FAQ on Inter-universal Teichmüller Theory

By Go Yamashita, RIMS, Kyoto University.

September 2018

Q1. Which papers do you recommend for people who wish to begin their study of inter-universal Teichmüller theory?

A1. I recommend

- [AbsTopIII] to learn *the mono-anabelian reconstruction algorithms* (which are important ingredients in inter-universal Teichmüller theory) arising from *the Belyi cuspidalisation* developed in [AbsTopII], and
- [EtTh] to learn *the theory of mono-theta environments* (which is also an important ingredient in inter-universal Teichmüller theory), especially the three important rigidities that they satisfy. The algorithm underlying constant multiple rigidity uses *the elliptic cuspidalisation* developed in [AbsTopII].

Both the Belyi cuspidalisation and the elliptic cuspidalisation mentioned above, which are absolute anabelian results, are obtained as consequences of the relative anabelian result of [pGC], i.e., *the relative Grothendieck Conjecture*.

It seems to me worthwhile to remark that it was very important, from the point of view of the application to inter-universal Teichmüller theory, that the relative Grothendieck Conjecture was proved over *local* fields[1] in [pGC], in spite of the implicit expectation at that time that, by analogy with the Tate conjecture, the relative Grothendieck Conjecture should hold only over *global* fields.

Q2. In inter-universal Teichmüller theory, one often considers the full poly-isomorphism[2] between distinct copies of mathematical objects such as, for example, copies ${}^{\dagger}G$ and ${}^{\ddagger}G$ of the absolute Galois group G_K of a local field K. Considering distinct copies seems to be useless and unnecessarily complicated. Why can't one simply replace ${}^{\dagger}G$ and ${}^{\ddagger}G$ by G_K and replace the full poly-isomorphism between ${}^{\dagger}G$ and ${}^{\ddagger}G$ by the identity morphism of G_K?

A2. Mathematical objects such as G_K are *a priori* equipped with the "histories of operations" that appear in their constructions. Thus, for example, G_K is equipped with its structure as a "Galois group", i.e., its structure as a group of automorphisms of some field. By taking an isomorph[3] ${}^{\dagger}G$ of G_K *as an abstract topological group*, we forget the "history" of G_K, i.e., ${}^{\dagger}G$ is no longer equipped with a "Galois group" structure (that is to say, structure as a group of automorphisms of some field). In particular, the *only properties* that ${}^{\dagger}G$ *shares* with G_K are properties that arise from the abstract topological group structure of these topological groups.

> *It is only after forgetting such "histories of operations" that appear in the constructions of these objects and considering them as abstract topological groups, abstract topological monoids etc., that one may obtain a kind of* ***symmetry*** *(i.e., a kind of "switchability" or "exchangeability" between two distinct copies of objects)* ***while maintaining the relationship between such "switchable" objects***

[1]More generally, he proved it over sub-p-adic fields.
[2]i.e., the set of all isomorphisms.
[3]i.e., isomorphic object.

> ***and "a priori non-switchable" objects***[4]. *This forms the essential content of the multiradiality of the algorithms that play a central role in the main results of inter-universal Teichmüller theory.*

Q3. Suppose that, in the situation of (Q2), we proceed (as suggested in (Q2)) to replace ${}^{\dagger}G$ and ${}^{\ddagger}G$ by G_K and replace the full poly-isomorphism between ${}^{\dagger}G$ and ${}^{\ddagger}G$ by the identity morphism of G_K, but that we also consent to treat G_K as an abstract topological group when required. Would inter-universal Teichmüller theory still function under these conditions?

A3. Once we forget their "histories of operations", we should never recall back them. (In fact, when we take an isomorph ${}^{\dagger}G$ of G_K as a topological group, ${}^{\dagger}G$ can no longer recall back the "history of operations".) The identity morphism from ${}^{\dagger}G$ to ${}^{\ddagger}G$ does not make sense, since we need their "histories of operations" as G_K to define the identity morphism, and ${}^{\dagger}G$ and ${}^{\ddagger}G$ are different objects! If we sometimes forget "histories of operations", and sometimes recall them back, then it destroys "the symmetry" mentioned in (A3), especially, if we take the identity morphism by recalling back their "histories of operations", then the final algorithm does not work.

Q4. Suppose that, in the situation of (Q2), we consider the full poly-isomorphism between ${}^{\dagger}G$ and ${}^{\ddagger}G$, but we replace (as suggested in (Q2)) ${}^{\dagger}G$ and ${}^{\ddagger}G$ by G_K, and we consent to treat G_K as an abstract topological group and to completely forget its "history of operations". Would inter-universal Teichmüller theory still function under these conditions?

A4. In theory, the proof still works under these conditions, so long as one is careful not to confuse oneself by the use of the *same notation* G_K *to denote different topological groups, which, moreover, are not to be regarded, a priori, as Galois groups of fields* (i.e., despite the fact that the notation "G_K" suggests that any topological group denoted by "G_K" is to be regarded as the absolute Galois group of some field K). If one takes sufficient care when using the notation "G_K", then *working with* G_K *is essentially equivalent to working with isomorphs* ${}^{\dagger}G$ *and* ${}^{\ddagger}G$ *of* G_K *that are treated as abstract topological groups.* In light of this state of affairs, it seems much less confusing to work with such isomorphs ${}^{\dagger}G$ and ${}^{\ddagger}G$ (treated as abstract topological groups) from the beginning. Moreover, it is not clear that there is anything to be gained by using the same notation "G_K" to denote different topological groups, which are not, *a priori*, supposed to be regarded as Galois groups of fields. Thus, in summary, the use of the notation "G_K" only creates unnecessary confusion. *It seems much more natural to use different notation for different objects, and to avoid notation for abstract topological groups that erroneously suggests that those groups are to be regarded as Galois groups of fields.*

This sort of phenomenon may be understood by considering a *"toy model"*, as follows[5]. Consider the following dialogue between a *student* and a *professor* concerning n-dimensional vector spaces over $\mathbb{R}$, for fixed $n > 0$:

> *Student*: People often speak of n-dimensional vector spaces V and W over $\mathbb{R}$, but this seems to be useless and unnecessarily complicated. Why can't one always just work with $\mathbb{R}^n$?
>
> *Professor*: Using the notation "$\mathbb{R}^n$" to denote *arbitrary* n-dimensional vector spaces over $\mathbb{R}$ would result in a substantial amount of confusion. For instance, it might lead one to conclude (erroneously) that there exists a canonical isomorphism between V and its dual V^*, or that it makes sense to consider composites

[4] More precisely, Frobenius-like objects.

[5] This toy model arose in discussions with Akio Tamagawa.

of the form "$f \circ f$", where f is an $\mathbb{R}$-linear map $f : V \to W$ between distinct V and W.

Student: I still don't understand. It seems that there should be no problem with using the notation "$\mathbb{R}^n$" to denote arbitrary n-dimensional vector spaces over $\mathbb{R}$, so long as one is careful not to confuse different "$\mathbb{R}^n$'s".

Professor: To regard some abstract n-dimensional vector space over $\mathbb{R}$ as "$\mathbb{R}^n$" is essentially equivalent to considering a basis of the vector space. The notion of a 'vector space with a basis' is more complicated than the notion of 'vector space'. In particular, one can often simplify arguments and avoid confusion by omitting the specification of a particular basis and working with abstract vector spaces "V" or "W".

Student: Working with such "V" and "W" seems to be useless and unnecessarily complicated. They seem to be simply alternative names for the same vector space $\mathbb{R}^n$! Moreover, it seems that we should be able to regard any isomorphism between V and W as the identity morphism, simply by choosing suitable bases for V and W.

Professor: If you take care not to get confused[6] by the notation after choosing bases for the various vector spaces that arise in an argument, then it is true that, in theory, one may always work with vector spaces equipped with a basis. On the other hand, doing this can often result in confusion or in making arguments unnecessarily complicated. Moreover, it is important to remember that the notion of the 'identity morphism' is only defined when the domain and codomain of the morphism are *equal*, i.e., it does not make sense to speak of the 'identity morphism' between *distinct* vector spaces V and W.

There are many other toy models that one may consider. For example, a manifold is defined by patching together open sets of Euclidean space. If one identifies such open subsets with one another, then the theory of manifolds collapses immediately[7]. Another example may be seen in the theory of group representations, where, if, for instance, a representation of a group contains multiple copies of an irreducible representation of the group, then it is of crucial importance to distinguish the notion of *equality* of subrepresentations from the notion of *isomorphism of representations*. Yet another example may be seen in the elementary theory of field extensions, where it is of crucial importance to distinguish the notion of *equality* of subextensions of a field from the notion of *isomorphism of fields*.

Q5. Can you give some sort of toy model to illustrate the phenomenon of changing universes?

A5. Perhaps the most fundamental example of such a toy model is the phenomenon of changes of coordinates in Euclidean space. Please see (A4) of [FAQ] for other examples. The philosophy of inter-universality also plays an important role in combinatorial anabelian geometry.

Q6. All of the constructions in the series of papers on inter-universal Teichmüller theory seem to be trivial and devoid of any nontrivial content. How can one deduce a nontrivial consequence

[6]For instance, one must take care not to confuse the approach proposed in (Q4) with the approaches proposed in (Q2) and (Q3).

[7]This toy model is compatible with the point of view mentioned in (A4) of [FAQ] to the effect that "changing coordinates is a toy model of changing universes". See also (A3) of [FAQ].

from such trivial constructions?

<u>A6.</u> Let me cite the following passage from the Introduction of [Y]:

> ... Indeed, once the reader admits the main results of the preparatory papers (especially [AbsTopIII], [EtTh]), the numerous constructions in the series of papers [IUTchI], [IUTchII], [IUTchIII], [IUTchIV] on inter-universal Teichmüller theory are likely to strike the reader as being somewhat trivial. On the other hand, the way in which the main results of the preparatory papers are *interpreted* and *combined* in order to perform these numerous constructions is *highly nontrivial* and based on very delicate considerations (cf. Remark 9.6.2 and Remark 12.8.1) concerning, for instance, the notions of multiradiality and uniradiality (cf. Section 11.1). Moreover, when taken together, these numerous trivial constructions, whose exposition occupies literally hundreds of pages, allow one to conclude a *highly nontrivial consequence* (i.e., the desired Diophantine inequality) practically effortlessly! Again, from the point of view of the author, the point of the proof seems to lie in the *establishment of a suitable framework* in which one may deform the structure of a number field by abandoning the framework of conventional scheme theory and working instead in the framework furnished by inter-universal Teichmüller theory (cf. also Remark 1.15.3).
>
> In fact, the main results of the preparatory papers [AbsTopIII], [EtTh], etc. are also obtained, to a substantial degree, as consequences of numerous constructions that are not so difficult. On the other hand, the *discovery of the ideas and insights* that underlie these constructions may be regarded as highly nontrivial in content. Examples of such ideas and insights include the "hidden endomorphisms" that play a central role in the mono-anabelian reconstruction algorithms of Section 3.2, the notions of arithmetically holomorphic structure and mono-analytic structure (cf. Section 3.5), and the distinction between étale-like and Frobenius-like objects (cf. Section 4.3). Thus, in summary, it seems to the author that, if one *ignores* the delicate considerations that occur in the course of interpreting and combining the main results of the preparatory papers, together with the ideas and insights that underlie the theory of these preparatory papers, then, in some sense, the only nontrivial mathematical ingredient in inter-universal Teichmüller theory is the classical result [pGC], which was already known in the last century!

<u>Q7.</u> Can you cite any nontrivial partial results that occur as consequences of intermediate steps in the proofs of the main results of inter-universal Teichmüller theory?

<u>A7.</u> The constructions in the first three papers of the series of papers on inter-universal Teichmüller theory, taken together, constitute a *single algorithm*, namely, the final multiradial algorithm of Theorem 3.11 in the third paper. In particular, the structure of inter-universal Teichmüller theory is such that it is only meaningful to search for "nontrivial partial results" in the preparatory papers. For example, as mentioned before, the mono-anabelian reconstruction algorithms arising from the Belyi cuspidalisation are very important results and have already yielded further results (that are not directly related to inter-universal Teichmüller theory).

<u>Q8.</u> Can you give examples of further research or results that arose from inter-universal Teichmüller theory?

<u>A8.</u> I myself am interested in pursuing the possibility of applying various ideas that appear in inter-universal Teichmüller theory to the study of the Riemann zeta function. At the present

time, I have obtained some interesting observations, but no substantive results. Hoshi is studying an application of inter-universal Teichmüller theory to the birational section conjecture in birational anabelian geometry, while Porowski and Minamide are studying numerical improvements of certain height inequalities in inter-universal Teichmüller theory. I also hear that Dimitrov is studying the possibility of applying inter-universal Teichmüller theory to the study of Siegel-zeroes.

References

[pGC] S. Mochizuki, *The Local Pro-p Anabelian Geometry of Curves.* Invent. Math. **138** (1999), p.319–423.

[EtTh] S. Mochizuki, *The Étale Theta Function and its Frobenioid-theoretic Manifestations.* Publ. Res. Inst. Math. Sci. **45** (2009), p.227–349.

[AbsTopII] S. Mochizuki, *Topics in Absolute Anabelian Geometry II: Decomposition Groups.* J. Math. Sci. Univ. Tokyo **20** (2013), p.171–269.

[AbsTopIII] S. Mochizuki, *Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms.* J. Math. Sci. Univ. Tokyo **22** (2015), p.939–1156.

[FAQ] G. Yamashita, *FAQ on "Inter-Universality"*, an informal note available at `http://www.kurims.kyoto-u.ac.jp/~motizuki/research-english.html`

[Y] G. Yamashita, *A proof of the abc conjecture after Mochizuki*, preprint available at `http://www.kurims.kyoto-u.ac.jp/~gokun/myworks.html`