“On tilts and Inter-universal Teichmüller theory”

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(The contents of these slides are essentially same as §3.5.3 of Mochizuki’s paper “On the Essential Logical Structure of Inter-universal Teichmüller Theory”.)

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We discuss $\begin{cases} \bullet \text{ similarities, and} \\ \bullet \text{ differences} \end{cases}$

between

$$\begin{cases} \bullet \quad \text{the unit group portions of the} \\ \qquad \mathcal{F}^{\Vdash \blacktriangleright \times \mu}\text{-prime-strips in the } \Theta\text{-link in IUT} \\ \qquad\qquad \text{and} \quad \left(\underset{\text{+gl.real'd Fr'd}}{\text{"} G_{\underline{v}} \curvearrowright \mathcal{O}^{\times \mu}_{\overline{K}_{\underline{v}}} \times \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}} \text{"}} \right) \\ \bullet \quad \text{the theory of tilts of perfectoid fields} \end{cases}$$

p : prime number

k : perfectoid field of char $= 0$ $\leftarrow$ $\left(\begin{array}{l} \text{i.e., equipped } w/ \ \underset{|p|<1}{|\cdot| : k \to \mathbb{R}_{\geq 0}} \\ \begin{cases} \bullet |\cdot| : \text{non-discrete} \\ \bullet k : \text{complete for } |\cdot| \\ \bullet \mathcal{O}_k/p \xrightarrow{\varphi} \mathcal{O}_k/p \quad x \mapsto x^p \\ \qquad \text{is surjective} \end{cases} \end{array} \right)$

$\overline{k}$: an algebraic closure of k

$G_k = \mathrm{Gal}(\overline{k}/k)$

$\swarrow$ completion

$\mathbb{C}_k = \widehat{\overline{k}}$ $\leftarrow$ again perfectoid field

$$|\cdot|_{k^\flat} = |(\cdot)^{(0)}|_k$$

$$\mathcal{O}_{k^\flat} := \varprojlim_{\varphi} \mathcal{O}_k/p\mathcal{O}_k, \quad k^\flat := \mathrm{Frac}\, \mathcal{O}_{k^\flat} : \text{ perfect(oid) CDVF of char} = p > 0$$

$$\ni (x_n)_n$$

$$\cong \varprojlim_{\varphi : x \mapsto x^p} \mathcal{O}_k \qquad \ni (x^{(n)} := \lim_{m \to \infty} \tilde{x}_{m+n}^{p^m})_n$$

$$\ni$$

$$(x^{(n)})_n, (y^{(n)})_n$$

$$(x+y)^{(n)} = \lim_{m \to \infty} (x^{(n+m)} + y^{(n+m)})^{p^m}, \ (xy)^{(n)} = x^{(n)} y^{(n)}$$

$$\mathcal{O}^{\times\mu} := \mathcal{O}^{\times}/(\text{torsion}), \quad \mathcal{O}^{\times\mu'} := \mathcal{O}^{\times}/(\text{prime-to-}p\text{ torsion})$$

We have

$$\mathcal{O}^{\times}_{k^\flat} \xrightarrow{\sim} \varprojlim \mathcal{O}^{\times}_{k}$$

$$\mathcal{O}^{\times\tilde{\mu}}_{\overline{k}} := \varprojlim \mathcal{O}^{\times\mu'}_{\overline{k}}$$

$$\mathcal{O}^{\times\mu}_{\mathbb{C}^\flat_k} = \mathcal{O}^{\times\mu'}_{\mathbb{C}^\flat_k} \quad \xrightarrow{\sim} \quad \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_k} := \varprojlim \mathcal{O}^{\times\mu'}_{\mathbb{C}_k}$$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_k} & \longrightarrow & \mathcal{O}^{\times\mu}_{\mathbb{C}_k} & \longrightarrow & 1 \\
 & & \| & & \cup \ \nparallel_{\text{dense}} & & \cup \ \nparallel_{\text{dense}} & & \\
1 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & \mathcal{O}^{\times\tilde{\mu}}_{\overline{k}} & \longrightarrow & \mathcal{O}^{\times\mu}_{\overline{k}} & \longrightarrow & 1
\end{array} \qquad (\text{exact})$$

multiplicative topological modules
compat. $w/$ natural actions of $G_{k^\flat} \xrightarrow{\sim} G_k$

$$\Uparrow$$

tilting equiv.

$^{\dagger}k,\ ^{\ddagger}k$: perfectoid fields of char $= 0$
s.t. $\quad {}^{\dagger}k^{\flat} \xrightarrow{\sim} {}^{\ddagger}k^{\flat}$
as topological fields

$$\left(\begin{array}{l} \text{e.g.} \quad {}^{\dagger}k = E(\zeta_{p^\infty})^{\wedge}, \quad \leftarrow p\text{-adic compl.} \\ \qquad\quad {}^{\ddagger}k = E(\pi^{1/p^\infty})^{\wedge} \\ \qquad\qquad\qquad \pi\text{:uniformizer} \\ \qquad E/\mathbb{Q}_p \text{ fin. unram. ext.} \end{array}\right)$$

$$\Downarrow$$

$$\left(\begin{array}{c} G_{{}^{\dagger}k}, G_{{}^{\ddagger}k} \subset G_{\mathbb{Q}_p} \\ \text{closed, } \underline{\text{not}} \text{ open} \end{array}\right)$$

$\Rightarrow$

$$\begin{array}{ccccccc} \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_{{}^{\dagger}k}} & \xleftarrow{\sim} & \mathcal{O}^{\times\mu}_{\mathbb{C}^{\flat}_{{}^{\dagger}k}} & \xrightarrow{\sim} & \mathcal{O}^{\times\mu}_{\mathbb{C}^{\flat}_{{}^{\ddagger}k}} & \xrightarrow{\sim} & \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_{{}^{\ddagger}k}} \\ \curvearrowright & & \curvearrowright & & \curvearrowright & & \curvearrowright \\ G_{{}^{\dagger}k} & \xleftarrow{\sim} & G_{{}^{\dagger}k^{\flat}} & \xrightarrow{\sim} & G_{{}^{\ddagger}k^{\flat}} & \xrightarrow{\sim} & G_{{}^{\ddagger}k} \end{array}$$

natural isom's of multiplicative top. modules
$w/$ continuous actions by top. groups.

For $\mathbb{Q}_p \subset K \subset K_1 \subset \cdots \subset \overline{\mathbb{Q}_p}$ fin ext's, $K_\infty := \bigcup_i K_i$

$$\begin{cases} \bullet\ i \gg 0 \quad K_{i+1}/K_i \quad \text{totally ramified of } \deg = p \\ \bullet\ K_\infty/K \text{ deeply ramified (i.e. } \nexists v > 0 \quad K_\infty \subset \overline{K}^{(v)}) \end{cases}$$

corr. to ram. fil.

$$\overset{\Rightarrow}{\text{LCFT}}\ G^{ab}_{K_\infty} \xrightarrow{\sim} \underset{\substack{i \\ \text{induced by incl.}}}{\varprojlim} G^{ab}_{K_i} \xrightarrow{\sim} \underset{\substack{i \\ \text{norms}}}{\varprojlim} \widehat{K_i^\times}$$

$$\begin{array}{ccc} G^{ab}_{K_\infty} & & \varprojlim \widehat{K_i^\times} \\ \bigcup & & \bigcup \\ \mu_{p'}(G^{ab}_{K_\infty}) & \xrightarrow{\sim} & \underline{K^\times_\infty} \quad \leftarrow \text{ residue field} \\ \bigcap & & \bigcap \\ \underset{\substack{H \subset G_{K_\infty} \\ \text{open}}}{\varinjlim} \mu_{p'}(H^{ab}) & \xrightarrow{\sim} & \overline{\mathbb{F}}_p^\times \curvearrowleft G_{K_\infty} \end{array}$$

$\ker = I_{K_\infty}$ inertia subgroup

$$\Rightarrow \quad G_{K_\infty} \underset{\substack{\text{functorial} \\ \text{gp. th'c. algorithm}}}{\rightsquigarrow} G_{K_\infty} \curvearrowright \overline{\mathbb{F}}_p^\times \rightsquigarrow \text{ the set } \left\{ \begin{array}{r} \chi : G_{K_\infty} \to \mathbb{Z}_p^\times \\ \text{totally ramified char's} \\ w/\text{ open image} \end{array} \right\}$$

For $k = \widehat{K}_\infty$ perfectoid field w/ $G_k = G_{K_\infty}$

$$\underset{\nearrow \text{use field str. of } k}{\log_p} : \mathcal{O}_{\mathbb{C}_k}^{\times\mu} \xrightarrow{\sim} \mathbb{C}_p$$

$$\begin{cases} \bullet\ {}^{\dagger}k := E(\zeta_{p^\infty})^\wedge & E/\mathbb{Q}_p \text{ fin. unram. ext} \\ \bullet\ {}^{\ddagger}k := E(\pi^{1/p^\infty})^\wedge & \pi \in E \text{ uniformizer} \end{cases}$$

$$\underset{\underset{\underline{\mathbf{1}}}{||}}{{}^{\dagger}\chi_{\mathrm{cyc}}} : G_{{}^{\dagger}k} \to \mathbb{Z}_p^\times, \quad \underset{\underset{\underline{\text{open image}}}{\uparrow}}{{}^{\ddagger}\chi_{\mathrm{cyc}}} : G_{{}^{\ddagger}k} \to \mathbb{Z}_p^\times$$

cyclotomic characters

via tilting isom.

$\Rightarrow$ isom. $\mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_{\dagger k}} \xrightarrow{\sim} \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_{\ddagger k}}$

<u>fails</u> to induce an isom. of exact seq.'s

$w/ G_{\dagger k} \simeq G_{\ddagger k}$ actions

via tilting isom.

$$
\begin{tikzcd}
 & {}^{\dagger}\chi_{\mathrm{cyc}} = \mathbf{1} & & & \\
1 \arrow[r] & \mathbb{Q}_p(1) \arrow[r] \arrow[d, dotted, "\nexists"'] \arrow[dr, phantom, "\curvearrowright"] & \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_{\dagger k}} \arrow[r, "alive"'] \arrow[d, "\wr"] \arrow[dr, phantom, "\curvearrowright"] & \mathcal{O}^{\times\mu}_{\mathbb{C}_{\dagger k}} \arrow[r] \arrow[d, dotted, "\nexists"'] & 1 \\
1 \arrow[r] & \mathbb{Q}_p(1) \arrow[r] \arrow[urr] & \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_{\ddagger k}} \arrow[r] & \mathcal{O}^{\times\mu}_{\mathbb{C}_{\ddagger k}} \arrow[r] & 1 \\
 & {}^{\ddagger}\chi_{\mathrm{cyc}} \text{ open image} & & &
\end{tikzcd}
$$

use field str. of ${}^{\dagger}k$

taking $\log_p : \mathcal{O}^{\times\mu}_{\mathbb{C}_{\dagger k}} \xrightarrow{\sim} \mathbb{C}_{\dagger k}$,

the set of such char's can be gp. th'cally reconstructed from $G_{\dagger k}$

$({}^{\dagger}\chi_{\mathrm{cyc}} \neq)\psi := {}^{\ddagger}\chi_{\mathrm{cyc}}|_{G_{\dagger k}} : G_{\dagger k} \to \mathbb{Z}_p^{\times}$ tot. ram. char. w/ open image

$$H^0(G_{\dagger k}, \mathbb{C}_{\dagger k}(\psi^{-1})) \neq 0$$

<u>sharp contrast</u> to the classical case of $k/\mathbb{Q}_p$ fin. $\downarrow$open image

$\rightharpoonup \quad H^0(G_k, \mathbb{C}_k(\psi^{-1})) = 0$

(due to Tate)

Via tilting isom

$$\begin{array}{ccc} G_{{}^{\dagger}k} & \xrightarrow{\sim} & G_{{}^{\ddagger}k} \\ \curvearrowright & & \curvearrowright \\ \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_{{}^{\dagger}k}} & \xrightarrow{\sim} & \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_{{}^{\ddagger}k}} \end{array}$$

- ${}^{\dagger}k, {}^{\ddagger}k$ do not share a <u>common cyclotomic character</u>
  (sharp contrast to the classical case $k/\mathbb{Q}_p$ fin.)
- ${}^{\dagger}k, {}^{\ddagger}k$ do not share a <u>"common p-adic Hodge theory"</u>

$$\uparrow$$

$$\left(\begin{array}{l} \text{classical } H^0(G_k, \mathbb{C}(\psi^{-1})) = 0 \\ \qquad \text{was a fundamental aspect} \\ \qquad\quad \text{of } p\text{-adic Hodge theory} \end{array} \right)$$

In particular, any aspects of anabelian geometry involving p-adic Hodge theory cannot be applied to the isom via tilting isom.

<u>cf.</u> $\begin{cases} \bullet \text{ IUT has two dimensionality} \\ \bullet \text{ construction of perfectoid fields involves killing "one dimension".} \end{cases}$

(adding "ζ_{p^∞}", "π^{1/p^∞}")

Similarity between IUT & the theory of tilts

$$
\begin{array}{cccccccc}
\mathcal{O}^{\times\mu}_{\mathbb{C}^\flat_k} & \xrightarrow{\sim} & \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_k} & & \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_{\dagger k}} & \xrightarrow{\sim} & \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_{\ddagger k}} \\
\curvearrowright & & \curvearrowright & , & \curvearrowright & & \curvearrowright \\
G_{k^\flat} & \xrightarrow{\sim} & G_k & & G_{\dagger k} & \xrightarrow{\sim} & G_{\ddagger k}
\end{array}
$$

$\longleftrightarrow \left(G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\mu}_{\overline{K}_{\underline{v}}} \right)$ the unit group portion of $\mathcal{F}^{\Vdash\blacktriangleright\times\mu}$-prime-strips in Θ-link

Differences

1. $G_k \hookrightarrow G_{\mathbb{Q}_p}$ closed, but NOT open, $\mathcal{O}^{\times\tilde{\mu}}_{\overline{\mathbb{Q}}_p} \hookrightarrow \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_p}$ dense, but NOT surjective

2. the tilt $k^\flat$ is of char $= p > 0$ $\quad \mathcal{O}^{\times\mu}_{\mathbb{C}^\flat_k} \hookrightarrow \mathcal{O}^{\times\tilde{\mu}}_{\mathbb{C}_k}$

$\mathrm{char} > 0$ local $\quad$ $\mathrm{char}\ 0$ local

$\uparrow \quad \nearrow$

two different notions of globality is required if we regard them as localization of global fields

Short digress

$$\mathcal{O}_{k^\flat} = \varprojlim(\mathcal{O}_k \xleftarrow{\varphi} \mathcal{O}_k \leftarrow \cdots$$

redundant copy school

If we identify them as in RCS,

then $\cong \{x \in \mathcal{O}_k \mid x^p = x\}$

$\neq \mathcal{O}_{k^\flat}$!

($\rightsquigarrow$ we should regard them as distinct rings)

Analogies between theories

	tilts	complex Teichmüller theory	IUT
1	$\mathcal{O}_k \xleftarrow{\varphi} \mathcal{O}_k$ $\cong$ as rings, i.e., if we do not require compat. w/φ . . . regard LHS, RHS as *distinct rings*	Teichmüller dilation $\mathbb{C} \overset{\Lambda}{\rightsquigarrow}$ $\cong$ as Riemann surfaces, i.e., if we do not require compat. w/Λ . . . regard LHS, RHS as *distinct hol. strs.*	$(\Theta^{\pm\mathrm{ell}}\mathrm{NF}\text{-})$ Hodge theater $\bullet \overset{\Theta}{\underset{\Theta\text{-link}}{\text{———}}} \bullet$ $\cong$ as Hodge theaters, i.e., if we do not require compat. w/Θ . . . regard LHS, RHS as *distinct arith. hol. strs.*
2	$\mathcal{O}_k \xleftarrow{\varphi} \mathcal{O}_k$ $\nexists \cong$ as rings compat. w/φ	$\mathbb{C} \overset{\Lambda}{\rightsquigarrow}$ $\nexists \cong$ as Riemann surfaces compat. w/Λ	$\bullet \overset{\Theta}{\text{———}} \bullet$ $\nexists \cong$ as Hodge theaters compat. w/Θ (This property is <u>NOT</u> logically necessary in IUT)

By contrast,

$$\begin{matrix} G_{k^\flat} & \cong & G_k \\ \curvearrowright & & \curvearrowright \\ \mathbb{C}^\flat_k & \ncong & \mathbb{C}_k \end{matrix} \quad , \quad \begin{pmatrix} G_{\dagger k} \\ \curvearrowright \\ \mathbb{C}_{\dagger k} \end{pmatrix} \ncong \begin{pmatrix} G_{\ddagger k} \\ \curvearrowright \\ \mathbb{C}_{\ddagger k} \end{pmatrix}$$

as topological fields
w/ cont. actions of
a top. gp

regardless of whether or not
one imposes a compatibility
conditon

(a counterexample of the analogue of Neukirch-Uchida for NF)

Riemann surfaces
w/ the same underlying
topological space

$X_1 \ncong X_2$

↑ as Riemann surfaces
regardless of whether or not
one imposes a compatibility
condition

This situation is useless in IUT from the point of view of constructing any sort of theory that is structurally analoguous to IUT.

Differences

③ char $k^\flat > 0 \Rightarrow \nexists$ p-adic log $\rightsquigarrow \nexists$ log-link

④ char $k^\flat > 0 \Rightarrow \nexists$ pro-p part of cyclotomic rigidity isom's $\left(\substack{\text{theta, LCFT,}\\ \text{global}}\right)$

$\nwarrow$ fundamental role in IUT

cannot switch

⑤ $\begin{pmatrix} G_{\dagger_k} \\ \curvearrowright \\ \mathbb{C}_{\dagger_k} \end{pmatrix} \ncong \begin{pmatrix} G_{\ddagger_k} \\ \curvearrowright \\ \mathbb{C}_{\ddagger_k} \end{pmatrix} \Rightarrow \nexists$ multiradiality of representation of the Θ -pilot

no symmetry $\nwarrow$ crucial role in IUT

⑥ k : <u>NOT</u> locally compact $\Rightarrow \nexists$ compatibility of cycl. rig. (theta, LCFT)

$\uparrow$ w/ profinite topology

perfectoid field $\nwarrow$ fundamental role in IUT

⑦ k : <u>NOT</u> locally compact $\Rightarrow \nexists$ log-volume

$\uparrow$

perfectoid field $\nwarrow$ fundamental role in IUT