution of difference problem)

Let $R > 1$, $0 < \epsilon < 1$ and $f \in \mathcal{F}^{\int}_{R,\epsilon}$. Assume that there exists $F \in \mathcal{F}^{\int}_{R,\epsilon}$ such that $f(B) = F(B+1) - F(B)$. Then, we have

$$\mathrm{Ber}^{\int}(f(B)) = \left.\frac{dF}{dB}\right|_{B=1}.$$

Proof: By replacing $F(B)$ by $F(B) - F(1)$, we may assume that $F(1) = 0$. Then, by the residue theorem, we have $\mathrm{Ber}^{\int}(f) = 2\pi i \oint \frac{F(B)e^{2\pi i B}dB}{(e^{2\pi i B}-1)^2} = \left.\frac{dF}{dB}\right|_{B=1}$, where the path of the integral is a small counterclockwise circle around $B = 1$ in $\Omega_{R,\epsilon}$.

Remark (necessity of holomorphicity for $\mathrm{Re}(B) \gg 0$)

For $R > 1$ and $0 < \epsilon < 1$, we define $\Omega'_{R,\epsilon} := \Omega_{R,\epsilon} \setminus \overline{1 + \Omega_{R+2\epsilon,\epsilon/4}}$, and $\Omega''_{R,\epsilon} := \Omega'_{R,\epsilon} \cap (-1 + \Omega'_{R,\epsilon})$. We write $\mathcal{F}'^{\int}_{R,\epsilon} := \mathcal{F}^{\int}(\Omega'_{R,\epsilon})$ and $\mathcal{F}''^{\int}_{R,\epsilon} := \mathcal{F}^{\int}(\Omega''_{R,\epsilon})$. Then we may extend the definition of the Bernoullization $\mathrm{Ber}^{\int}(f)$ for $f \in \mathcal{F}''^{\int}_{R,\epsilon}$ by the same formula, and Proposition 4 holds for any $f \in \mathcal{F}''^{\int}_{R,\epsilon}$ and $F \in \mathcal{F}'^{\int}_{R,\epsilon}$ satisfying $f(B) = F(B+1) - F(B)$ for $B \in \Omega''_{R,\epsilon}$. However, this extension of the Bernoullization for $f \in \mathcal{F}''^{\int}_{R,\epsilon}$ might **depend on the choices** of R and ϵ, since f might have poles on $\Omega_{R,\epsilon} \setminus \overline{\Omega_{R',\epsilon'}}$ for $R < R'$ and $\epsilon > \epsilon'$.

Examples : (i) nonnegative integral Powers

For $f(B) = (B-1+a)^k$ $(k \in \mathbb{N})$ with $a \in \mathbb{C}$, put $F(B) := \frac{B_{k+1}(B-1+a)}{k+1}$. Since $B_{k+1}(x+1) - B_{k+1}(x) = (k+1)x^k$, we have $f(B) = F(B+1) - F(B)$, and $F(B) \in \mathcal{F}^{\int}_{R,\epsilon}$ for any $R > 1$, $0 < \epsilon < 1$. Hence, by $\frac{dB_{k+1}(x)}{dx} = (k+1)B_k(x)$ and Proposition 4, we have

$$\mathrm{Ber}^{\int}((B-1+a)^k) = \frac{1}{k+1} \frac{d(B_{k+1}(B-1+a))}{dB}\Big|_{B=1} = B_k(a).$$

In particular, for $a = 1$, we have $\mathrm{Ber}^{\int}(B^k) = B_k$, as intended.

Examples : (i) negative integral Powers

For $f(B) = (B-1+a)^{-k}$ $(k \in \mathbb{Z}_{>0})$ with $|1-a| < 1$, put $F(B) := \frac{(-1)^{k-1}}{(k-1)!}\psi^{(k-1)}(B-1+a)$. Since $\Gamma(s+1) = s\Gamma(s)$, we have $\psi^{(k-1)}(s+1) - \psi^{(k-1)}(s) = \frac{(-1)^{k-1}(k-1)!}{s^k}$, hence $f(B) = F(B+1) - F(B)$. By Stirling's asymptotic formula for gamma function, $F(B) \in \mathcal{F}^{\int}_{R,\epsilon}$ for some $R > 1$, $0 < \epsilon < 1$. Hence, by Proposition 4, we have

$$\begin{aligned}\mathrm{Ber}^{\int}((B-1+a)^{-k}) &= \frac{(-1)^{k-1}}{(k-1)!}\frac{d\psi^{(k-1)}(B-1+a)}{dB}\Big|_{B=1} \\ &= \frac{(-1)^{k-1}}{(k-1)!}\psi^{(k)}(a) = k\zeta(k+1,a).\end{aligned}$$

Examples : (ii) Exponentials

For $f(B) = e^{(B-1+a)t}$ $(\mathrm{Re}(t) < 0)$ with $a \in \mathbb{C}$, put $F(B) := \frac{e^{(B-1+a)t}}{e^t-1}$. Since $\frac{e^{(B+a)t}}{e^t-1} - \frac{e^{(B-1+a)t}}{e^t-1} = e^{(B-1+a)t}$, we have $f(B) = F(B+1) - F(B)$, and $F(B) \in \mathcal{F}^{\int}_{R,\epsilon}$ for any $R > 1$, $0 < \epsilon < 1$. Hence, by Proposition 4, we have

$$\mathrm{Ber}^{\int}(e^{(B-1+a)t}) = \frac{d}{dB}\frac{e^{(B-1+a)t}}{e^t - 1}\bigg|_{B=1} = \frac{te^{at}}{e^t - 1}.$$

Examples : (iii) Logarithms

For $f(B) = -\log(B-1+a)\,(a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0})$, put $F(B) := -\log\Gamma(B-1+a)$. Since $\Gamma(s+1) = s\Gamma(s)$, we have $f(B) = F(B+1) - F(B)$. By Stirling's asymptotic formula for gamma function, we have $F(B) \in \mathcal{F}^{\int}_{R,\epsilon}$ for some $R > 1$, $0 < \epsilon < 1$. Hence, by Proposition 4, we have

$$\mathrm{Ber}^{\int}(-\log(B-1+a)) = -\frac{d}{dB}\log\Gamma(B-1+a)\Big|_{B=1} = -\psi(a).$$

Family version of $\mathrm{Ber}^{\int}$

For a domain $X \subset \mathbb{C}^N$, we define

$$\mathcal{M}^{\Omega,\int}(X) := \left\{ \begin{array}{c|} \text{mero. fct. } f \text{ on } \Omega \times X \\ \forall x_0 \in X, \exists U \text{ open nbd. of } x_0, R_U > 1, 0 < \epsilon_U < 1 \text{ s.t.} \\ \Omega_{R_U,\epsilon_U} \subset \Omega, f|_{\Omega_{R_U,\epsilon_U} \times U} : \text{hol. and} \\ \text{of polynomial growth uniformly w.r.t. } U \end{array} \right\}$$

and extend $\mathrm{Ber}^{\Omega,\int}$ to the $\mathcal{O}(X)$-linear homomorphism

$$\mathrm{Ber}_X^{\Omega,\int} : \mathcal{M}^{\Omega,\int}(X) \to \mathcal{O}(X) := \{\text{hol. fct. on } X\}.$$

Extension by meromorphic continuation

Let $\widetilde{X} \subset \mathbb{C}^N$ be a domain containing X. For $f \in \mathcal{M}(\Omega \times \widetilde{X})$, if there exist $g \in \mathcal{M}(\widetilde{X})$ such that $\mathrm{Ber}_X^{\Omega,\int}(f|_{\Omega\times X}) = g|_X$, then we define $\mathrm{Ber}_{\widetilde{X}}^{\Omega,\int}(f) := g$.

Solution of difference problem via Bernoullization

Proposition 5 (Solution of difference problem via Bernoullization)

Let $\Omega \subset \mathbb{C}$ be a simply connected domain. Assume that $\Omega_{R,\epsilon} \subset \Omega' := \{B' \in \mathbb{C} \mid \Omega_{R,\epsilon} + B' - 1 \subset \Omega\}$ for some $R > 1$, $0 < \epsilon < 1$.
For $f \in \mathcal{F}^{\int}(\Omega)$ and $B \in \Omega$, we set $g(B) := \int_*^B f(x)dx$.
(Note that $g(B + B' - 1)|_{\Omega_{R,\epsilon} \times \Omega'} \in \mathcal{M}^{\Omega_{R,\epsilon},\int}(\Omega')$.)
We set $F(B') := \mathrm{Ber}_{\Omega'}^{\Omega_{R,\epsilon},\int}(g(B + B' - 1))$.
Then $F(B')$ is a holomorphic function on Ω', $F(B)|_{\Omega_{R,\epsilon}} \in \mathcal{F}^{\int}_{R,\epsilon}$, and $f(B) = F(B+1) - F(B)$ on $\Omega \cap \Omega'$.

Proof of Proposition 5

Proof: On $\Omega \cap \Omega'$, we have

$$F(B+1) - F(B) = \mathrm{Ber}^{\Omega_{R,\epsilon},\int}_{B',\Omega\cap\Omega'}(g(B+B') - g(B+B'-1))$$
$$= \left.\frac{dg(B+B'-1)}{dB'}\right|_{B'=1} = f(B)$$

by Proposition 4. The holomorphicity of $F(B)$ on Ω' is clear since we have $g(B+B'-1) \in \mathcal{M}^{\Omega_{R,\epsilon},\int}(\Omega')$ as mentioned. The growth condition for $f(B)$ implies that $g(B-1)$ is of polynomial growth for $B \in \Omega_{R,\epsilon} + \Omega_{R,\epsilon} := \{B + B' \mid B, B' \in \Omega_{R,\epsilon}\}$. From this the condition for the growth of $F(B)|_{\Omega_{R,\epsilon}}$ follows. $\square$

Comparison Theorem

Theorem 2 (Comparison theorem)

Let $\Omega \subset \mathbb{C}$ and $X \subset \mathbb{C}^N$ be domains, and $f \in \mathcal{M}^{\Omega,\Sigma}(X) \cap \mathcal{M}^{\Omega,\int}(X)$. Then we have $\mathrm{Ber}_X^{\Omega,\Sigma}(f) = \mathrm{Ber}_X^{\Omega,\int}(f)$.

Proof: Let $x \in X$. It suffices to prove the equality at x. There exist $R > 1$ and $0 < \epsilon < 1$ such that $\Omega_{R,\epsilon} \subset \Omega$, and $f_x(B) = f(B,x) \in \mathcal{F}_{R,\epsilon}^{\int}$.
We set $R' = (R+1)/2$ and $\epsilon' = \epsilon/2$. We note that, for any $B, B' \in \Omega_{R',\epsilon'}$, we have $B + B' - 1 \in \Omega_{R,\epsilon}$. Let F be as in Proposition 5 applied to the datum $(R, \epsilon, \Omega, f) = (R', \epsilon', \Omega_{R,\epsilon}, f_x)$. Then, by Proposition 4, we have $\mathrm{Ber}_X^{\Omega,\int}(f)(x) = \left.\frac{dF(B)}{dB}\right|_{B=1} = -\sum_{n=1}^{\infty} \frac{df_x}{dB}(n) = \mathrm{Ber}_X^{\Omega,\Sigma}(f)(x)$. $\square$

§4, Proof of complex KP-conjecture

Idea of the proof of Theorem 3 : $\Theta_{\mathbf{s}} = \frac{m!\mathbf{s}!}{(m+|\mathbf{s}|)!}$

Idea: We will apply Carlson's theorem:

Theorem (Carlson, 1914)

$f(z)$ is a holomorphic function on $\mathrm{Re}(z) \geq 0$ of order $e^{k|z|}$ with $k < \pi$, and $f(n) = 0$ for $n \in \mathbb{N}$, then $f(z) \equiv 0$.

Note that, if one fixes B, then $P_{\mathbf{s}}$ is of polynomial growth on s by Stirling's asymptotic formula, however, we need a polynomial estimate of $\int_{1/2-\infty i}^{1/2+\infty i} |P_{\mathbf{s}}| e^{-2\pi|\mathrm{Im}(B)|} dB$ with respect to $\mathrm{Im}(s)$ in order to apply Carlson's theorem, hence we need a polynomial estimate of $|P_{\mathbf{s}}| e^{-(2\pi-\epsilon')|\mathrm{Im}(B)|}$ $(0 < \epsilon' \ll 1)$ with respect to $\mathrm{Im}(s)$ in a uniform manner with respect to $\mathrm{Im}(B)$.

Key Proposition

Proposition 6 (Uniform polynomial estimate of $|P_{\mathbf{s}}|e^{-(2\pi-\epsilon')|\mathrm{Im}(B)|}$ w.r.t. $\mathrm{Im}(s)$)

For $\mathbf{s} = (s, s_1, \ldots, s_m) \in (\mathbb{C} \setminus \mathbb{R}_{\leq -1})^{m+1}$ there exists a constant C depending on $s_1, \ldots, s_m$ such that the inequality

$$|P_{\mathbf{s}}|e^{-2\pi|\mathrm{Im}(B)|} \leq \frac{Ce^{-\pi|\mathrm{Im}(B)|}2^{\mathrm{Re}(s)}}{|s|^{1/2}}\left(1 + |2\mathrm{Im}(B)|^N + \frac{1 + |2\mathrm{Im}(B)|^{N+\mathrm{Re}(s)}}{|2^{1/2}s|^{\mathrm{Re}(s)}}\right)$$

holds for any $B, s \in \mathbb{C}$ with $\mathrm{Re}(B) = \frac{1}{2}$ and $\mathrm{Re}(s) \geq \frac{1}{2}$.

Proof: omit (a bit complicated). $\square$

Proof of Theorem 3 : $\Theta_{\mathbf{s}} = \frac{m!\mathbf{s}!}{(m+|\mathbf{s}|)!}$

Induction on the number $r(\mathbf{s})$ of $0 \leq i \leq m$ such that $s_i \notin \mathbb{Z}_{\geq 0}$.

For $r(\mathbf{s}) = 0$, OK by Theorem 1. For $r \geq 1$, we assume Theorem 3 for $r(\mathbf{s}) = r - 1$, and we will show Theorem 3 for $r(\mathbf{s}) = r$. By the symmetry, we may assume that $s_0 \notin \mathbb{Z}_{\geq 0}$.

We consider the holomorphic function $\Theta_{(s,s_1,\ldots,s_m)}$ on $s \in \mathbb{C} \setminus \mathbb{R}_{\leq -1}$. Let $f(s)$ be an arbitrary entry of the matrix in the definition of $\Theta_{\mathbf{s}}$. We fix $s_1, \ldots, s_m \in \mathbb{C} \setminus \mathbb{R}_{\leq -1}$ and regard $f(s)$ as a function of $s \in \mathbb{C} \setminus \mathbb{R}_{\leq -1}$. It follows from the definition that $f(s)$ is holomorphic on $\mathrm{Re}(s) > 0$.

Proof of Theorem 3 : $\Theta_{\mathbf{s}} = \frac{m!\mathbf{s}!}{(m+|\mathbf{s}|)!}$

We apply Key Proposition after replacing m by $m+1$, and $s_1, \ldots, s_m$ by $s_1, \ldots, s_m, j$ with $j \in \mathbb{Z}$, $0 \leq j \leq m-1$, then, there exists a positive constant C depending on $s_1, \ldots, s_m$ such that we have

$$|f(s)| \leq C 2^{\mathrm{Re}(s)}$$

for $\mathrm{Re}(s) \geq \frac{1}{2}$. From this it follows that there exists a positive constant C' depending on $s_1, \ldots, s_m$ such that we have

$$|\Theta_{(s, s_1, \ldots, s_m)}| \leq C' 2^{m(\mathrm{Re}(s))}$$

for $\mathrm{Re}(s) \geq \frac{1}{2}$. By the induction hypothesis, Theorem 3 holds if $s \in \mathbb{N}$.

Proof of Theorem 3 : $\Theta_{\mathbf{s}} = \frac{m!\mathbf{s}!}{(m+|\mathbf{s}|)!}$

By applying Carlson's theorem to the function $G(s+1)$ where

$$G(s) = 2^{-ms}\left(\Theta_{(s,s_1,\dots,s_m)} - \frac{m!\mathbf{s}!}{(m+|\mathbf{s}|)!}\right),$$

we see that Theorem 3 holds for $\mathrm{Re}(s) > 1$, hence for $s \in \mathbb{C} \setminus \mathbb{R}_{\leq -1}$ by the identity theorem.

Since Theorem 3 holds for any $\mathbf{s} \in (\mathbb{C} \setminus \mathbb{R}_{\leq -1})^{m+1}$, the function $\Theta_{\mathbf{s}}$ has an analytic continuation to $\mathbf{s} \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^{m+1}$ as a single valued holomorphic function, and Theorem 3 holds for any $\mathbf{s} \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^{m+1}$. This completes the proof. □

§5, Relations for general maps

Contiguous relation

Theorem 4 (Contiguous relation, Kawashima for $m = 1, 2$)

For $m \geq 1$ and $\mathbf{n} = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$, it holds that

$$\left(\prod_{0\leq i<j\leq m}(n_i - n_j)\right) \Theta^{\varphi}_{\mathbf{n}} = \sum_{0\leq r\leq m}(-1)^r \left(\prod_{\substack{0\leq i<j\leq m,\\ i,j\neq r}}(n_i - n_j)\right)(n_r + 1)^m \Theta^{\varphi}_{\mathbf{n}+(0,\ldots,0,1\,(r\text{-th}),0,\ldots,0)}.$$

(For $m = 1$, we understand that $\prod$ with the empty index set is equal to 1.)
Equivalently, if we symmetrize the equality with respect to $n_0, \ldots, n_m$,

$$\Theta^{\varphi}_{\mathbf{n}} = \sum_{0\leq r\leq m} \frac{(n_r+1)^m}{\prod_{\substack{0\leq i\leq m,\\ i\neq r}}(n_r - n_i)} \Theta^{\varphi}_{\mathbf{n}+(0,\ldots,0,1\,(r\text{-th}),0,\ldots,0)}$$

for distinct $n_0, \ldots, n_m$.

Idea of proof

We interpret the equality

$$\begin{aligned}&(n+1)\frac{(B)_{n+1}}{(n+1)!}\frac{(B)_m}{m!}-(m+1)\frac{(B)_n}{n!}\frac{(B)_{m+1}}{(m+1)!}\\&=\frac{(B)_n}{n!}\frac{(B)_m}{m!}((B+n)-(B+m))=(n-m)\frac{(B)_n}{n!}\frac{(B)_m}{m!}\end{aligned} \tag{1}$$

in terms of *the generating function and differential operators*.

Interpretation in terms of the generating fct. and diff. operators

For $T := (1-t_0)\cdots(1-t_m)$ and $0 \le i, j \le m$, we have

$$\left(\frac{d}{dt_i} - \frac{d}{dt_j}\right)\frac{1}{T^B} = \left(\frac{1}{1-t_i} - \frac{1}{1-t_j}\right)\frac{B}{T^B}$$
$$= \left(\frac{t_i}{1-t_i} - \frac{t_j}{1-t_j}\right)\frac{B}{T^B} = \left(t_i\frac{d}{dt_i} - t_j\frac{d}{dt_j}\right)\frac{1}{T^B}. \qquad (2)$$

Note that $\frac{d}{dt_i}\sum_{n_i\ge 0}\frac{(B)_{n_i}}{n_i!}t_i^{n_i} = \sum_{n_i\ge 0}(n_i+1)\frac{(B)_{n_i+1}}{(n_i+1)!}t_i^{n_i}$, and $\sum_{\mathbf{n}\in\mathbb{N}^{m+1}} P_{\mathbf{n}}\mathbf{t}^{\mathbf{n}} = \frac{1}{T^B}$. Hence (2) is *an interpretation of the relation* (1) *in terms of partial differential operators*.

Notation of differntial operators

For $0 \leq i, j \leq m$, we write

$$\langle i \rangle := \frac{d}{dt_i}, \quad [i] := t_i \frac{d}{dt_i}, \quad [ij] := t_i \frac{d}{dt_i} - t_j \frac{d}{dt_j}.$$

Since $\frac{d}{dB}$ and $\frac{d}{dt_i}$ commute for $0 \leq i \leq m$, we have moreover

$$(\langle i \rangle - \langle j \rangle) \frac{d^k}{dB^k} \frac{1}{T^B} \frac{(B)_\ell}{\ell!} = [ij] \frac{d^k}{dB^k} \frac{1}{T^B} \frac{(B)_\ell}{\ell!} \tag{3}$$

for $0 \leq i, j \leq m$ and $0 \leq k, \ell \leq m - 1$.

Notation of copies etc.

Let ${}^{\ell}(-)$ denote a copy of $(-)$ for $0 \leq \ell \leq m-1$. For partial differential operators $D_0, \dots, D_{m-1}$ with respect to $t_0, \dots, t_m$, we write

$$(D_0, \dots, D_{m-1}) := \det \left({}^{\ell}D_{\ell} \frac{d^k}{d^{\ell}B^k} \frac{1}{{}^{\ell}T^{\ell B}} \frac{({}^{\ell}B)_{\ell}}{\ell!} \right)_{0 \leq k, \ell \leq m-1}$$

$$\in \mathbb{Q}\left[{}^{\ell}B \Big|_{0 \leq \ell \leq m-1} \right] \left[\left[{}^{\ell}t_i \Big|_{\substack{0 \leq i \leq m, \\ 0 \leq \ell \leq m-1}} \right]\right].$$

Notation $\overset{\Delta}{=}$

We write ${}^{\Delta}t_i := \prod_{0\leq\ell\leq m-1} {}^{\ell}t_i$ for $0 \leq i \leq m$.
For $g_1, g_2 \in \mathbb{Q}\left[{}^{\ell}B \mid_{0\leq\ell\leq m-1}\right]\left[\left[{}^{\ell}t_i \mid_{0\leq i\leq m,\, 0\leq\ell\leq m-1}\right]\right]$, we write

$$g_1 \overset{\Delta}{=} g_2,$$

if the coefficient of $({}^{\Delta}\mathbf{t})^{\mathbf{n}} := ({}^{\Delta}t_0)^{n_0} \cdots ({}^{\Delta}t_m)^{n_m}$ in g_1 is equal to the one in g_2 for any $\mathbf{n} \in \mathbb{N}^{m+1}$ (Δ stands for d of "diagonal").

Then we have

$$([i]\cdot,\ldots,\cdot) \triangleq \cdots \triangleq (\cdot,\ldots,[i]\cdot,\ldots,\cdot) \triangleq \cdots \triangleq (\cdot,\ldots,[i]\cdot) \triangleq [i](\cdot,\ldots,\cdot),$$

$$([ij]\cdot,\ldots,\cdot) \triangleq \cdots \triangleq (\cdot,\ldots,[ij]\cdot,\ldots,\cdot) \triangleq \cdots \triangleq (\cdot,\ldots,[ij]\cdot) \triangleq [ij](\cdot,\ldots,\cdot)$$

for $0 \leq i,j \leq m$. Note that there is no ambiguity for $[i]$ (resp. $[ij]$) in the most right-hand sides in the above formulae even if we do not specify which copy it is, since the action of ${}^{\ell}[i]$ (resp. ${}^{\ell}[ij]$) on ${}^{\ell}\mathbf{t}^{\mathbf{n}}$ for $\mathbf{n} = (n_0,\ldots,n_m)$ is the multiplication by n_i (resp. $n_i - n_j$) for any $0 \leq \ell \leq m-1$.

Note also that $[i]$ and $[ij]$ commute to each others for $0 \leq i,j \leq m$.

Vandermonde orthogonality

We identify the sub-$\mathbb{Q}$-algebra $\mathbb{Q}\left[\left[{}^{\Delta}t_0, \ldots, {}^{\Delta}t_m\right]\right]$ with $\mathbb{Q}[[t_0, \ldots, t_m]]$ by sending ${}^{\Delta}t_i$ to t_i for $0 \leq i \leq m$.

Lemma 3 (Vandermonde orthogonality)

For $m \geq 2$ and $0 \leq s \leq m-1$, we have

$$\sum_{1 \leq r \leq m} (-1)^r \left(\prod_{\substack{1 \leq i < j \leq m, \\ i,j \neq r}} [ij] \right) [0r]^s = \begin{cases} 0, & 0 \leq s \leq m-2, \\ (-1)^m \prod_{1 \leq i < j \leq m} [ij], & s = m-1 \end{cases}$$

as partial differential operators acting on $\mathbb{Q}[[t_0, \ldots, t_m]]$. (For $m = 2$, we understand that $\prod$ with the empty index set is equal to 1.)

Proof of Lemma 3

Proof: We show the equality

$$\sum_{1\le r\le m}(-1)^{r-1}\left(\prod_{1\le i<j\le m,\, i,j\ne r}[ji]\right)[0r]^s=\begin{cases}0, & \text{for } 0\le s\le m-2,\\ \prod_{1\le i<j\le m}[ji], & \text{for } s=m-1,\end{cases} \tag{4}$$

which is equivalent to Lemma 3. Note that, by replacing $[ij]$ by $[ji]$, the left-hand (resp. right-hand) side of Lemma 3 is multiplied by $(-1)^{(m-1)(m-2)/2}$ (resp. $(-1)^{m(m-1)/2}$).

Proof of Lemma 3

The cofactor expansion of the determinant

$$(-1)^{m-1}\det\begin{pmatrix} \mathrm{id} & \mathrm{id} & \cdots & \mathrm{id} \\ [1] & [2] & \cdots & [m] \\ [1]^2 & [2]^2 & \cdots & [m]^2 \\ \vdots & \vdots & \vdots & \vdots \\ [1]^{m-2} & [2]^{m-2} & \cdots & [m]^{m-2} \\ [01]^s & [02]^s & \cdots & [0m]^s \end{pmatrix} \tag{5}$$

in the last row is equal to the left-hand side of (4), by using Vandermonde determinants. Thus, it suffices to show that (5) is equal to the right-hand side of (4).

Proof of Lemma 3

When $0 \leq s \leq m-2$, for $1 \leq r \leq m$, $[0r]^s = ([0]-[r])^s$ can be written as a linear combination of $[r]^{s'}$ for $0 \leq s' \leq s$, hence (5) is equal to 0, which is equal to the right-hand side of (4).

When $s = m-1$, for $1 \leq r \leq m$, if we subtract a linear combination of $[r]^{s'}$ for $0 \leq s' \leq m-2$, from $[0r]^{m-1} = ([0]-[r])^{m-1}$, then it becomes $(-1)^{m-1}[r]^{m-1}$. Hence (5) is equal to the right-hand side of (4), by using Vandermonde determinants again. This completes the proof of Lemma 3. $\square$

Proof of Theorem 4

Now, we prove Theorem 4 (Contiguous relation). It suffices to show the relation

$$\left(\prod_{0\le i<j\le m}[ij]\right)(\mathrm{id},\ldots,\mathrm{id}) \stackrel{\Delta}{=} \sum_{0\le r\le m}(-1)^r\left(\prod_{0\le i<j\le m,\, i,j\neq r}[ij]\right)(\langle r\rangle,\ldots,\langle r\rangle),$$

because taking the operation ${}^{0}\varphi\circ\cdots\circ{}^{m-1}\varphi$ on the above equality and comparing the coefficients of $({}^{\Delta}\mathbf{t})^{\mathbf{n}}$ on the both sides give us the desired relation in Theorem, where ${}^{\ell'}\varphi$ denotes (with no risk of confusion) the $R\left[{}^{0}B,\ldots,{}^{\ell'-1}B\right]\left[\left[{}^{\ell}t_i\mid_{\substack{0\le\ell\le m-1,\\ 0\le i\le m}}\right]\right]$-linear extension of the copy ${}^{\ell'}\varphi : R\left[{}^{l'}B\right]\to R$ of φ for $0\le\ell'\le m-1$.

By Lemma 3 for $s = 0$ ($\mathbb{Q}[[t_1, \dots, t_m]]$ ($\subset \mathbb{Q}[[t_0, \dots, t_m]]$) in Lemma 3 is $\mathbb{Q}[[t_0, \dots, t_m]]$ here), we have

$$\sum_{0\le r\le m}(-1)^r \left(\prod_{0\le i<j\le m,\, i,j\ne r}[ij]\right)(\langle r\rangle, \dots, \langle r\rangle)$$
$$= \sum_{1\le r\le m}(-1)^r \left(\prod_{0\le i<j\le m,\, i,j\ne r}[ij]\right)((\langle r\rangle, \dots, \langle r\rangle) - (\langle 0\rangle, \dots, \langle 0\rangle)).$$

By the formula (3), this is equal to

$$= \sum_{1\le r\le m}(-1)^r \left(\prod_{\substack{0\le i<j\le m,\\ i,j\ne r}}[ij]\right)((\langle 0\rangle + [r0], \dots, \langle 0\rangle + [r0]) - (\langle 0\rangle, \dots, \langle 0\rangle))$$
$$= \sum_{1\le r\le m}(-1)^r \left(\prod_{\substack{0\le i<j\le m,\\ i,j\ne r}}[ij]\right)\sum(\langle 0\rangle \text{ or } [r0], \dots, \langle 0\rangle \text{ or } [r0]\,(\text{except all } \langle 0\rangle))$$
$$= \sum_{1\le r\le m}(-1)^r \left(\prod_{\substack{1\le i<j\le m,\\ i,j\ne r}}[ij]\cdot\prod_{1\le j\le m,\, j\ne r}[0j]\right)$$
$$\cdot\sum(\langle 0\rangle \text{ or } [r0], \dots, \langle 0\rangle \text{ or } [r0]\,(\text{except all } \langle 0\rangle)).$$

Let $0 \leq s \leq m-1$. In the most right-hand side of the above equality, we consider the sum of the terms where there are $(s+1)$ $[r0]$'s, $(m-s-1)$ $\langle 0 \rangle$'s, and the places of $\langle 0 \rangle$ coincide. For each sum,

$$\sum_{1 \leq r \leq m} (-1)^r \left(\prod_{\substack{1 \leq i < j \leq m, \\ i,j \neq r}} [ij] \right) \left(\prod_{\substack{1 \leq j \leq m, \\ j \neq r}} [0j] \right) ((s+1)\ [r0]\text{'s, else } \langle 0 \rangle) \triangleq$$

$$(-1)^{s+1} \left(\sum_{1 \leq r \leq m} (-1)^r \left(\prod_{\substack{1 \leq i < j \leq m, \\ i,j \neq r}} [ij] \right) [0r]^s \right) \left(\prod_{1 \leq j \leq m} [0j] \right) ((s+1)\, \mathrm{id}\text{'s, else } \langle 0 \rangle)$$

is equal to

$$= \begin{cases} 0, & \text{for } 0 \leq s \leq m-2, \\ \left(\prod_{1 \leq i < j \leq m} [ij] \right) \left(\prod_{1 \leq j \leq m} [0j] \right) (\mathrm{id}, \ldots, \mathrm{id}) & \\ = \left(\prod_{0 \leq i < j \leq m} [ij] \right) (\mathrm{id}, \ldots, \mathrm{id}), & \text{for } s = m-1, \end{cases}$$

by Lemma 3 again. Theorem follows from this. □

Dual of a formal power series

Let A be a commutative ring with the unity (not necessarily containing $\mathbb{Q}$). Let $f(t) = \sum_{m \geq 0} a_m t^m \in A[[t]]$ be a formal power series with $a_0 = 1$. For $m < 0$, we put $a_m := 0$. Since f is invertible, we define the **dual** f^* of f to be

$$f^*(t) := 1/f(-t) \in A[[t]].$$

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ be a tuple of nonnegative integers. Then λ is a partition of $|\lambda| := \lambda_1 + \cdots + \lambda_n$ with length less than or equal to n. We use the correspondence between the partitions and the Young diagrams, which are the sets of boxes arranged in left-justified rows. We write λ^t for the *conjugate partition* of λ, i.e., the partition corresponding to the Young diagram obtained by exchanging the columns and the rows of the Young diagram corresponding to λ.

skew Young determinant

Definition (skew Young determinant)

For $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ and $\mu = (\mu_1 \geq \cdots \geq \mu_n \geq 0)$, we define the **skew Young determinant** of f for λ, μ to be

$$Y_f(\lambda, \mu) := \det\left(a_{\lambda_i - \mu_j - i + j}\right)_{1 \leq i,j \leq n}.$$

If $\lambda = (\overbrace{k \geq \cdots \geq k}^{n})$ (i.e., the corresponding Young diagram is the rectangle with k columns and n rows) and $\mu = 0$, then the skew Young determinant

$T_f(k, n) := Y_f((\overbrace{k \geq \cdots \geq k}^{n}), 0) = \det\left(a_{k-i+j}\right)_{1 \leq i,j \leq n}$ is called the **Toeplitz determinant** with center k of size n associated with f.

Remark (dependence only on the partitions)

In the definition of $Y_f(\lambda, \mu)$, if we replace n by a larger integer N than n and add $\lambda_{n+1} = \cdots = \lambda_N = 0$ and $\mu_{n+1} = \cdots = \mu_N = 0$, then, in the entries in the added rows, the diagonal entries are 1 and the entries below the diagonal are 0, hence the determinant is unchanged, and **depends only on the partitions** of $|\lambda|, |\mu|$.

Remark (vanishing unless $\lambda \geq \mu$)

The sequence $\lambda_i - \mu_j - i + j$ is strictly increasing with respect to j, and strictly decreasing with respect to i. Thus, if there exists k such that $\lambda_k < \mu_k$, then the (i, j)-entry of $Y_f(\lambda, \mu)$ is 0 for $i \geq k$ and $j \leq k$, hence we have $Y_f(\lambda, \mu) = 0$. Therefore, $Y_f(\lambda, \mu)$ is **nonzero only if** $\lambda_k \geq \mu_k$ for any k (we write $\lambda \geq \mu$ for this condition).

Remark (decomposition into the connected comp's and dependence only on λ/μ)

Assume $\lambda \geq \mu$. The diagram obtained from the Young diagram corresponding to λ by deleting the Young diagram corresponding to μ is denoted by λ/μ and called as **skew Young diagram** of shape λ/μ.

Similarly as in the previous remark, if there exists k such that $\lambda_{k+1} \leq \mu_k$, then the (i,j)-entry of $Y_f(\lambda, \mu)$ is 0 for $i \geq k+1$ and $j \leq k$. This implies that $Y_f(\lambda, \mu)$ is **the product of** the det of the upper-left $k \times k$ block and the det of the lower-right $(n-k) \times (n-k)$ block.

Moreveor, each integer k with $\lambda_{k+1} \leq \mu_k$ corresponds, in the skew Young diagram λ/μ, to a place where the diagram is cut into two connected components of the diagram. (Here we understand that two boxes sharing only one point (corner) in the skew Young diagram are in different connected components).

Remark (decomposition into the connected comp's and dependence only on λ/μ)

By these observations, we obtain the following:

- $Y_f(\lambda, \mu)$ is **the product of** the $Y_f(-, -)$'s for the connected components of the skew Young diagram λ/μ.
- $Y_f(\lambda, \mu)$ **depends only on** f and λ/μ, and does not depend on the choice of λ and μ giving the same skew Young diagram. Therefore, we may write $Y_f(\lambda/\mu) := Y_f(\lambda, \mu)$.

Duality of skew Young determinants

Theorem 5 (Duality of skew Young determinants, essentially Macdonald)

Let $f(t) \in A[[t]]$ be a formal power series with constant term 1, and λ, μ partitions of nonnegative integers. Then, we have $Y_f(\lambda, \mu) = Y_{f^*}(\lambda^t, \mu^t)$.

Remark (duality of Toeplitz determinants)

If $\lambda = (\overbrace{k \geq \cdots \geq k}^{n})$ (i.e. the corresponding Young diagram is a rectangle) and $\mu = 0$, then Theorem says the duality $T_f(k, n) = T_{f^*}(n, k)$ of the Toeplitz determinant **with center k of size n** associated with f, and the Toeplitz determinant **with center n of size k** associated with f^*.

Remark (skew Shur polynomial, and Theorem 5 is essentially due to Macdonald)

If $A = \mathbb{Z}[x_1, \ldots, x_N]$, and $f(t) = \prod_{1 \le i \le N} \frac{1}{1 - x_i t} = \sum_{n \ge 0} h_n t^n$, $f^*(t) = \prod_{1 \le i \le N}(1 + x_i t) = \sum_{n \ge 0} e_n t^n$, where h_n is the n-th *complete symmetric polynomial*, and e_n is the n-th *elementary symmetric polynomial*, then $Y_f(\lambda/\mu)$ is the **skew Schur polynomial** for λ/μ, and Theorem 5 is due to Macdonald.
In particular, if $\mu = 0$, then Theorem 5 implies the equality obtained by combining **Jacobi-Trudy identity** (which relates a Schur polynomial with a det of complete symmetric polynomials) and **Giambelli's formula** (which relates a Schur polynomial with a det of elementary symmetric polynomials).
It is easy to show Theorem 5 from Macdonald's formula, by considering that h_i's are formal variables, and regarding the case of $A = \mathbb{Z}[x_1, \ldots, x_N]$ as the "universal" case. However, for the sake of completeness, we give a direct and self-contained proof of Theorem 5, since it is also easy and short.

Example (duality for Toeplitz determinant)

If we apply the duality for Toeplitz det's to $f(t) = \frac{te^t}{e^t-1} = \sum_{n\geq 0} \frac{B_n}{n!} t^n$, $f^*(t) = \sum_{n\geq 0} \frac{1}{(n+1)!} t^n$, then, we have

$$\det\left(\frac{B_{k-i+j}}{(k-i+j)!}\right)_{0\leq i,j\leq n-1} = \det\left(\frac{1}{(n+1-i+j)!}\right)_{0\leq i,j\leq k-1},$$

where $\frac{B_\ell}{\ell!} = \frac{1}{(\ell+1)!} := 0$ for $\ell < 0$. Similar determinantal relations hold for **Euler numbers** E_n, i.e., $f(t) = \frac{2}{e^t+e^{-t}} = \sum_{n\geq 0} \frac{E_n}{n!} t^n$ and $f^*(t) = \sum_{n\geq 0} \frac{t^{2n}}{(2n)!}$ as well. Note that, in the case $k = 1$ or $n = 1$, the above formula is a classical determinantal identity for Bernoulli numbers (and Euler numbers).

Jacobi's lemma

To show Theorem 5, we use the following lemma

Lemma 4 (Jacobi)

Let M be an $m \times m$ invertible matrix with entries in A. Let $I, J \subset \{1, \ldots, m\}$ be subsets with $\#I = \#J$. We write $M_{I,J}$ for the matrix made of the rows in I and the columns in J of M. Put $I^c := \{1, \ldots, m\} \setminus I$, and $J^c := \{1, \ldots, m\} \setminus J$. Then, we have

$$\det M_{I,J} = (-1)^{\sum_{i \in I} i + \sum_{j \in J} j} (\det M) \det(M^{-1})_{J^c, I^c}.$$

Remark ($\#I = \#J = m - 1$ case)

When $\#I = \#J = m - 1$, it is a well known formula for the (i, j)-entry of M^{-1} .

Proof: If $I = J = \{1, \ldots, k\} =: K$, then Lemma 4 follows by taking the determinants of the both sides of

$$M \begin{pmatrix} 1_k & (M^{-1})_{K,K^c} \\ 0 & (M^{-1})_{K^c,K^c} \end{pmatrix} = \begin{pmatrix} M_{K,K} & 0 \\ M_{K^c,K} & 1_{m-k} \end{pmatrix}.$$

For general $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$, we write $I^c = \{i_1^c < \cdots < i_{m-k}^c\}$ and $J^c = \{j_1^c < \cdots < j_{m-k}^c\}$, and let P (resp. Q) be the permutation matrix such that the multiplication of P (resp. Q) from the left sends the a-th row to the i_a-th (resp. j_a-th) row for any $1 \leq a \leq k$, and the b-th row to the i_{b-k}^c-th (resp. j_{b-k}^c-th) row for any $k+1 \leq b \leq m$. Note that the multiplication of P^{-1} (resp. Q^{-1}) from the right sends the a-th column to the i_a-th (resp. j_a-th) column for any $1 \leq a \leq k$, and the b-th column to the i_{b-k}^c-th (resp. j_{b-k}^c-th) column for any $k+1 \leq b \leq m$.

Then, by applying the above special case to PMQ^{-1}, we obtain

$$\begin{aligned}\det M_{I,J} &= \det(PMQ^{-1})_{K,K} = \det(PMQ^{-1})\det(QM^{-1}P^{-1})_{K^c,K^c}\\ &= \det(PMQ^{-1})\det(M^{-1})_{J^c,I^c}.\end{aligned}$$

By noting that $\det P = (-1)^{(i_1-1)+\cdots+(i_k-k)}$ and $\det Q = (-1)^{(j_1-1)+\cdots+(j_k-k)}$, we complete the proof of Lemma 4. $\square$

Proof of Therem 5

We prove Theorem 5. We write $f(t) = \sum_{m\geq 0} a_m t^m$ and $f^*(t) = \sum_{m\geq 0} b_m t^m$. For $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ and $\mu = (\mu_1 \geq \cdots \geq \mu_n \geq 0)$, we write $\lambda^t = (\lambda_1^t \geq \cdots \geq \lambda_k^t \geq 0)$ and $\mu^t = (\mu_1^t \geq \cdots \geq \mu_k^t \geq 0)$.

We claim that, for $I = \{\lambda_i + k + 1 - i \mid 1 \leq i \leq n\} \subset \{1, 2, \ldots, n+k\}$, we have $I^c = \{-\lambda_j^t + k + j \mid 1 \leq j \leq k\}$.

Proof of the claim

We show the claim. We embed the Young diagram corresponding to λ into the $n \times k$ rectangular Young diagram, and put a numbering $1, 2, \dots, n+k$ on the edges of the boundary from the bottom-left to the top-right. Then, the numbers on the vertical edges are exactly I and, by considering the transposition of the Young diagram, the numbers on the horizontal edges are $\{n+k+1-(\lambda_j^t+n+1-j) \mid 1 \leq j \leq k\} = \{-\lambda_j^t+k+j \mid 1 \leq j \leq k\}$. The claim is proved.

Proof of Theorem 5

By the same manner, for $J=\{\mu_i+k+1-i \mid 1\leq i\leq n\}\subset\{1,2,\dots,n+k\}$, we have $J^c=\{-\mu_j^t+k+j \mid 1\leq j\leq k\}$.
By the definition of f^*, for any $l>0$, we have $\sum_{0\leq i\leq l}(-1)^{l-i}a_ib_{l-i}=0$. Hence, the unipotent lower-triangular matrices $(a_{i-j})_{1\leq i,j\leq n+k}$ and $((-1)^{i-j}b_{i-j})_{1\leq i,j\leq n+k}$ are inverse matrices to each other. Apply Lemma 4 to $M=(a_{i-j})_{1\leq i,j\leq n+k}$, $I=\{\lambda_i+k+1-i \mid 1\leq i\leq n\}$, $I^c=\{-\lambda_j^t+k+j \mid 1\leq j\leq k\}$, $J=\{\mu_i+k+1-i \mid 1\leq i\leq n\}$, and $J^c=\{-\mu_j^t+k+j \mid 1\leq j\leq k\}$. We complete the proof of Theorem 5, by reversing the order of the rows and columns (which does not change the signs of the det) of the matrix $M_{I,J}$, by taking the transpose (which does not change the det) of the matrix $(M^{-1})_{J^c,I^c}$, and by noting $(-1)^{\sum_{i\in I}i+\sum_{j\in J}j}(-1)^{\sum_{1\leq i,j\leq k}(\lambda_i^t-\mu_j^t-i+j)}=(-1)^{|\lambda|+|\mu|+|\lambda^t|+|\mu^t|}=1$. $\square$

Dual of a map

When $\varphi(1) = 1$, we define the **dual** φ^* of φ by the R-linear homomorphism $\varphi^* : R[B] \to R$ determined by the condition $f_{\varphi^*}(t) = f_\varphi^*(t) \, (= 1/f_\varphi(-t))$.

Examples (dual of φ)

- $\mathrm{Ber}^*(B^n) = \frac{1}{n+1}$, since $\frac{e^{-t}-1}{(-t)e^{-t}} = \frac{e^t-1}{t} = \sum_{n\geq 0} \frac{t^n}{(n+1)!}$,
- ev_a for $a \in R$ is self-dual, i.e., $\mathrm{ev}_a^* = \mathrm{ev}_a$, since $f_{\mathrm{ev}_a}(t) = e^{at}$.

Remark (self-dual maps)

There are self-dual φ's other than ev_a's. In fact, the self-dual φ's are: $f_\varphi(t) = \frac{g(t)}{g(-t)}$ $(g(t) \in 1 + tR[[t]])$, or $f_\varphi(t) = e^{\sum_{i\geq 0} a_i t^{2i+1}}$ $(a_i \in R)$.

Duality relation for some KP-determinants

Corollary (Duality relation for some KP-determinants)

Assume that $\varphi(1) = 1$. For $m, m^* \geq 1$, $\mathbf{n} = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$, and $\mathbf{n}^* = (n_0^*, \ldots, n_{m^*}^*) \in \mathbb{N}^{m^*+1}$ with $0 \leq n_0, n_0^*, \ldots, n_m, n_{m^*}^* \leq 1$ satisfying $m^* = |\mathbf{n}|$, and $m = |\mathbf{n}^*|$. Then it holds that $\Theta_{\mathbf{n}}^{\varphi} = \Theta_{\mathbf{n}^*}^{\varphi^*}$.

Proof: We apply the duality of Toeplitz det's for the rectangle $m \times m^*$. Since $T_{f_\varphi}(m^*, m) = \left(\prod_{0 \leq i \leq m-1} \frac{i!}{(m^*+i)!}\right) \Theta_{\mathbf{n}}^{\varphi}$ by Corollary of Wronskian relation, and $\prod_{0 \leq i \leq m-1} \frac{i!}{(m^*+i)!} = \prod_{0 \leq i \leq m^*-1} \frac{i!}{(m+i)!}$, Corollary follows. $\square$

Elliptic analogue is a work in progress

- real analytic extensions of Eisensteinization,
- p-adic L-function for CM elliptic curves and Coates-Wiles homomorphism,
- construction of Padé approximants and irrational number theory for special values of p-adic L-function for CM elliptic curves, etc.

Thank you very much!