IUT IV sect 1

w/ some remarks on the language of species

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11/Dec/2015 at Oxford

The author expresses his sincere gratitude to RIMS secretariat for typesetting his hand-written manuscript.

[IUT IV, Prop 1. 2] $k_i/\mathbb{Q}_p$ fin with ram. index $=: e_i$ $(i \in I, \sharp I < \infty)$
For autom. $\forall \phi : (\otimes_{i\in I} \log_p O_{k_i}^{\times}) \otimes \mathbb{Q}_p \xrightarrow{\sim} (\otimes_{i\in I} \log_p O_{k_i}^{\times}) \otimes \mathbb{Q}_p$

$\uparrow$

(Ind 1) étale transport indet. $\curvearrowleft$	&	(Ind 2) ${}^{\dagger}\mathcal{O}^{\times\mu} \simeq {}^{\ddagger}\mathcal{O}^{\times\mu}$ hor. indet. $\rightarrow$

of $\mathbb{Q}_p$ -vect. sp. which induces an autom. of the submodule
$\otimes_{i\in I} \log_p O_{k_i}^{\times}$,
put

$$a_i := \begin{cases} \frac{1}{e_i}\lceil \frac{e_i}{p-1} \rceil & (p > 2) \\ 2 & (p = 2), \end{cases} \quad b_i := \lfloor \frac{\log \frac{pe_i}{p-1}}{\log p} \rfloor - \frac{1}{e_i}$$

$\delta_i :=$ ord (different of $k_i/\mathbb{Q}_p$)
$a_I := \Sigma_{i\in I} a_i, \quad b_I := \Sigma_{i\in I} b_i, \quad \delta_I := \Sigma_{i\in I} \delta_i$

$\Rightarrow$ Then, we have $p^{\lfloor \lambda \rfloor} \otimes_{i \in I} \frac{1}{2p} \log_p O_{k_i}^{\times}$

(Ind 1)(Ind 2) $\qquad \bigcap\!| \longleftarrow$ (Ind 3) $\uparrow$ vert. indet.

$\downarrow$

$$\phi(p^{\lambda} O_{k_{i_o}} \otimes_{O_{k_{i_o}}} \underbrace{(\otimes_{i \in I} O_{k_{i_o}})^{\sim}}_{\text{normalisation}}) \subseteq p^{\lfloor \lambda \rfloor - \lceil \delta_I \rceil - \lceil a_I \rceil} \otimes_{i \in I} \log_p O_{k_i}^{\times}$$

$$\subseteq p^{\lfloor \lambda \rfloor - \lceil \delta_I \rceil - \lceil a_I \rceil - \lceil b_I \rceil} (\otimes_{i \in I} O_{k_i})^{\sim}$$

$\uparrow$ its hol. upper bound

this contains

the union of all possible images of Θ-pilot objects for $\lambda \in \frac{1}{e_{i_0}} \mathbb{Z}$.

(For a bad place, $\lambda = ord(q_{\underline{v}_{i_0}})$)

e.g. $e < p - 2$

$\mathcal{O} \subseteq \frac{1}{p} \log_p \mathcal{O}^\times = \frac{1}{p}\mathfrak{m}$

$\uparrow$

$\mathbb{Z}_p$ -basis $\pi, \pi^2, \ldots, \pi^e$

$\nearrow$

cannot distinguish if we have no ring str.

"differential / $\mathbb{F}_1$ "

cf. Teichmüller dilation

$$\left(\begin{array}{l} k/\mathbb{Q}_p \text{ fin.} \\ G_k \xrightarrow{\sim} G_k \\ \exists \text{ non-sch. th'c autom. also } \textit{cf.}\ [\mathbb{Q}_p\mathrm{GC}] \text{ main thm} \\ G_k/I_k : \text{rigid} \\ I_k : \text{non-rigid} \end{array} \right)$$

It's a THEATRE OF ENCOUNTER of

anab. geom.

↙↗ ↖↘

Teich. point of view ⟷ Hodge-Arakelov

(& "diff. / $\mathbb{F}_1$ ")

$\rightsquigarrow$ Diophantine conseq. !

By this upper bound,

([IUT IV, Th 1.10])

main thm. of IUT $\quad -|\log(\underline{\underline{\Theta}})|$

$\searrow \quad \lesssim \quad\quad \wedge\!\parallel$

$$-|\log(\underline{\underline{q}})| \qquad \frac{\ell+1}{4}\Big\{\Big(\boxed{1} + \frac{36 d_{\mathrm{mod}}}{\ell}\Big)\underline{\big(\log \mathfrak{d}^{F_{tpd}} + \log \mathfrak{f}^{F_{tpd}}\big)}$$

$\searrow \quad \swarrow$

log-diff + log-cond

(“(almost zero) ≤ - (large)”) $\quad +10(d^{*}_{\mathrm{mod}} \cdot \ell + \eta_{prm}$ (← abs. const. given by prime number thm.)

$$-\underline{\frac{1}{6}}\Big(1 - \frac{12}{l^2}\Big)\underline{\log(\mathfrak{q})}\Big\} - \log(\underline{\underline{q}})$$

$\searrow \quad \swarrow$

ht

$\rightsquigarrow$ ht $\lesssim (\boxed{1} + \varepsilon)$(log-diff + log-cond))

$\mathrm{ht} \lesssim (\boxed{1} + \varepsilon)(\text{log-diff} + \text{log-cond})$

$\uparrow$

miracle equality

already appeared in Hodge Arakelov theory.

$$\Gamma((E/N)^{\dagger}, \mathcal{O}(P)|_{(E/N)^{\dagger}})^{<\ell} \tilde{\to} \otimes_{j=-\ell^{*}}^{\ell^{*}} \underline{\underline{q}}^{j^2} O_K \otimes K$$

$P \in (E/N)[2](F)$

polar coord $\quad \frac{1}{\ell}\deg(LHS) \approx -\frac{1}{\ell}\sum_{i=0}^{\ell-1} i[\omega_E] \approx -\frac{1}{2}[\omega_E]$

$\|$

cartesian coord $\quad \frac{1}{\ell}\deg(RHS) \approx -\frac{1}{\ell^2}\sum_{j=1}^{\ell^{*}} j^2[\log q] \approx -\frac{1}{24}[\log q]$

i.e. discretisation of

$$“\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}”$$

cartesian coord | polar coord

On the ε - term

$$\mathrm{ht} \leq \delta + *\delta^{\boxed{\frac{1}{2}}} \log(\delta)$$

↑

it appears as a kind of
"quadratic balance"

$$\left(\begin{array}{l} \mathrm{ht} := \frac{1}{6} \log \mathfrak{q}^{\forall} \\ \delta := \log\text{-diff} + \log\text{-cond.} \end{array} \right)$$

↑

(*cf.* Masser, Stewart-Tijdeman
analytic lower bound)

$\frac{1}{2} \leftrightarrow$ Riemann zeta ?

calculation of the intersection number

$\downarrow$

IUT : $\Delta.\Delta$ for "$\Delta \subset \mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$"

More precisely $\Delta.(\Delta + \varepsilon\Gamma_{\mathsf{Fr}})$

$\uparrow$

the graph of "abs. Frobenius"

cf. Θ - link $\leftrightarrow$ abs. Frob.

$\updownarrow$

"mod p^2 lift"

$$\Delta.(\Delta + \varepsilon\Gamma_{\mathrm{Fr}})$$

$$= \underline{\underline{\Delta.\Delta}} + \underline{\Delta.\varepsilon\Gamma_{\mathrm{Fr}}}$$

$\downarrow$ main term of abc

$\downarrow$ ε -term

$\uparrow$ $\frac{1}{2}$ appeared

Question

Can we "integrate" it to

$$\Delta.(\Delta + \varepsilon\Gamma_{\mathsf{Fr}} + \frac{\varepsilon^2}{2}\Gamma_{\mathsf{Fr}}^2 + \dots) = \Delta.\Gamma_{\mathsf{Fr}}$$

$\downarrow$

Riemann !!?

some remarks on the language of species

Recall

bi-anabelian

e.g. $\quad {}^{\dagger}\Pi \cong {}^{\ddagger}\Pi \Longrightarrow ({}^{\dagger}\Pi \curvearrowright \mathcal{O}^{\triangleright}({}^{\dagger}\Pi)) \cong ({}^{\ddagger}\Pi \curvearrowright \mathcal{O}^{\triangleright}({}^{\ddagger}\Pi))$

mono-anabelian

e.g. $\quad {}^{\dagger}\Pi \rightsquigarrow_{\text{alg'm}} ({}^{\dagger}\Pi \curvearrowright \mathcal{O}^{\triangleright}({}^{\dagger}\Pi))$

Problems

1. How to rigorously formulate
   an "algorithm"?

2. Do we really need
   the "mono-anabelian philosophy"?
   (*i.e.* "bi-anabelian" is enough?)

① In bi-anabelian,

a "group theoretical" reconstruction means

$$ {}^{\dagger}\Pi \cong {}^{\ddagger}\Pi \Longrightarrow \text{output of } {}^{\dagger}\Pi $$

$$ \uparrow \qquad\qquad \cong \text{output of } {}^{\ddagger}\Pi $$

we consider them

as abstract top. gps.

How to formulate it
in the mono-anabelian case?

To state the output object is the desired one WITHOUT mentioning the content of the algorithm,

it seems inevitable to state
like ${}^{\dagger}\Pi \cong {}^{\mathrm{model}}\Pi \Longrightarrow$ output of ${}^{\dagger}\Pi$
$\cong$ output of ${}^{\mathrm{model}}\Pi$

i.e. we need to introduce a model object

$\rightsquigarrow$ essentially, it is bi-anabelian

Thus, currently we need to
state all of the contents of the algorithm
to rigorously formulate
a mono-anabelian proposition!

(the algorithm itself should be
the content of the proposition.)

Then, it often lengthy

$$\left(\begin{array}{l} \text{and a proof is also} \\ \qquad\qquad \text{the statement itself!} \end{array} \right)$$

To rigorously settle the meaning

of “algorithm”,

Mochizuki introduced

the notions of species & mutations

“species-objects” & “species-morphism”
(& mutations of them)

are formulated in terms of
a collection of RULES

(set-theoretic formula)

(NOT a specific sets)

the construction of such sets in
an unspecified “indeterminate” ZFC model

Note also that
the category theory is
not sufficient.

e.g. By Neukirch-Uchida

$$\begin{pmatrix} \text{the cat. of} \\ \quad\text{the number fields} \end{pmatrix}$$

$$\underset{\text{cat. equiv.}}{\cong} \begin{pmatrix} \text{the cat. of profinite gps} \\ \quad\text{of NF-type} \end{pmatrix}$$

$\nearrow$

cannot see the axioms of fields

② Do we really need "mono-anabelian philosophy"?

For example,

how about taking a quasi-inverse qi

$$(\Pi \curvearrowright \mathcal{O}^{\triangleright}) \underset{\text{forget}}{\longmapsto} \Pi$$

$$\Pi \longmapsto \Pi \curvearrowright \mathrm{qi}(\Pi)$$

of the forgetful functor?

In a mono-anabelian alg'm

"post-anabelian"

$${}^{\dagger}\Pi \overset{\text{recon}}{\rightsquigarrow} {}^{\dagger}\Pi \curvearrowright \mathcal{O}^{\triangleright}({}^{\dagger}\Pi) \rightsquigarrow {}^{\dagger}\Pi \curvearrowright {}^{\dagger}\mathcal{O}^{\triangleright}$$

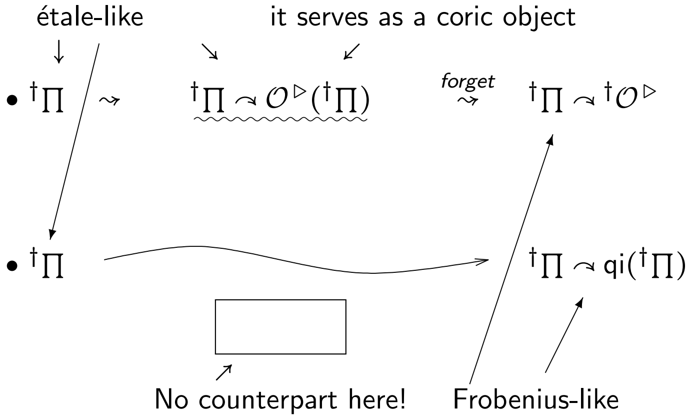
étale-like

Frobenius-like

(forget that $\mathcal{O}^{\triangleright}({}^{\dagger}\Pi)$ is the reconstructed object)

In the case of quasi-inverse

$$
\begin{array}{ccc}
{}^{\dagger}\prod & \rightsquigarrow & {}^{\dagger}\prod \curvearrowright \mathrm{qi}({}^{\dagger}\prod) \\
\uparrow & & \uparrow \\
\text{étale-like} & & \text{Frobenius-like}
\end{array}
$$


étale-like
it serves as a coric object
$\bullet\ {}^{\dagger}\Pi$
$\rightsquigarrow$
${}^{\dagger}\Pi \curvearrowright \mathcal{O}^{\triangleright}({}^{\dagger}\Pi)$
forget
${}^{\dagger}\Pi \curvearrowright {}^{\dagger}\mathcal{O}^{\triangleright}$
$\bullet\ {}^{\dagger}\Pi$
${}^{\dagger}\Pi \curvearrowright \mathrm{qi}({}^{\dagger}\Pi)$
No counterpart here!
Frobenius-like

$\rightsquigarrow$ "qi" does not work.

$$\xrightarrow{\hspace{2em}} \overset{n-1}{\bullet} \xrightarrow[\log]{\hspace{2em}} \overset{n}{\bullet} \xrightarrow[\log]{\hspace{2em}} \overset{n+1}{\bullet} \xrightarrow{\hspace{2em}} \cdots$$

since we need a coric object
(shared)

Then, how about

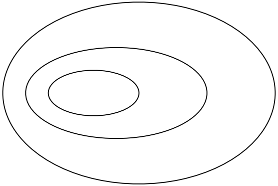
$$^{\mathrm{coric}}\mathcal{O}^{\triangleright} := \left(\underset{\substack{\text{full}\\ \text{poly}}}{\overset{\sim}{\to}} \mathrm{qi}({}^{n-1}\Pi) \underset{\substack{\text{full}\\ \text{poly}}}{\overset{\sim}{\to}} \mathrm{qi}({}^{n}\Pi) \overset{\sim}{\to} \cdots \right) \qquad ?$$

$\rightsquigarrow$ then ${}^{\text{coric}}\mathcal{O}^{\triangleright}$ is not an object

of the original log-sequence.

$\rightsquigarrow$ we need to extend the domain of qi

$\rightsquigarrow$ extensions of universes



...

In the language of species, it is the RULES,

not specific sets,

(sometime arbitrarily assigned *e.g.* quasi-inverse)

that are given.