IUTch III–IV with remarks on the function-theoretic roots of the theory

Go Yamashita

RIMS, Kyoto

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from Hodge-Arakelov theory

de Rham's thm $/\mathbb{C}$

$\rightsquigarrow$

p-adic Hodge comparison $/\mathbb{Q}_p$

$\rightsquigarrow$

Hodge-Arakelov comparison $/\mathrm{NF}$

$\rightsquigarrow$

(A motivation of) Θ-link

$$
\begin{array}{ccc}
H_1(\mathbb{C}^\times, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} H^1_{\mathrm{dR}}(\mathbb{C}^\times) & \longrightarrow & \mathbb{C} \\
\circlearrowleft \otimes \dfrac{dT}{T} & \longmapsto & \displaystyle\int_{\circlearrowleft} \frac{dT}{T} = 2\pi i
\end{array}
$$

induces a comparison isom.

$$H^1_{\mathrm{dR}}(\mathbb{C}^\times) \xrightarrow{\sim} (H_1(\mathbb{C}^\times, \mathbb{Z}) \otimes \mathbb{C})^*$$

$/\mathbb{Q}_p \quad \mathbb{G}_{\mathrm{m}}$-case

étale side $\qquad\qquad$ dR side

$$
\begin{array}{ccc}
T_p\mathbb{G}_{\mathrm{m}} \otimes_{\mathbb{Z}_p} H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}/\mathbb{Q}_p) & \longrightarrow & B_{\mathrm{crys}} \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
\underline{\varepsilon} \otimes \dfrac{dT}{T} & \longmapsto & \text{“}\displaystyle\int_{\underline{\varepsilon}} \frac{dT}{T}\text{”} = \log[\underline{\varepsilon}] = t
\end{array}
$$

$$t \longleftrightarrow 2\pi i$$

$$\underline{\varepsilon} = (\varepsilon_n)_n$$

$$\uparrow \quad \varepsilon_0 = 1, \varepsilon_1 \neq 1, \varepsilon_{n+1}^p = \varepsilon_n$$

“analytic path” around the origin

$/\mathbb{Q}_p$ $\quad E$: elliptic curve $/\mathbb{Z}_p$

$$0 \to \mathrm{coLie} E_{\mathbb{Q}_p} \to E^{\dagger}_{\mathbb{Q}_p} \to E_{\mathbb{Q}_p} \to 0 \quad \text{univ. ext'n}$$

étale side $\qquad$ dR side

$$T_p E_{\mathbb{Q}_p} \underset{\mathbb{Q}_p}{\otimes} \underbrace{H^1_{\mathrm{dR}}(E_{\mathbb{Q}_p}/\mathbb{Q}_p)}_{\substack{\parallel \\ \mathrm{coLie}\widehat{E^{\dagger}_{\mathbb{Q}_p}}}} \longrightarrow B_{\mathrm{crys}}$$

$$\begin{array}{ccc} \Psi & & \Psi \\ \underline{P} \otimes \omega & \longmapsto & \text{“} \int_{\underline{P}} \omega \text{”} = \text{“} \log_\omega(\underline{P}) \text{”} \end{array}$$

$\underline{P} = (P_n)_n \quad P_n \in E(\overline{\mathbb{Q}_p}), pP_{n+1} = P_n,$ “analytic path” on E

$$\begin{array}{ccc} \mathrm{coLie}\widehat{E^{\dagger}_{\mathbb{Q}_p}} \cong \mathrm{Hom}(\mathrm{Lie}\widehat{E^{\dagger}_{\mathbb{Q}_p}}, \mathrm{Lie}\widehat{\mathbb{G}_{a/\mathbb{Q}_p}}) \cong & \mathrm{Hom}(\widehat{E^{\dagger}_{\mathbb{Q}_p}}, \widehat{\mathbb{G}_{a/\mathbb{Q}_p}}) & \log^*_\omega(dT) = \omega \\ \Psi & \Psi & \\ \omega \longmapsto & \log_\omega & \end{array}$$

/NF

Hodge-Arakelov theory

"discretise" & "globalise "

the p-adic Hodge comparison map

$E \,/\, F \leftarrow \mathrm{NF} \qquad \ell > 2$ prime

assume $\quad 0 \neq P \in E(F)[2]$

$\mathcal{L} := \mathcal{O}(\ell[P])$

$E[\ell]$: approximation of “underlying mfd.”

Zar. locally $E^\dagger \cong \mathbb{G}_{\mathrm{a}/E} \cong \mathcal{O}_E[T]$
relative degree

<u>Roughly</u>

$$\dim_F = \ell^2 \to \Gamma(E^\dagger, \mathcal{L}\,|_{E^\dagger})^{\deg<\ell} \xrightarrow{\sim} \mathcal{L}\,|_{E^\dagger[\ell]} \; (= \bigoplus_{E[\ell]} F)$$

$\dim_F = \ell^2$

dR side (fcts) étale side (values)

an isom. of F-vector spaces

& preserves specified integral str.'s (omit)

at non-Arch. & Arch. places

cf. degenerate ($\mathbb{G}_{\mathrm{m}}$) case

$$\begin{array}{ccc} F[T]^{\deg<\ell} & \xrightarrow{\sim} & \bigoplus_{\zeta\in\mu_\ell} F \\ \Psi & & \Psi \\ f & \longmapsto & (f(\zeta))_{\zeta\in\mu_\ell} \end{array}$$

(Vandermonde $\det \neq 0$)

(LHS) (*i.e.* dR side) has filtration by rel. deg.

s.t. $\mathrm{Fil}^{-j}/\mathrm{Fil}^{-j+1} \cong \omega_E^{\otimes(-j)}$

(RHS) (*i.e.* étale side)

in the specific integral str.

we have a Gaussian pole $q^{j^2/8\ell} O_F$

the map: (derivatives of) theta fcts

$\longmapsto$ theta values

Consider both sides as vector bdl's over the moduli $\mathcal{M}_{\mathrm{ell}}$

degree comparison

$$(\mathrm{LHS}) = -\sum_{j=0}^{\ell-1} j[\omega_E] \approx -\frac{\ell^2}{2}[\omega_E]$$

$$! \quad \big\| \quad \text{by } [\omega_E^{\otimes 2}] = [\Omega_{\mathcal{M}_{\mathrm{ell}}}] = \tfrac{1}{6}[\log q]$$

$$(\mathrm{RHS}) = -\frac{1}{8\ell}\sum_{j=0}^{\ell-1} j^2[\log q] \approx -\frac{\ell^2}{24}[\log q]$$

Motivation of Θ-link

Assume $\quad$ a global mult. subspace $M \subset E[\ell]$

take $N \subset E[\ell]$ a gp scheme s.t. $\quad M \times N \cong E[\ell]$

apply Hodge-Arakelov for $E' := E/N$

over $K := F(E[\ell])$

$$\Gamma((E')^{\dagger}, \mathcal{L}|_{(E')^{\dagger}})^{\deg<\ell} \xrightarrow{\sim} \bigoplus_{-\frac{\ell-1}{2} \leqq j \leqq \frac{\ell-1}{2}} (q^{\frac{j^2}{2\ell}} O_K) \otimes_{O_K} K$$

$$q = (q_v)_{v:\mathrm{bad}}$$

incompatibility of Hodge fil. on (LHS)

w/ the $\oplus$ decomp. on (RHS)

$$\rightsquigarrow \quad \mathrm{Fil}^0 = \underline{\underline{q}}\,O_K \underset{\circledast}{\hookrightarrow} \underline{\underline{q}}^{j^2} O_K$$

"arith. Kodaira-Spencer morph."

deg ≈ 0 deg $\ll 0$

$$\Rightarrow 0 \lesssim -(\text{large number})(\approx -ht)$$

(scheme theoretic)

Hodge-Arakelov: use scheme theory

cannot obtain $\circledast$

IUTch: abandon scheme theory

use (non-scheme theoretic)

"$\circledast$" $\{\underline{\underline{q}}^{j^2}\}_j \longmapsto \underline{\underline{q}}$

Θ-link

$^{\exists}2$ ways of cracking a nut

— crack it in one breath by a nutcracker,

— soak it in a large amount of water,
soak, soak, and soak.
then it cracks by itself.

an example of the 2nd one :

rationality of congruent zeta by Lefschetz trace formula :

many commutative diagrams

& proper base change, smooth base change

- proper & smooth base change $\leftarrow$ not the "point" of the proof
- each commutative diagram

In some sense, the "point" of the proof was to establish the scheme theory & étale cohomology theory

(*i.e.*, the circumstances where a topological (not coherent) cohomology theory works (in positive char.))

IUTch also goes in the 2nd way of nutcracking.

Before IUTch, the essential ingredients
already appeared.

What was remained was

to put them together

(in a very delicate manner) !

$\forall$ constructions are (locally) trivial

After many (locally) trivial constructions

(in several hundred pages),

highly non-trivial inequality follows!

The “point” was to establish

the circumstances,

in which non-arith. hol. operations work!

IUTch III

In short, in IUT II,
we performed "Galois evaluation"

$$\begin{array}{ccc} \text{theta fct} & \longmapsto & \text{theta values} \\ \text{"env" labels} & & \text{"gau" labels} \end{array}$$

$$\left(\begin{array}{l} \mathcal{MF}^{\nabla}\text{-objects} \\ \text{(filtered } \varphi\text{-modules)} \end{array} \quad \longmapsto \quad \text{Galois rep'ns} \right)$$

Two Problems

1. Unlike “theta fcts”, “theta values” DO NOT admit a multiradial alg’m in a NAIVE way.

2. We need ADDITIVE str. for (log-) height fcts. $\mu^{\log}$

On 1.

$$\begin{tikzcd}
\text{theta fcts} \arrow[r, mapsto, "\text{Gal. eval.}"] & \text{theta values} \arrow[d, phantom, "\cap"] \\
\left(\begin{array}{c}\text{admit multirad. alg'm}\\ \textit{cf. } \text{[IUT II, Prop 3.4(i)]}\end{array}\right) \arrow[u] & \prod_{\mathbb{F}_\ell^{\circledast}} \left(\begin{array}{c}\text{constant}\\ \text{monoids}\end{array}\right) \\
& \text{requires cycl. rig. via LCFT} \arrow[u]
\end{tikzcd}$$

(*cf.* [IUT II, Prop 3.4(ii), Cor 3.7(ii), Rem 4.10.2(ii)])

Recall cycl. rig. via LCFT uses

$$\mathcal{O}^{\triangleright} = (\text{unit portion}) \times (\text{value gp portion})$$

$$\begin{tikzcd}
\boxed{{}^{\dagger}\mathcal{O}^{\triangleright}} \arrow[d, dash] & \\
\boxed{\text{core}} \arrow[r, dash] & \boxed{{}^{\ddagger}\mathcal{O}^{\triangleright}}
\end{tikzcd}$$

theta values

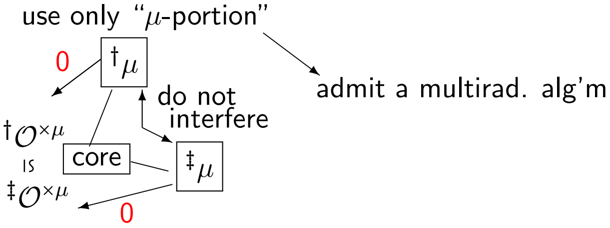
We DO NOT share it
in both sides of Θ-link!

"$\{\underline{\underline{q}}^{j^2}\}_j \longmapsto \underline{\underline{q}}$"

DO NOT admit a multirad. alg'm in a NAIVE way.

cf.

$$\begin{cases} \text{cycl. rig. via mono-theta env.} \\ \text{cycl. rig. via } \widehat{\mathbb{Z}}^{\times} \cap \mathbb{Q}_{>0} = \{1\} \end{cases}$$



To overcome these problems,
$\longrightarrow$ use log link!

$$\left(\begin{array}{c} \text{\& allowing } \underset{\uparrow}{\text{mild}} \text{ indet's} \\ \\ \text{non-interference etc. (later)} \end{array}\right)$$

log-Θ lattice

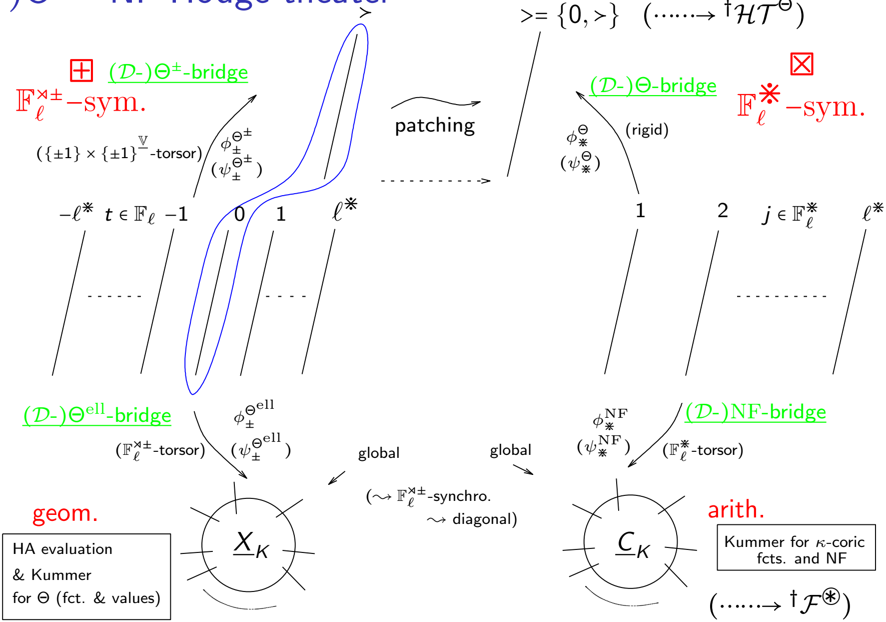
$$
\begin{tikzcd}
& \vdots & \vdots & \\
& \bullet & \bullet & \\
\cdots \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \cdots \\
& \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

$\bullet$: ($\Theta^{\pm\mathrm{ell}}$NF-) Hodge theater

$\uparrow$: log-link

$\rightarrow$: $\Theta^{(\times\mu)}_{(\mathrm{LGP})}$-link

$(\mathcal{D}\text{-})\Theta^{\pm\mathrm{ell}}$ NF Hodge theater



pTeich	IUTch
hyperb. curve / char $= p > 0$	an NF
indigenous bdl. over a hyperb. curve / char $= p > 0$	once punctured ell. curve over an NF
Frob. in char = $p > 0$	log - link
"Witt" lift $p^n/p^{n+1} \rightsquigarrow p^{n+1}/p^{n+2}$	Θ - link
can. lift of Frob.	log-Θ lattice

want to see alien ring str.

$$\begin{tikzcd}
\boxplus & \bullet \arrow[r, "\Theta\text{-link}"] & \bullet \arrow[l, bend right] & \vartriangleright \ \text{eye} \\
\boxtimes \ \mathcal{O}^{\times\mu} \arrow[u] & \bullet \arrow[u, "\text{log-link}"'] & &
\end{tikzcd}$$

(Note $\mathbb{F}_{\ell}^{\rtimes\pm}$-symm. isom's
are compatible w/ log-links
$\rightsquigarrow$ can pull-back Ψ_{gau} via log-link)

However,

$$
\begin{tikzcd}
\boxplus & \bullet \arrow[r, "\Theta"] & \bullet \arrow[l, bend right, shift left=6] \\
\boxtimes\ \mathcal{O}^{\times\mu} \arrow[u] & \bullet \arrow[u, "\log"'] \arrow[r, "\Theta"'] & \bullet \arrow[u, "\log"']
\end{tikzcd}
$$

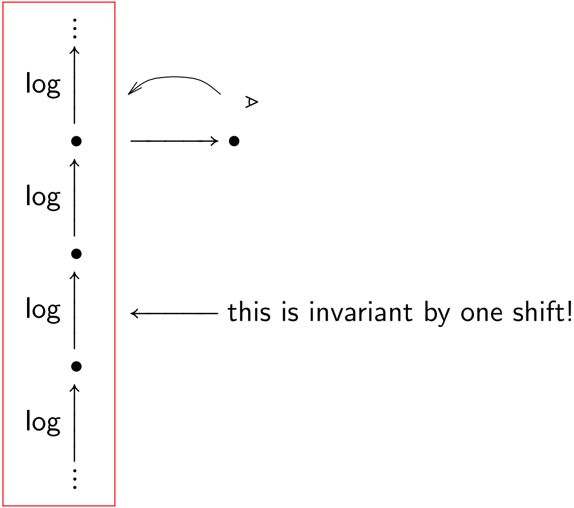
is highly <u>non</u>-commutative

(*cf.* $\log(a^N) \neq (\log a)^N$)

$\rightsquigarrow$ cannot see from the right

$$
\begin{tikzcd}
\bullet \\
\bullet \arrow[u, "\log"']
\end{tikzcd}
$$

We consider the infinite chain of log-links



Important Fact

$k/\mathbb{Q}_p$ fin.

$$
\begin{array}{ccl}
 & & \hspace{6em}\underline{\text{log shell}} \\
\log \mathcal{O}_k^\times & \subset & \frac{1}{2p}\log \mathcal{O}_k^\times = \mathcal{I}_k \\
\uparrow \log & \circlearrowleft & \text{the domain \& codomain} \\
 & & \text{of log are} \\
\mathcal{O}_k^\times & & \text{contained in the log-shell}
\end{array}
$$

upper semi-compatibility

(Note also: log-shells are rigid)

Besides theta values, we need
another thing :

we need NF (:= number field)

to convert $\boxtimes$-line bdles

into $\boxplus$-line bdles

and vice versa.

$\boxtimes$ -line bdles

$\leftarrow$ def'd in terms of <u>torsors</u>

$\boxplus$ -line bdles

$\leftarrow$ def'd in terms of <u>fractional ideals</u>

$\exists$ natural

cat. equiv. in a scheme theory

$\boxtimes$ -line bdles

$\leftarrow$ def'd only in terms of $\boxtimes$-str's

$\rightarrow$ admits precise log-Kummer corr.

But, difficult to compute log-volumes

$\boxplus$ -line bdles

$\leftarrow$ def'd by both of $\boxtimes$ & $\boxplus$-str's

$\rightarrow$ only admits upper semi-compatible log-Kummer corr.

But, suited to <u>explicit estimates</u>

We also include NFs as data

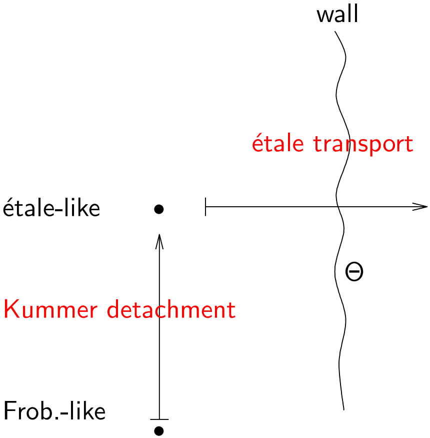
$$(\text{an NF})_j \subset \prod_{v_{\mathbb{Q}}} \log(\mathcal{O}^{\times})$$

$$\left.\begin{array}{l}\text{theta values}\\ \text{NFs}\end{array}\right\} \longleftarrow \quad \text{story goes } \underline{\text{in a parallel way}} \text{ in some sense}$$

(of course ∃ essential difference
cf. [IUT III, Rem 2.3.2, 2.3.3])

To obtain the final multirad. alg'm:

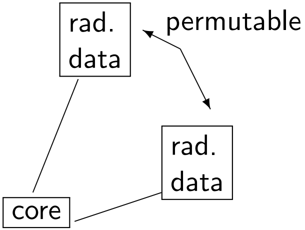
$$\begin{tikzcd}
\text{Frob.-like} \arrow[r] \arrow[d, mapsto] & \bullet\ \text{data} \arrow[dd, squiggly, "\text{Kummer theory}"] & \text{assoc. to } \mathcal{F}\text{-prime-strips} \\
\text{étale-like} \arrow[dr] & & \\
 & \bullet\ \text{data} \arrow[dd, squiggly, "\text{forget arith. hol. str.}"] & \text{assoc. to } \mathcal{D}\text{-prime-strips} \\
\text{arith.-hol.} \arrow[ur] \arrow[d, mapsto] & & \\
\text{mono-an.} \arrow[r] & \bullet\ \text{data} & \text{assoc. to } \mathcal{D}^{\vdash}\text{-prime-strips}
\end{tikzcd}$$


wall
étale transport
étale-like
Kummer detachment
Frob.-like
Θ

Frobenius-picture

$$\begin{array}{cccccc}
 & \vdots & & \vdots & & \\
 & \cdot & & \cdot & & \\
 & \uparrow & & \uparrow & & \\
 & \cdot & & \cdot & & \\
\log & \uparrow & & \uparrow & \log & \longrightarrow \\
 & \cdot & & \cdot & & \text{Kummer} \\
\log & \uparrow & & \uparrow & \log & \text{theory} \\
 & \cdot & \xrightarrow[\Theta]{} & \cdot & & \uparrow \\
\log & \uparrow & & \uparrow & \log & \text{(cycl. rig.'s)} \\
 & \vdots & & \vdots & &
\end{array}$$

étale-picture



3 portions of Θ-link

$$\begin{array}{ccc} {}^{\dagger}G_{\underline{v}} & \xrightarrow{\sim} & {}^{\ddagger}G_{\underline{v}} \\ \circlearrowright & & \circlearrowright \\ {}^{\dagger}\mathcal{O}^{\times\mu} & \xrightarrow{\sim} & {}^{\ddagger}\mathcal{O}^{\times\mu} \end{array} \leftarrow \text{share } (\rightsquigarrow \text{ht} + \text{fct})$$

local $\left\{ \begin{array}{l} \bullet\ \boxed{\text{unit}} \\ \\ \bullet\ \boxed{\text{value gp}} \end{array} \right.$

$${}^{\dagger}\underline{\underline{q}}^{\begin{pmatrix} 1^2 \\ \vdots \\ j^2 \end{pmatrix}^{\mathbb{N}}} \xrightarrow{\sim} {}^{\ddagger}\underline{\underline{q}}^{\mathbb{N}} \leftarrow \text{drastically changed}$$

- $\underbrace{\boxed{\text{global realified}}}_{\underline{\mathbb{V}} \ni \underline{v}}$ $\quad (\mathbb{R}_{\geq 0})_{\underline{v}} (\cdots, j^2, \cdots) \xrightarrow{\sim} (\mathbb{R}_{\geq 0})_{\underline{v}} \log \underline{\underline{q}}$

$\rightsquigarrow$ (ht fct)

Kummer theory

unit portions

$^{\dagger}G_{\underline{v}} \curvearrowright {}^{\dagger}\mathcal{O}^{\times\mu} := {}^{\dagger}\mathcal{O}^{\times}_{\bar{k}}/\mu$ $\mathbb{Q}_p$-module

$+$ integral str. *i.e.* $\mathrm{Im}(\mathcal{O}^{\times}_{\bar{k}^H}) \subseteq (\mathcal{O}^{\times\mu}_{\bar{k}})^H$ $^{\forall}H \subset G_{\underline{v}}$ open

$\uparrow$ fin. gen. $\mathbb{Z}_p$–mod.

Θ wall $\downarrow \wr \leftarrow$ non-ring theoretic $\rightsquigarrow$ log-shell

$\left(\rightsquigarrow \text{computable log-vol.}\right)$

$^{\ddagger}G_{\underline{v}} \curvearrowright {}^{\ddagger}\mathcal{O}^{\times\mu}$

$$({}^\dagger G_{\underline{v}} \curvearrowright {}^\dagger \mathcal{O}^\times_{\bar{k}}) \underset{\text{Kummer}}{\overset{\sim}{\longrightarrow}} ({}^\dagger G_{\underline{v}} \curvearrowright \mathcal{O}^\times_{\bar{k}}({}^\dagger G_{\underline{v}}))$$

$\uparrow$

unlike the case of $\mathcal{O}^{\triangleright}_{\bar{k}}$,

$\left(\begin{array}{c} \leftarrow \text{now, we cannot} \\ \text{use } \mathcal{O}^{\triangleright}_{\bar{k}}. \\ \text{use only } \mathcal{O}^\times_{\bar{k}} \end{array} \right)$

$\widehat{\mathbb{Z}}^\times$- indet. occurs

$\uparrow$ $\searrow$

$\left(\begin{array}{c} \text{container is invariant} \\ \text{under this } \widehat{\mathbb{Z}}^\times\text{-indet.} \end{array} \right)$

OK

cycl. rig. $\mu(G_{\underline{v}}) \tilde{\to} \mu(\mathcal{O}^\times_{\bar{k}})$

via LCFT ?

does not hold.

We want to protect

$$\begin{cases} \text{value gp portion} \\ \text{global real'd portion} \end{cases}$$

from this $\widehat{\mathbb{Z}}^{\times}$- indet!

$$\left(\begin{array}{ll} \text{sharing } {}^{\dagger}\mathcal{O}^{\times\mu} & \xrightarrow{\sim} {}^{\ddagger}\mathcal{O}^{\times\mu} \quad \text{w/ int. str.} \\ & \rightsquigarrow \ \color{red}{\text{(Ind 2)}} \\ & \bullet \underset{\ominus}{\longrightarrow} \bullet \quad \text{horizontal indet.} \end{array} \right)$$

value gp portion

$\mathcal{O}^{\triangleright}$

$\downarrow$

mono-theta cycl. rig. (unlike LCFT cycl. rig.)

only μ is involved

0

$\boxed{{}^{\dagger}\mu}$

${}^{\dagger}\mathcal{O}^{\times\mu}$

shared $\wr\!\!\parallel$ $\boxed{\text{core}}$ do not obstruct each others

${}^{\ddagger}\mathcal{O}^{\times\mu}$

$\boxed{{}^{\ddagger}\mu}$

0

NF portion

$\widehat{\mathbb{Z}}^{\times} \cap \mathbb{Q}_{>0} = \{1\} \rightsquigarrow$ cycl. rig.

multirad.

(on the function level)

Note also

mono-theta cycl. rig.

is compat. w/ prof. top.

$\rightsquigarrow$ $\mathbb{F}_\ell^{\rtimes\pm}$- sym. (conj. synchro.)

$\bullet \boxplus$

log$\uparrow$ $\mathbb{F}_\ell^{\rtimes\pm}$- sym. is compat. w/ log-links

$\bullet \boxtimes$

$\rightsquigarrow$ can $\underline{\text{pull-back}}$ coric (diagonal) obj.

via log-links

later

$\rightsquigarrow$ LGP monoid (Logarithmic Gaussian Procession)

$$
\begin{array}{cccccc}
 & \bullet & & & & \\
 & \uparrow & & & & \\
 & \bullet & \text{Kummer} & & & \\
\log & \uparrow & \searrow & & & \\
 & \bullet & \rightarrow & \circ & \text{coric} \\
\log & \uparrow & \text{Kummer} & & & \\
 & \bullet & \nearrow & \nwarrow & & \\
 & \uparrow & \text{Kummer} & \rightsquigarrow & & \\
\end{array}
$$

value gp portion

After

(Kummer) $\circ(\log)^n \quad (n \geqq 0)$,

take the action of "$\underline{\underline{q}}^{j^2}$" on $\mathcal{I} \otimes \mathbb{Q}$

log-Kummer correspondence

unit portion

not compat.

consider of a common rigid

$\downarrow$

upper bound given by log-shell

$$
\begin{array}{ll}
\bullet & \text{(Ind 3)} \\
\uparrow \log & \text{vertical indet.} \\
\bullet &
\end{array}
$$

value gp portion

- const. multiple rig.

$$\log \uparrow \quad \overset{\longrightarrow\!\!\leadsto}{\underset{\nearrow\ \text{hor. core}}{\text{label } 0}} \quad \overset{\exists}{}\ \underline{\text{splitting modulo } \mu} \text{ of}$$

- $0 \to \mathcal{O}^\times \overset{\curvearrowleft}{\to} \mathcal{O}^\times \cdot \underline{\underline{q}}^{j^2} \to \mathcal{O}^\times \cdot \underline{\underline{q}}^{j^2} / \mathcal{O}^\times \to 0$

&

$\log_p(\mu) = 0$

$\leadsto$ No <u>new</u> action appears

by the iteraions of log.'s

No interference

Note also

$$\mu^{\log}(\log_p(A)) = \mu^{\log}(A)$$

$$\text{if } A \underset{\text{bij}}{\tilde{\to}} \log_p(A)$$

$$\begin{pmatrix} \text{compatibility of log-volumes} \\ \hfill \text{w/ log-links} \end{pmatrix}$$

$\rightsquigarrow$ do not need to care about
$\underline{\text{how many times}}$ log.'s are applied.

In the Archimedean case,

we use a system (*cf.* [IUT III, Rem 4.8.2(v)])

$$\{\cdots \twoheadrightarrow \mathcal{O}^{\times}/\mu_N \twoheadrightarrow \mathcal{O}^{\times}/\mu_{N'} \twoheadrightarrow \cdots\}$$

& μ_N is killed in $\mathcal{O}^{\times}/\mu_N$

& constructions (of log-links, $\cdots$)

start from $\mathcal{O}^{\times}/\mu_N$'s, not $\mathcal{O}^{\times}$ (*cf.* [IUT III, Def 1.1(iii)])

& we put "weight N" on $\mathcal{O}^{\times}/\mu_N$

for the log-volumes (*cf.* [IUT III, Rem.1.2.1(i)])

NF portion

as well, consider the actions of $(F^{\times}_{\mathrm{mod}})_j$
after (Kummer) $\circ(\log)^n \quad (n \geqq 0)$

By $F^{\times}_{\mathrm{mod}} \cap \prod_v \mathcal{O}_v = \mu$

$\rightsquigarrow$ No new action appears

in the interation of log.'s

No interference

<u>*cf*</u>	multirad. geom. container		contained in a mono-analytic container
<u>val gp</u>	theta fct	eval $\rightsquigarrow$ (depends on labels & hol. str.)	theta values $\underline{\underline{q}}^{j^2}$
<u>NF</u>	$^{(\infty)}\kappa$-coric fcts	eval $\rightsquigarrow$ (indep. of labels dep. on hol. str.) Belyĭ cusp'tion	NF $F_{\mathrm{mod}}^{\times}$ (up to $\{\pm 1\}$)

	cycl. rig	log-Kummer
<u>theta</u>	mono-theta cycl. rig.	no interference by const. mult. rig.
<u>NF</u>	$\widehat{\mathbb{Z}}^\times \cap \mathbb{Q}_{>0} = \{1\}$ cycl. rig	no interference by $F^\times_{\mathrm{mod}} \cap \prod_v \mathcal{O}_v = \mu$

vicious cycles

$\bullet^{\dagger}\mu_{Fr}$ $\searrow^{\text{Kummer}}$

$\uparrow$ log $\xrightarrow[\text{Kummer}]{}$ $\circ\mu^{\forall}_{\acute{e}t}$

$\bullet^{\ddagger}\mu_{Fr}$ <u>theta</u>

$\uparrow$ log $\nearrow$ <u>NF</u>

$\left(\begin{array}{l} cf.\ \underline{\underline{q}}^{j^2}\text{'s are} \\ \text{not inv.} \\ \text{under } \{\pm 1\} \end{array}\right)$ $\searrow$

$\mu^{\forall}_{\acute{e}t}$

$\downarrow$

indet. $=: \mathbb{I}^{\mathrm{ord}}$

$\cap$

$\mathbb{N}_{\geq 1} \times \{\pm 1\}$

$\mathbb{I}^{\mathrm{ord}} = \{1\}$

by zero of order $= 1$

at each cups

$\mathbb{I}^{\mathrm{ord}} \twoheadrightarrow \mathrm{Im} \subset \mathbb{N}_{\geq 1}$

$\|$

$\{1\}$

by $\widehat{\mathbb{Z}}^{\times} \cap \mathbb{Q}_{>0} = \{1\}$

we have $(F^{\times}_{\mathrm{mod}}) \curvearrowleft \{\pm 1\}$-indet.

However

the totally of

$F^{\times}_{\mathrm{mod}}$ is invariant $\longleftarrow$

under $\{\pm 1\}$

OK. $\leftarrow$

$\begin{matrix} {}^{-1}_{-2} & \circlearrowright & {}_{1}^{2} \\ & 0 & \\ & \uparrow & \end{matrix}$

0 is also permuted

cf. [IUT III, Fig 2.7]

$\mathbb{F}_\ell^{\rtimes\pm}$- sym. <u>theta</u>	local & transcendental $q = e^{2\pi iz}$ compat. w/ prof. top.	$\longleftarrow$ theta fct zero of order $= 1$ at each cusp "only one valuation" $\rightsquigarrow$ cycl. rig. (Note theta fcts/ theta values do not have $\mathbb{F}_\ell^{\rtimes\pm}$- sym. But, the cycl. rig. DOES. $\uparrow$ use [,])
<u>NF</u>	global & algebraic rat. fcts. <u>incompat.</u> w/ prof. top. $\widehat{\mathbb{Z}}^\times \cap \mathbb{Q}_{>0} = \{1\}$ "many valuations" $\leftarrow$ global	Never for alg. rat. fcts sacrifice the compat. w/ prof. top.

$\mathbb{F}_\ell^{\circledast}$-sym.

0 is isolated

$\begin{matrix} & \downarrow & \\ & 0 & \\ {}^{-1}_{-2} & \circlearrowright & {}^{1}_{2} \end{matrix}$

Note also Gal. eval. $\leftarrow$ use hol. str.

labels

<u>theta</u> Gal. eval. & Kummer

$\leftarrow$ compat. w/ labels

<u>NF</u> $\leftarrow$ $\begin{cases} \text{the output } F^{\times}_{\text{mod}} \text{ does not depend on labels.} \\ \\ \text{global real'd monoids are} \\ \qquad \text{mono-analytic nature } (\leftarrow \text{units are killed}) \\ \quad \rightsquigarrow \text{do not depend on hol. str.} \end{cases}$

$$\left.\begin{array}{lllr}
\text{unit} & {}^{\dagger}\mathcal{O}^{\times\mu} \cong {}^{\ddagger}\mathcal{O}^{\times\mu} & & \text{(Ind 2)} \rightarrow \\
\text{val. gp} & \{\underline{\underline{q}}^{j^2}\} & & \text{w/ (Ind 3)} \uparrow \\
 & \curvearrowright & \mathcal{I} \otimes \mathbb{Q} & \\
 & \curvearrowright & & \\
\text{NF} & \overset{(-)}{\mathbb{M}}_{\mathsf{mod}} & &
\end{array}\right\} \text{Kummer detachement}$$

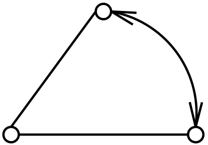
$$\rightsquigarrow$$

étale-like objects

étale transport

full poly

$${}^{\dagger}G_{\underline{v}} \xrightarrow{\sim} {}^{\ddagger}G_{\underline{v}} \qquad \text{(Ind 1)}$$



permutative indet.

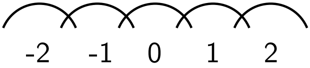
$\rightsquigarrow$ we can transport the data over the Θ-wall

Another thing

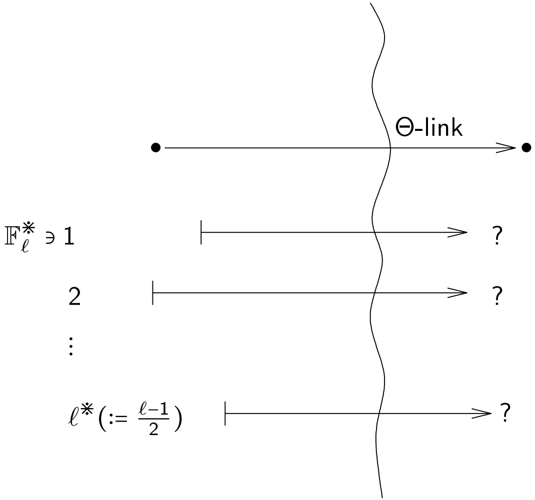
$$\Psi_{\mathrm{gau}} \subset \prod_{t \in \mathbb{F}_\ell^{\divideontimes}} \quad \text{(const. monoids)}$$

$$\uparrow$$

labels come from
arith. hol. str.



<u>cannot</u> transport the labels for Θ-link



0 $\bullet$ $\longmapsto$? $\leftarrow$ $\ell^{\pm}$-possibilities

1 $\bullet$ $\longmapsto$?

$\vdots$

$\ell^{\text{※}}$ $\bullet$ $\longmapsto$?

in total

$(\ell^{\pm})^{\ell^{\pm}}$-possibilities

the final inequality is weak.

$\left(\ell^{\pm} := \ell^{\text{※}} + 1 = \frac{\ell+1}{2}\right)$

use <u>processions</u>

$$
\begin{array}{ccccccccc}
\{0\} & \subset & \{0,1\} & \subset & \{0,1,2\} & \subset \cdots \subset & \{0,\cdots,\ell^*\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{?\} & \subset & \{?,?\} & \subset & \{?,?,?\} & \subset \cdots \subset & \{?,\cdots,?\}
\end{array}
$$

$\longrightarrow$ then, in total $(\ell^{\pm})!$-possibilities

$\nearrow$

gives more strict inequality

than the former case

A rough picture of the final multirad. rep'n:

$$
\begin{array}{ccccccc}
 & & (F^{\times}_{\mathrm{mod}})_1 & & \cdots & & (F^{\times}_{\mathrm{mod}})_{\ell^{\divideontimes}} \\
\{\mathcal{I}^{\mathbb{Q}}_0\} & \subset & \{\mathcal{I}^{\mathbb{Q}}_0, \mathcal{I}^{\mathbb{Q}}_1\} & \subset \quad \cdots \quad \subset & & & \{\mathcal{I}^{\mathbb{Q}}_0, \cdots, \mathcal{I}^{\mathbb{Q}}_{\ell^{\divideontimes}}\}
\end{array}
$$

$\underline{\underline{q}}^{1^2}$ $\quad$ $\underline{\underline{q}}^{2^2}$ acts $\quad$ $\underline{\underline{q}}^{(\ell^{\divideontimes})^2}$

const. mult. rig.

$\swarrow$

$\Psi^{\perp}_{\mathrm{LGP}} \longleftarrow$ value gp portion via canonical splitting modulo μ

Recall

$\mathcal{R}$, $\mathcal{C}$: groupoids (*i.e.* $^\forall$morph's are isom's) s.t. $^\forall$objects are isomorphic

$\Phi : \mathcal{R} \longrightarrow \mathcal{C}$: functor ess. surj.

If Φ is <u>full</u> (*i.e.*, multiradial)

$$\Rightarrow \text{sw} : \begin{array}{ccc} \mathcal{R} \times_{\mathcal{C}} \mathcal{R} & \longrightarrow & \mathcal{R} \times_{\mathcal{C}} \mathcal{R} \\ \Psi & & \\ (R_1, R_2, \alpha : \Phi(R_1) \xrightarrow{\sim} \Phi(R_2)) & \longmapsto & (R_2, R_1, \alpha^{-1}) \end{array}$$

preserves the isom. class.

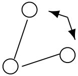
By this multirad. rep's & the compatibility w/ Θ-link :

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\{\underline{\underline{q}}^{j^2}\}_{1\le j\le \ell^{*}} \arrow[r, leftrightarrow] & \underline{\underline{q}} \\
\deg \ll 0 \arrow[u, "\text{w/ indet.'s}"'] & \deg \approx 0 \arrow[u, "\text{w/ indet.'s}"']
\end{tikzcd}$$

w/ indet.'s

we cannot

distinguish them!!

(Ind1) permutative indet.  ${}^{\dagger}G_{\underline{v}} \cong {}^{\ddagger}G_{\underline{v}}$

in the étale transport

(Ind2) horizontal indet. $\bullet \xrightarrow{\Theta} \bullet$ ${}^{\dagger}\mathcal{O}^{\times\mu} \cong {}^{\ddagger}\mathcal{O}^{\times\mu}$

in the Kummer detach. w/ int. str.

(Ind3) vertical indet.

$$\begin{tikzcd} \bullet \\ \bullet \arrow[u, "\log"'] \end{tikzcd} \qquad \begin{tikzcd} \log(\mathcal{O}^{\times}) \arrow[r, phantom, "\subset"] & \frac{1}{2p}\log(\mathcal{O}^{\times}) \\ \mathcal{O}^{\times} \arrow[u, "\log"'] \arrow[ur, phantom, "\subset"] & \end{tikzcd}$$

in the Kummer detach.

can be considered as a kind of

“descent data from $\mathbb{Z}$ to $\mathbb{F}_1$”

$$\begin{array}{ccc} \mathbb{Z} & \otimes_{\mathbb{F}_1} & \mathbb{Z} \\ \searrow & & \nearrow \end{array}$$

(Ind1) hol. hull

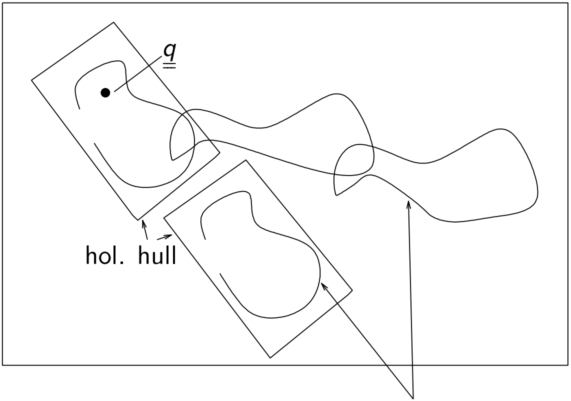
(Ind2)

(Ind3)

mono-analytic container

$\parallel$

log-shell



$\mathcal{I}^{\mathbb{Q}}$

possible images of "$\{\underline{\underline{q}}^{j^2}\}_j$"
somewhere, it contains a region
with the same log-volume as $\underline{\underline{q}}$

Recall $\{\underline{\underline{q}}^{j^2}\}_j \longmapsto \underline{\underline{q}}$

$$\longrightarrow \quad 0 \lesssim -(\mathrm{ht}) + \underbrace{(\text{indet})}_{(\boxed{1} + \varepsilon)\left(\begin{array}{c}\text{log-diff}\\ (+\text{log-cond})\end{array}\right)}$$

$$\longrightarrow \quad (\mathrm{ht}) \lesssim (1 + \varepsilon)(\text{log-diff} + \text{log-cond})$$

calculation in Hodge-Arakelov

miracle equality $\left(\begin{array}{l} \dfrac{1}{\ell^2}\sum_j j^2[\ \log q\ ] \approx \dfrac{\ell^2}{24}[\ \log q\ ] \\ \dfrac{1}{\ell}\sum j[\ \omega_E\ ] \approx \dfrac{\ell^2}{24}[\ \log q\ ] \end{array}\right)$

$\underline{cf.}$ Hodge-Arakelov

IF a global mult. subspace existed

$$\begin{array}{ccccc} \Longrightarrow & \underline{\underline{q}}\mathcal{O} & \hookrightarrow & \underline{\underline{q}}^{j^2}\mathcal{O} \\ & \uparrow & & \uparrow \\ & \deg \risingdotseq 0 & & \deg \ll 0 \end{array}$$

$$\Longrightarrow \qquad -(\text{large}) \geq 0$$

What was needed was

the circumstances, in which

this calculation of the miracle equality works!!

$\left(\begin{array}{l} \textit{i.e.}\text{, to abandan the scheme theory,} \\ \text{and to go to IU !!} \end{array} \right)$

[IUT III, Th 3.11] In summary,

tempered conj. vs prof. conj. (semi-graphs of anbd.)

$\mathbb{F}_\ell^{\rtimes\pm}$-conj. synchro ($\rightsquigarrow$ diagonal, $\rightsquigarrow$ hor. core)

	(i)(objects)	(ii)(log-Kummer)	(iii) (compat. w/ $\Theta^{\times\mu}_{\mathrm{LGP}}$-link)
$\mathbb{F}_\ell^{\rtimes\pm}$-symm. ⊞	$\mathcal{I} \hookleftarrow$ unit	invariant after admitting (Ind3) ↑	invariant after admitting (Ind2) → ↑ $\widehat{\mathbb{Z}}^\times$-indet.
	$\Psi^\perp_{\mathrm{LGP}}$ val gp compat. of log-link w/ $\mathbb{F}_\ell^{\rtimes\pm}$-symm.	no interference by const. mult. rig. ell. cusp'tion ← pro-p anab. + hidden endom.	only μ is involved $\rightsquigarrow$ multirad. protected from $\widehat{\mathbb{Z}}^\times$-indet. by mono-theta cycl. rig. quadratic str. of Heisenberg gp
$\mathbb{F}_\ell^{\circledast}$-symm. ⊠	$\overset{(-)}{\mathbb{M}}_{\mathrm{mod}}$ NF Belyĭ cusp'tion ↑ pro-p anab. + hidden endom.	no interference by $F^\times_{\mathrm{mod}} \cap \prod_v \mathcal{O}_v = \mu$	protected from $\widehat{\mathbb{Z}}^\times$-indet. by $\widehat{\mathbb{Z}}^\times \cap \mathbb{Q}_{>0} = \{1\}$
others	(compat. of log-volumes w/ log-links)	(arch. theory: Aut-hol. space ell. cusp'tion)	(étale picture: permutable after admitting (Ind1)) ↑ (autom. of processions are included)

Some questions

How about the following variants of Θ -link ?

i) $\{\underline{\underline{q}}^{j^2}\}_j \longmapsto \underline{\underline{q}}^{N} \qquad (N > 1)$

ii) $\{(\underline{\underline{q}}^{j^2})^{N}\}_j \longmapsto \underline{\underline{q}} \qquad (N > 1)$

$$\text{i)}\ \{\underline{\underline{q}}^{j^2}\}_j \longmapsto \underline{\underline{q}}^N$$

$$\begin{array}{l} \uparrow \\ \deg \fallingdotseq 0 \end{array} \begin{pmatrix} \ell \approx \mathrm{ht} \\ \& \\ \leftarrow \deg \ll \ell \end{pmatrix}$$

<u>it works</u>

$$\longrightarrow N \cdot 0 \lesssim -(\mathrm{ht}) + (\mathrm{indet.})$$

(as for $N \ll \ell$)

(When $N > \ell \Rightarrow$ the inequality is weak)

ii) $\{(\underline{\underline{q}}^{j^2})^N\}_j \longmapsto \underline{\underline{q}}$

it <u>DOES NOT</u> work !

Because

① $\Theta \underset{\text{replace}}{\rightsquigarrow} \Theta^N \Rightarrow \cancel{\begin{matrix}\text{mono-theta}\\ \text{cycl. rig.}\end{matrix}}$

mono-theta cycl. rig. comes

from the quadraticity of [,]

cf. [EtTh, Rem2.19.2]

$\longrightarrow \Theta^N \ (N > 1) \longrightarrow \nexists$ Kummer compat.

② vicious cycles

Θ^N zero of order $= N > 1$ at cusps

various Frob-like $\mu \overset{\text{Kummer theory}}{\simeq}$ étale-like $\mu \leftarrow$ cusp

$$
\begin{tikzcd}
\bullet \arrow[dr, "\mathrm{Kummer}"] & \\
\bullet \arrow[u, "\log"] \arrow[r, "\mathrm{Kummer}"] & \circ \\
\bullet \arrow[u, "\log"] \arrow[ur, "\mathrm{Kummer}"'] &
\end{tikzcd}
$$

cf. [IUT III, Rem.2.3.3(vi)]

$\circlearrowleft$ loop $\rightarrow$ one loop gives once N-power

IF it WORKED

$$\longrightarrow \quad 0 \leq -N(\mathrm{ht}) + (\mathrm{indet.})$$

$$\longrightarrow \quad (\mathrm{ht}) \leq \frac{1}{N}(1+\varepsilon)(\text{log-diff.} + \text{log-cond.})$$

$\longrightarrow$ contradition to a lower bound

given by analytic number theory

(Masser, Stewart-Tijdeman)

IUTch IV

[IUT IV, Prop 1. 2] $k_i/\mathbb{Q}_p$ fin with ram. index $=: e_i$ $(i \in I, \sharp I < \infty)$
For autom. $\forall \phi : (\otimes_{i \in I} \log_p O^{\times}_{k_i}) \otimes \mathbb{Q}_p \xrightarrow{\sim} (\otimes_{i \in I} \log_p O^{\times}_{k_i}) \otimes \mathbb{Q}_p$

$\uparrow$

(Ind 1) étale transport indet. $\curvearrowleft$	&	(Ind 2) ${}^{\dagger}\mathcal{O}^{\times\mu} \simeq {}^{\ddagger}\mathcal{O}^{\times\mu}$ hor. indet. $\to$

of $\mathbb{Q}_p$ -vect. sp. which induces an autom. of the submodule
$\otimes_{i \in I} \log_p O^{\times}_{k_i}$,
put

$$a_i := \begin{cases} \frac{1}{e_i}\lceil \frac{e_i}{p-1} \rceil & (p > 2) \\ 2 & (p = 2), \end{cases} \quad b_i := \lfloor \frac{\log \frac{pe_i}{p-1}}{\log p} \rfloor - \frac{1}{e_i}$$

$\delta_i := \operatorname{ord}$ (different of $k_i/\mathbb{Q}_p$)
$a_I := \Sigma_{i \in I} a_i, \quad b_I := \Sigma_{i \in I} b_i, \quad \delta_I := \Sigma_{i \in I} \delta_i$

$\Rightarrow$ Then, we have $p^{\lfloor\lambda\rfloor} \otimes_{i\in I} \frac{1}{2p} \log_p O_{k_i}^{\times}$

$$\begin{array}{c} \cap\!\! \mid \end{array} \longleftarrow \boxed{\begin{array}{l} \text{(Ind 3)} \uparrow \\ \text{vert. indet.} \end{array}}$$

(Ind 1)(Ind 2)

$\downarrow$

$$\phi(\underbrace{p^{\lambda} O_{k_{i_o}} \otimes_{O_{k_{i_o}}} (\otimes_{i\in I} O_{k_{i_o}})^{\sim}}_{\text{normalisation}}) \subseteq p^{\lfloor\lambda\rfloor - \lceil\delta_I\rceil - \lceil a_I\rceil} \otimes_{i\in I} \log_p O_{k_i}^{\times}$$
$$\subseteq p^{\lfloor\lambda\rfloor - \lceil\delta_I\rceil - \lceil a_I\rceil - \lceil b_I\rceil} (\otimes_{i\in I} O_{k_i})^{\sim}$$

$\uparrow$

its hol. upper bound

this contains

the union of all possible images of Θ-pilot objects for $\lambda \in \frac{1}{e_{i_0}}\mathbb{Z}$.

(For a bad place, $\lambda = ord(q_{\underline{v}_{i_0}})$)

$\underline{e.g.}$ $\quad e < p - 2$

$\mathcal{O} \subseteq \frac{1}{p} \log_p \mathcal{O}^\times = \frac{1}{p}\mathfrak{m}$

$\uparrow$

$\mathbb{Z}_p$ -basis $\pi, \pi^2, \ldots, \pi^e$

$\nearrow$

cannot distinguish if we have no ring str.

"differential / $\mathbb{F}_1$ "

cf. Teichmüller dilation

$\rightsquigarrow$

$$\left(\begin{array}{l} k/\mathbb{Q}_p \text{ fin.} \\ G_k \xrightarrow{\sim} G_k \\ \exists \text{ non-sch. th'c autom. also } \textit{cf.}\ [\mathbb{Q}_p\mathrm{GC}] \text{ main thm} \\ G_k/I_k : \text{rigid} \\ I_k : \text{non-rigid} \end{array} \right)$$

It's a THEATRE OF ENCOUNTER of

$$\begin{tikzcd} & \text{anab. geom.} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{Teich. point of view} \arrow[rr, leftrightarrow] & & \text{Hodge-Arakelov} \end{tikzcd}$$

(& “diff. / $\mathbb{F}_1$ ”)

$\rightsquigarrow$ Diophantine conseq. !

By this upper bound,

([IUT IV, Th 1.10])

main thm. of IUT $\quad -|\log(\underline{\underline{\Theta}})|$

$$\wedge\!\!\!\parallel$$

$$-|\log(\underline{\underline{q}})| \quad \frac{\ell+1}{4}\Big\{\Big(\boxed{1} + \frac{36d_{\mathrm{mod}}}{\ell}\Big)\ \underline{\big(\log \mathfrak{d}^{F_{tpd}} + \log \mathfrak{f}^{F_{tpd}}\big)}$$

log-diff + log-cond

("(almost zero) ≤ - (large)") $\quad +10\big(d^{*}_{\mathrm{mod}} \cdot \ell + \eta_{prm}$ (← abs. const. given by prime number thm.)

$$\underline{-\frac{1}{6}}\Big(1 - \frac{12}{\ell^2}\Big)\underline{\log(\mathfrak{q})}\Big\} - \log(\underline{\underline{q}})$$

ht

$$\rightsquigarrow \mathrm{ht} \lesssim \big(\boxed{1} + \varepsilon\big)(\text{log-diff} + \text{log-cond})\ \big)$$

$$\mathrm{ht} \lesssim (\boxed{1} + \varepsilon)(\text{log-diff} + \text{log-cond})$$

$\uparrow$

miracle equality

already appeared in Hodge Arakelov theory.

$$\Gamma((E/N)^{\dagger}, \mathcal{O}(P)|_{(E/N)^{\dagger}})^{<\ell} \tilde{\to} \otimes_{j=-\ell^{*}}^{\ell^{*}} \underline{\underline{q}}^{j^2} O_K \otimes K$$

$$P \in (E/N)[2](F)$$

polar coord $\quad \frac{1}{\ell} \deg(LHS) \approx -\frac{1}{\ell} \sum_{i=0}^{\ell-1} i[\omega_E] \approx -\frac{1}{2}[\omega_E]$

$\|$

cartesian coord $\quad \frac{1}{\ell} \deg(RHS) \approx -\frac{1}{\ell^2} \sum_{j=1}^{\ell^{*}} j^2 [\log q] \approx -\frac{1}{24}[\log q]$

i.e. discretisation of

"$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$"

cartesian coord　　　　polar coord

On the ε - term

$$\mathrm{ht} \leq \delta + *\delta^{\boxed{\frac{1}{2}}} \log(\delta)$$

$\uparrow$

it appears as a kind of
"quadratic balance"

$$\left(\begin{array}{l} \mathrm{ht} := \frac{1}{6} \log \mathfrak{q}^{\forall} \\ \delta := \log\text{-diff} + \log\text{-cond.} \end{array} \right)$$

(*cf.* Masser, Stewart-Tijdeman
analytic lower bound)

$\frac{1}{2} \leftrightarrow$ Riemann zeta ?

calculation of the intersection number

$\downarrow$

IUT : $\Delta.\Delta$ for "$\Delta \subset \mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$"

More precisely $\Delta.(\Delta + \varepsilon\Gamma_{\mathsf{Fr}})$

$\uparrow$

the graph of "abs. Frobenius"

cf. Θ - link $\leftrightarrow$ abs. Frob.

$\updownarrow$

"mod p^2 lift"

cf.

$$\Delta.\Delta \longleftrightarrow \text{Gauss-Bonnet}$$
$$\left|\log(\underline{\underline{\Theta}})\right| \lesssim \left|\log(\underline{\underline{q}})\right| \doteqdot 0$$

expresses the hyperbolicity of NF

$$\left(\begin{array}{lll} \underline{p\text{Teich}} & \text{der.} & \text{of can. lift of Frob.} \overset{\Theta\text{-link}}{\swarrow\!\!\nearrow} \\ & \Rightarrow & \omega \hookrightarrow \Phi^*\omega \Rightarrow (1-p)(2g-2) \leqq 0 \end{array} \right)$$

$$\Delta.(\Delta + \varepsilon\Gamma_{\mathsf{Fr}})$$

$$= \underline{\underline{\Delta.\Delta}} \quad + \quad \underline{\Delta.\varepsilon\Gamma_{\mathsf{Fr}}}$$

$\downarrow$ main term of abc

$\downarrow$ ε -term

$\uparrow$ $\frac{1}{2}$ appeared

Question

Can we "integrate" it to

$$\Delta.(\Delta + \varepsilon\Gamma_{\mathrm{Fr}} + \frac{\varepsilon^2}{2}\Gamma_{\mathrm{Fr}}^2 + \dots) = \underset{\displaystyle \text{Riemann !!?}}{\underset{\big\downarrow}{\Delta.\Gamma_{\mathrm{Fr}}}}$$

Recall

$$\Theta \quad \overset{\text{Mellin}}{\rightsquigarrow} \quad \zeta$$

étale Θ also plays crucial roles in IUTch.

[IUTchI]	arith.	upper & lower half plane
[IUTchII]	arith. funct. eq.	(étale theta)
[IUTchIII]	arith. analytic cont.	(log-shell & log-link)

a phenomenon of Fourier transf. in IUTch

$$\circlearrowright \quad \mathcal{O}^{\times\mu} \qquad \longleftrightarrow \qquad \int e^{-\frac{x^2}{2}} e^{-2\pi i \xi x} dx$$

$$\widehat{\mathbb{Z}}^{\times} \quad \text{indet.}$$

$$\begin{array}{ccc} \text{quadricity} & \nearrow & \uparrow \\ & & \text{(gp. str.)} \end{array}$$

$$\rightsquigarrow$$

<u>mono-theta rigidities</u>

$$\text{multiradiality} \longleftrightarrow \int$$

$\exists$? IU-Mellin transf.