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IUT IV sect 1

w/ some remarks on the language of species

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[IUT IV, Prop 1.2] $k_i/\mathbb{Q}_p$ fin. ($i \in I$, $\# I < \infty$), $\leftarrow$ ram. index $=: e_i$

For $\forall \phi : \left(\bigotimes_{i \in I} \log_p \mathcal{O}_{k_i}^\times\right) \otimes \mathbb{Q}_p \xrightarrow{\sim} \left(\bigotimes_{i \in I} \log_p \mathcal{O}_{k_i}^\times\right) \otimes \mathbb{Q}_p$
autom. of $\mathbb{Q}_p$-vect. sp. which induces an autom. of the submodule $\bigotimes_{i \in I} \log_p \mathcal{O}_{k_i}^\times$,

(Ind 1) étale transport indet. ↺

& (Ind 2) $\pm \mathcal{O}^{\times\mu} \simeq \pm \mathcal{O}^{\times\mu}$

hor. indet.

put
$$a_i := \begin{cases} \frac{1}{e_i}\left\lceil \frac{e_i}{p-1} \right\rceil & (p > 2) \\ 2 & (p = 2) \end{cases}, \quad b_i := \left\lfloor \frac{\log \frac{p e_i}{p-1}}{\log p} \right\rfloor - \frac{1}{e_i}$$

$\delta_i :=$ ord (different of $k_i/\mathbb{Q}_p$)

$$a_I := \sum_{i \in I} a_i, \quad b_I := \sum_{i \in I} b_i, \quad \delta_I := \sum_{i \in I} \delta_i$$

$\Rightarrow$ Then, we have

normalization

$$\Phi\Big(\big(p^{\lambda}\,\mathcal{O}_{k_{i_0}} \underset{\mathcal{O}_{k_{i_0}}}{\otimes} \big(\underset{i\in I}{\otimes}\mathcal{O}_{k_i}\big)\big)^{\sim}\Big) \subseteq p^{\lfloor\lambda\rfloor-\lceil\delta_I\rceil-\lceil a_I\rceil}\underset{i\in I}{\otimes}\log_p \mathcal{O}^{\times}_{k_i}$$

(Ind 1)

(Ind 2)

$$\subseteq p^{\lfloor\lambda\rfloor-\lceil\delta_I\rceil-\lceil a_I\rceil-\lceil h_I\rceil}\Big(\underset{i\in I}{\otimes}\mathcal{O}_{k_i}\Big)^{\sim}$$

its hol. upper bound

$$p^{\lfloor\lambda\rfloor-\lceil\,\rceil}\underset{i\in I}{\otimes}\frac{1}{2p}\log_p\mathcal{O}^{\times}_{k_i}$$

$\bigcap\big|$ (Ind 3) $\uparrow$ vert. indet

this contains

the union of all possible images of Θ-pilot objects

for $\lambda \in \frac{1}{e_{i_0}}\mathbb{Z}$.

(For a bad place, $\lambda = \mathrm{ord}\big(\underbrace{q^{j^2}}_{=\underline{v}_{i_0}}\big)$)

e.g. $e < p-2$

$$\mathcal{O} \subseteq \frac{1}{p}\underbrace{\log_p \mathcal{O}^\times}_{\partial_p} = \mathfrak{m}$$

$\mathbb{Q}_p$-basis

$\pi, \pi^2, \ldots, \pi^e$

cannot distinguish if we have no ring str.

cf. Teichmüller dilation

$k/\mathbb{Q}_p$ fin.

$G_k \xrightarrow{\sim} G_k$

$\exists$ non-sch. th'c autom.

G_k / I_k : rigid

I_k : non-rigid

also cf. [$\mathbb{Q}_p$GC] main thm

"differential / $\mathbb{F}_1$"

It's a THEATRE OF ENCOUNTER of

$$\begin{tikzcd}
& \text{anab. geom.} \arrow[dl, leftrightarrow] \arrow[dr] & \\
\text{Teich. point of view} \arrow[rr, leftrightarrow] & & \text{Hodge-Arakelov}
\end{tikzcd}$$

(& "diff. / $\mathbb{F}_1$") !

$\rightsquigarrow$ Diophantine conseq. !

By this upper bound,

([IUT IV, Th 1.10])

main thm of IUT

$$-|\log(\underline{q})| \;\le\; \frac{l+1}{4}\left\{\left(\boxed{1}+\frac{36\, d_{mod}}{l}\right)\overbrace{\left(\log \mathfrak{d}^{F_{tpd}} + \log \mathfrak{f}^{F_{tpd}}\right)}^{\text{log-diff + log-cond}} + 10\left(d^{*}_{mod}\cdot l + \eta_{prm}\right) - \underbrace{\frac{1}{6}\left(1-\frac{12}{l^{2}}\right)\log(\underline{q})}_{ht}\right\} - \log(\underline{q})$$

$-|\log(\underline{q})|$

abs. consts (given by prime number thm)

("|almost zero| ≤ −(large)")

$$\rightsquigarrow \quad ht \lesssim (\boxed{1}+\varepsilon)(\text{log-diff} + \text{log-cond})$$

$$ht \lesssim (\boxed{1} + \varepsilon)(\text{log-diff} + \text{log-cond})$$

↑
miracle equality

already appeared in Hodge-Arakelov theory:

$$\Gamma\left((E/N)^{+}, \mathcal{O}(P)|_{(E/N)^{+}}\right)^{<\ell} \xrightarrow{\sim} \bigoplus_{j=-\ell^*}^{\ell^*} \underline{q^{j^2}} \mathcal{O}_K \otimes K$$

$P \in (E/N)[2](F)$

polar coord $\quad \frac{1}{\ell} \deg(LHS) \approx -\frac{1}{\ell} \sum_{i=0}^{\ell-1} i\,[\omega_E] \approx -\frac{1}{2}[\omega_E]$

cartesian coord $\quad \frac{1}{\ell} \deg(RHS) \approx -\frac{1}{\ell^2} \sum_{j=1}^{\ell^*} j^2 [\log q] \approx -\frac{1}{24}[\log q]$

i.e. discretisation of

"
$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$
"

cartesian coord

polar coord.

On the ϵ- term

$$ht \leq \delta + * \, \delta^{\boxed{\frac{1}{2}}} \log(\delta)$$

it appears as a kind of "quadratic balance"

$$\left(\begin{array}{l} ht := \frac{1}{6} \log \mathfrak{q}^{\forall} \\ \delta := \text{log-diff} + \text{log-cond.} \end{array} \right)$$

(cf. Masser, Stewart - Tijdeman analytic lower bound)

$\frac{1}{2}$ $\longleftrightarrow$ Riemann zeta ?

calculation of the intersection number

IUT : $\Delta \cdot \Delta$

for "$\Delta \subset \mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$"

More precisely

$$\Delta \cdot (\Delta + \varepsilon \Gamma_{Fr})$$

the graph of "abs. Frobenius" $\longleftrightarrow$ abs. Frob.

cf. W-lift

$\updownarrow$

"mod p^2 lift"

$$\Delta \cdot \left(\Delta + \varepsilon \Gamma_{Fr} \right)$$

$$= \underbrace{\Delta \cdot \Delta}_{\text{main term of abc}} + \underbrace{\Delta \cdot \varepsilon \Gamma_{Fr}}_{\varepsilon\text{-term}}$$

$\frac{1}{2}$ appeared

Question

Can we "integrate" it

A_σ

$$\Delta \cdot \left(\Delta + \varepsilon \Gamma_{F_r} + \frac{\varepsilon^2}{2} \Gamma_{F_r}^2 + \cdots \right) = \Delta \cdot \Gamma_{F_r}$$

Riemann !! ?

Some remarks
on the language
of species

Recall

bi-anabelian

e.g. $${}^{\dagger}\Pi \cong {}^{\ddagger}\Pi \Longrightarrow \left({}^{\dagger}\Pi \curvearrowright \mathcal{O}^{\triangleright}({}^{\dagger}\Pi)\right) \cong \left({}^{\ddagger}\Pi \curvearrowright \mathcal{O}^{\triangleright}({}^{\ddagger}\Pi)\right)$$

mono-anabelian

e.g. $${}^{\dagger}\Pi \underset{\text{alg'm}}{\rightsquigarrow} {}^{\dagger}\Pi \curvearrowright \mathcal{O}^{\triangleright}({}^{\dagger}\Pi)$$

Problems

1. How to rigorously formulate an "algorithm"?

2. Do we really need the "mono-anabelian philosophy"?
   (i.e. the "bi-anabelian" is enough?)

1) In bi-anabelian,

a "group theoretical" reconstruction

means

$$ {}^{\dagger}\Pi \cong {}^{\ddagger}\Pi \implies \text{output of } {}^{\dagger}\Pi \cong \text{output of } {}^{\ddagger}\Pi $$

we consider them

as abstract top. gps.

How to formulate it
in the mono-anabelian case?

To state the output object is
the desired one

WITHOUT mentioning the content of
the algorithm,

it seems inevitable to state

like ${}^{\dagger}\Pi \xrightarrow{\sim} {}^{\text{model}}\Pi \rightsquigarrow$ output of ${}^{\dagger}\Pi$

$\xrightarrow{\sim}$ output of ${}^{\text{model}}\Pi$

i.e. we need to introduce
a model object

essentially,
it is bi-anabelian

Thus, currently we need to state <u>all of the contents of the algorithm</u>

to rigorously formulate a mono-anabelian proposition!

(the algorithm itself should be the content of the proposition.)

Then, it often lengthy

(and ~~the~~ a proof is also

the statement itself!)

To rigorously settle the meaning of "algorithm",

Mochizuki introduced the notions of

species & mutations

"species-objects" & "species-morphisms"

(& mutations of them)

are formulated in terms of

a collection of RULES

(set-theoretic formula)

(NOT a specific sets)

the construction of such sets in

an unspecified "indeterminate" ZFC

model

Note also that

the category theory is

not sufficient.

<u>e.g.</u> By Neukirch–Uchida

(the cat. of the number fields) $\cong$ (the cat. of profinite gps of NF-type)

cat. equiv.

cannot see the axioms of fields

② Do we really need "mono-anabelian philosophy"?

For example,

how about taking a quasi-inverse qi

$$(\Pi \curvearrowleft \mathcal{O}^{\triangleright}) \xmapsto[\text{forget}]{} \Pi$$

$$\Pi \longmapsto \Pi \curvearrowleft qi(\Pi)$$

of the forgetful functor?

In the mono-anabelian alg'm

"post-anabelian"

$$ {}^{\dagger}\Pi \overset{\text{recon}}{\rightsquigarrow} {}^{\dagger}\Pi \curvearrowright \mathcal{O}^{\triangleright}({}^{\dagger}\Pi) \rightsquigarrow {}^{\dagger}\Pi \curvearrowright {}^{\dagger}\mathcal{O}^{\triangleright} $$

Étale-like

Frobenius-like

(forget that $\mathcal{O}^{\triangleright}({}^{\dagger}\Pi)$ is the reconstructed object)

In the case of quasi-inverse,

$${}^{\dagger}\Pi \rightsquigarrow {}^{\dagger}\Pi \cong qi({}^{\dagger}\Pi)$$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \text{étale-like} & & \text{Frobenius-like} \end{array}$$

étale-like

it serves as a core object

$$\circledast \quad {}^\dagger\Pi \quad \leadsto \quad \underbrace{{}^\dagger\Pi \curvearrowright \mathcal{O}^{\triangleright}({}^\dagger\Pi)} \quad \overset{\text{forget}}{\leadsto} \quad {}^\dagger\Pi \curvearrowright {}^\dagger\mathcal{O}^{\triangleright}$$

$$\circledast \quad {}^\dagger\Pi \quad \leadsto \quad \boxed{} \quad {}^\dagger\Pi \curvearrowright \mathcal{G}({}^\dagger\Pi)$$

NO counterpart here!

Frobenius-like

$\longrightarrow$ "qi" does not work.

$$\longrightarrow {}^{n-1}\bullet \xrightarrow[\log]{} {}^{n}\bullet \xrightarrow[\log]{} {}^{n+1}\bullet \longrightarrow \cdots$$

since

we need a coric object
(shared)

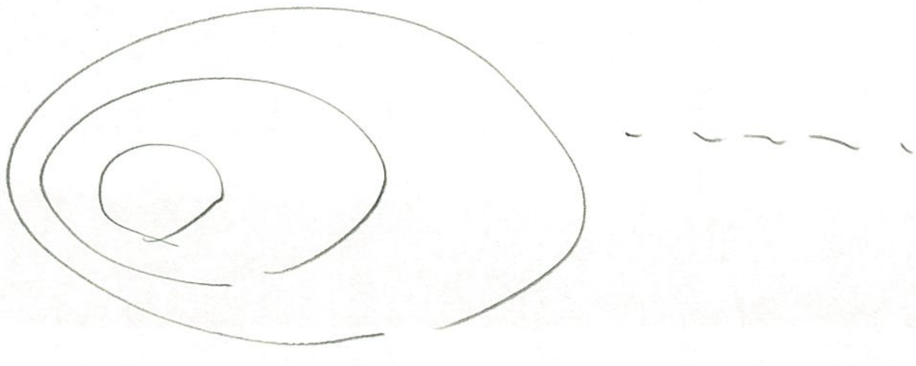
Then, how
about $^{\text{coric}}\Theta^{\Delta} := \left(\xrightarrow[\text{full poly}]{\sim} qi({}^{n-1}\pi) \xrightarrow[\text{full poly}]{\sim} qi({}^{n}\pi) \xrightarrow{\sim} \cdots \right)$

?

→ then conic $\Theta^{\triangleright}$ is not an object of the original log-sequence.

→ we need to extend the domain of q_i

→ extensions of universes



In the language of species,

it is the RULES,

not specific sets,

(sometimes arbitrarily assigned
e.g. quasi-inverse)

that are given.