モチーフの勉強会第2回

English Page
モチーフの勉強会第1回(2005年12月19ー22日)

日時: 2006年9月25日(月)ー29日(金)
場所: 東京大学大学院数理科学研究科、大講義室

講演者:
朝倉政典(九大数理), 萩原啓(東大数理), 橋本義武(大阪市大),
池田京司(阪大理学部), 木村健一郎(筑波大), 木村俊一(広大理学部),
蔵野和彦(明治大理工), 南範彦(名工大), 望月哲史(東大数理),
Ambrus Pal(IHES), Joel Riou(Jussieu), 佐藤周友(名大多元),
寺杣友秀(東大数理), 柳田伸顕(茨城大), 山下剛(RIMS), 安田健彦(RIMS)
(アルファベット順)

なお、本集会は日本学術振興会先端研究拠点事業「数論幾何・モチーフ理論・ガロア理論の新展開と、その社会的実用」(コーディネーター 松本眞)及び科学研究費 基盤研究(B)一般 課題番号17340008「数論的多様体のp進的手法による研究」(代表 都築暢夫)から補助を受けております。

日程:
25日(月)
10:00-11:00 山下剛(RIMS) Review and Overview
11:15-12:15 佐藤周友(名大多元) Motivic cohomology groups with finite coefficients 1
13:30-14:30 橋本義武(大阪市大) 代数トポロジー入門1 Eilenberg-MacLane空間, Steenrod代数, Adamsスペクトル系列
14:45-15:45 朝倉政典(九大数理) Introduction to regulator
16:00-17:00 Ambrus Pal(IHES) The torsion of Drinfeld modules of rank two

26日(火)
10:00-11:00 佐藤周友(名大多元) Motivic cohomology groups with finite coefficients 2
11:15-12:15 望月哲史(東大数理) Reading Morel-Voevodsky 1
13:30-14:30 橋本義武(大阪市大) 代数トポロジー入門2 一般コホモロジー, スペクトラム, 無限ループ空間
14:45-15:45 南範彦(名工大) Symmetric Spectra (following Hovey-Shipley-Smith)
16:00-17:00 Joel Riou(Jussieu) Spanier-Whitehead duality in algebraic geometry

27日(水)
10:00-11:00 望月哲史(東大数理) Reading Morel-Voevodsky 2
11:15-12:15 山下剛(RIMS) Reduced Power Operations 1
13:30-14:30 柳田伸顕(茨城大) Milnor's primitive operations Q_i
14:45-15:45 安田健彦(RIMS) モティヴィク積分とホモロジカルMcKay対応
16:00-17:00 Joel Riou(Jussieu) Operation on algebraic K-theory and regulators via the homotopy theory of schemes
懇親会

28日(木)
10:00-11:00 山下剛(RIMS) Reduced Power Operations 2
11:15-12:15 萩原啓(東大数理) Milnor-Bloch-Kato予想の周辺
13:30-14:30 池田京司(阪大理学部) Algebraic cycles and differential forms
14:45-15:45 木村俊一 Introduction to finite dimensional motives
16:00-17:00 木村健一郎 Nori's category of motives

29日(金)
10:00-11:00 萩原啓(東大数理) Milnor予想の証明 (1)

11:15-12:15 萩原啓(東大数理) Milnor予想の証明 (2)
13:30-14:30 蔵野和彦(明治大理工) SerreによるIntersection multilicityの代数的な記述について
14:45-15:45 寺杣友秀(東大数理) Bar construction and its application
16:00-17:00 TBA

この勉強会では、午前中はMilnor予想の証明に向けて、山下剛氏、佐藤周友氏、望月哲史氏、萩原啓氏による連続講義が行われます。午後の講演は、原則として専門家による初心者のための入門的な話となります。Joe"l Riou氏の二つの講演はResearch talkですが、ともに最初の半分は初心者のための入門的な話となるようにお願いしております。また、トポロジーの専門家である橋本義武氏と南範彦氏によるモチーフのためのホモトピー理論入門の講演をお願いしております。皆様、ふるってご参加いただきますよう、お願い申し上げます。

随時更新中

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戻る

山下剛

- morning session

<u>goal</u>:
Voevodsky's proof of
Milnor Conj.
Sato, Mochizuki, Yamashita
Hagihara

- afternoon session
  - Review from (classical) algebraic topology
  - topics

Review and Overview

Voevodsky's idea

transport concepts and techniques from alg. top.

- spectrum
- coh. operation
- Margolis coh.

last year : motivic (co)homology

this year : motivic homotopy

homological alg. $\rightsquigarrow$ homotopical alg.
abel. cat $\rightsquigarrow$ model cat
derived cat $\rightsquigarrow$ homotopy cat
inj. obj $\rightsquigarrow$ fibrant obj
$[+1]$ $\rightsquigarrow$ $S_s^1 \wedge$
sphere

Review k: perfect field

Sm/k cat. of smooth sch. /k

$$\begin{tikzcd} DM^{eff}_{gm}(k) \arrow[r, phantom, "\subset"] \arrow[d, squiggly, "\text{Tate obj } \mathbb{Z}(1) \text{ invert}"'] & DM^{eff}_-(k) \\ DM_{gm}(k) & \end{tikzcd}$$

← last year

↑ tensor triag. cat

↑ rigid tensor triag. cat

$DM^{eff}_-(k)$ = the derived cat of bounded above complex of Nisnevich sheaves with transfers with homotopy invariant coh sheaves

Def. $\{X_i \xrightarrow{f_i} X\}_{i\in I} \subset Sm/k$: Nisnevich cov

$\overset{\text{def}}{\iff}$
- · étale cov.
- · $X \ni x$, $\exists i \in I$, $\exists y \in X_i$ s.t $f_i(y) = x$, $k(x) \xrightarrow{\sim} k(y)$

↑ residue field □

coarse fine

Zar < Nis < ét

$\forall x \in X \in Sm/k$

Nis. loc. $X \simeq \mathbb{A}^n$

⇝ Nis. top, Nis. sheaf

Def. F : presheaf of abel. gp.

homotopy inv $\overset{\mathrm{def}}{\Longleftrightarrow}$ $\forall X \in Sm/_k$ $\quad F(X) \xrightarrow[pr]{\sim} F(X \times \mathbb{A}^1)$

$\square$

$SmCor(k)$ $\quad$ obj. $X \in Sm/_k$

reduced irr. closed subsch.

Morph: $$\mathrm{Hom}_{SmCor(k)}(X,Y) := \left\langle Z \;\middle|\; \begin{tikzcd} Z \arrow[r, phantom, "\subset"] \arrow[dr, dashed] & X \times Y \arrow[d] \\ & X \end{tikzcd} \right\rangle_{\mathbb{Z}}$$

finite surj over a component of X

composition

intersect properly

$$p_{13*}\left(p_{12}^{*}(Z) \cap p_{23}^{*}(Z')\right)$$

$$\begin{tikzcd} Sm/_k \arrow[r] & SmCor(k) \\ X \arrow[r, mapsto] & X \\ f \arrow[r, mapsto] & \Gamma_f \end{tikzcd}$$ graph

Def. F : presheaf with transfers

$\overset{\mathrm{def}}{\Longleftrightarrow}$ $F : SmCor(k) \longrightarrow (Ab)$

$F \begin{pmatrix} Nis \\ Zar \\ ét \\ cdh \end{pmatrix}$ sheaf with trans $\Longleftrightarrow$ F : presheaf with trans $+ \; F|_{Sm/k} : \begin{pmatrix} Nis \\ \vdots \end{pmatrix}$ sheaf

motivic cpx. $\in DM^{eff}_{-}(k)$

$\mathbb{Z}(n)$ $\quad (n \geq 0)$

$M(X)$ $\quad (X \in Sm/_k)$

Def. $\mathbb{Z}_{tr}(X)(-) := \mathrm{Hom}_{SmCor(k)}(-, X)$ presheaf with trans

$\rightsquigarrow \begin{array}{c} \text{Zar} \\ \text{Nis} \end{array}$ sheaf □

$$\Delta^n = \mathrm{Spec}\, k[T_0, \dots, T_n]/\left(\sum_i^n T_i - 1\right)$$

Def. (Suslin complex)

F presheaf $\quad C_n(F)(-) = F(\Delta^n \times -)$

$\rightsquigarrow C_*(F)$ associated complex to the simplicial presheaf $n \longmapsto C_n(F)$

Fact $C_*(F)$: homologues are homotopy inv. presheaf

Def. $X \in Sm/k$

$M(X) \in DM_-^{eff}(k)$ motive of X

$\parallel$

$C_*(\mathbb{Z}_{tr}(X)) \longleftrightarrow$ alg analogue of singular chain complex

$C^m(F) = C_{-m}(F)$

Def. $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}) := \mathrm{Cok}\left(\bigoplus_{i=0}^{n-1} \mathbb{Z}_{tr}(\mathbb{G}_m^{\times(n-1)}) \xrightarrow[\uparrow]{} \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})\right)$

$n \geq 0$ direct summand

Def. $\mathbb{Z}(n) := C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))[-n] \in DM_-^{eff}(k)$

motivic complex

$$\left(\begin{array}{c} H^p_{Nis}(X, \mathbb{Z}(q)) = \mathrm{Hom}_{DM_-^{eff}(k)}(M(X), \mathbb{Z}(q)) \\ \wr\! \wr \\ H^p_{Zar}(X, \mathbb{Z}(q)) \end{array}\right)$$

properties

① $\mathbb{Z}(0) = \mathbb{Z}$ ② $\mathbb{Z}(1) = \mathcal{O}_X^*[-1]$

Milnor's K-grp

③ F f.g. field$/k$ $\Rightarrow H^n(\operatorname{Spec} F, \mathbb{Z}(n)) \simeq K_n^M(F)$

④ $X \in Sm/k \Rightarrow H^{2n}_{\underset{(Nis)}{Zar}}(X, \mathbb{Z}(n)) \simeq CH^n(X)$
Chow grp.

$(H^p_{Zar}(X, \mathbb{Z}(q)) \simeq CH^q(X, 2q-p))$

⑤ $X \in Sm/k$

$\exists$ spec seq Atiyah-Hirzebruch

$$E_2^{p,q} = H^{p-q}_{\underset{(Nis)}{Zar}}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

(motivic homotopy)

- Bloch-Lichtenbaum, Friedlander-Suslin
- Levin-Voevodsky
- Grayson-Suslin

<u>Beilinson Suslin vanishing conjecture</u>

$X \in Sm/k$ $n \geq 0$

$H^i_{Zar}(X, \mathbb{Z}(n)) = 0$ for $i > 0$

<u>Beilinson Lichtenbaum Conj</u> $\ell \neq$ char k

$\mathbb{Z}/\ell(q) := \mathbb{Z}(q) \otimes \mathbb{Z}/\ell$

$BL(q, \mathbb{Z}/\ell)$ $\alpha : (Sm/k)_{et} \to (Sm/k)_{Zar}$

$$\mathbb{Z}/\ell(q) \xrightarrow{qis} \tau_{\leq q} R\alpha_* \overset{\mu_\ell^{\otimes q}}{\overset{\parallel}{\overline{\alpha^* \mathbb{Z}/\ell(q)}}}$$

BL$(q, \mathbb{Z}/\ell)$ → $\ell = 2$ Milnor conj

→ Bloch-Kato conj — coincides with the classical map by symbol

F: f.g. field /k

$$K^M_q(F)/\ell \xrightarrow{\sim} H^q_{et}(\mathrm{Spec}\, F, \mu_\ell^{\otimes q})$$

(only surj ⇝ weak Bloch-Kato)

weak BK(q, ℓ)

⇒ vanishing of Bochstein BV(q, ℓ)

F: f.g. /k $\forall j$ $\beta_{q,j}: H^q(F, \mu_\ell^{\otimes q}) \to H^{q+1}(F, \mu_{\ell^i}^{\otimes q})$ trivial

BL$(q, \mathbb{Z}_{(\ell)})$: $\mathbb{Z}_{(\ell)}(q) := \mathbb{Z}(q) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)}$

$$\mathbb{Z}_{(\ell)}(q) \xrightarrow[qis]{} \tau_{\le q+1} R\alpha_* \alpha^* \mathbb{Z}_{(\ell)}(q)$$

⇒ generalized Hilbert 90 H90(q, ℓ)

$\forall$ F: f.g. /k ⇒ $H^{q+1}_{et}(F, \mathbb{Z}_{(\ell)}(q)) = 0$

$$\begin{tikzcd}
\overset{\forall q \le n}{BL(q, \mathbb{Z}/\ell)} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow, "\forall q \le n"] & \overset{\forall q \le n}{BL(q, \mathbb{Z}_{(\ell)})} \arrow[d, Rightarrow, "\forall q \le n"] \\
BK(q, \ell) \arrow[d, Rightarrow, "\forall q \le n"] & H90(q, \ell) \arrow[ldd, Rightarrow] \\
wBK(q, \ell) \arrow[d, Rightarrow, "\forall q \le n"] & \\
BV(q, \ell) \arrow[uuu, Rightarrow, bend left, "\text{Suslin-Voevodsky, Geisser Levine}"] &
\end{tikzcd}$$

佐藤周友

Motivic cohomology with finite coefficents

k. perfect field

<u>Aim</u> (1) Define $\mathbb{Z}(q)$ $(q \geq 0)$ in $D^-(Shv_{zar}(Sm/k))$

(2) Study the canonical adjunction map

$$\mathbb{Z}(q) \longrightarrow R\varepsilon_* \varepsilon^* \mathbb{Z}(q) \qquad \begin{tikzcd}(Sm/k)_{et} \arrow[r] \arrow[rr, bend right, "\varepsilon"'] & (Sm/k)_{Nis} \arrow[r, "\beta"] & (Sm/k)_{zar}\end{tikzcd}$$

using Voevodsky's framework

Background.

(1) $\mathbb{Z}(q)$ is a strong candidate for $\Gamma(q)$ conjectured by Beilinson - Lichtenbaum

(2) comparison between

$$H^i_{zar}(X, \mathbb{Z}(q)) \dashrightarrow H^i_{et}(X, \varepsilon^* \mathbb{Z}(q))$$

(should be the same for any $i \leq q+1$)

§1. $\mathbb{Z}(q)$

Presheaf with trans. $F : (SmCor(k))^{op} \longrightarrow (Ab)$

$$\begin{tikzcd}\text{Presheaf with trans.} \arrow[d, squiggly] \arrow[dd, squiggly, bend right] & \\ \text{Zariski sheafification } F_{zar} & \\ \text{Nisnevich sheafification } F_{Nis} & \end{tikzcd}$$

(prserves transfer str.)

<u>Thm</u> (Voevodsky) $F : PST(k)$

(1) $F_{zar} = F_{Nis}$ as presheaves over (Sm/k)

If F [illegible] inv.

(2) F_{Nis} has canonical str. of $PST(k)$

(3) $X \in Sm/k$ $\quad H^i_{zar}(X, F_{zar}) \xrightarrow{\sim} H^i_{Nis}(X, F_{Nis})$ for $\forall i \geq 0$

$\square$

<u>Def.</u> $\mathbb{Z}(q)^{Nis} \in DM^{eff}_{-}(k)$

$\mathbb{Z}(0)^{Nis} = \mathbb{Z} \qquad \mathbb{Z}(1)^{Nis} := \mathcal{O}^\times[-1] \; (\simeq C^*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge 1})[-1]))$

$q \geq 2 \quad \mathbb{Z}(q)^{Nis} = \underbrace{\mathbb{Z}(1)^{Nis} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathbb{Z}(1)^{Nis}}_{q\text{-times}} \; (\simeq C^*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q]))$

$\mathbb{Z}(q) :=$ image of $\mathbb{Z}(q)^{Nis}$ via $R\beta_* \circ$ (forgetful)

$\cap$

$D^-(Shv_{zar}(Sm/k))$

<u>Rem</u> $\mathbb{Z}(q) \underset{\substack{\uparrow \\ \text{Voevodsky}}}{\xrightarrow{qis}}$ Zariski sheafification of Bloch's cycle cpx.

<u>Thm A</u> (Suslin-Voevodsky / Geisser-Levine ×2 / Voevodsky)

Assume $wBK(k, q, l)$ for all prime $l \neq ch(k)$

Then $\begin{cases} R^{q+1} \varepsilon_* \varepsilon^* \mathbb{Z}(q) = 0 \\ \mathbb{Z}(q) \underset{\alpha^q}{\xrightarrow{\sim}} \tau_{\leq q} R\varepsilon_* \varepsilon^* \mathbb{Z}(q) \end{cases}$

in $D^-(Shv_{zar}(Sm/k))$

Thm A' (essential part of Thm A)

Assume wBK(k, q, l) for all $l \neq ch(k)$

Then $\mathbb{Z}(q) \otimes^{\mathbb{L}} \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \tau_{\leq q} R\varepsilon_* \varepsilon^* (\mathbb{Z}(q) \otimes^{\mathbb{L}} \mathbb{Q}/\mathbb{Z})$

in $D^-(Shv_{Zar}(Sm/k))$

Rem on Thm A'

$ch(k) = p > 0 \Rightarrow$ p-primary part : Geisser-Levine (unconditional

prime to ch(k) part : S-V : assuming res. of sing.
and wBK(k, m, l) for $m \leq q$

G-L : assuming only wBK(k, q, l)

§2 Galois symbol and wBK(k, q, l)

< Milnor K-grp >

k : field $q \geq 0$

$$K_q^M(F) = \begin{cases} \mathbb{Z} & (q = 0) \\ F^\times & (q = 1) \\ \underbrace{F^\times \otimes \cdots \otimes F^\times}_{q\text{-times}} / I \end{cases}$$

I = subgp of $(F^\times)^{\otimes q}$ generated by elements of the form
$x_1 \otimes \cdots \otimes x_q$ $x_i \in F^\times$
$x_i + x_j = 1$ for some $i \neq j$

< Galois symbol > $n \in \mathbb{N}$, $ch(k) \nmid n$

$\mu_n :=$ group of n-th roots of 1 in $(F^{sep})^\times$

$\circlearrowleft$

$G_F = Gal(F^{sep}/F)$

$$0 \to \mu_n \to (F^{sep})^\times \to (F^{sep})^\times \to 0$$

(exact seq.)

$$\rightsquigarrow \quad F^\times \xrightarrow{d_{F,n}} H^1_{Gal}(G_F, \mu_n) = H^1(F, \mu_n)$$

$$(F^\times)^{\otimes q} \overset{(d_{n,F})^{\otimes q}}{\dashrightarrow} H^1(F, \mu_n)^{\otimes q} \xrightarrow{\text{cup prod.}} H^q(F, \mu_n^{\otimes q})$$

Lem (Tate) The above map factors through $K_q^M(F)$

(∵) essential case : $q = 2$ $\quad \zeta_\ell \in F$

Have to show. $\forall a \in F^\times \setminus \{1\}$ $\quad d_{\ell^r}(a) \cup d_{\ell^r}(1-a) = 0$

in $H^2(F, \mu_{\ell^r}^{\otimes 2})$

We show : For $\forall r' \leq r$, $\exists A \in H^2(F, \mu_{\ell^r}^{\otimes 2})$

s.t. $d_{\ell^r}(a) \cup d_{\ell^r}(1-a) = \ell^{r'} A$

Induction on $r' \geqslant 0$

$a \in (F^\times)^\ell \Rightarrow \exists b \in F^\times$ s.t. $a = b^\ell$

$$d_{\ell^r}(a) \cup d_{\ell^r}(1-a)$$
$$= \ell \cdot \sum_{i=0}^{\ell-1} d_{\ell^r}(\zeta_\ell^i b) \cup d_{\ell^r}(1 - \zeta_\ell^i b)$$

$a \notin (F^\times)^\ell \Rightarrow$ Put $E = F(\sqrt[\ell]{a})$, $N_{E/F}(1 - \sqrt[\ell]{a}) = 1 - a$

$$d_{\ell^r}(a) \cup d_{\ell^r}(1-a) = d_{\ell^r}(a) \cup \mathrm{Cor}_{F/F}(d_{\ell^r,E}(1 - \sqrt[\ell]{a})$$
$$= \ell \cdot \mathrm{Cor}_{E/F}(d_{\ell^r,E}(\sqrt[\ell]{a}) \cup d_{\ell^r,E}(1 - \sqrt[\ell]{a}))$$

▨

$\chi^q_{n,F} : K^M_q(F)/n \longrightarrow H^q(F, \mu_n^{\otimes q})$ (Galois symbol)

Know to be bij in the following cases :

- $q = 0$ (clear)
- $q = 1$ (Hilbert 90, $H^1(F, (F^{sep})^\times) = 0$)
- $q = 2$ (Merkurjev - Suslin. 1983)
- $n = 2^r$, $ch(F) \neq 2$ ($q = 3$: M-S, $q \geq 4$ Voevodsky)

<u>Conj</u> (B-K) $\chi^q_{n,F}$ is always bijective

BK(k, q, ℓ) = $\chi^q_{\ell,F}$ is surjective for all finitely generated fields F/k

§3 Relation between $\chi^q_{n,F}$ and α^q

<u>Lem</u>: $\mathbb{Z}(q)^{ét} := \varepsilon^* \mathbb{Z}(q) \in D^-(Shv_{ét}(Sm/k))$

$ch(k) \nmid n \Rightarrow \mathbb{Z}(q)^{ét} \overset{L}{\otimes} \mathbb{Z}/n \simeq \mu_n^{\otimes q}$

(.) $\mathbb{Z}(1)^{ét} \simeq \mathcal{O}^\times[-1]$

$0 \to \mu_n \to \mathcal{O}^\times \xrightarrow{\times n} \mathcal{O}^\times \to 0$ exact in the étale top.

$q \geq 2$ $DM^{eff}_{-,ét}(k, \mathbb{Z}[1/p])$ $p = ch(k) > 0$

$$\mathbb{Z}(q)^{ét} \overset{L}{\otimes} \mathbb{Z}/n = \mathbb{Z}(1)^{ét} \overset{L}{\otimes} \cdots \overset{L}{\otimes} \underbrace{\mathbb{Z}(1)^{ét} \overset{L}{\otimes} \mathbb{Z}/n}_{\mu_n},$$

$$= \mu_n^{\otimes q}$$ ▨

<u>Prop</u> $ch(k) \nmid n$ $\exists$ a commutative diagram

F/k f.g. field

induced by $F^\times = H^1_{Zar}(F, \mathbb{Z}(1))$ and prod.

adj. + lem

$$\begin{tikzcd} K^M_q(F)/n \arrow[r, "\chi^q_{n,F}"] \arrow[d, "\simeq (S\text{-}V)"] & H^q_{Gal}(G_F, \mu_n^{\otimes q}) \arrow[d, equal] \\ H^q_{\mathcal{M}}(F, \mathbb{Z}(q) \overset{L}{\otimes} \mathbb{Z}/n) \arrow[r] & H^q_{ét}(\mathrm{Spec}(F), \mu_n^{\otimes q}) \end{tikzcd}$$

Rem

$$\begin{tikzcd}
CH^q(F, q) \arrow[dd, "\wr"', "\sim"] & & \\
& & K^M_q(T) \arrow[llu, "\theta"', "\sim"] \arrow[lld, "\theta", "\sim"'] \\
H^q_{zar}(F, \mathbb{Z}(q)) & &
\end{tikzcd}$$

Voevodsky

Nesterenko - Suslin

S-V

$$wBK(k, q, \ell) \underset{\text{Thm } A'}{\Longrightarrow} BK(k, q, \ell^r) \quad (\forall r > 0)$$

朝倉政典

Higher Chern maps

$$c_{ij} : K_i(X) \longrightarrow H^{2j-i}(X, \Gamma(j)) \qquad \begin{matrix} j \geq 0 \\ i \geq 1 \end{matrix}$$

$c_{3,2}$ $K_3(\mathbb{C}) \dashrightarrow \mathbb{R}(1)$ is written by D_2

Bloch Wigner function

S : a base scheme

$V \subset (Sch/S)$ full sub cat

s.t.

(i) $X \in ob(V)$ $U \underset{open}{\subset} X \Rightarrow U \in ob(V)$

(ii) $E \longrightarrow X$: vector bundle $\Rightarrow \mathbb{P}(E) \in ob(V)$

V_{ZAR} the big Zariski site

$\Gamma = \bigoplus_{j \in \mathbb{Z}} \Gamma(j)$ complex of abel sheaves on V_{ZAR}

$$H(X, \Gamma(j)) \underset{def}{=} H(X_{Zar}, \Gamma(j))$$

(i) Γ is a unitary graded commutative

$e : \mathbb{Z} \longrightarrow \Gamma(0)$ $\qquad \cup : \Gamma(j) \overset{L}{\otimes} \Gamma(j') \longrightarrow \Gamma(j+j')$

(ii) $i : X \hookrightarrow Y$ closed imm. $\Rightarrow i_* : H^*(Y, \Gamma) \longrightarrow H^*(Y, \Gamma)$

s.t. • if X is pure codim r $\quad i_* H^k(X, \Gamma(j)) \subset H^{k+2r}(Y, \Gamma(j+r))$

• $i_*(x \cdot i^* y) = i_* x \cdot y$

cup product

(iii) $\mathbb{G}_m[-1] \longrightarrow \Gamma(1)$ in $D(V_{ZAR})$

s.t.

(1) $X \overset{i}{\hookrightarrow} Y$ codim 1 Y: regular

$$\begin{tikzcd} \Rightarrow H^0(X, \Gamma(0)) \arrow[r, "i_*"] & H^2(Y, \Gamma(1)) \\ & H^1(Y, \mathbb{G}_m) = \mathrm{Pic}\, Y \arrow[u, "c"] \end{tikzcd}$$

$i_*(1_X) = c(\mathcal{O}_Y(X))$

(2) projective bundle formula

$E \longrightarrow X$ vector bundle of rk r

$\mathbb{P}(E) \xrightarrow[\pi]{} X$ $\quad \xi_E := c(\mathcal{O}_{\mathbb{P}(E)}(1))$

$$\Rightarrow \bigoplus_{k=0}^{r-1} H^{*-2k}(X, \Gamma(j-k)) \xrightarrow[\oplus \pi^*(\ )\cdot \xi_E^k]{\sim} H^*(\mathbb{P}(E), \Gamma(j))$$

Examples

(de Rham)

$V =$ (smooth scheme$/_k$) ch $k = 0$ $\Gamma(j) = \Omega^{\cdot}_{/k}$

(Betti)

$V =$ (sep. schemes of finite type $/ \mathbb{C}$)

$\Gamma(j) = Ru^{an}_* \underbrace{\mathbb{Z}(j)}_{\overset{\parallel}{(2\pi i)^j \mathbb{Z}}}$ $\quad u^{an}$ $V_{an} \to V_{ZAR}$

(Etale) $n \geq 2$ integer

$V =$ (sep. schemes $/ \mathbb{Z}[1/n]$) $\quad \Gamma(j) := Ru^{et}_* \mathbb{Z}/n(j)$

$u^{et} : V_{et} \longrightarrow V_{ZAR}$

(Deligne - Beilinson)

$V = $ (smooth schemes / $\mathbb{C}$) $\qquad \mathrm{cone}(Rj_*\mathbb{Z}_X(j) \to \Omega_{\bar{X}}^{\leq j-1}(\log D))[-1]$

$\Gamma(j) = \mathbb{Z}(j)_D \qquad \mathbb{R}(j)_D$

$$H_D(X, \mathbb{Z}(j)) = H(X_{Zar}, \mathbb{Z}(j)_D) \simeq H(\bar{X}, \ \ )$$

$\bar{X} \underset{j}{\supset} X$ smooth compactification

$D = \bar{X} - X$ NCD

$X.$ simplicial scheme (in V)

$\longrightarrow H(X., \Gamma(j))$ cohomology gp of simp. sch.

e.g. projective bundle formula O.K.

$E. \longrightarrow X.$ simp. vector bundle $rk = r$

$\Rightarrow c_j(E.) \in H^{2j}(X., \Gamma(j))$ Chern class

defined by

$$\xi_E^r + \pi^* c_1(E.) \cdot \xi_E^{r-1} + \cdots + c_r(E.) = 0 \text{ in } H^{2r}(\mathbb{P}(E.), \Gamma(r))$$

G/S group sch. $\qquad X$: sch.

$G \times_S X \longrightarrow X$ left action

$[X/G]_n = \underbrace{G \times \cdots \times G}_{n} \times X \qquad \Longrightarrow [X/G].$

$\partial. : [X/G]_n \longrightarrow [X/G]_{n-1} \qquad s_i (g_1, \cdots, g_n, x)$

$(g_1, \cdots g_n, x) \xmapsto{\partial_0} (g_2, \cdots, g_n, x) \qquad \xmapsto{s_0} (1, g_1, \cdots g_n, x)$

$\xmapsto{\partial_1} (g_1 g_2, \cdots, g_n, x)$

$\xmapsto{s_n} (g_1, \cdots g_n, 1, x)$

$\xmapsto[\partial_n]{} (g_1, \cdots, g_{n-1}, g_n x)$

$$\begin{tikzcd}
{[\mathbb{A}^n_S / GL_{n,S}]} \arrow[r] \arrow[d, equal] & {[S / GL_{n,S}]} \arrow[d, equal] \\
E_n^{univ} & B.GL_{n,S}
\end{tikzcd}$$

the univ vector bundle — the classifying space

$c_j(E_n^{univ}) \in H^{2j}(B.GL_{n,S}, \Gamma(j))$ $\qquad c_0 = 1$

$GL_{n,S} \hookrightarrow GL_{n+1,S} \qquad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$

$E_{n+1}^{univ}|_{B.GL_n} = E_n^{univ} + [\mathcal{O}]$ in $K_0(B.GL_n)$

Def. $c_j^{univ} = \varprojlim_n c_j(E_n^{univ}) \in \varprojlim_n H^{2j}(B.GL_n, \Gamma(j))$
the univ Chern class

$$\begin{tikzcd}
c_j^{univ} \arrow[dddd, bend right] & \in & \varprojlim H^{2j}(B.GL_n, \Gamma(j)) \arrow[d, equal] \\
 & & \varprojlim \mathrm{Hom}_{D(V_{ZAR})}(\mathbb{Z}\underline{B.GL_n}, \Gamma(j)[2j]) \arrow[d] \\
 & & \mathrm{Hom}_{D(V_{ZAR})}(\mathbb{Z}\underline{B.GL}, \Gamma(j)[2j]) \arrow[d] \\
 & & \bigoplus \mathrm{Hom}(H^{-i}(X, \mathbb{Z}\underline{B.GL}), H^{2j-i}(X, \Gamma(j))) \arrow[d] \\
c_{i,j} & \in & \bigoplus \mathrm{Hom}(K_i(X), H^{2j-i}(X, \Gamma(j)))
\end{tikzcd}$$

$\mathbb{Z}\underline{X}$: sheaf ass. to $U \mapsto \mathbb{Z}\mathrm{Hom}_S(U, X)$

$C_{3,2}$: $K_3(\mathbb{C}) \longrightarrow \mathbb{R}(1)$

Block group of $\underset{\text{infinite field}}{F}$

$$P(F) := \bigoplus_{x\in F^\times - \{1\}} \mathbb{Z}[x] \Big/ \left\langle [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] \right\rangle$$

scissors congruence

$$\begin{tikzcd} P(F) \arrow[r, "\lambda"] & F^\times \wedge F^\times \\ {[x]} \arrow[r, mapsto] & x \wedge (1-x) \end{tikzcd}$$

Def. $B(F) := \operatorname{Ker} \lambda$ Bloch grp.

Thm (Suslin)

$K_3^{ind}(F) = K_3(F) / K_3^M(F)$ indecomp K_3

$K_3^{ind}(F)_{\mathbb{Q}} \simeq B(F)_{\mathbb{Q}}$ $\square$

$$K_3^{ind}(F)_{\mathbb{Q}} \xrightarrow{\sim} H_3(SL_2(F), \mathbb{Q}) \longrightarrow B(F)_{\mathbb{Q}}$$

$\uparrow$ induced by Hurewicz

$\infty \in \mathbb{P}^1(F)$

$$\sum [g_0 : g_1 : g_2 : g_3] \longmapsto \sum [g_0\infty : g_1\infty : g_2\infty : g_3\infty]$$

$g_i \in SL_2(F)$

$$[a_0 : a_1 : a_2 : a_3] \underset{\text{def}}{=} \left[\frac{a_0 - a_2}{a_0 - a_3} \cdot \frac{a_1 - a_3}{a_1 - a_2}\right]$$

Thm (Bloch-Wigner)

$$D_2(z) = \arg(1-z)\log|z| - \operatorname{Im} \int_0^z \log(1-t)\,\frac{dt}{t} \quad z \in \mathbb{C} - \{0, 1\}$$

$(D_2(0) = D_2(1) = D_2(\infty) = 0)$ C^∞-func. on $\mathbb{P}^1(\mathbb{C})$

$$\begin{tikzcd}
K_3^{ind}(\mathbb{C})_{\mathbb{Q}} \arrow[rr, "\sim"] \arrow[dr, "c_{3,2}"'] & & B(\mathbb{C})_{\mathbb{Q}} \arrow[dl] & \ni [z] \arrow[dl, dashed, mapsto] \\
 & \mathbb{R}(1) \arrow[d, equal] & \ni \ iD_2(-z) & \\
 & H^1_D(\mathbb{C}, \mathbb{R}(2)) & &
\end{tikzcd}$$

proof (Sketch)

$$\begin{tikzcd}
c_2^{univ} \arrow[dddd, dashed, mapsto] & \in & H^4_D(B.GL_2(\mathbb{C}), \mathbb{R}(2)) \arrow[d] \\
 & & H^4_D(B.SL_2(\mathbb{C}), \mathbb{R}(2)) \arrow[d, "\star"] \\
 & & H^3_{cont}(SL_2(\mathbb{C}), \mathbb{R}(1)) \arrow[d, equal, "def"] \\
 & & H^3(B.SL_2(\mathbb{C}), A^0_{\mathbb{R}} \otimes \mathbb{R}(1)) \arrow[d] \\
c_{3,2} & \in & Hom(H_3(SL_2(\mathbb{C}), \mathbb{Z}), \mathbb{R}(1))
\end{tikzcd}$$

in simp. top. sp.

$\star$ is induced from

$$\begin{tikzcd}
\mathbb{R}(2) \arrow[r] & \mathcal{O} \arrow[r, dashed] & \Omega^1 \\
 & A^0_{\mathbb{R}} \otimes \mathbb{R}(1)[-1] \arrow[from=1-2, to=2-2, "Im"] &
\end{tikzcd}$$

↗ sheaf of C^∞-func

$H^3 = SL_2(\mathbb{C})/SO(2)$ hyperbolic 3-space

$$0 \rightarrow \mathbb{R} \dashrightarrow A^0(H^3) \rightarrow \underset{\text{1-form}}{A^1(H^3)} \rightarrow A^2(H^3) \rightarrow A^3(H^3) \dashrightarrow 0$$

acyclic res. of cont $SL_2(\mathbb{C})$-mod

$$H^3_{cont}(SL_2(\mathbb{C}), \mathbb{R}) = A^3(H^3)^{SL_2(\mathbb{C})} / dA^2$$

$= \mathbb{R} \cdot \omega$ $\quad \omega$: hyperbolic volume

$$\omega \longmapsto \left[[g_0 : g_1 : g_2 : g_3] \longmapsto \int_{\Delta(g_0\omega, \cdots, g_3\omega)} \omega \right]$$

$$\omega \longmapsto \left[[z] \longmapsto \int_{\Delta(z, 0, 1, \infty)} \omega \right] \underset{\exists \lambda}{=} \lambda D_2(z) \quad \text{(Lobachevsky)}$$

Ambrus Pal

F : any field $\qquad p(X) \in F[X]$

$p^{(n)} = \underbrace{p \circ \cdots \circ p}_{n\text{-times}}$ ⇐ n-th iterate

$x \in F$ a pre periodic point for p if the set

$\{ p^{(n)}(x) \in F \mid n \in \mathbb{N} \}$ is finite

Assume now that $F = \mathbb{F}_2(T)$, $p(X) = TX + gX^2 + \Delta X^4$

$g, \Delta \in \mathbb{F}_2(T)$

Thm (A.P) : The set of preperiod points p in F has cardinality at most 4

Crutial property of p = additivity

$p(X+Y) = p(X) + p(Y)$

Let $\mathcal{B}$ be an $\mathbb{F}_q$-alg $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a)/_{\mathcal{B}} = \mathcal{B}\{\tau\}$

$\mathcal{B}\{\tau\} = \{ \sum_{i=0}^{n} a_i \tau^i \mid a_i \in \mathcal{B} \}$ $\qquad \tau \underset{\underset{\mathcal{B}}{\cap}}{b} = a^q \tau$

$x \mapsto x^q$ has τ

If $\quad p = 1 + \sum a_i \tau^i$, $\quad a_n \in \mathbb{F}_q(T)^*$

then $\quad \varphi : \underset{\overset{\shortparallel}{A}}{\mathbb{F}_q[T]} \longrightarrow \mathbb{F}_q(T)\{\tau\}$ $\qquad$ s.t. $\varphi(T) = p$

1) $\partial_0(\varphi(p)) = p \quad \forall p \in A$ $\qquad \partial_0 : \underset{\text{constant}}{B\{\tau\} \longrightarrow B}$

2) $\deg(\varphi(p)) = \deg(p) \cdot \deg(p) \quad \forall p \in A$

⇐ def. of Drinfeld module of rank $\deg(p)$

⇐ sort of a motivic over $\mathcal{B}$ with coef in $\mathbb{F}_q[T]$

$\varphi : A \to F\{\tau\}$ of rk. d, $m \triangleleft A$

$(F = \mathbb{F}_q(T))$

$\varphi[m] = \{ x \in \bar{F} \mid \varphi(P)(x) = 0 \ \forall P \in m \}$

Prop As an A-mod. $\varphi[m] = (A/m)^{\oplus d}$

Tate module of φ $\quad T(\varphi) = \varprojlim_{m \triangleleft A} \varphi[m] \circlearrowleft Gal(\bar{F}/F)$

Def. : $\lambda : \varphi \to \psi$, if $\lambda \in F\{\tau\}$ s.t.

$\lambda \circ \varphi = \psi \circ \lambda$ (isogeny) $\quad$ isomorphism $\Leftrightarrow \lambda \in F^\times$

Rem Tate's conjecture on isogeny holds

(due to A Tamagawa, Taguchi)

$\varphi(T) = T + g\tau + \Delta\tau^2$ $\quad$ its F-rational torsion

$\varphi_{tor}(F) = \{ x \in F \mid \exists\, 0 \neq P \in A \ \ \varphi(P)(x) = 0 \}$

$|\varphi_{tor}(F)| \leq 4 \qquad (q = 2)$

j-invariant of φ : $\quad j(\varphi) = \dfrac{g^{q+1}}{\Delta} \in F \qquad \overset{\in F}{\Delta \neq 0}$

$j(\varphi)$ depends only on the isomorphism class of φ

$\Rightarrow$ " j-line "

$Y_1(P)$ = Drinfeld modular curve parametrizing Drinfeld modules $\varphi : A \to L\{\tau\}$ $\quad (A \hookrightarrow L)$ injection

with a level str. $\quad i : A/p \to \varphi[p](L)$

$(p \triangleleft A$: prime ideal$)$

Thm $Y_1(p)(F) = \emptyset$ when $\deg(p) \geq 3$

$\nexists\ \varphi : A \longrightarrow F\{\tau\}$ of rk. two with non-trivial F-rational p-torsion.

counter example $\Longrightarrow$

$$0 \to A/p \to \varphi[p] \longrightarrow \varphi_0[p] \longrightarrow 0$$

$\varphi_0(T) = T - \Delta\tau$ (unique Drinfeld module with this property).

Hamahata (⊗-cat. of t-motives)

$$\det(T(\varphi)) = T(\varphi_0)$$
$$\det(\varphi) = \varphi_0$$

K = field of def. of $\varphi_0[p] \subset \overline{F}$

$\varphi_0[p] \underset{Gal(\overline{K}/K)}{\simeq} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $\quad * : Gal(\overline{K}/K) \longrightarrow (A/p)^\times$

<u>Prop A</u>: $*$ above is unramified for $\forall$ place v of K for $\forall v$ which is above ∞, the place of F corresponding to the valuation deg

<u>Prop B</u>: this class group of p-torsion homomorphism is trivial when <u>$q=2$</u> $\quad \varphi(T) = T+\tau$

◎ sketch of the proof of the second half of Prop A

F_∞ = completion of F w.r.t. ∞

<u>def.</u> An A-submod $\Lambda \subset \overline{F}_\infty$ is called an F-lattice if

(1) fin. gen and free A-mod

(2) Λ is $Gal(\overline{F}_\infty/F_\infty)$-invariant

(3) $|\Lambda \cap \{z \in \bar{F}_\infty \mid |z| < c\}| < \infty$

Rem $\mathrm{Gal}(\bar{F}_\infty/F_\infty) \circlearrowright \Lambda$ is through a finite quotient

Def. two F-lattice Λ, Λ' are isom if $c \in F^\times$ $\Lambda = c\Lambda'$

Thm $\exists$ equivalence of categories

$$\{F\text{-lattices}\} \xrightarrow{c} \{\varphi : A \to F_\infty\{\tau\}\}$$

s.t. $T(c(\Lambda)) = \Lambda \otimes_A \hat{A}$ ← profinite completion of A

Lem $G < GL_2(\Lambda)$ finite subgroup $\Rightarrow$

G is conjugate to subgroup of $GL_2(\mathbb{F}_2)$ or

〃 of $U(A) = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in A\}$

proof: $GL_2(\Lambda) \circlearrowright$ Bruhat-Tits tree

Fact finite $G \circlearrowright \mathcal{J} \Rightarrow G$ has a fixed vertex

$\Lambda = A^2$, $c(\Lambda)[p]$ has a non-trivial $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$-inv subgrp. $\Rightarrow$ $\Lambda \simeq \langle 1, \tau \rangle_A$ $\tau \in \bar{F}_\infty - F_\infty$

$$\begin{tikzcd} \Lambda \arrow[d, phantom, "\cup"] & \tau \in \bar{F}_\infty - F_\infty \arrow[d, phantom, "\cap"] \\ A\cdot 1 & U(A)\backslash\Omega \\ & U(A)\backslash\hat{\Omega} \arrow[u] \end{tikzcd}$$

Drinfeld's upper half plane

主藤 周友

ℓ: prime number $\neq ch(k)$

Thm A'' (S-V)

Assume RS(k) and wBK(k, q', ℓ) for $^\forall q' \leq q$. Then

$$\alpha^{q}_{\ell^r} \quad \mathbb{Z}(q)^{sh} \overset{\mathbb{L}}{\otimes} \mathbb{Z}/\ell^r \longrightarrow \tau_{\leq q} R\gamma_* \underbrace{\mathbb{Z}(q)^{\acute{e}t} \overset{\mathbb{L}}{\otimes} \mathbb{Z}/\ell^r}_{\simeq \mu_{\ell^r}^{\otimes q}}$$

is an isom. in $DM^{eff}_{-}(k)$ for $^\forall r \geq 1$ $\square$

Thm B (S-V)

Assume RS(k) and BV(k, q', ℓ) for $^\forall q' \leq q$.

Then $\alpha^q_{\ell^r}$ is an isom for $^\forall r \geq 1$ $\square$

Rem $\quad H^i_{\acute{e}t}(X, \mu_{\ell^r}^{\otimes q}) \xrightarrow{\simeq} H^i_{\acute{e}t}(X \times \mathbb{A}^1, \mu_{\ell^r}^{\otimes q}) \quad \begin{pmatrix} \forall X \in Sm/k \\ \forall i \in \mathbb{Z}_{\geq 0} \end{pmatrix}$

Rem $\quad \alpha^q_\ell$ is an isom $\Leftrightarrow$ $\alpha^q_{\ell^r}$: isom for $\forall r \geq 1$

Thm A'' $\quad \Leftrightarrow \alpha^q_{\ell^\infty} : \mathbb{Z}(q) \overset{\mathbb{L}}{\otimes} \mathbb{Q}_\ell/\mathbb{Z}_\ell \xrightarrow{\simeq} \tau_{\leq q} R\gamma_* \mathbb{Z}(q)^{\acute{e}t} \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$

Thm B

Def. RS(k) : the following two conditions

(1) $^\forall X \in Sch/k$ integral $\quad \exists\, Y \longrightarrow X$ proper birational
$\quad$ Y smooth $/k$

(2) $^\forall X \in Sm/k$ integral $\quad \forall\, Y \longrightarrow X$ proper birational

$$\begin{tikzcd} \exists\, X_n \arrow[r] \arrow[dr] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X \\ & Y \arrow[urr] & & \end{tikzcd}$$

a blow up at smooth centers

§5 Proof of Thm A″ (outline)

$\mathbb{Z}/\ell(q) := \mathbb{Z}(q)^{Nis} \overset{L}{\otimes} \mathbb{Z}/\ell$, $B_\ell(q) := \tau_{\leq q} R\gamma_* \mu_\ell^{\otimes q} \quad \in DM^{eff}_-(k)$

<u>Key fact</u> (Voevodsky) For $f : F_1 \to F_2$ homomorphism of homotopy inv PST

f is an isom $\Longleftrightarrow$ $f(E) : F_1(E) \to F_2(E)$ is bij for any f.g. fields E/k □

Suffices to show:

$\alpha_\ell^{i,q}(F) : H^i(F, \mathbb{Z}/\ell(q)) \longrightarrow H^i(F, B_\ell(q))$ is bij for $\forall i \leq q$, $\forall F/k$ f.g. field

Nis is omitted

<u>Surjectivity</u> of $\alpha_\ell^{i,q}(F)$ easy

Reduced the case $\mu_\ell \in F$. Then $0 \leq i \leq q$

$$\begin{tikzcd} H^i(F, \mathbb{Z}/\ell(i)) \otimes H^0(F, \mathbb{Z}/\ell(q-i)) \arrow[r, "cup"] \arrow[d, Rightarrow, "wBK(k,i,\ell)"'] & H^i(F, \mathbb{Z}/\ell(q)) \arrow[d] \\ H^i(F, B_\ell(i)) \otimes H^0(F, B_\ell(q-i)) \arrow[r, "\simeq"] & H^i(F, B_\ell(q)) \end{tikzcd}$$

<u>Injectivity</u> of $\alpha_\ell^{i,q}(F)$: hard

Use induction on $q > 0$. Necessary staffs:

(1) For $X \in Sch/k$ define $M(X) \in DM^{eff}_-(k)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; \| \; C^*(\mathbb{Z}_{tr}(X))$

(2) For $\mathcal{K} \in DM^{eff}_-(k)$

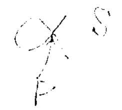
$H^i(X, \mathcal{K}) := \mathrm{Hom}_{DM^{eff}_-(k)}(M(X), \mathcal{K})$

$X \in Sm/k \quad \cong \; H^i_{Nis}(X, \mathcal{K})$

(3) $\Delta^\bullet$: standard cosimplicial scheme.

$\partial\Delta^n$ = union of faces $\mathrm{Im}(\partial^i \cdot \Delta^{n-1} \to \Delta^n)$ $(i\ 0, \cdots, n)$

S := $\mathbb{A}^1_k$ with 0 and 1 identified



$P \in S$: singular pt

(4) $Z \hookrightarrow X$: closed immersion of schemes / k

$M_Z(X) = C^*(\mathbb{Z}_{tr}(X) / \mathbb{Z}_{tr}(X \setminus Z)) \in DM^{eff}_-(k)$

Step 1 $\mathcal{K} \in DM^{eff}_-(k)$, $q \geq 0$ $H^i(F, \mathcal{K}) \hookrightarrow H^{q+1}(\partial\Delta^{q-i+1}_F \times S, \mathcal{K})$ (functorial in $\mathcal{K}$)

Step 2. U := semi-localization of $\partial\Delta^{q-i+1}_F \times S$ at $v_1 \times P, v_2 \times P, \cdots, v_{q-i+1} \times P$

($v_1, \cdots v_n \in \partial\Delta^n_F$ vertices of $\partial\Delta^n_F$ intersecting pt of $(n-1)$ components)

Show

$H^i(F, \mathbb{Z}/\ell(q)) \hookrightarrow H^{q+1}(\partial\Delta^{q-i+1}_F \times S, \mathbb{Z}/\ell(q)) \xrightarrow{(*)} H^{q+1}(U, \mathbb{Z}/\ell(q$

is zero map.

Step 3. $H^{q+1}(\partial\Delta^{q-i+1}_F \times S, \mathbb{Z}/\ell(q)) \xrightarrow{\alpha^q_\ell} H^{q+1}(\partial\Delta^{q-i+1}_F \times S, B_\ell(q))$

is injective on $\mathrm{Ker}(*)$

- induction on q · cancellation of $\mathbb{Z}(1)$
- Gabber's base change for henselian pairs

§ Proof of Thm B

BV (k, q, ℓ) $\beta_{q,j} : H^q_{Gal}(F, \mu_\ell^{\otimes q}) \underset{\uparrow}{\longrightarrow} H^{q+1}_{Gal}(F, \mu_{\ell^j}^{\otimes q})$

$(0 \to \mu_{\ell^j}^{\otimes q} \to \mu_{\ell^{j+1}}^{\otimes q} \to \mu_\ell^{\otimes q} \to 0)$

is zero for $\forall j > 0$ and $\forall F/k$ f.g field

<u>Lem</u> BV(k, q, ℓ) + (BL(q', ℓ) for $\forall q' < q$) + RS(k)

$\Rightarrow$ χ^q_{F,ℓ^∞} : $K^M_q(F) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow H^q_{Gal}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(q))$ is surj.

(Lem $\Rightarrow$ Thm B)

望月 哲史 Reading Morel-Voevodsky

Contents

1. Raw material
2. Introduction to stable homotopy theory
   - 2.1 to do homotopy theory
   - 2.2 to analyse (co)homology theories
   - 2.3 a recipe for a general frame work in stable homotopy theory

(3. Brief survey for Algebraic K theory)

1. Raw material

Mumford : Lect. on curves on alg. surface

<u>Ex.</u> $\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1} \hookrightarrow \quad \hookrightarrow \mathrm{Pic}$

$$h_{\mathbb{P}^n} \to \mathrm{Pic} \qquad \begin{tikzcd} h_{\mathbb{P}^n}(X) \arrow[r] & \mathrm{Pic}X \\[-15pt] \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\[-15pt] f \arrow[r, mapsto] & f^*\mathcal{O}(1) \end{tikzcd}$$

$$\underset{\text{up to homotopy}}{\mathrm{Pic} \doteqdot \mathbb{P}^\infty}$$

In algebraic topology.

Hot = " the homotopy cat. of CW-complexes "

$$\mathrm{Hot} \longrightarrow \mathrm{Set}$$

$$X \longmapsto \{\text{top } \overset{\mathbb{C}}{\text{line}} \text{ bundle}/X\}/\text{isom}$$

is represented by $\mathbb{P}^\infty_{\mathbb{C}}$

This statement is true in the context of alg geom. using the theory of Morel-Voevodsky

2. Q1. What is the meaning of phrase

"to do homotopy theory"

Q2. What is the meaning of the phrase

"to analyse (co)homology theory"

2.1 Q Throughout the geom. realyzation functor, how approximated homotopical property of topological space by combiractorial str.?

$$\begin{tikzcd}
\mathrm{Cat} \arrow[r, "\mathrm{Yoneda}"] \arrow[rd, dashed, "N"] & (\mathrm{Cat})^{\Delta} \arrow[d] \\
\Delta \arrow[u, hook] \arrow[r, dashed, "\mathrm{Yoneda}"'] \arrow[d] & \Delta^{op}\mathrm{Set} \arrow[ld, dashed, "|\ |"'] \\
\mathrm{Top} \arrow[r, "\mathrm{Sing}"] & \mathrm{Top}^{\Delta} \arrow[u]
\end{tikzcd}$$

Prop (Kan)

X, Y simplicial sets (Kan comp

f: X → Y simplical homotopic

⇔ weak homotopy equiv

Homological alg analogue $\mathcal{A}$ abelian cat

X. Y. bounded below complex consisting of proj obj

f: X. → Y. chain homotopic ⇔ f qis

sketch of proof Considering Cone(f). We reduce to the following statement

" Z. bounded below complex ⇔ consisting of proj obj.

Z. contractible (chain homotopic to 0 ⇔ id_Z chain homotopic to 0

⇔ acyclic

$\Leftarrow$) uniqueness of (lifting up to homotopy

$$\begin{tikzcd} Z. \arrow[r] \arrow[d, shift right, "id_Z"'] \arrow[d, shift left, "0"] & 0 \arrow[d, "id=0"] \\ Z. \arrow[r] & 0 \end{tikzcd}$$

$id_Z \sim 0$,

Axiomized !!

Weak answer for Q1

$S_{cat} = \{ f \in Mor\, Cat : |Nf| : weak\ eq \}$

$S_{top} = \{ f \in Mor\, Top : f\ weak\ eq. \}$

$S_{\Delta^{op}Set} \ \{ f \in Mor \Delta^{op} Top : |f| : weak\ eq \}$

$$S_{cat}^{-1}\, Cat \xrightarrow[\sim]{N} S_{\Delta^{op}Set}^{-1}\, \Delta^{op} Set \xrightarrow[\sim]{|\ |} S_{Top}^{-1}\, Top \qquad S \subset Mor\, \mathcal{C}$$

Provisional def. : (1) A "homotopy theory" is a pair $(\mathcal{C}, S)$

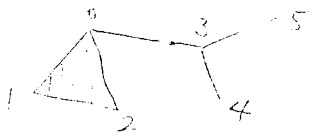
$\mathcal{H}o(\mathcal{C}) := S^{-1}\mathcal{C}$ $\quad$ $\mathcal{C}$ is called model category or

do homotopy theory

Ex. X simplicial complex

$Face(X) = \{$ ordered set of faces in $X \}$ is

beyantric decomposition of X



$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] \arrow[rr, dotted] & & \{0,3\} & 3 \arrow[l] \arrow[r] \arrow[dd] & \{3,5\} & 5 \arrow[l] \\ \{0,1\} \arrow[dr] & & \{0,2\} \arrow[dl] & & & & \\ & \{0,1,2\} & & & \{3,4\} & & \\ 1 \arrow[uu] \arrow[r] & \{1,2\} \arrow[u] & 2 \arrow[l] \arrow[uu] & & 4 \arrow[u] & & \end{tikzcd}$$

(Cat, $\mathcal{W}_{Cat}$), ($\Delta^{op}Set$, $\mathcal{W}_{\Delta^{op}Set}$), ($Top$, $\mathcal{W}_{Top}$) have a same homotopy theory

Q. Is $S^{-1}C$ actually local small category?

we need more axioms!!

2.2

<u>Thm</u> (A. Neeman)

$\mathcal{T}$ compactly generated triangulated cat

$H : \mathcal{T}^{op} \to Ab$ cohomological functor

the following is equivalent

(1) H : representable

(2) H is commutative with small coproducts □

cohomology theory $\Leftrightarrow$ obj in $\mathcal{T}$

<u>Weak</u> <u>answer</u> <u>for</u> <u>Q2</u>

$\mathcal{C}$ the category of spaces

$\rightsquigarrow$

$\mathcal{C}$: (compactly generated) triangulated cat

Control higher homotopical str → Model str.

(the cat. of (symmetric) spectra)

← homotopy cat lose many info

<u>Toy's model</u> V : fin. dim vector sp. / $\mathbb{C}$ with inner prod

$\cup$

W subspace $\boxed{V/W \xrightarrow{\sim} W^{\perp}\,!}$

<u>cf.</u> $D^{-1}(Shv_{Nis}(SmCor(k))) / \langle [X \times \mathbb{A}^1] - [X] \rangle$

$\xrightarrow{\sim} \langle [X \times \mathbb{A}^1] - [X] \rangle^{\perp} \simeq \mathbb{A}^1$- <u>local obj.</u>

$\mathcal{C}$: cat $\hookrightarrow \mathcal{C}^\wedge$

Grothendieck $\rightsquigarrow$ J

$$\begin{tikzcd} \mathcal{C}^\wedge/J \arrow[r, dashed, "\simeq"] \arrow[d, phantom, "\parallel"] & J^\perp \\ S_J^{-1}\mathcal{C} & \end{tikzcd}$$

$$J^\perp \overset{\text{full subcat}}{\hookrightarrow} \mathcal{C}^\wedge$$

sheaf

$S_J = \{ f \in \operatorname{Mor}\mathcal{C} \mid a(f) \text{ isom} \}$

<u>classical</u>

Topos

$\rightsquigarrow$

$\mathcal{O}_X$-Mod Abelian cat

$\rightsquigarrow$ lose info

D($\mathcal{O}_X$-Mod), triag cat

<u>Morel-Voevodsky</u> (Bousfield localization

site with interval theory

the category of spaces

$\downarrow$

the category of spectrum $\circlearrowleft$ Model str.

$\downarrow$

SH( )

<u>Def.</u> (Model Category)

$\mathcal{M}$ category $\quad \begin{matrix} W \\ F \\ \mathrm{Cof} \end{matrix} \subset \mathcal{M}$: subcat

$(\mathcal{M}, W, F, \mathrm{Cof})$

W cat. of weak eq.

F cat of fibrations

Cof " cofibrations

(M0) $\mathcal{M}$ is closed under limits and colimits

(M1) (retraction) W, F, Cof are closed under retraction

$$(f \in \operatorname{Mor} W) \quad \begin{tikzcd} A \arrow[r, dashed] \arrow[d, "g"'] \arrow[rr, bend left, "id"] & X \arrow[r, dashed] \arrow[d, "f"] & A \arrow[d, "g"] \\ B \arrow[r, dashed] \arrow[rr, bend right, "id"'] & Y \arrow[r, dashed] & B \end{tikzcd} \Rightarrow g \in \operatorname{Mor} W$$

(M2) (2-out of 3) $X \xrightarrow{f} Y \xrightarrow{g} Z$

If 2 of f, g, gf are w.e then so the 3rd

(M3) (lifting property)

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "j"'] \arrow[ur, dashed, "\exists h"] & Y \end{tikzcd}$$

$f \in Cof$
$g \in Fib$

If $f \in W$ or $g \in W$

$\Rightarrow \exists h,\ B \to X$ s.t
$g h = j,\ hf = i$

(M4) (factorisation) $X \xrightarrow{f} Y$

$$\underbrace{X \xrightarrow{g} Z \xrightarrow{h} Y}_{f}$$

$\left(\begin{array}{l} h : fib \ \& \text{ weak} \\ g : cof \end{array} \right.$ or

$\left(\begin{array}{l} h : fib \\ g : cof \ \& \text{ w.e.} \end{array} \right.$

This factorisation is functorial

□

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[dr, "X\times\{0\}"'] & Y \\ & X\times[0,1] \cup Y \arrow[u, "\wr"'] \end{tikzcd}$$

$(x,t) \mapsto f(x)$

$$\begin{tikzcd} X & \subset & Y \\ & \{\gamma : [0,1] \to Y \mid \gamma(0) \in X\} \arrow[ul, "\gamma(0)"] \arrow[ur, "\gamma(1)"'] & \end{tikzcd}$$

No 1

Date

Joel Riou

Spanier - Whitehead duality in alg. geom.

hard written notes available at URL:

http://www.math.jussieu.fr/~riou/notes/

1 - Lemma on vector spaces

Lem Let A be a commutative ring

M, N be two A-modules

$I : A \to M \otimes N$ $\qquad I(1) = \sum_i m_i \otimes n_i$

$E : N \otimes M \to A$

such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "I \otimes id_M"'] \arrow[rr, bend left, "id_M"] & M \otimes N \otimes M \arrow[r, "id_M \otimes E"'] & M \end{tikzcd}$$

$$\begin{tikzcd} N \arrow[r, "id_N \otimes I"'] \arrow[rr, bend left, "id_N"] & N \otimes M \otimes N \arrow[r, "E \otimes id_N"'] & N \end{tikzcd}$$

Then M, N are f.g projective A-modules

(∵) ∀ A-mod T, we have a can. isom

$$\mathrm{Hom}_A(M,T) \xrightarrow{\simeq} T \otimes N$$

$$\begin{tikzcd} f \arrow[r, mapsto] & f \otimes id_N : M \otimes N \arrow[r] & T \otimes N \\ & A \arrow[u, "I"] \arrow[ur, dashed] & \end{tikzcd}$$

$\Rightarrow$ $\mathrm{Hom}_A(M,A) \xrightarrow{\simeq} N$

☐

Def. (Dold-Puppe) Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal cat.

M, N : obj in $\mathcal{C}$
$I : \mathbb{1} \to M \otimes N$
$E : M \otimes N \to \mathbb{1}$

it is strong duality if the previous diag commutes.

$$\forall T \in \mathcal{C} \quad \underline{Hom}(M, T) \xrightarrow{\sim} T \otimes N$$

$$N \simeq \underline{Hom}(M, \mathbb{1})$$

⇝ notion of strong dualizable obj

$E : \underline{Hom}(M, \mathbb{1}) \otimes M \longrightarrow \mathbb{1}$ evaluation map

$I : \mathbb{1} \xrightarrow[id_M]{} \underline{Hom}(M, M) \xleftarrow{\sim} M \otimes \underline{Hom}(M, \mathbb{1})$

A: commutative ring In $(D(A\text{-mod}), \otimes, A)$ strong dualizable objects are perfect complexes

1 SW in alg. top.

Thm (Poincaré duality)

Let X be an oriented connected compact smooth mfd. of dim d.

$\forall$ field k $\forall i \in \mathbb{Z}$ $H^i(X, k) \times H^{d-i}(X, k) \longrightarrow H^d(X, k) \xrightarrow{\sim} k$

is a perfect pairing

Idea divide this into two statements

(a) orientation
(b) duality

(a) <u>Thom</u> isom

X: smooth mfd E vector bundle of rk r

$\|\cdot\|$ a Euclidean metric on E

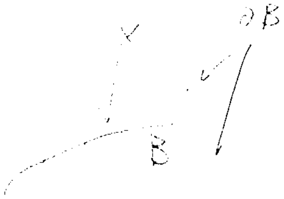
$$\begin{tikzcd} B = \{e \in E,\ \|e\| < 1\} \arrow[d, phantom, "\cup"] \\ \partial B \end{tikzcd} \qquad B/\partial B = Th_X E$$

trivial bundle

$$Th_X(\varepsilon^n) = S^n \wedge X_+$$

Thm: If F is oriented, there is a class $\mu \in H^r(Th_X F)$

inducing isom. $H^*(X) \xrightarrow{\simeq} \widetilde{H}^{*+r}(Th_X F)$



(b) X smooth compact mfd

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d] & \mathbb{R}^n \\ \partial B \subset B \arrow[ur, hook] & \end{tikzcd}$$

normal bundle ν

tubular nbd.

$$Th_X \nu = B/\partial B$$

$$\begin{aligned} B \times X &\longrightarrow \mathbb{R}^n \\ (b, x) &\longmapsto b - x \end{aligned}$$

if $b \in \partial B$, then $b - x \neq 0$

$$B \times X / \partial B \times X \longrightarrow \mathbb{R}^n / \mathbb{R}^n - \left(\begin{array}{l} \text{small open ball} \\ \text{centered on } 0 \end{array} \right)$$

$$Th_X \nu \wedge X_+ \dashrightarrow S^n$$

$\big\downarrow$ $\widetilde{C}_*$ singular chain complex functor

$$\widetilde{C}_*(Th_X \nu) \otimes C_* X \longrightarrow \mathbb{Z}[n] \quad \text{in } D(Ab)$$

$$\widetilde{C}_*(Th_X \nu)[-n] \otimes C_* X \longrightarrow \mathbb{Z}$$

<u>Thm</u>: This map induces a strong duality in D(Ab)

between $C_* X$ and $\widetilde{C}_*(Th_X \nu)[-n]$

part ⓐ §1 qis Thom isom

$C_* X[-d]$

Refinement $\mathcal{H}^{ft}_{\bullet}$ the pointed homotopy cat. of finite CW-complexes

$$\wedge : \mathcal{H}^{ft}_{\bullet} \times \mathcal{H}^{ft}_{\bullet} \longrightarrow \mathcal{H}^{ft}_{\bullet}$$

$$\underset{\overset{\parallel}{- \wedge S^1}}{S} \; \mathcal{H}^{ft}_{\bullet} \longrightarrow \mathcal{H}^{ft}_{\bullet}$$

Def. The Spanier Whitehead cat SW is the category

objects (X, n) $X \in \mathcal{H}^{ft}_{\bullet}$ $n \in \mathbb{Z}$

hom $\mathrm{Hom}_{SW}((X,n), (Y,m)) = \varinjlim_{r \gg 0} (S^{r+n} \wedge X, S^{r+m} \wedge Y)$ $\square$

$\mathcal{H}^{ft}_{\bullet} \longrightarrow SW : X \longmapsto (X, 0)$

SW is a symmetric monoidal triangulated cat

$(X, n)[1] = (X, n+1)$

Prop All objects in SW are strongly dualizable

(i.e. SW is a rigid $\otimes$-cat)

$SW = \langle S^0 \rangle$

the proof uses Ayoub's four functors

Thm X: smooth compact mfd $X \hookrightarrow \mathbb{R}^n$

ν: normal bundle

There is a strong duality in SW between

X_+ and $\mathrm{Th}_X \nu[-n]$ $\square$

$0 \to TX \to \underset{\overset{\parallel}{T\mathbb{R}^n|_X}}{\mathcal{E}^n} \to \nu \to 0$

part ⓔ of Poincaré duality

$\mathrm{Th}_X \nu[-n] = \mathrm{Th}_X(-TX)$

$SW \xrightarrow{\tilde{C}_*} D(Ab)$

§ – SW in alg. geom.

Lefschetz motive

$h(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{L}$

– k. field CHM(k) : Chow motives

X proj. smooth var $h(X)$ has a strong dual $h(X) \otimes \mathbb{L}^{-d_X}$

1.F corresponds to elements in $CH^{d_X}(X \times X)$, they are given by $[\Delta_X]$.

S : noetherian scheme $\rightsquigarrow$ $(\mathcal{SH}(S), \wedge, S^0)$ ← stable $\mathbb{A}$-ho cat

<u>Def:</u> X : scheme / S, E / X : vector bundle
$Th_X E = E/(E - \text{zero section}) \in \mathcal{SH}(S)$

<u>Rem</u> a short exact seq $0 \to E' \to E \to E'' \to 0$ gives a can isom. $Th_X E \xrightarrow{\sim} Th_X(E' \oplus E'')$
(Deligne's virtual cat) $\longrightarrow \mathcal{SH}(S)$ $E \mapsto Th_X E$

<u>Thm</u> If X is proj smooth / S. there is a strong duality between X_+ and $Th_X(-TX)$ □

several sub-triangulated cat of $\mathcal{SH}(S)$

$\mathcal{SH}(X)^{pf} = \{ X \in \mathcal{SH}(S) \mid Hom_{\mathcal{SH}(S)}(X, -) : \mathcal{SH}(S) \to Ab$ commutes with $\oplus \}$

$\cup$

$= \langle X_+ \wedge (\mathbb{P}^1)^{\wedge n}, Y : \text{smooth}/S, n \in \mathbb{Z} \rangle^{4}$ ← pseudo ab. hull

$\mathcal{SH}(X)^{proj} = \langle$ 〃 X proj smooth 〃 $\rangle^{4}$

$\mathcal{SH}(X)^{rig} = \{ X \in \mathcal{SH}(S) \mid X : \text{strong. dual.} \}$

$$\mathcal{SH}(S)^{proj} \underset{Thm}{\subset} \mathcal{SH}(S)^{rig} \underset{triv}{\subset} \mathcal{SH}(S)^{pf}$$

<u>Thm</u> Let k be a field admitting res of sing. (weak RS(k))
then $\mathcal{SH}(k)^{proj} = \mathcal{SH}(k)^{rig} = \mathcal{SH}(k)^{pf} \underset{def}{=} SW(k)$ □

If k is perfect, replace $\mathcal{SH}(k)$ by $\mathcal{SH}(k)_{\mathbb{Q}}$ (use de Jong)

$k = \mathbb{C}$ $\quad \mathbb{H}_{dR.diff} \xleftarrow{\sim} \mathbb{H}_{dR.alg}$

$\mathbb{H}_{dR.diff} \in \mathcal{SH}(\mathbb{C})$ representing de Rham coh of $X(\mathbb{C})$

$\xleftarrow{\sim}$ ↑ thm by Grothendieck

$\mathbb{H}_{dR.alg}$ rep. alg de Rham coh.

reduce to proj smooth and use GAGA.

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S: noether scheme finite Krull dim

$Sp(S) := \Delta^{op} Shv_{Nis}(Sm/S)$

(simplicial Nisnevich sheaf over Sm/S)

$Sp(S)_\bullet := \Delta^{op} Shv_{Nis}(Sm/S)_\bullet$

$$\begin{tikzcd} Sm/S \arrow[r] & Sp(S)_\bullet \\ X \arrow[r, mapsto] & X_+ \\ \Delta^{op}Set_* \arrow[ru] & \end{tikzcd}$$

$E \to X$: vector bundle

$Th\,E = E/E - s(X)$

zero section

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & Y/X \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

$Th(E \boxtimes F) = Th\,E \wedge Th\,F$ (一般の presheaf では X)

$E \xrightarrow{f} X$ (X, x), (Y, y)

$F \xrightarrow{g} Y$ $(X,x) \wedge (Y,y) = \left(X \times Y / x \times Y \cup X \times y, \mathrm{can} \right)$

Define model str. on $Sp(S)$

Step 1 (W_s, C, F_s)

$W_s = \{ f \in Mor\,Sp(S) \mid x^*(f)$ is w.e. for every points $x \}$

$C = \{$monomorphism$\}$

F_s : defined by left lifting property

$Sp(S)$ has an internal hom obj

$\mathcal{X}, \mathcal{Y} \in Sp(S)$

$S(\mathcal{X}, \mathcal{Y}) := Hom_{Sp(S)}(\mathcal{X} \times \Delta^{\cdot}, \mathcal{Y})$

$\mathcal{H}_s(S) := W_s^{-1} Sp(S)$

Step 2. $\mathcal{X} \in Sp(S)$ is called $\mathbb{A}^1$-local

$Hom_{\mathcal{H}_s(S)}(\mathcal{Y}, \mathcal{X}) \to Hom_{\mathcal{H}_s(S)}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$

is isom $\forall \mathcal{Y}$

$W_{\mathbb{A}^1} := \left\{ \begin{array}{l|l} f \in Mor\, Sp(S) & S(\mathcal{Y}, \mathcal{Z}) \to S(\mathcal{X}, \mathcal{Z}) \text{ w.e. for} \\ f : \mathcal{X} \to \mathcal{Y} & \forall \mathcal{Z} \text{ fibrant } \mathbb{A}^1\text{-local obj} \end{array} \right\}$

$\mathcal{Z} \to *$ fibration

$C := \{\text{monomorphism}\}$

$F_{\mathbb{A}^1}$ defined by left lifting property

Thm (Morel-Voevodsky)

$(Sp(S), W_{\mathbb{A}^1}, C, F_{\mathbb{A}^1})$ proper model category □

$\mathcal{H}^{\mathbb{A}^1}(S) := W_{\mathbb{A}^1}^{-1} Sp(S)$

$\mathcal{H}^{\mathbb{A}^1}(S)_{\bullet} = W_{\mathbb{A}^1}^{-1} Sp(S)_{\bullet}$.

S^1_s -- simplicial circle

$\cap$

$\Delta^{op}Set_* \xrightarrow{\Sigma} Sp(S)$

$S^1_t = (\mathbb{A}^1 - \{0\}, 1)$

[illegible] case

$M(\mathbb{A}^1 - \{0\}, 1) = \mathbb{Z}(1)[1]$

$M(\mathbb{P}^1, \infty) = \mathbb{Z}(1)[2]$

$\nearrow$ $M(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$

$C^*(\mathbb{Z}_{tr}(\mathbb{P}^1)/\mathbb{Z}_{tr}(\operatorname{Spec} k))$

$(\mathbb{P}^1, \infty) = S^1_s \wedge S^1_t$ in $\mathcal{H}^{\mathbb{A}^1}(S)$. $\quad M(\mathbb{P}^1, \infty) = \mathbb{Z}[1] \otimes \mathbb{Z}(1)[1] = \mathbb{Z}(1)[2]$

$S^{p,q} := S_s^{p-q} \wedge S_t^q \qquad \mathbb{Z}(q)[p]$

(X, Z) smooth pair

Thm (homotopy purity theorem)

$X/X-Z \xrightarrow{\simeq} Th\, N_{X,Z}$

(considering Nisnevich local we may assume $(X, Z) = (\mathbb{A}^n, \mathbb{A}^d)$)

$$\begin{tikzcd} M_Z X \arrow[d, equal] \simeq M\, Th\, N_{X,Z} \xrightarrow{\simeq} M(Z)(r)[2r] \\ C^*(\mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X-Z)) \end{tikzcd}$$ $(Z \hookrightarrow X \text{ cod } r)$

G : sheaf of group $\in Sp(S)$

we can define BG_*

(similar to simplicial construction)

$U \in Sm/S$

$$\mathrm{Hom}_{\mathcal{H}_s(S)_\bullet}(S_s^i \wedge U_+, BG) = \begin{cases} H^1_{Nis}(U, G) & i=0 \\ G(U) & i=1 \\ 0 & i \geq 1 \end{cases}$$

If G $\mathbb{A}^1$-local $\longrightarrow \|$

$$\mathrm{Hom}_{\mathcal{H}^{\mathbb{A}^1}(S)}(S_s^i \wedge U_+, BG)$$

$G = \mathbb{G}_m$ or abelian var $\Rightarrow$ $\mathbb{A}^1$-local

$$\mathrm{Hom}_{\mathcal{H}^{\mathbb{A}^1}(S)_\bullet}(U_+, B\mathbb{G}_m) = H^1_{Nis}(U, \mathbb{G}_m) = \mathrm{Pic}\, U$$

$BGL \leftarrow$ Grassman

$B\mathbb{G}_m = (\mathbb{P}^\infty, \infty)$ in $\mathcal{H}^{\mathbb{A}^1}(S)$.

Over $\mathcal{H}^{\mathbb{A}^1}(S)_\bullet$ $- \wedge S_t^1$ is not an isom.

Dold - Puppe :

$$\begin{tikzcd} \Delta^{op}(Ab) \arrow[r, "\sim"] & C_{\geq 0}(Ab) \arrow[d, phantom, "\cap"] \\ & C(Ab) \end{tikzcd}$$

$(\mathbb{P}^1, \infty)$ - spectra

$$\{X_i,\ X_i \wedge (\mathbb{P}^1, \infty) \xrightarrow{e_i} X_{i+1}\}_i$$

$\Rightarrow$ the category of $(\mathbb{P}^1, \infty)$ - spectra $Spc(S)$.

$\downarrow$ localizing

$\mathcal{SH}(S) \leftarrow$ symmetric monoidal category
triangulated cat.

$\Sigma^\infty : Sm/S \longrightarrow \mathcal{SH}(S)$

$E \in \mathcal{SH}(S)$ $\quad (X, x) \in (Sm/S)_*$

$$\begin{cases} E^{p,q}(X, x) := Hom(\Sigma^\infty(X, x), E \wedge S^{p,q}) \\ E_{p,q}(X, x) := Hom(S^{p,q}, \Sigma^\infty(X, x) \wedge E) \end{cases}$$

$H\mathbb{Z} \in \mathcal{SH}((X)$

$S = Spec\, k$

$H_{\mathbb{Z}}^{p,q}(X_+) = H^p_{\mathcal{M}}(X, \mathbb{Z}(q))$

Ex. X pointed simplicial set

$Symm^\infty(X) := \varinjlim Symm^n(X)$

$\Big(Symm^n(X) = \underbrace{X \times \cdots \times X}_{n} / \mathfrak{S}_n$

abelian monoid

In topological homotopy category

$Symm^\infty(X)^+ = \prod K(\hat{H}_n(X, \mathbb{Z}), n)$

$Symm^\infty(S^n)^+ = K(\mathbb{Z}, n)$

In the context of motivic homotopy theory,

ch $k = 0$ U : smooth connected

$$\mathrm{Hom}(U, \mathrm{Symm}^{\infty}(X)) = \bigoplus_{\substack{Z \subset U \times X \\ \downarrow \ \text{finite surj} \\ U}} \mathbb{N} Z$$

$$(\mathrm{Symm}^{\infty})^{+} \doteqdot \mathbb{Z}_{tr}(\ \ )$$

$$K(\mathbb{Z}(n), 2n) = \mathbb{Z}_{tr}((\mathbb{P}^1, \infty)^{\wedge n})$$

$$\Downarrow$$

$$H\mathbb{Z} = \{ K(\mathbb{Z}(n), 2n) \}$$

Reduced Power Operation:

<u>topology</u> stable homotopy operation

$$\begin{tikzcd}
\tilde{H}^*(X_+, \mathbb{Z}/\ell) \arrow[r] \arrow[d, "S"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{H}^{*+k}(X_+, \mathbb{Z}/\ell) \arrow[d, "S"] \\
\tilde{H}^{*+1}(\Sigma X_+, \mathbb{Z}/\ell) \arrow[r] & \tilde{H}^{*+k+1}(\Sigma X_+, \mathbb{Z}/\ell)
\end{tikzcd}$$

<u>Steenrod alg</u>

$\mathcal{A}^* = \left(\begin{array}{c}\text{stable} \\ \text{coh. operation}\end{array}\right)$

$= \mathrm{Hom}(H_{\mathbb{Z}}, \Sigma^* H_{\mathbb{Z}})$

$\underset{\text{Serre}}{\underset{\uparrow}{=}} \langle \beta, P^i \rangle_{\mathbb{Z}/\ell\text{-alg}}$

<u>Milnor</u>

H^* : cup prod $\rightsquigarrow$ $\mathcal{A}^*$: coprod.

$\exists !\ \psi^*$ ← graded co-commutative

$\mathcal{A}_*$ dual Steenrod alg.

← graded commutative

<u>Th</u> (Milnor)

$\ell > 2 \Rightarrow \mathcal{A}_* = \mathbb{Z}/\ell[\xi_1, \xi_2, \cdots] \underset{\mathbb{Z}/\ell}{\otimes} \Lambda^*\left(\bigoplus_{i=0}^{\infty} \mathbb{Z}/\ell \cdot \tau_i\right)$

$\deg \tau_i = 2\ell^i - 1$

$\deg \xi_i = 2(\ell^i - 1)$

$\ell = 2 \Rightarrow \mathcal{A}_* = \mathbb{Z}/\ell[\tau_0, \tau_1, \cdots]$ $(\ell = 2 \Rightarrow \tau_i^2 = \xi_{i+1})$

$\Rightarrow \{\prod_{i\ge0}\tau_i^{\varepsilon_i}\prod_{j\ge1}\xi_j^{r_j}\}_{\substack{\varepsilon_i=0,1\\ r_j\ge0}}$ formes a basis of $\mathcal{A}_*$

$$\begin{tikzcd} \xi_i \arrow[d, leftrightarrow] & \tau_i \arrow[d, leftrightarrow] \\ \varepsilon_i & Q_i \end{tikzcd}$$

Milnor's primitive operation

$\updownarrow$ dual basis of $\mathcal{A}^*$

$\{P(E,R)\}$

$\deg Q_i = 2l^i-1 \qquad \deg \varepsilon_i = 2l^i-2$

$Q_0 = \beta \qquad Q_{i+1} = [Q_i, P^{l^i}]$

($\uparrow$ NOT holds in the motivic case)

$Q_i^2 = 0 \Rightarrow \{H^*(X,\mathbb{Z}/l), Q_i\}$: complex

$\Rightarrow$ Margolis coh. $\widetilde{MH}^*{}_i(X)$

Want to construct a motivic analogue

How to use in the last stage of the proof of Milnor conj?

(a glance)

<u>H90(n,2)</u> $H^{n+1,n}(\check{C}(Q_{\underline{a}}),\mathbb{Z}/2) = 0$

$\uparrow$ want to prove

$\cong$

$\tilde{H}^{n+2,n}(\check{C}(Q_{\underline{a}}))$

3 differences

① bigraded $\rightsquigarrow$ harmless

② $H^{*,*}(\mathrm{Spec}\,k,\mathbb{Z}/l)$: non-trivial

$a = (\cup a) \in \mathcal{A}^{*,*}$

→ $l > 2$, harmless

$l = 2 \Rightarrow$ a little complicated

$$\begin{tikzcd} \tau \in H^{0,1}(k) & & \rho \in H^{1,1}(k) \\ -1 \in \mu_2(k) = \{\pm 1\} \arrow[u, mapsto] \arrow[u, phantom, "\cong"'] & & -1 \in k^\times/(k^\times)^2 \arrow[u, mapsto] \arrow[u, phantom, "\cong"'] \end{tikzcd}$$

② We don't know

$$\mathcal{A}^{*,*} \overset{?}{=} \mathrm{Hom}(H_{\mathbb{Z}/l}, \Sigma_s^* \Sigma_t^* H_{\mathbb{Z}/l})$$

→ $l = 2$ harmless

$l > 2 \Rightarrow$ harm!

<u>strategy of construction of P^i</u>

Rem bistable coh. op.

$\tilde{H}^{*,*} \longrightarrow \tilde{H}^{*+i,*+j}$ functorial

$S^1_s \wedge$, $S^1_t \wedge$ compatible with suspension isom

$\Leftrightarrow \tilde{H}^{2n,n} \longrightarrow \tilde{H}^{2n+i,n+j}$ compatible with $\wedge T = \mathbb{A}^1/\mathbb{A}^1 - \{0\}$

$$\begin{tikzcd} T \arrow[d, squiggly] \\ \mathbb{Z}(1)[2] \end{tikzcd}$$

$T = S^1_s \wedge S^1_t$

$$\left(\tilde{H}^{p,q}(X_+) = \tilde{H}^{p+a+b,q+b}(S^a_s \wedge S^b_t \wedge X_+) \right)$$

S^a_s ↕ $(1,0)$, S^b_t ↕ $(1,1)$

を用いる

$$\tilde{H}^{2n,n}(F_\cdot, \mathbb{Z}/l) \xrightarrow{P} \tilde{H}^{2nl,nl}(F_\cdot \wedge (B\mathfrak{S}_l)_+, \mathbb{Z}/l)$$

$F_\cdot$: pointed simplicial sheaf

$B\mathfrak{S}_l$: classifying space of l-th sym. grp.

$\tilde{H}^{*,*}(F_\cdot \wedge (B\mathfrak{S}_\ell)_+, \mathbb{Z}/\ell)$

$\tilde{H}^{*,*}(F_\cdot)[\![c, d]\!] / (c^2)$ $\quad \deg(c) = (2\ell-3, \ell-1)$ $\quad (\ell > 2)$

$\quad \deg(d) = (2\ell-2, \ell-1)$

$\tilde{H}^{*,*}(F_\cdot)[\![c, d]\!] / (c^2 - \tau d - \rho c)$ $\quad \ell = 2$

$$\begin{array}{ll} \hat{H}^{2n,n}(F_\cdot) & \hat{H}^{2n\ell, n\ell}(F_\cdot \wedge (B\mathfrak{S}_\ell)_+) \\ \quad \Psi & \quad \Psi \\ \alpha & P(\alpha) = \sum B^i(\alpha)\, c d^i + \sum P^i(\alpha)\, d^i \end{array}$$

property of $P \rightsquigarrow$ property of P^i, B^i

$\rightsquigarrow \quad \mathcal{A}^{*,*}$: left $H^{*,*}$-mod

$H^{*,*}$: cup prod $\longrightarrow \mathcal{A}^{*,*}$: coprod

$\mathcal{A}_{*,*} := \mathrm{Hom}_{H^{*,*}(k)}(\mathcal{A}^{*,*}, H^{*,*}(k))$ $\curvearrowleft$ prod

<u>calculate</u> $\quad \tilde{H}^{**}(F_\cdot \wedge (BG_\ell)_+, \mathbb{Z}/\ell)$

$k \ni \zeta_\ell \neq 1 \qquad \mu_\ell \overset{p_\zeta}{\simeq} \mathbb{Z}/\ell \hookrightarrow \mathfrak{S}_\ell$

$$\tilde{H}^*(F_\cdot \wedge (B\mathfrak{S}_\ell)_+) \xrightarrow{p_\zeta^*} \tilde{H}(F_\cdot \wedge (B\mu_\ell)_+)$$

$\uparrow$

splitting injection

<u>Lem</u> $\quad B\mu_\ell = \mathcal{O}(-\ell)_{\mathbb{P}^\infty} - \mathbb{P}^\infty$

$\uparrow$

zero section

$$\begin{tikzcd} \text{(∵)} \quad \mathbb{A}^{n+1} - \{0\} \arrow[rr, two heads] \arrow[dr] & & \mathbb{P}^n & \mathcal{O}(-\ell)_{\mathbb{P}^\infty} - \mathbb{P}^\infty \arrow[dll, dashed, "n \to \infty"] \\ & (\mathbb{A}^{n+1} - \{0\})/\mu_\ell \arrow[ur] & & \end{tikzcd}$$

$$B\mu_\ell \hookrightarrow \mathcal{O}(-\ell)_{\mathbb{P}^\infty} \twoheadrightarrow Th_{\mathbb{P}^\infty}(\mathcal{O}(-\ell)_{\mathbb{P}^\infty})$$

cofibration seq.

$c = [\mathcal{O}(-\ell)] \in H^{2,1}(\mathbb{P}^\infty)$

$$\begin{tikzcd}
{} \arrow[r] & H^{*,*}(Th(\mathcal{O}(-\ell))) \arrow[r] & (H(k)[\![\sigma]\!])^{*,*} \arrow[r] & H^{*,*}(B\mu_\ell) \arrow[r] & H^{*+1,*}(Th) \\
x\,t_{\mathcal{O}(-\ell)} & (H(k)[\![\sigma]\!])^{*-2,*-1} \arrow[u, "\cong"] & , x\ell\sigma & & \\
x \arrow[u, mapsto] & & \exists x \arrow[u, mapsto] & &
\end{tikzcd}$$

$t_{\mathcal{O}(-\ell)}$ Thom class of $\mathcal{O}(-\ell)$

$$0 \to (H(k)[\![\sigma]\!])^{*,*} \to H^{*,*}(B\mu_\ell) \to (H(k)[\![\sigma]\!])^{*-1,*-1} \to 0$$

(exact)

$H^{m,n}(X;\mathbb{Z}/p) = H^m_{Zar}(X; \mathbb{Z}/p(n))$

$ch(k)=0 \qquad \zeta_p \in k \qquad X$: smooth

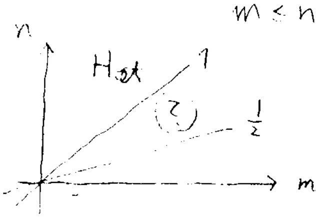
$H^{m,n}(X;\mathbb{Z}/p) = 0$ if $m \geqslant 2n$

$H^{2n,n}(X, \mathbb{Z}/p) = CH^n(X)/p$

(Bloch-Kato)

$H^{m,n}(X;\mathbb{Z}/p) = H^m_{et}(X;\mathbb{Z}/p)$

$m \leq n$



$Q_i : H^{*,*'}(X;\mathbb{Z}/p) \longrightarrow H^{*+2p^i-1,\, *'+(p^i-1)}(X;\mathbb{Z}/p)$

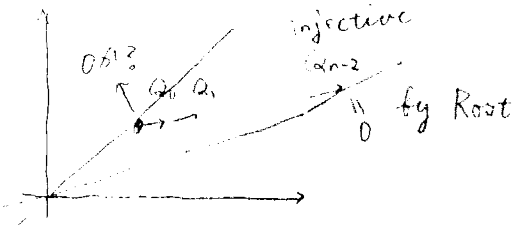
if $p \geqslant 3 \qquad Q_i = [Q_{i-1}, P^{p^{i-1}}]$

$\{-1\} \in K_1^M(k)/2 = H^1(\mathrm{Spec}\,k, \mathbb{Z}/2)$

$p=2 \qquad Q_i =$ 〃 mod $\overset{\prime\prime}{\rho}$

ρ^i 傾き $\frac{1}{2}$ $(CH^*(X)/p)$

Q_i 〃 $\frac{1}{2}$ 少し<u>小</u>



$x \in H^{*,*}(X(\mathbb{C}), \mathbb{Z}/p)$

$Q_i(x) \neq 0$

$\Rightarrow x$ は cl

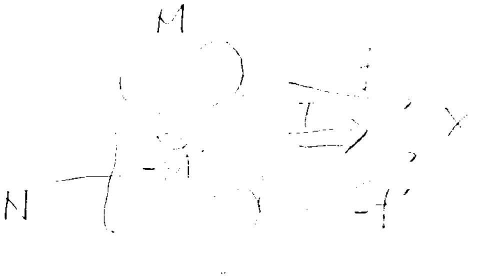
2. Cobordism theory

complex cobordism

$$MU^*(X) = [f: M \to X] / (\text{cobordism relation})$$

$= [X, MU]$, $MU = Th_{BU}(EU)$

M: weak complex mfd $\quad T_M \oplus \varepsilon$ complex bundle
(U-mfd) $\quad$ ↑ trivial

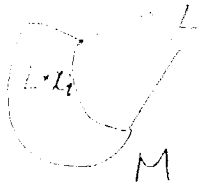


$[M, f] \sim [M', f']$
$\Leftrightarrow \exists$ U-mfd N $\quad F: N \to X$
$\partial N = M \cup (-M')$
$F|M = f \quad F|_{M'} = -f'$

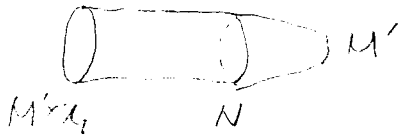
$MU^*(pt) = \mathbb{Z}[x_1, \cdots]$ $\quad |x_i| = 2i$
(Milnor, Novikov)

bordism theory of singularity of type x_i

$MU(x_i)^*(X)$
by $\hat{M} = M \cup L \times \text{cone}\, x_i$
$\partial M \simeq L \times x_i$



$M = M' \times x_i$



$M' \times x_i = 0$ in $MU(x_i)^*(X)$

<u>Th</u> (Sullivan)

$$\begin{tikzcd}
MU^*(X) \arrow[rr, "\times x_i"] & & MU^*(X) \arrow[dl, "\rho"] \\
& MU(x_i)^*(X) \arrow[ul, "\delta"] &
\end{tikzcd}$$

$MU(x_{i_1}, x_{i_2}, \quad)^*(X)$ も考えられる

$i \neq p^i - 1$ $\qquad MU(x_i \mid i \neq p^i - 1)^*(X)_{(p)} = MU^*(X)/(x_i \mid i \neq p^i - 1)$

$\underset{\text{def}}{=} BP^*(X)$ (Brown-Peterson theory)

$(p, \underset{\overset{\shortparallel}{v_1}}{x_{p-1}}, \underset{\overset{\shortparallel}{v_2}}{x_{p^2-1}}, \quad)$ $\qquad (v_n, l)$-mfd

$BP^*(pt) = \mathbb{Z}_{(p)}[v_1, v_2, \quad]$

norm variety

$MU(p, x_1, x_2, \quad)^*(X) = BP(p, v_1, \quad)^*(X)$

$MU(p, x_1, x_2, \cdots)^*(pt) = MU^*/(p, x_1, \quad) = \mathbb{Z}/p$

<u>Th</u> (Sullivan)

$MU(p, x_1, \quad)^*(X) = H^*(X; \mathbb{Z}/p)$

$MU(x_1, x_2, \cdots)^*(X) = H^*(X; \mathbb{Z})$

<u>Cor</u> (Yagita)

$\underset{\overset{\shortparallel}{\hat{M}}}{x} \in H^*(X; \mathbb{Z}/p) = BP(p, v_1, \cdots)^*(X)$

$\hat{M} = M \cup \hat{M}_0 \times \mathrm{cone}(p) \cup \hat{M}_1 \times \mathrm{cone}(v_1) \cup \cdot$

$Q_{v_i}(\hat{M}) = \hat{M}_i \Rightarrow$ cohomology operation $Q_{v_i} = Q_i =$

Morava K-theory

$$BP(p, \ldots, \overset{\text{omit}}{\widehat{v_n}}, \ldots)^*(X) \overset{おく}{=:} k(n)^*(X)$$ correcti Morava K-theory

$$k(n)^*(pt) = \mathbb{Z}/p[v_n]$$

$$[v_n^{-1}]\, k(n)^*(X) = K(n)^*(X)$$

$$\begin{tikzcd}
k(n)^*(X) \arrow[rr, "v_n"] & & k(n)^*(X) \arrow[dl, "\rho"] \\
& H^*(X;\mathbb{Z}/p) \arrow[ul, "\delta"] &
\end{tikzcd}$$

<u>Cor</u> $k(n)^*(X)$: v_n-tor. $\Rightarrow$

$$H^*(X;\mathbb{Z}/p) \simeq \mathbb{Z}/p\{1, Q_n\} \otimes k(n)^*(X)/p$$

$$\Rightarrow MH(H^*(X;\mathbb{Z}/p); Q_n) = 0$$

$\operatorname{Ker} Q_n / \operatorname{Im} Q_n$

$MU^*(X)$ $\qquad MGL^{*,*}(X) = AMU^{*,*}(X)$

$$k(n)^*(X) \Rightarrow Ak(n)^{2*,*}(X)$$

<u>Thm</u> (Voevodsky) , v_n

$$Ak(n)^*(\tilde{C}(V_a)) \text{ is } v_j\text{-torsion}$$

$0 \neq a \in K_n^M(k)/2$ $\quad V_a$ norm variety

$$\tilde{C}(X) \longrightarrow C(X) \longrightarrow \operatorname{Spec}(k)$$

$$C(X) \simeq X \times C(X)$$

$$Ak(n)^{2*,*}(\tilde{C}(X)) \xrightarrow[p^*]{} \underset{\overset{\|}{0}}{Ak(n)^{2*,*}(\tilde{C}(X)\times X)} \xrightarrow[p_*]{} Ak(n)^{2*,*}(\tilde{C}(X))$$

$$p_* p^*(x) = v_n \times x$$
$$= 0$$

No. 1

Date

安田 健彦

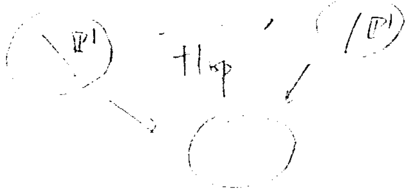
intro $/\mathbb{C}$

flop

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr] & \\ X : 3\text{-dim} & & Y \end{tikzcd}$$

$\Rightarrow K_{Z/X} = K_{Z/Y}$

$K_Z - f^*K_X$



<u>Def.</u> X, Y birational smooth var.

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

X & Y are K-equivalent

$\underset{def}{\Longleftrightarrow} K_{Z/X} = K_{Z/Y}$ □

K-eq varieties have the same "invariant"

⇑

motivic integration

generalize ↓

K-eq. orbifold (smooth Deligne Mumford stack) have the same "orbifold inv" (Homological McKay)

⇑

motivic int. over DM stack

generalize

normal, $\mathbb{Q}$-div

(X, D) KLT (Kawamata log terminal)

K-eq. KLT pairs have the same "stringy inv."

motivic int.

the Grothendieck ring of MHS

X : var. Hodge char.

$$\chi_h(X) := \sum (-1)^i [H^i_c(X,\mathbb{Q})] \in K_0(HS)$$

$\mathrm{Arcs}(X) = \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[[t]], X)$

↑ $\widehat{K}_0$-valued measure

$K_0(HS) =: K_0 \rightsquigarrow \widehat{K}_0$

weight completion

$F : \mathrm{Arcs}(X) \to \widehat{K}_0$ measurable function

↓ $\int F \, d\mu_X$

X : sm $\int 1 \, d\mu_X = \chi_h(X)$

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \arrow[ll, bend left, "K\text{-eq}"] \end{tikzcd}$$

$f_* : \mathrm{Arcs}(Z) \to \mathrm{Arcs}(X)$

Key fact : f_* is almost bij (outside measure zero subsets)

$$\left(\begin{tikzcd} \mathrm{Spec}\,\mathbb{C}((t)) \arrow[r, dashed] \arrow[d] & Z \arrow[d] \\ \mathrm{Spec}\,\mathbb{C}[[t]] \arrow[r] \arrow[ur, dashed, "\exists !"] & X \end{tikzcd} \right)$$

$$\chi_h(X) = \int d\mu_X \underset{\substack{\uparrow \\ \text{change of} \\ \text{variables}}}{=} \int \underbrace{\mathbb{L}^{F_{K_{Z/X}}}}_{\substack{\text{depend only on} \\ K_{Z/X} = K_{Z/Y}}} d\mu_Z = \int d\mu_Y = \chi_h(Y)$$

$\mathbb{L} = \chi_h(\mathbb{A}^1) = [\mathbb{Q}(-1)]$

$F_{K_{Z/X}} \ \mathrm{Arcs}(Z) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$

Furthermore if X, Y : proj, then $\underset{\text{Hodge str.}}{H^*(X) \simeq H^*(Y)}$

Homological McKay correspondence

finite subgroup

$$\begin{tikzcd} & \mathbb{C}^d \arrow[d] & G \subset SL_d \\ \underset{\text{smooth}}{Y} \arrow[r, "\pi", "\substack{\text{crepant} \\ \text{resol}}"'] & \mathbb{C}^d/G = X & \end{tikzcd}$$

$\pi^* K_X = K_Y$

<u>Th</u> (Batyrev)

$$H^i(Y, \mathbb{Q}) = \begin{cases} 0 & (i : \text{odd}) \\ \mathbb{Q}(-i/2)^{\oplus n_i} & (i : \text{even}) \end{cases}$$

($g \in G$ for suitable basis, $g = \mathrm{diag}(\zeta_\ell^{a_1}, \dots, \zeta_\ell^{a_d})$ $0 \le a_i \le \ell$

$$\mathrm{age}(g) := \frac{1}{\ell} \sum_i a_i \in \mathbb{Z} \quad (\Leftarrow G \subset SL_d)$$

$$n_i = \# \{ (g) \in \mathrm{Conj}(G) \mid \mathrm{age}(g) = i/2 \}$$) $\square$

$$\begin{tikzcd} \mathbb{A}^d \arrow[r] \arrow[loop below, "G"] & {[\mathbb{A}^d/G]} \arrow[r] & \mathbb{A}^d/G \end{tikzcd}$$

quotient stack — quotient var.

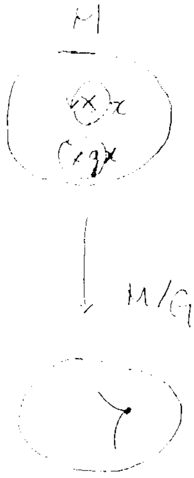
smooth DM stack — have quotient singularity

Points have automorphism groups

manifold ⇝ orbifold (Satake's V-mfd)

scheme (alg. space) ⇝ DM stack

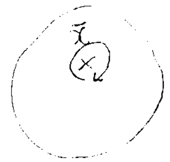
$\forall$ DM stack is locally isom. to a quotient stack



scheme (mfd) ↻ G : finite gp.

stab(x) ⊂ G

stab(gx) = g stab(x) $\cdot g^{-1}$ ⇝ [M/G]



quotient singularity

Aut($\bar{x}$)
≃ stab(x)
≃ stab(gx)

$$\begin{tikzcd}
{[\mathbb{A}^d/G]} \arrow[r, "\pi"] & \mathbb{A}^d/G \\
{[U/G]} \arrow[u, hook] \arrow[r, equal, "\text{birational}"'] & U/G \arrow[u, hook]
\end{tikzcd}$$

$\mathbb{A}^d \supset U := \{x \mid stab(x) = 1\}$

$G \subset SL$

⇒ no reflection

$\mathrm{cod}\ \mathbb{A}^d \setminus U \geq 2$

π has no exceptional div.

⇒ crepant

$$\begin{tikzcd}
& {[\mathbb{A}^d/G]} \arrow[d] \\
Y \arrow[ur, dash, "K\text{-eq}"] \arrow[r] & X
\end{tikzcd}$$

motivic integration over DM stacks

⇓

$\chi_h(Y) = \chi_h^{orb}([\mathbb{A}^d/G])$

Th $\mathcal{X}, \mathcal{Y}$: smooth DM-stack K-eq.

⇒ $\chi_h^{orb}(\mathcal{X}) = \chi_h^{orb}(\mathcal{Y})$ □

DM stack

$\mathcal{X} \rightsquigarrow I\mathcal{X}$ inertia stack $\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X}$

$I\mathcal{X} = \mathcal{X} \times_{\Delta, \mathcal{X}\times\mathcal{X}, \Delta} \mathcal{X} = \{(x, [g]) \mid x \in \mathcal{X}, [g] \in \mathrm{Conj}(\mathrm{Aut}(x))\}$

$$\chi_h^{orb}(\mathcal{X}) = \sum_{\substack{D \subset I\mathcal{X} \\ \text{conn comp}}} \chi_h(\bar{D})\, \mathbb{L}^{sht(D)}$$

$\bar{D}$: coarse moduli space

$sht(D) \doteq age(g) \in \mathbb{Q}$

$\mathrm{Spec}\,\mathbb{C}[[t]] \longrightarrow \mathcal{X}$

$$\begin{tikzcd} & \mathrm{Spec}\,\mathbb{C}((t)) \arrow[r, dashed] \arrow[d] & \mathcal{Y} \arrow[d, "\begin{array}{l}\text{proper}\\ \text{fin}\end{array}"] \\ \mathrm{Spec}\,\mathbb{C}[[t^{1/n}]] \arrow[r] \arrow[urr, dashed, bend left] & \mathrm{Spec}\,\mathbb{C}[[t]] \arrow[r] \arrow[ur, phantom, "\times"] & \mathcal{X} \end{tikzcd}$$

γ induces a hom of fin grp. $\mu_\ell \to G$

$\ell \in \mathbb{N}$, $\mu_\ell \subset \mathbb{C}^\times$ $\mathcal{D}_\ell = [\mathrm{Spec}\,\mathbb{C}[[t]]/\mu_\ell] \xrightarrow{\gamma} \mathcal{X}$

representable twisted stack

injective

$\mathrm{Tw\,Arcs}(\mathcal{X})/\mathrm{isom}$

$\supset \mu_X \quad \int F\, d\mu_X$

$$\begin{tikzcd} \mathrm{Tw\,Arcs}(\mathcal{X}) \arrow[d] & \ni \gamma \arrow[d, mapsto] \\ I\mathcal{X} & \ni (\gamma(\text{special pt}), \ ) \end{tikzcd}$$

Arc lift

$$\chi_h^{orb}(\mathcal{X}) = \int \mathbb{L}^{sht(\cdot)}\, d\mu_X$$

Joel Riou

Operations on algebraic K-theory and regulators via the homotopy theory of schemes

K_0 SGA 6

Let X be a scheme

Def: $K_0(X)$ is the abelian group

$$\begin{cases} \text{generators} & [M] \quad M: \text{isom class of vector bundles on } X \\ \text{relations} & [M']+[M'']=M \text{ if } 0\to M'\to M\to M''\to 0 \text{ is exact} \end{cases}$$

G: any coh. group

$$K_0(X) \longrightarrow G$$

$$\Updownarrow$$

maps $\left\{ \begin{matrix} \text{isom. classes} \\ \text{of vector bundles}/X \end{matrix} \right\} \longrightarrow G$

which are additive on short exact seq

$K_0(X)$ is a commutative ring

$[M]\cdot[N] = [M\otimes N]$

Def. M: vector bundle$/X$, $\underset{\underset{K_0(X)[[t]]}{\cap}}{\lambda_t M} = \sum_{n\geq 0} [\Lambda^n M] t^n$

Lem If $0\to M'\to M\to M''\to 0$ is exact seq. of bundles then $\lambda_t M = \lambda_t M' \cdot \lambda_t M''$

$\Rightarrow$ morphism $K_0(X) \xrightarrow{\lambda_t} 1+K_0(X)[[t]]$

on $\Lambda^* M$ there is a filtration such that

$$Gr^p \Lambda^n M \simeq \Lambda^p M' \otimes \Lambda^{n-p} M''$$

$\rightsquigarrow$ $K_0(X)$ is a λ-ring
(pré-λ-anneau)

Thm (SGA 6) $K_0(X)$ is a special λ-ring
(λ-anneau)

there are formulas for $\lambda^n(1)$, $\lambda^n(uv)$, $\lambda^n(\lambda^m(u))$ □

- X : compact space $\rightsquigarrow$ $K_0^{top}(X)$ use top. complex v.b.

Grassman varieties

$(d,r) \in \mathbb{N}^2$

$Gr_{d,r}(\mathbb{C}) = \{ V \subset \mathbb{C}^{d+r} \mid V : \text{sub } \mathbb{C}\text{-v.sp. of dim } d \}$

$\mathcal{M}'_{d,r} \to Gr_{d,r}$ tauto. vector bundle of rk d

$$\begin{tikzcd}
Gr_{d,r}(\mathbb{C}) \arrow[r, hook] \arrow[d] & Gr_{d,r+1}(\mathbb{C}) \arrow[r, hook] & \cdots & Gr_{d,\infty}(\mathbb{C}) \arrow[l, hook'] \arrow[d, dotted] \\
Gr_{d+1,r}(\mathbb{C}) \arrow[r, hook] & Gr_{d+1,r+1}(\mathbb{C}) \arrow[r, hook] & \cdots & Gr(\mathbb{C})
\end{tikzcd}$$

$\mathcal{M}_{d,r} = [\mathcal{M}'_{d,r}] - d \in K_0^{top}(Gr_{d,r}(\mathbb{C}))$

$(\mathcal{M}_{d,r})_{(d,r)\in\mathbb{N}^2} \in \varprojlim_{(d,r)} (Gr_{d,r}(\mathbb{C}))$

Thm X : compact sp.

$$[X, \mathbb{Z} \times Gr(\mathbb{C})] \simeq K_0^{top}(\mathbb{C})$$ □

<u>def.</u> $\Rightarrow$ $K_n^{top}(X) = [S^n \wedge X_+, \mathbb{Z} \times Gr(\mathbb{C})]$

X: space

Thm (Morel Voevodsky) S: regular scheme, $X/_S$: smooth

$\mathrm{Hom}_{\mathcal{H}(S)}(X, \mathbb{Z} \times Gr) \simeq K_0(X)$

$\mathrm{Hom}_{\mathcal{H}(S)}(S^n \wedge X_+, \mathbb{Z} \times Gr) \simeq K_n(X)$ (defined by Quillen)
$\forall n \geq 0$ $\square$

more precisely, any pointed endomorphism of $\mathbb{Z} \times Gr$ in $\mathcal{H}_\cdot(S)$ gives map $K_n(X) \to K_n(X)$ $\forall n$, $\forall X$: smooth

<u>Thm</u> S: regular scheme, $K_0(-): (Sm/S)^{opp} \to (Sets)$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{H}(S)}(\mathbb{Z}\times Gr, \mathbb{Z}\times Gr) \arrow[r, "\sim", "\alpha"'] \arrow[dr, dashed, "\gamma"] & \mathrm{Hom}_{(Sm/S)^{opp} \to (Sets)}(K_0(-), K_0(-)) \arrow[d, "\beta", "\wr"'] \\
 & \left[\varprojlim_{d,r} K_0(Gr_{d,r})\right]^{\mathbb{Z}} \arrow[d, phantom, "\wr"] \\
 & \left(K_0(S)[[c_1, c_2, \cdots]]\right)^{\mathbb{Z}}
\end{tikzcd}$$

$\tau: K_0(-) \to K_0(-)$
$\downarrow$
(evaluate on $u_{d,r} + n$, $n \in \mathbb{Z}$)

<u>proof</u> γ: <u>bijection</u>. Milnor's exact seq. $\rightsquigarrow$ γ: surj.

$\rightsquigarrow$ $\mathrm{Ker}\,\gamma = \left[R^1 \varprojlim_{(d,r)} K_1(Gr_{d,r})\right]^{\mathbb{Z}}$

$d \leq d'$
$r \leq r'$

$$\begin{tikzcd}
K_1(Gr_{d',r'}) \arrow[r, two heads] \arrow[d, phantom, "\wr"] & K_1(Gr_{d,r}) \\
K_1(S) \otimes_{K_0(S)} K_0(Gr_{d',r'}) &
\end{tikzcd}$$

K-theory of Grassmannian (SGA6)

β: injective $\tau, \tau' : K_0(-) \longrightarrow K_0(-)$ s.t.

$$\tau(\mathcal{M}_{d,r}+n) = \tau'(\mathcal{M}_{d,r}+n)$$

$X \in Sm/S$ $\quad x \in K_0(X)$

- X: affine and connected

$$x = [M] - \underset{\overset{\shortparallel}{d}}{\mathrm{rk}\, M} + n \qquad n \in \mathbb{Z}$$

M: subbundle of $\mathcal{E}^{d+r}$, $r \gg 0$

$\exists\, f : X \longrightarrow Gr_{d,r}$ s.t. $f^*\mathcal{M}_{d,r} = M$

$x = f^*(\mathcal{M}_{d,r}+n) \Rightarrow \tau(x) = \tau'(x)$

- X: general use Jouanolou's trick

$$\begin{tikzcd} \exists\ T \arrow[d] \\ X \end{tikzcd}$$

torsor under some vector bundle s.t. T affine

$\simeq$ in $\mathcal{H}(S)$ $\quad K_0(X) \simeq K_0(T)$ ▨

Variant with several variables

maps
$$\begin{tikzcd} (\mathbb{Z}\times G)^n \arrow[r] \arrow[d, leftrightarrow, "1:1"] & \mathbb{Z}\times G \text{ in } \mathcal{H}(S) \\ (K_0(-))^n \arrow[r] & K_0(-) \text{ in } (Sm/S)^{opp} \to Sets \end{tikzcd}$$

$\lambda^n : K_0(-) \longrightarrow K_0(-) \rightsquigarrow \lambda^n : \mathbb{Z}\times Gr \longrightarrow \mathbb{Z}\times Gr$

$$\begin{tikzcd} \cdot : K_0(-)\times K_0(-) \longrightarrow K_0(-) \rightsquigarrow \times : (\mathbb{Z}\times G)^2 \arrow[r] \arrow[d] & \mathbb{Z}\times G \\ (\mathbb{Z}\times G)\wedge(\mathbb{Z}\times G) \arrow[ur, dashed] & \end{tikzcd}$$

$K_i(X)\times K_j(X) \longrightarrow K_{i+j}(X)$ (the same as those of Quillen, Loday ~~Waldhausen~~)

Thm $\mathbb{Z}\times Gr$ is a special λ-Ring in $\mathcal{H}(S)$ □

Soulé defined operation for elements of $\begin{array}{c} R_{\mathbb{Z}}GL \\ \downarrow \\ End(\mathbb{Z}\times Gr)\end{array}$

Additive operations.

understand $\tau : K_0(-) \longrightarrow K_0(-)$ that are additive

example Adam's operation $\psi^* : K_0(-) \longrightarrow K_0(-)$

$$\forall x \in K_0(X) \qquad \frac{1}{\lambda_t(x)} \frac{d\lambda_t(x)}{dt} = \sum_{k=1}^{\infty} (-1)^{k-1} \psi^k(x)\, t^{k-1}$$

L line bundle $\quad \psi^k([L]) = [L^{\otimes k}]$

Thm S : regular sch.

$$\begin{tikzcd}
\mathrm{Hom}_{(Sm/S)^{opp}\to(Ab)}(K_0(-),K_0(-)) \arrow[rr] \arrow[ddr, "c^*"', "\wr"] & & \mathrm{Hom}_{(Sm/S)^{opp}\to(Sets)}(\mathrm{Pic}(-),K_0(-)) \arrow[d, phantom, sloped, "\cong"] \\
& & \varprojlim_n K_0(\mathbb{P}^n) \arrow[d, phantom, "||"] \\
& K_0(S)[[U]] \arrow[r, phantom, "="] & \varprojlim_n K_0(S)[U]/(U^{n+1})
\end{tikzcd}$$

$U = [\mathcal{O}(1)] - 1$

proof : injectivity . "splitting principle"

M : v. bundle on X

$\exists$ Flag(M) $\quad [f^*M] = \sum [$line bundles$]$

$\downarrow f$

$X \quad M \nearrow$

$K_0(X) \hookrightarrow K_0(\mathrm{Flag}\, M)$

<u>surjectivity</u> : $c^*(\psi^k) = (1+U)^k$

$x \in K_0(S)$

$x\psi^k : K_0(-) \longrightarrow K_0(-)$ $c^*(x\psi^k) = x(1+U)^k$

$y \longmapsto x \cdot \psi^k(y)$

▨

<u>Rem</u> : there is a composition law

$* : K_0(S)[[U]] \times K_0(S)[[U]] \longrightarrow K_0(S)[[U]]$ $\mathbb{Z}$-bilinear continuous

$x(1+U)^k * y(1+U)^{k'} = (x\psi^k(y))(1+U)^{kk'}$ □

- <u>Regulators</u>

k: perfect field $K(\mathbb{Z}(n), 2n) \in \mathcal{H}_\cdot(k)$

$\mathrm{Hom}_{\mathcal{H}(k)}(X, K(\mathbb{Z}(n), 2n)) \simeq CH^n(X)$

<u>Prop</u> : $\mathrm{Hom}_{\mathcal{H}_\cdot(k)}(\mathbb{Z} \times Gr, K(\mathbb{Z}(n), 2n))$

$$\xrightarrow{\sim} \underbrace{\mathrm{Hom}_{(Sm/S)^{opp} \to (Sets)}(K_0(-), CH^n(-))}_{\text{polynomials in Chern classes}}$$

<u>Prop</u> : $\mathrm{Hom}_{(Sm/S)^{opp} \to (Ab)}(K_0(-), CH^n(-)) \simeq \mathbb{Z}\chi_n$

$\chi_n : K_0(-) \dashrightarrow CH^n(-)$ additive

$\underset{\substack{\text{line bundle} \\ \text{divisor } D}}{[L]} \longmapsto [D]^n$

⇝ computes maps

$$\mathbb{Z}\times G \longrightarrow \mathbb{H}_A(q)[p]$$

$A\cdot\mathbb{Q}$ Chern character $K_0(-) \to \bigoplus CH^*(-)_{\mathbb{Q}}$

$$BGL \longrightarrow \bigoplus_p \mathbb{H}_{\mathbb{Q}}(p)[2p]$$

higher Chern character $K_i \longrightarrow H^{2n+i}_{\mu}(X.\mathbb{Q}(n))$

No. 1

Date

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$$B\mu_\ell \rightarrowtail \mathcal{O}(-\ell)_{\mathbb{P}^\infty} \twoheadrightarrow Th_{\mathbb{P}^\infty}(\mathcal{O}(-\ell))$$

$\rightsquigarrow$ $\qquad\qquad$ $\mathbb{Z}/\ell$ - coef!

$$0 \to \tilde{H}^{*,*}(k)[\![\sigma]\!] \to H^{*,*}(B\mu_\ell) \to (\tilde{H}(k)[\![\sigma]\!])^{*-1,*-1} \to 0$$

$$\sigma \longmapsto v$$

$$\sigma \in [\mathcal{O}(-1)] \in H^{2,1}(\mathbb{P}^\infty)$$

$v :=$ c(taut. line bundle on $B\mu_\ell$) $\in H^{2d}$

<u>Lem</u> $\quad H^{1,1}(B\mu_\ell, \mathbb{Z}/\ell) \xrightarrow{\delta} H^{2,1}(B\mu_\ell, \mathbb{Z})$

$$\exists! \; \overset{\rotatebox{90}{$\in$}}{u} \longmapsto \overset{\rotatebox{90}{$\in$}}{v} \qquad u|_* = 0$$

<u>Prop</u> $\quad v^i, uv^i \; (i \geq 0)$ forms a basis of $\tilde{H}^{*,*}(F_\cdot \wedge (B\mu_\ell)_+, \mathbb{Z}/\ell)$ over $\tilde{H}^{*,*}(F_\cdot, \mathbb{Z}/\ell)$

□

$u^2 = ?$ $\qquad \ell > 2 \Rightarrow$ graded commutativity $\quad u^2 = 0$

<u>$\ell = 2$</u> $\quad H^{2,2}(B\mu_\ell, \mathbb{Z}/2) = H^{0,1}(k, \mathbb{Z}/2)\, v \oplus H^{1,1}(k, \mathbb{Z}/2)\, u \oplus H^{2,2}(k, \mathbb{Z}/2)$

$$\begin{array}{ccccccc} \rotatebox{90}{$\in$} & & (2,1) & & (1,1) & & u|_* = 0 \\ u^2 & = & x v & + & y u & + & 0 \end{array}$$

<u>Fact</u> $\quad x = \tau \in H^{0,1} \simeq \mu_2 \qquad y = \rho \in H^{1,1} \simeq k^\times / (k^\times)^2$

<u>Prop</u> $\quad \tilde{H}^{*,*}(F_\cdot \wedge (B\mu_\ell)_+, \mathbb{Z}/\ell)$

$$\simeq \begin{cases} \tilde{H}^{*,*}(F_\cdot, \mathbb{Z}/\ell)[\![u, v]\!]/(u^2) & \ell > 2 \\ \tilde{H}^{*,*}(F_\cdot, \mathbb{Z}/\ell)[\![u, v]\!]/(u^2 - \tau v - \rho u) & \end{cases}$$

□

$\rightsquigarrow B\mathfrak{S}_\ell$ $\quad \xi_\ell$: taut. vect. bundle on $B\mathfrak{S}_\ell$

$(\mathfrak{S}_\ell \xrightarrow{id} \mathfrak{S}_\ell)$

$d := c(\xi_\ell / \mathcal{O}) \in H^{2(\ell-1), \ell-1}(B\mathfrak{S}_\ell, \mathbb{Z}/\ell)$

<u>Thm</u> $\exists ! \; c \in H^{2\ell-3, \ell-1}(B\mathfrak{S}_\ell) \quad \delta(c) = d, \; c|_* = 0$ $\square$

graded commutative

$\cup$

$\ell > 2 \Rightarrow c^2 = 0$

$\ell = 2 \Rightarrow B\mu_2 = B\mathfrak{S}_2$

$$\left(k \ni \zeta \qquad B\mu_\ell \xrightarrow[p_\xi]{} B\mathfrak{S}_\ell \qquad \begin{array}{l} p_\xi^*(d) = \prod_{i=1}^{\ell-1}(iv) = -v^{\ell-1} \\ p_\xi^*(c) = -uv^{\ell-2} \end{array} \right)$$

$\ell > 2$

$$\begin{tikzcd}
\tilde{H}(F_\cdot)[\![c, d]\!]/(c^2) \arrow[r] \arrow[ddr, bend right, "\sim"] & \tilde{H}^{*,*}(F_\cdot \wedge (B\mathfrak{S}_\ell)_+, \mathbb{Z}/\ell) \arrow[d, hook, "p_\xi^*"] \\
& \tilde{H}^{*,*}(F_\cdot \wedge (B\mu_\ell)_+, \mathbb{Z}/\ell)^{Aut(\mu_\ell)} \arrow[d, phantom, "="] \\
& \tilde{H}^{*,*}(F_\cdot)[\![x, y]\!]/(x^2)
\end{tikzcd}$$

$x = uv^{\ell-2}, \quad y = v^{\ell-1}$

<u>Thm</u> $\tilde{H}^{*,*}(F_\cdot \wedge (B\mathfrak{S}_\ell)_+, \mathbb{Z}/\ell)$

$$\begin{cases} \tilde{H}^{*,*}(F_\cdot, \mathbb{Z}/\ell)[\![c, d]\!]/(c^2) & \ell > 2 \\ \tilde{H}^{*,*}(F_\cdot, \mathbb{Z}/\ell)[\![c, d]\!]/(c^2 - \tau d - \rho c) & \ell = 2 \end{cases}$$

$\deg c = (2\ell-3, \ell-1)$

$\deg d = (2\ell-2, \ell-1)$

$\square$

total power operation

$$K_{n,R} = K(\mathbb{Z}(n) \otimes R, 2n) = R \otimes \mathbb{Z}_{tr}((\mathbb{P}^1, \infty)^{\wedge n})$$
$$= R \otimes \mathbb{Z}_{tr}(\mathbb{A}^n/(\mathbb{A}^n - \{0\}))$$

construction ①

$X \in Sm/k$, R: commutative ring

$$\begin{tikzcd}
\left[\begin{array}{l} \text{input } E: \text{vect. bundle} \\ \text{cycle } Z \subset E \\ \text{fin. equidim } \searrow \ \downarrow \\ \qquad X \end{array}\right] \arrow[r, squiggly] \arrow[d] & \left[\begin{array}{l} \text{output} \\ \lambda(Z) \in H^{2rk(E), rk(E)}(X, R) \\ \text{Not depend on } L, \phi \end{array}\right] \\
\left[\begin{array}{l} \text{auxiliary} \\ L: \text{v.b on } X \\ \phi: E \oplus L \xrightarrow{\sim} \mathcal{O}^N \end{array}\right] \arrow[r, squiggly] & \left[Th_X(L) \xrightarrow[\lambda(Z)]{} K_{N,R}\right] \arrow[u, "\text{Thom isom.}", "\wr"']
\end{tikzcd}$$

$$\begin{tikzcd}
Z \subset E \{(\ell' \text{ over } x), Z_x \times \ell'\}_{(\ell' \text{ over } x)} \arrow[d] & \subset & L \times_X E \times_X L \arrow[d] \overset{\phi}{\underset{\wr}{/}} \mathbb{A}^N_X \\
(\ell' \text{ over } \underset{X}{\overset{\cap}{x}}) & & L
\end{tikzcd}$$

$Z \xrightarrow{\text{fin. equidim}} X$, $\{\ldots\}_{(\ell' \text{ over } x)} \xrightarrow{\text{fin. equidim}} L$

$$\begin{tikzcd}
 & L \arrow[r] & \mathbb{Z}_{tr}(\mathbb{A}^N_X) \\
Th_X(L) \xrightarrow[\lambda(Z)]{} K_{N,R} & L - X \arrow[u, phantom, "\cup"] \arrow[r] \arrow[l, dashed] & \mathbb{Z}_{tr}(\mathbb{A}^N_X - \{0\}) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

input. G: fin. grp.

$r : G \longrightarrow \mathfrak{S}_n$ hom

$U \in Sm/k$

$\overset{\curvearrowleft}{G}$ freely

(ξ_n vector bundle on V/G)

$\rightsquigarrow$

output

$P : K_{i,R} \wedge (U/G)_+ \longrightarrow K_{in,R}$

Not depend on ϕ, L

Thom iso.

($G = \mathfrak{S}_\ell$ $n = \ell$ $r = id$

$U = \mathbb{A}^{m\ell} - \Theta$ $R = \mathbb{Z}/\ell$ $V/G \sim B\mathfrak{S}_\ell$)

auxiliary.

L: vect. b. on U/G

$\phi : \xi_n \oplus L \xrightarrow{\sim} \mathcal{O}^N$

$\tilde{P} : K_{i,R} \wedge Th_{U/G}(L^i) \to K_{iN,R}$

auxiliary

$$\begin{tikzcd} Z \arrow[r, phantom, "\subset"] \arrow[dr, "\text{fin equi dim}"'] & X \times \mathbb{A}^i \arrow[d] \\ & X \end{tikzcd}$$

$\rightsquigarrow$

$X_+ \wedge Th(L^i) \xrightarrow[a(Z)]{} K_{iN,R}$

$$\begin{tikzcd} Z \arrow[r, phantom, "\subset"] \arrow[dr, "\text{fin equi dim}"'] & X \times \mathbb{A}^i \arrow[d] \\ & X \end{tikzcd}$$

$$\begin{tikzcd} p^*(Z^{\times n}) \arrow[r, phantom, "\subset"] \arrow[d] & (X \times \mathbb{A}^i)^n \times U \arrow[d] \\ Z' \arrow[r, phantom, "\subset"] \arrow[dr, "\text{fin equi dim}"'] & ((X \times \mathbb{A}^i)^n \times U)/G \arrow[d] \\ & (X^n \times U)/G \end{tikzcd}$$

$X \times \xi_n^i$

$\|$

$$\begin{tikzcd} Z'' \arrow[r, phantom, "\subset"] \arrow[dr, "\text{fin equi dim}"'] & X \times (\mathbb{A}^{in} \times U)/G \arrow[d] \\ & X \times U/G \end{tikzcd}$$

Δ^*

$(\Delta : X \to X^n)$

$a(Z'') : X_+ \wedge Th_{U/G}(L^i) \to K_{iN,R}$

Construction (1)

$G = \mathfrak{S}_\ell \quad n = \ell \quad r = id \quad U = \underset{(m \to \infty)}{\mathbb{A}^{\ell m} - \Delta}, \ R = \mathbb{Z}/\ell$

$P : \tilde{H}^{2d,d}(F_\cdot, \mathbb{Z}/\ell) \longrightarrow \tilde{H}^{2d\ell, d\ell}((F_\cdot \wedge B\mathfrak{S}_\ell)_+, \mathbb{Z}/\ell)$

total power operation

$\rightsquigarrow P^i, B^i$

property of $P \rightsquigarrow$ property of P^i, B^i

$P(a \wedge b)$
$= \Delta^*(P(a) \wedge P(b)) \rightsquigarrow$ Cartan formula

symm. thm $\rightsquigarrow$ Adem relation

$\beta P = 0 \rightsquigarrow \beta B^i = 0, \ \beta P^i = B^i$

$m < n \Rightarrow \tilde{H}^{*,m}(K_{n,A}, B) = 0 \rightsquigarrow P^i = 0 \ B^i = 0 \ i < 0$

$\mathcal{A}^{*,*} := \langle \beta, P^i \ (i \geq 0), \ \lambda \in H^{*,*}(k) \rangle_{\text{left } H^{*,*}(k)\text{-alg}}$

$\subset$ (bistable coh. operation)

$\overset{?}{=} \mathrm{Hom}_{Spt}(H\mathbb{Z}/\ell, \Sigma_s^* \Sigma_t^* H\mathbb{Z}/\ell)$

$\mathcal{A}_{*,*} = \mathrm{Hom}_{\text{left } H^{*,*}(k)\text{-mod}}(\mathcal{A}^{*,*}, H^{*,*}(k))$

$H^{*,*}$ cup prod $\rightsquigarrow \mathcal{A}^{*,*}$: coprod. $\rightsquigarrow \mathcal{A}_{*,*}$. prod.

杉原望

On the original Milnor conj.

§ 1937 Witt

<u>Def.</u> k field $\mathrm{ch} \neq 2$

$$\text{quadratic space} = (\overset{\text{fin. dim } k\text{-v. sp.}}{V}, \overset{\text{symm. bilin form}}{\mu})$$

□

(V, μ), (V', μ') : quad sp.

$$\underset{\text{isom}}{V \simeq V'} \overset{}{\underset{\text{def}}{\iff}} \exists\, V \xrightarrow[\sim]{\tau} V' \text{ isom of v sp}$$
$$\text{s.t. } \mu' \circ \tau = \mu$$

Classify (V, μ)'s

<u>ex.</u> a, $a \in k$ $\langle a \rangle \cdots$ 1-dim k-v sp $k\overset{\text{base}}{e}$
$\mu(e, e) = a$

□

$V = (V, \mu)$, $V' = (V', \mu')$

$\rightsquigarrow$ $V \perp V'$: v. sp. $V \oplus V'$ $\mu \oplus \mu'(v \oplus v', w \oplus w')$
$= \mu(v, w) + \mu'(v', w')$

$V \otimes V'$: v sp. $V \otimes V'$ $\mu \otimes \mu'(v \otimes v', w \otimes w')$
$= \mu(v, w) \cdot \mu'(v', w')$

$\langle a_1, \cdots, a_n \rangle := \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$

$\rightsquigarrow$ every (V, μ) is isomorphic to some $\langle a_1, \cdots, a_n \rangle$

$\langle a_1, \cdots, a_n \rangle \otimes \langle b_1, \cdots, b_n \rangle = \langle \cdots a_i b_j \cdots \rangle$

Def $v \in V$: isotropic $\underset{\text{def}}{\iff}$ $\mu(v,v)=0$

V: isotropic $\underset{\text{def}}{\iff}$ $\exists \underset{\neq 0}{v} \in V$, v: isotropic

V: an " $\iff$ $\forall \underset{\neq 0}{v} \in V$ v not —

Lem $V \simeq \mathbb{H} = \langle 1,-1 \rangle$

$\iff \exists \{x,y\}$ basis of V s.t. $\begin{cases} x, y : \text{isot.} \\ \mu(x,y) \neq 0 \end{cases}$

$\iff V \simeq \langle a, -a \rangle$ $(\exists a \in k^\times)$ □

Witt's decomp. thm (1937)

$\forall (V,\mu)$

$(V,\mu) \simeq (V_{an}, \mu_{an}) \perp \mathbb{H}^m \perp (V_t, \mu_t)$

isom class of anisot.

totally isot. (i.e. $\forall v \in V$ $\mu(v,v)=0$)

$\left.\begin{matrix}(V_{an}, \mu_{an}) \\ m \\ (V_t, \mu_t)\end{matrix}\right\}$ uniquely det. by (V,μ) □

Witt's cancellation thm

$(V_1,\mu_1) \perp (V,\mu) \simeq (V_2,\mu_2) \perp (V,\mu)$

$\Rightarrow (V_1,\mu_1) \simeq (V_2,\mu_2)$

Def. (Witt ring)

$$M(k) = \{\text{non deg quad sp.}\}/_{\simeq} \hookrightarrow \widehat{W}(k) = \text{the Groth. gp. of } M(k)$$

$\perp, \otimes$: semiring / cancellation ring

Lem $\mathbb{Z}\cdot\mathbb{H} \subset \widehat{W}(k)$: ideal
$(\langle a, -a\rangle \simeq \langle 1, -1\rangle)$

Def. $W(k) := \widehat{W}(k)/\mathbb{Z}\cdot\mathbb{H}$: Witt ring

$$\updownarrow 1:1$$

$$\{\text{anisot.}\}/\simeq$$

classical invariant

$(V,\mu) \rightsquigarrow m := \dim V \in \mathbb{Z}$

non-deg. $\rightsquigarrow (-1)^{\frac{m(m-1)}{2}} \det\mu \in k^\times/(k^\times)^2$

$\quad /\!/$
$d(V,\mu)$

Clifford alg $T(V)/(v\otimes v - \mu(v,v))$

$$\rightsquigarrow c(V,\mu) := \begin{cases} [C(V)] & (\dim V : \text{even}) \\ [C_0(V)] & (\dim V : \text{odd}) \end{cases}$$

C_0: even part

$\in {}_2\mathrm{Br}(k)$

$\xrightarrow{\text{induces}}$ a map

$$\begin{tikzcd} W(F) \arrow[r, "e_0"] \arrow[dr, "e_1"] \arrow[ddr, "e_2"'] & \mathbb{Z}/2 \\ & k^\times/(k^\times)^2 \\ & {}_2\mathrm{Br}(k) \end{tikzcd}$$

(Rem: not a hom)

Def. $I := \mathrm{Ker}(e_0)$, $\mathrm{Gr}_I W := \bigoplus_{n \geq 0} I^n / I^{n+1}$ □

$\rightsquigarrow$ $e_1|_I$, $e_2|_{I^2}$ grp hom

$\bar{e}_0 : W/I \xrightarrow{\simeq} \mathbb{Z}/2$

$\bar{e}_1 : I/I^2 \xrightarrow{\simeq} k^\times/(k^\times)^2$

§ 1970 Milnor

$$\begin{tikzcd}
 & K^M_n k/2 \arrow[dl, "s_n"'] \arrow[dr, "h_n"] & \\
I^n/I^{n+1} \arrow[rr, dashed, "\bar{e}_n"'] & & H^n_{Gal}(k, \mathbb{Z}/2)
\end{tikzcd}$$

Milnor conj : s_n, h_n isom

$$s_n(\{a_1, \cdots, a_n\}) := \langle\langle a_1, \cdots, a_n \rangle\rangle := \overset{2^n\text{-dim}}{\bigotimes_{i=1}^{n} \langle 1, -a_i \rangle}$$

(Pfister form).

Rem $s_n, h_n, \bar{e}_n$ are compatible $(n=0,1,2)$

- Milnor also conjectured that $\bigcap_{n \geq 0} I^n = 0$

This was proven by Arason-Pfister (1971) □

s_2 : isom … Milnor (1970)

h_2 : isom … Merkurjev (1981)

h_n : Voevodsky (1996)

s_n : Orlov-Vishik-Voevodsky (2001), Morel (2004)

§ 2001 Voevodsky

Idea of the proof by OVV

Elman-Lam (1971): $\langle\langle a_1,\dots,a_n\rangle\rangle = \langle\langle b_1,\dots b_n\rangle\rangle$ mod I^{n+1}

$\Leftrightarrow$ —— = ...

$\Leftrightarrow \{a_1,\dots,a_n\} = \{b_1,\dots,b_n\}$ in $K_n^M k/2$

In particular

$s_n(\{a_1,\dots,a_n\}) = 0 \Rightarrow \{a_1,\dots,a_n\} = 0$

It suffices to prove k: field, $x \in K_n^M k/2$, $x \neq 0$

$\Rightarrow \exists E/k$ s.t. $x_F \in K_n^M F/2$ is of the form $\{a_1,\dots,a_n\}$

& $x_E \neq 0$

Def. $\underline{a} = (a_1,\dots,a_n) \in (k^\times)^n$ dim $2^{n-1}-1$

$Q_{\underline{a}} := \left(\begin{array}{c}\text{the quadric ass. to the quad. sp.} \\ \langle\langle a_1,\dots a_{n-1}\rangle\rangle \perp \langle -a_n\rangle\end{array}\right)$

$K_{\underline{a}} = k(Q_{\underline{a}})$

ex $n=2$

$Q_{\underline{(a_1,a_2)}} = (x_0^2 - a_1x_1^2 - a_2x_2^2 = 0)$

$n=3$

$Q_{\underline{(a_1,a_2,a_3)}} = (x_0^2 - a_1x_1^2 - a_2x_2^2 + a_1a_2x_3^2 - a_3x_4^2 = 0)$ □

Prop $\{a_1,\dots a_n\} = 0$ in $K_n^M(K_{\underline{a}})/2$
(calculation)

Key Thm (OVV)

$$\{a_1, \ldots, a_n\} \neq 0 \Rightarrow 0 \to \mathbb{Z}/2 \to K_n^M k/2 \to K_n^M K_{\underline{a}}/2 \quad \text{(exact)}$$

generated by $\{a_1, \ldots, a_n\}$

(calculated by the technique of $\mathbb{A}^1$-ho cat.
DM
+ Assume $BK(n,2)$)

Assuming Key Thm

$x \neq 0$, $\in K_n^M k/2$ given, $x = y_1 + y_2 + \cdots + y_r$

y_i is of the form
$\{\alpha_1, \cdots, \alpha_n\}$

$E_i := k(Q_{y_1} \times \cdots \times Q_{y_i})$

Then

$$\begin{tikzcd}
K_n^M k/2 \arrow[r] & \cdots \arrow[r] & K_n^M E_j/2 \arrow[r] & K_n^M E_{j+1}/2 \arrow[r] & \cdots \arrow[r] & K_n^M E_r/2 \\
x \neq 0 \arrow[u, phantom, "\in"] \arrow[rr, mapsto] & & x' \arrow[u, phantom, "\in"] \arrow[r, mapsto] & 0 \arrow[u, phantom, "\in"] \arrow[rr, mapsto] & & 0 \arrow[u, phantom, "\in"] \\
 & & 0 \arrow[u, phantom, "\neq"] & & &
\end{tikzcd}$$

$\xRightarrow{\text{Key Thm}} x' = y_j$ ∎

proof of Key thm

(sketch) $\mathbb{Z}(n) \in D^-(Shv_{zar}(Sm/k))$ A abel

$$H^{p,q}(X, A) := H^p_{zar}(X, \mathbb{Z}(q) \overset{L}{\otimes} A)$$

$$H_L^{p,q}(X, A) := H^p_{et}(X, \mathbb{Z}(q) \overset{L}{\otimes} A)$$

$$\begin{tikzcd}
\rightsquigarrow & \mathbb{Z}/2(n-1) \arrow[r, "\tau"] & \mathbb{Z}/2(n) \arrow[r] & H^n(\mathbb{Z}/2(n))[-n] \arrow[r, "+1"] & {} \\
& & \tau : \mathbb{Z}/2 \to \mathbb{Z}/2(1) \arrow[ul, squiggly] & R^n\varepsilon_*\cancel{\mu}_2^{\otimes n} \arrow[u, phantom, "\shortparallel"] &
\end{tikzcd}$$

Def $Y \in Sm/k$

$$\check{C}(X) = \left([n] \longmapsto \overbrace{X \times \cdots \times X}^{n+1}\right) \in \Delta^{op}\, Sm/k$$

(Čech s. sch)

Lem ① If $\mathrm{Hom}(Y, X) \neq \phi$

then $\check{C}(X) \times Y \xrightarrow{\simeq} Y$ in $\mathcal{H}^{\mathbb{A}^1}$

In particular $H^{*,*}(Y) \simeq H^{*,*}(\check{C}(X) \times Y)$

② $H_L^{*,*}(\mathrm{Spec}\, k) \simeq H_L^{*,*}(\check{C}(X))$ (Hochschild-Serre / If $k = k^{sep}$, use the above) □

Then (exact)

$$\begin{tikzcd}
0 \arrow[r] & H^{n,n-1}(\check{C}(Q_a), \mathbb{Z}/2) \arrow[r] \arrow[d, hook, "\text{vanishing of Margolis coh}"'] & H^{n,n}(\check{C}(Q_a), \mathbb{Z}/2) \arrow[r] \arrow[d, phantom, "\wr\ BK(n,2)"] & H^0(\check{C}(Q_a), H^n(\mathbb{Z}/2(n))) \arrow[d, hook, "\text{Gersten conj for preth}"] \\
& H^{2^n-1, 2^{n-1}-1}(\check{C}(Q_a), \mathbb{Z}/2) & K_n^M k/2 & H^n(\mathbb{Z}/2(n))(k(Q_a)) \arrow[d, phantom, "\shortparallel\ BK"] \\
& \mathbb{Z}/2 \arrow[u, "\text{Rosts result}"'] & & K_n^M(k(Q_a))/2
\end{tikzcd}$$

池田京司

X : proj smooth var / $\mathbb{C}$

$$Z_i(X) = \bigoplus_{\substack{Z \subset X \\ \text{sub var} \\ \dim i}} \mathbb{Q}\, Z$$

$$\cup$$

$Z_i(X)_{rat}$ $\qquad CH_i(X) = Z_i(X)/Z_i(X)_{rat}$

properties of alg. cycles

- characterized by differential forms
- influenced by definition field

1. topological cycles and alg. cycles

$\varphi : Z_i(X) \longrightarrow H_{2i}(X, \mathbb{Q})$

(cycle map)

γ : top. $2i$-cycle

$$[\gamma] = 0 \in H_{2i}(X, \mathbb{Q}) \Longleftrightarrow \int_\gamma \omega = 0 \quad \forall \omega \in \underset{\substack{\uparrow \\ \text{Hodge filt.}}}{F^i} H^{2i}_{dR}(X)$$

$$[\gamma] : \text{algebraic} \quad \Longrightarrow \quad \int_\gamma \omega = 0 \quad \forall \omega \in F^{i+1} H^{2i}_{dR}(X)$$

$$([\gamma] \in \mathrm{Im}\,\varphi) \qquad \underset{\substack{\uparrow \\ \text{Hodge conj.}}}{(\Leftarrow)}$$

$Z_i(X)_{hom} = \mathrm{Ker}(\varphi)$

$\cup$

$Z_i(X)_{rat}$

2. Abel-Jacobi map

$$\begin{array}{ccl} \rho : Z_i(X)_{hom} & \longrightarrow & \mathrm{Hom}(F^i H^{2i+1}_{dR}(X), \mathbb{C}) \big/ H_{2i+1}(X, \mathbb{Q}) \\ \rotatebox{90}{$\in$} & & \qquad\quad \rotatebox{90}{$\in$} \\ \alpha & & \Big[\omega \longmapsto \int_\Gamma \omega \Big] \\ \| & & \\ \partial\Gamma & & \end{array}$$

$$\alpha \in Z_i(X)_{rat} \quad \Rightarrow \quad \rho(\alpha) = 0$$

$$\not\Leftarrow \left(\text{e.g. Mumford: } \dim X = 2 \quad H^0(\Omega^2_X) \neq 0 \atop \Rightarrow Z_0(X)_{hom} / Z_0(X)_{rat} : \underset{\mathbb{C}}{\text{big}} \right)$$

<u>Conj</u> (Bloch Beilinson)

If X and α are defined $/ \bar{\mathbb{Q}}$ then $\rho(\alpha) = 0 \Rightarrow \alpha \in Z(X)_{rat}$

3. Differential forms / $\mathbb{Q}$

$\alpha \in Z_i(X)$ $\qquad \mathbb{Q} \subset {}^{\exists}K \subset \mathbb{C}$ $\qquad \exists X_K$: proj smooth $/K$

$\exists\, \alpha_K \in Z_i(X_K)$ $\quad$ s.t. $\dim_K \Omega^1_{K/\mathbb{Q}} =: m < \infty$ $\quad X_K \otimes_K \mathbb{C} = X$

$K = \bar{K}$ $\qquad \alpha_K \otimes_K \mathbb{C} = \alpha$

<u>natural pairing</u>

$$\langle\ ,\ \rangle : \underset{\substack{Y_K \subset X_K \\ \text{smooth} \\ i\text{-dim}}}{Z_i(X_K)} \times H^i(X_K, \Omega^{i+m}_{X_K/\mathbb{Q}}) \longrightarrow \Omega^m_{K/\mathbb{Q}} \ (\ni k)$$

$$\langle Y_K, * \rangle \quad H^i(\Omega^{i+m}_{X_K/\mathbb{Q}}) \longrightarrow H^i(\Omega^{i+m}_{Y_K/\mathbb{Q}})$$
$$\cong H^i(\Omega^i_{Y_K/K}) \otimes \Omega^m_{K/\mathbb{Q}}$$
$$\cong \Omega^m_{K/\mathbb{Q}}$$

$$Z_i(X)_\Omega := \{ \alpha \in Z_i(X) \mid \langle \alpha_K, \omega \rangle = 0 \quad \forall \omega \in H^i(\Omega^{i+m}_{X_K}) \}$$

$$Z_i(X)_{rat} \subset Z_i(X)_\Omega \subset Z_i(X)_{hom}$$

$$0 \to \Omega^1_{K/\mathbb{Q}} \otimes \mathcal{O}_{X_K} \to \Omega^1_{X_K/\mathbb{Q}} \to \Omega^1_{X_K/K} \to 0$$

$$Fil^p\,\Omega^j_{X_K/\mathbb{Q}} := Im(\Omega^p_{K/\mathbb{Q}} \otimes \Omega^{j-p}_{X_K/\mathbb{Q}} \to \Omega^j_{X_K/\mathbb{Q}})$$

$$\rightsquigarrow F^p_\Omega Z_i(X) = \{\alpha \in Z_i(X) \mid \langle \alpha, \omega \rangle = 0 \ \forall \omega \in Fil^{m+1-p} H^i(\Omega^{i+m}_{X_K/\mathbb{Q}})\}$$

- $F^0_\Omega Z_i(X) = Z_i(X)$
- $F^1_\Omega Z_i(X) = Z_i(X)_{hom}$
- $p > \dim X - i \quad F^p_\Omega Z_i(X) = Z_i(X)_\Omega$

Infinitesial inv.

$$Gr^p_{F\Omega} Z_i(X_K) \times \underbrace{Gr^{m-p}_{Fil} \ H^i(\Omega^{i+m}_{X_K/\mathbb{C}})} \longrightarrow \Omega^m_{K/\mathbb{Q}}$$

$$\Vert$$

$$H(\cdot \to \Omega^{m-p}_{\mathbb{Q}} \otimes H^i(\Omega^{i+p}_{X_K/K}) \to \Omega^{m-p+1}_{K/\mathbb{Q}} \otimes H^{i+1}(\Omega^{i+p+1}_{X_K/K}))$$

non trivial $\alpha \in Gr^*_{F\Omega} Z_i(X)$ tr.deg of $\alpha \geq p$

4. Motive and filtr. on Chow grp.

Conj. (Bloch - Beilinson)

$\exists$ F_M filt. on Chow gp s.t.

$CH_i(X) = CH^r(X) = F^0_M\, CH^r(X) \qquad r + i = \dim X$

$CH^r(X)_{hom} = F^1_M\, CH^r(X)$

$Gr^p_{F_M} CH^r(X) \simeq Ext^p(h(\mathrm{Spec}\, \mathbb{C}), h^{2r-p}(X)(r))$

5. Algebraic equivalence

$$CH_i(X)_{alg} := \sum_{\substack{Y:\text{proj sm}/\mathbb{C} \\ T \in CH_{i+\dim Y}(Y \times X)}} \operatorname{Im}\left(CH_0(Y)_{hom} \xrightarrow{T_*} CH_i(X)\right)$$

$\cap \; \neq$

$CH_i(X)_{hom}$

(e.g. (Griffiths)
$X \subset \mathbb{P}^4$: generic quintic hypersurf
$Griff(X) := CH_1(X) / CH_1(X)_{alg} \neq 0$)

$$F^p_\Omega CH_i(X)_{alg} := \sum_{Y, T} \operatorname{Im}\left(F^p_\Omega CH_0(Y) \longrightarrow CH_i(X)\right)$$

$$Griff^p_i(X) = F^p_\Omega CH_i(X) / F^p_\Omega CH_i(X)_{alg}$$

6. Examples

C : curve $\quad y^e = x^d + a_1 x^{d-1} + \cdots + a_d \quad (e > d)$

$a_1, \dots, a_d \in \mathbb{C}$ alg. indep. $/\mathbb{Q}$

J : jacobian of C $\Rightarrow$ $i \geq 1,\ p \geq 1,\ d \geq i+p+1$

$\Rightarrow 0 \neq Griff^p_i(J)$

No. 1

Date

木村俊一

$CH_* X = \{\text{alg cycles}\} \ni \sum n_i [V_i]$

$\sim$ deformation over $\mathbb{P}^1$
$\updownarrow$
$[0,1]$

X smooth proj $/\mathbb{C}$

C alg. curve $\quad Z_0 C \ni [P_1] + \cdots + [P_n]$

$$\underbrace{C \times \cdots \times C}_{n} / \mathfrak{S}_n = \mathrm{Sym}^n C$$

$\mathrm{Sym}^n C \longrightarrow J(C)$ fibers are proj. sp.

$\Rightarrow \quad CH_0(C) \simeq \mathbb{Z} \oplus J(C)$ ← finite dimensional

$X = C \times D$, $g(C) > 0$, $g(D) > 0$ universal abel. var.

$$\begin{tikzcd} X \arrow[r] & \mathrm{Alb}(X) = J(C) \times J(D) \\ \mathrm{Sym}^n X \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} CH_0 C \otimes_{\mathbb{Z}} CH_0 D \arrow[r, two heads] & CH_0(C \times D) \\ {[P] \otimes [Q]} \arrow[r, mapsto] & {[(P,Q)]} \\ \mathbb{Z} \arrow[r, "\sim"] & \mathbb{Z} \\ & \oplus \\ J(C) \oplus J(D) \arrow[r] & \mathrm{Alb}(C \times D) \\ & \oplus \\ J(C) \otimes_{\mathbb{Z}} J(D) \arrow[r, two heads] & ? \end{tikzcd}$$

Albanese Kernel
$\mathrm{Ker}(CH_0(C \times D)$
$\to \mathbb{Z} \oplus \mathrm{Alb}($

Thm (Mumford 1969) $\quad (P_g(C \times D) = P_g(C) \cdot P_g(D) > 0)$

$CH_0(C \times D)$ is "infinite dimensional" in the following sense.

$\forall N > 0$, $\exists U \underset{\text{dense open}}{\subset} \mathrm{Sym}^N(C \times D)$ s.t.

$$\begin{tikzcd} U \arrow[r] & CH_0(C \times D) \\ {[P_1] + \cdots + [P_n]} \arrow[u, phantom, "\in"] & \\ U \arrow[u, phantom, "\text{sub var}"] & \\ W \arrow[u, phantom, "\cup"] \arrow[r] & \{*\} \arrow[uuu, phantom, "\in"] \end{tikzcd}$$

$\Rightarrow \dim W \leq N \quad (\Leftrightarrow \operatorname{cod}_{\mathrm{Sym}^N(C\times D)} W \geq N)$ $\square$

If $\dim T = d \Rightarrow {}^{\forall}P \in T$ is defined by d equasions (locally, set-theoretically

${}^{\forall} S \xrightarrow{\varphi} T$, $\varphi^{-1}(P)$ is defined by d equations

$\Rightarrow \operatorname{cod}(\varphi^{-1}(P)) \leq d$

If $CH_0(C\times D)$ has a "scheme structure"

$\Rightarrow \dim CH_0(C\times D) \geq N \; ({}^{\forall}N) \Rightarrow \dim CH_0(C\times D) = \text{"}\infty\text{"}$

Because $J(C)$ and $J(D)$ are finite dimensional, should Alb. kernel behave finite dimensionally?

Goal 1. $N \gg 0$ $\quad \beta_1, \cdots, \beta_N \in \mathrm{Alb\ Ker} \subset CH_0(C\times D)$

$(N = p_g(C\times D)+1)$ $\quad \beta_1 \wedge \cdots \wedge \beta_N := \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma)(\beta_{\sigma(1)} \times \cdots \times \beta_{\sigma(N)})$

$\in CH_0((C\times D)^N))$

$\Downarrow$

$\beta_1 \wedge \cdots \wedge \beta_N = 0$

Tool: Pontrjagin Product of Chow group of Abelian var.

A: abelian var $\quad \mu : A \times A \to A$ multiplication morphism

$$\begin{tikzcd}
A \times A \arrow[r] & A \\
(P, Q) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & P+Q \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$$\begin{tikzcd}
CH_* A \otimes CH_* A \arrow[r] & CH_* A \\
\alpha \otimes \beta \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto, dashed] & \mu_*(\alpha \times \beta) =: \alpha * \beta
\end{tikzcd}$$
Pontrjagin produc

$P, Q \in A \Rightarrow [P] + [Q] = [P+Q]$ $\quad CH_0 A$ is a subring

$$\begin{tikzcd}
CH_0 A \arrow[r, "\deg"] & \mathbb{Z} \\
\sum n_i [P_i] \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \sum n_i \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$
is a ring hom.

$\mathrm{Ker}(\deg) = \underset{\substack{\| \\ CH_0(A)_{hom}}}{I}$ is an ideal generated by $[P]-[O]$

$CH_0A \supset I \supset I^{*2} \supset \cdots$ (Rem $I/I^{*2} \simeq A$)

Thm (Bloch: $I^{*(\dim A+1)} = 0$

(∵) outline : Mukai-Beauville Fourier transform

$\hat{A}$: dual Abelian var. $\mathcal{P}/A\times\hat{A}$: Poincaré line bundle

$\underset{\substack{\Cap \\ CH_*(A\times\hat{A})}}{\mathcal{F}} := \exp(c_1(\mathcal{P})) = \sum_{i=0}^{\infty} c_1(\mathcal{P})^i/i!$ ← considered as a correspondence $A \vdash \hat{A}$

① $\mathcal{F}_* : CH_*A \xrightarrow{\sim} CH_*\hat{A}$ $\left((∵)\ (\hat{\mathcal{F}}\circ\mathcal{F})_* = (-1)^{\dim A}(-1_A)_* \right)$

② $\mathcal{F}(\alpha * \beta) = (-1)^{\dim A}(-1_{\hat{A}})_*(\mathcal{F}(\alpha)\cdot\mathcal{F}(\beta))$
↑ intersection product

③ $\alpha \in I \subset CH_0(A) \Rightarrow \mathcal{F}(\alpha) \in \bigoplus_{i=0}^{\dim A-1} CH_i(\hat{A})$

$\left(\mathcal{F}([P]) = [\hat{A}] + c_1(L_p) + c_1(L_p)^2/2! + \cdots \right)$
$\quad \rotatebox{90}{$\in$}\ L_p/\hat{A}$

$N = \dim A+1$. $\alpha_1, \cdots, \alpha_N \in I$

$$\begin{tikzcd}
\mathcal{F}(\alpha_1 * \cdots * \alpha_N) \arrow[r, phantom, "\underset{②}{=}"] \arrow[d, Rightarrow, "①"] & (-1)^0(-1_{\hat{A}})_*\ \mathcal{F}(\alpha_1)\cdots\ \mathcal{F}(\alpha_N) \arrow[d, phantom, "\rotatebox{90}{$\in$}\ ③"] \\
\alpha_1 * \cdots * \alpha_N = 0 & \bigoplus_{i<0} CH_i\hat{A} = 0
\end{tikzcd}$$

▨

Cor C : curve $g(C) = g$, $N > g$ $\alpha_1, \cdots, \alpha_N \in CH_0C$, $\deg \alpha_i = 0$

$$\mathrm{Sym}(\alpha_1, \cdots, \alpha_N) := \frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N} \alpha_{\sigma(1)}\times\cdots\times\alpha_{\sigma(N)} \in CH_0(C^N)$$

If $N > g \Rightarrow \mathrm{Sym}(\alpha_1, \cdots, \alpha_N) = 0$

(∵) $$\begin{tikzcd} C^N \arrow[r, "\pi"] & \mathrm{Sym}^N C \arrow[r, "\varphi"] \arrow[d, phantom, "\shortparallel"] & J(C) \\ & C^N/\mathfrak{S}_N & \end{tikzcd}$$

$$\mathrm{Sym}(\alpha_1, \cdots, \alpha_N) = \frac{1}{N!}\pi^* \overset{\overset{0}{/\!/ \text{ want}}}{\pi_*(\alpha_1 \times \cdots \times \alpha_N)}$$

fibers of φ are proj. sp $\Rightarrow \varphi_* \; CH_0(\mathrm{Sym}^N C) \simeq CH_0 J(C)$

It is sufficient to prove $\varphi_* \pi_*(\alpha_1 \times \cdots \times \alpha_N)$

$$= \underset{\underset{I}{\uparrow}}{(\varphi \circ \pi)_* \alpha_1} * \cdots * \underset{\underset{I}{\uparrow}}{(\varphi \circ \pi) \alpha_N} = 0$$

▨

Claim Because $CH_0 C_{hom}$ and $CH_0 D_{hom}$ are "finite dimensional" as above, $(CH_0 C_{hom}) \otimes_{\mathbb{Z}} (CH_0 D_{hom})$ is also finite dim

(∵) Step 1. Prove that $V \otimes W$ is fin. dim v. sp when V and W are fin. gen using rep. theory

Step 2. Mimic it ▨

Can lift to motives $h^1(C) := (C, [\Delta_C] - [P \times C] - [C \times P])$

$$\mathrm{Sym}^{2g+1} h^1(C) = 0 \Rightarrow \mathrm{Sym}(\alpha_1, \cdots, \alpha_d) \in CH_*(\mathrm{Sym}^N \underset{\overset{\#}{0}}{h^1(C)})$$

$$\Downarrow$$

$$\bigwedge^{4g(C)g(D)+1} h^1(C) \otimes h^1(D) = 0$$

Def M is evenly fin dim $\Leftrightarrow$ for $N \gg 0$ $\overset{N}{\Lambda} M = 0$

" oddly " $\Leftrightarrow$ " $\mathrm{Sym}^N M = 0$

M is fin. dim $\Leftrightarrow$ $M = \underset{\text{even}}{M^+} \oplus \underset{\text{odd}}{M^-}$

Thm ① $h(C)$ is fin. dim

② fin. dim. is stable under $\otimes, \oplus$, direct summand $\square$

Rem All known examples are obtained by above

Thm If M is fin dim. motive, then $M = 0 \Leftrightarrow H^*(M) = 0$

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Construction

a diagram D set of objects $\mathcal{O}(D)$
$\forall p, \forall q \in \mathcal{O}(D)$ set of morphisms $M(p,q)$

(don't consider composition)

ex $\mathbb{C} \supset k$ fix $Sch/_k$: cat. of schemes of fin. type $/_k$

$H_* Sch_k$. Obj (X, Y, i) $X, Y \in Sch/_k$, $X \supset Y$ closed subsch, $i \in \mathbb{Z}$

for
$$\begin{tikzcd} X' \arrow[r, "f"] & X \\ \cup & \cup \\ Y' \arrow[r] & Y \end{tikzcd} \Rightarrow f_* \ (X', Y', i) \to (X, Y, i)$$

$X \supset Y \supset Z$ (closed, closed) $X, Y, Z \in Sch_k$
$$\Rightarrow \partial : (X, Y, i) \longrightarrow (Y, Z, i-1)$$

if we reverse the arrow $\Rightarrow H^* Sch_k$

$\mathcal{C}$: cat a representation T of D in $\mathcal{C}$ is given by a map
$$\mathcal{O}(D) \longrightarrow obj(\mathcal{C}) \quad \forall p, \forall q \in \mathcal{O}(D)$$
$$p \longmapsto Tp \quad M(p,q) \to Mor_{\mathcal{C}}(Tp, Tq)$$

ex. $H_* : H_* Sch_k \longrightarrow (Ab)$
$$(X, Y, i) \longmapsto H_i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Z})$$

$H^* \ H^* Sch_k \longrightarrow (Ab)$
$$(X, Y, i) \longmapsto H^i(\downarrow \quad \downarrow \quad \cdot)$$

Thm (Nori)

cat. of fin gen. R-mod

D : diag   T : D → (R-mod) repr.

(1) ∃ $\mathcal{C}(T)$: R-linear abelian cat.

$ff_T : \mathcal{C}(T) \longrightarrow (R\text{-mod})$

(R-lin exact faithful functor)

$\tilde{T} : D \longrightarrow \mathcal{C}(T)$ repr. $T = ff_T \circ \tilde{T}$

(2) $(\tilde{T}, \mathcal{C}(T))$ is universal for $\mathcal{A}$: R-lin abel. cat.

$f : \mathcal{A} \longrightarrow (R\text{-mod})$ R-lin exact faithful functor

$F : D \longrightarrow \mathcal{A}$ repr. s.t. $T = f \circ F$

Then ∃! $L(F) : \mathcal{C}(T) \to \mathcal{A}$ R-lin. functor s.t.

$$\begin{tikzcd}
 & & \mathcal{C}(T) \arrow[dd, "L(F)"] \arrow[rrd, "ff_T"] & & \\
D \arrow[rru, "\tilde{T}"] \arrow[rrd, "F"'] & \circlearrowleft & & \circlearrowleft & R\text{-mod} \\
 & & \mathcal{A} \arrow[rru, "f"'] & &
\end{tikzcd}$$

construction of $\mathcal{C}(T)$:

first assume $\mathcal{O}(D)$ is finite

$$\mathrm{End}(T) := \left\{ \prod_{p \in \mathcal{O}(D)} e_p \in \prod_p \mathrm{End}(Tp) \;\middle|\; \begin{tikzcd} T_p \arrow[r, "Tm"] \arrow[d, "e_p"'] \arrow[dr, phantom, "\circlearrowleft"] & T_q \arrow[d, "e_q"] \\ T_p \arrow[r, "Tm"'] & T_q \end{tikzcd} \quad \begin{matrix} \forall p, \forall q \in \mathcal{O}(D) \\ \forall m \in M(p,q) \end{matrix} \right\}$$

$\mathcal{C}(T)$: cat. of f.g. End(T)-modules.

$\hat{T}_p = T_p$ as End(T)-mod.

$\exists\ R \longrightarrow \mathrm{End}(T)$ ff_T: forget $\mathrm{End}(T)$-mod str

In general $\mathcal{C}(T) = \varinjlim_{\substack{F \subset D \\ \text{finite}}} \mathcal{C}(T|_F)$ ▨

$(H_*\mathrm{Sch}_k, H_*) \rightsquigarrow \mathrm{EHM}(k)$ cat. of effective homological motives

$(H^*\mathrm{Sch}_k, H^*) \rightsquigarrow \mathrm{ECM}(k)$

universality

$$\begin{tikzcd} \Rightarrow \mathrm{ECM}(k) \arrow[r] \arrow[dr] & \{ \mathbb{Q}_\ell\text{-}\mathrm{Gal}(\bar{k}/k) \text{ repr.} \} \\ & \mathrm{MHS} \end{tikzcd}$$

$\mathrm{Gal}(\bar{k}/k) \longrightarrow \mathrm{End}(H^*) \otimes \mathbb{Q}$

$$\begin{tikzcd} \text{Construction of} \quad \mathrm{Sch}_k \arrow[r] \arrow[dr] & D^b(\mathrm{EHM}(k)) \\ & D^b(\mathrm{ECM}(k))^{op} \end{tikzcd}$$

<u>Basic Lemma</u> (Beilinson - Nori)

X/k. affine scheme f.t. $/k$ $n = \dim X$

$X \supset Z$ closed subset, $\dim Z \leq n-1$.

Then $X \supset {}^\exists Y$: closed subset $\dim Y \leq n-1$ s.t.

(1) $Y \supset Z$ (2) $H^j(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Z}) = 0$ for $j \neq n$ □

$\rightsquigarrow$ X has a "cellular decomposition" i.e.

a filtration by closed subsets

$\phi = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$ s.t. $H^j(X_i(\mathbb{C}), X_{i-1}(\mathbb{C}), \mathbb{Z}) =$

$\dim X_i \leq i$ for $i \neq j$

$m^* : Sch/_k \longrightarrow D^b(ECM(k))^{op}$

$\quad X$: affine $\longmapsto \{ 0 \to H^0(X_0) \xrightarrow{\partial} H^1(X_1, X_0) \to \quad \to H^n(X, X_{n-1}) \to 0 \}$

X: separated $\Rightarrow$ Čech construction

Assume $X = U_1 \cup U_2$ $\quad U_i$: affine open $\quad U_1 \cap U_2 = U_{12}$

Take a cell decomp of U_{12} : $\phi = (U_{12})_{-1} \hookrightarrow (U_{12})_0 \hookrightarrow \cdots \hookrightarrow (U_{12})_n = U_{12}$

Then take cell decomp. of U_i s.t.

$\quad (U_{12})_j \subset (U_i)_j \quad 0 \le j \le n \quad i = 1, 2$

$\exists$ restriction $\quad m^*(U_i) \to m^*(U_{12})$

Čech cpx. $\quad 0 \to m^*(U_1) \oplus m^*(U_2) \to m^*(U_{12}) \to 0$

$$m^*(X) = \mathrm{Tot}\ (\text{——})$$

$\rightsquigarrow \quad m^* : Sch_k \longrightarrow D^b(FCM(k))^{op}$

$\quad m_* : Sch_k \longrightarrow D^b(EHM(k))$

$X/_k$: variety $\quad X \xrightarrow{f} \mathrm{Spec}\,k$

$\quad \rightsquigarrow \quad m^*(X) \otimes \mathbb{Q}_\ell \simeq Rf_* \mathbb{Q}_\ell$ in $D^b(Sh(\mathrm{Spec}\,k_{ét})^{\mathbb{Q}_\ell})$

Nori extends

$\quad m_*|_{Aff_k \cap Sm_k}$ to $\Pi : DM^{eff}_{gm}(k) \longrightarrow D^b(EHM(k))$

<u>Conj.</u> (someone) $\Pi \otimes \mathbb{Q}$ is fully faithful.

An application : second l-adic Abel-Jacobi map

X/k : sm proj var $\dim X = n$

$z \in CH^i(X)$ $\exists$ a class $[z] \in H^{2i}_{cont}(X, \mathbb{Z}_l(i))$

usual class $cl(z)$ is the restriction of $[z]$ under

$H^{2i}_{cont}(X, \mathbb{Z}(i)) \to H^{2i}(\bar{X}, \mathbb{Z}_l(i))$

Hochschild - Serre spec. seq : $E_2^{p,q} = H^p(G_k, H^q(\bar{X}, \mathbb{Z}_l(i))) \Rightarrow H^{p+q}_{cont}(X, \mathbb{Z}_l(i))$

induces l-adic Abel-Jacobi map

$$\begin{aligned} cl^1 : CH^i(X)_{hom} &\longrightarrow H^1(G_k, H^{2i-1}(\bar{X}, \mathbb{Z}_l(i))) \\ &\qquad \parallel \\ &\quad \mathrm{Ext}^1_{G_k}(\mathbb{Z}_l, H^{2i-1}(\bar{X}, \mathbb{Z}_l(i))) \end{aligned}$$

description $Y = X - |z|$

$$\begin{tikzcd} 0 \arrow[r] & H^{2i-1}(\bar{X}, \mathbb{Z}_l(1)) \arrow[r] & H^{2i-1}(\bar{Y}, \mathbb{Z}_l(1)) \arrow[r] & H^{2i}_{|z|}(\bar{X}, \mathbb{Z}_l(i)) \arrow[d, phantom, "\rotatebox{-90}{$\ni$}"] \\ & & & cl^1(z) \end{tikzcd}$$

$cl^1(z)$ is pull back of that by $\mathbb{Z}_l \, cl(z)$

second A-J map

$\mathrm{Ker}(cl^1) \longrightarrow H^2(G_k, H^{2i-2}(\bar{X}, \mathbb{Z}_l(i)))$

a description : $g : Y \longrightarrow \mathrm{Spec}\, k$ $Y = X - |z|$ $Rg_*\mathbb{Q}_l(i)$

$\underset{\rightsquigarrow}{\text{truncation}}$

$$X_{2i-2}(Y) : \quad 0 \to H^{2i-2}(\bar{Y}, \mathbb{Q}_l(i)) \to \frac{(Rg_*\mathbb{Q}_l(i))^{2i-2}}{\mathrm{Im}\, \partial^{2i-3}} \to \mathrm{Ker}\, \partial^{2i-1} \to H^{2i-1}(\bar{Y}, \mathbb{Q}_l(i)) \to 0$$

If $cl^1(z) = 0 \Rightarrow \exists$ a splitting $\mathbb{Q}_l \longrightarrow H^{2i-1}(\bar{Y}, \mathbb{Q}_l(i))$

Thm (Jannsen)

$cl^2(z)$ is the pull-back of $X_{2i-2}(Y)$ by the splitting $\Box$

Want more explicit description

Assume $i=n$ $\quad z = CH_0(X)_{\deg 0}$

$\rightsquigarrow$ $\exists H\ (\subset X)$ proj sm curve, intersection of $n-1$ hypersurfaces $i_H : H \hookrightarrow X$ s.t. $z \in i_* CH_0(H)$

Then $U := X - H$, $Y = X - |z|$ $(U \subset Y)$

$$0 \to \frac{H^{2n-2}(\bar{Y}, \mathbb{Q}_\ell(n))}{H^{2n-2}_{Y\cap H}(\bar{Y}, \mathbb{Q}_\ell(n))} \to H^{2n-2}(\bar{U}, \mathbb{Q}_\ell(n))$$

$$\to H^{2n-1}_{Y\cap H}(Y, \mathbb{Q}_\ell(n)) \to Q\ (\text{quotient}) \to 0$$

$(cl'(z)=0 \Rightarrow \exists$ a splitting $\mathbb{Q}_\ell \to Q$ $)$

Thm The push out of $cl^2(z)$ by the quotient

$H^{2n-2}(\bar{Y}) \to \dfrac{H^{2n-2}(\bar{Y})}{H^{2n-2}_{Y\cap H}(\bar{Y})}$ is given by the pull-back by the splitting.

萩原啓

Reference : On 2-torsion in motivic coh. (V. Voevodsky)

$n \geq 1$, ℓ prime $H90(n,\ell)$ $\forall k$ field $ch \neq \ell$ $H_L^{n+1,n}(k,\mathbb{Z}_{(\ell)})=0$

Aim : Prove $H90(n,2)$

< Why "Norm Varieties" > ℓ prime

Thm   Assume $H90(n-1,\ell)$

If k: field ① No fin ext of deg prime to ℓ, $ch \neq \ell$

② $K_n^M k/\ell = 0$

$\Rightarrow$ $H^n_{et}(k,\mathbb{Z}/\ell)=0$

Cor   Assume $H90(n-1,\ell)$

If $\forall k$ field, $\forall (a_1,\cdots,a_n) \in (k^\times)^n$

$$\exists E/k \text{ f.g. ext. s.t. } \begin{cases} \underline{a}=0 \text{ in } K_n^M E/\ell \\ H_L^{n+1,n}(k,\mathbb{Z}_{(\ell)}) \hookrightarrow H_L^{n+1,n}(E,\mathbb{Z}_{(\ell)}) \end{cases}$$

$\Rightarrow$ $H90(n,\ell)$ holds

Thm $\Rightarrow$ Cor

k given   By assumption + transfinite induction

$$\exists k'/k \text{ s.t. } \begin{cases} K_n^M k/\ell \xrightarrow{0} K_n^M k'/\ell \\ H_L^{n+1,n}(k,\mathbb{Z}_{(\ell)}) \hookrightarrow H_L^{n+1,n}(k',\mathbb{Z}_{(\ell)}) \end{cases} \quad (*)$$

$\exists \underset{\text{prime to } \ell}{k''/k'}$: s.t. k'' has no ext. of deg. prime to ℓ

($\Rightarrow$ (*) is still satisfied with k' replaced by k'')

Iterate this procedure ($k \rightsquigarrow k''$) : $\underset{k}{\overset{\parallel}{K_0}} \subset \underset{k''}{\overset{\parallel}{K_1}} \subset \underset{K_1''}{\overset{\parallel}{K_2}} \subset \cdots$

$K_\infty = \bigcup K_n$

$$\Rightarrow \begin{cases} K_n^M K_\infty / \ell = 0 \\ H_L^{n+1,n}(k, \mathbb{Z}_{(\ell)}) \hookrightarrow H_L^{n+1,n}(K_\infty, \mathbb{Z}_{(\ell)}) \\ K_\infty \text{ has no ext. of deg. prime to } \ell \end{cases}$$

$$\underset{\text{Thm}}{\Rightarrow} \quad H^n_{\text{ét}}(K_\infty, \mathbb{Z}/\ell) = 0$$

$$\begin{tikzcd}[column sep=small] & & & 0 \arrow[d, equal] & \\ & & & H^{n+1,n}(K_\infty, \mathbb{Q}_\ell) \arrow[dl, "\simeq"'] & \\ \Rightarrow \arrow[r] & H_L^{n,n}(K_\infty, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r] \arrow[d, equal] & H_L^{n+1,n}(K_\infty, \mathbb{Z}_{(\ell)}) \arrow[r] & H_L^{n+1,n}(K_\infty, \mathbb{Q}_\ell) \arrow[r] & {} \\ & 0 & & & \end{tikzcd}$$

$$\Rightarrow \quad H_L^{n+1,n}(K_\infty, \mathbb{Z}_{(\ell)}) = H_L^{n+1,n}(k, \mathbb{Z}_{(\ell)}) = 0 \qquad ▨$$

Rem To prove $H90(n, \ell)$, it is sufficient to prove

$H_L^{n+1,n}(k, \mathbb{Z}_{(\ell)}) = 0 \quad \forall k$: perfect

Def. $X \in Sm/k \quad \check{C}(X) := ([n] \mapsto \overbrace{X \times \cdots \times X}^{n+1}) \in \Delta^{op} Sm/k$

Def. $K(n) := \text{Cone}(\mathbb{Z}(n) \longrightarrow \tau_{\le n+1} R\varepsilon_* \mathbb{Z}(n)^{\text{ét}}) \in D^-(Shv_{zar}(Sm/k))$

Rem $H90(n, \ell) \Leftrightarrow K(n) \otimes \mathbb{Z}_{(\ell)}$: acyclic

Then $\underset{\substack{\| \\ k(X)}}{\text{Spec } K} \longrightarrow X \longrightarrow \check{C}(X) \longrightarrow \text{Spec } k$ induces

$$\begin{tikzcd} & & H^{n+1,n}(\check{C}(X), \mathbb{Z}_{(\ell)}) \arrow[d] & \text{yesterday} \arrow[d] & \\ H_L^{n+1,n}(K, \mathbb{Z}_{(\ell)}) \arrow[d] & H_L^{n+1}(X, \mathbb{Z}_{(\ell)}) \arrow[l] \arrow[d] & H_L^{n+1,n}(\check{C}(X), \mathbb{Z}_{(\ell)}) \arrow[l] \arrow[d] & \simeq & H_L^{n+1,n}(k, \mathbb{Z}_{(\ell)}) \\ H^{n+1}_{zar}(K, K(n)_{(\ell)}) & H^{n+1}_{zar}(X, K(n)_{(\ell)}) \arrow[l, "①"', "\simeq"] & H^{n+1}_{zar}(\check{C}(X), K(n)_{(\ell)}) \arrow[l, "⊛"'] & & \\ & H90(n-1, \ell) \arrow[ul, shorten <=2pt] & (\text{exact}) & & \end{tikzcd}$$

① birational invariant of $K(n)_{(\ell)}$

$\left[\; U \subset X \underset{\text{codd}}{\supset} Z = X \setminus U \right.$: smooth

$\lfloor \Rightarrow \quad \to H^{*-2d}(Z, K(*'-d)_{(\ell)}) \to H^*(X, K(*')_{(\ell)}) \to H^*(U, K(*')_{(\ell)}) \to \cdots$

Thus to prove H90(n, ℓ) (under H90(n-1, ℓ))

For $\underline{a} = (a_1, \cdots, a_n) \in (k^\times)^n$, it suffices to find a variety $X_{\underline{a}}$ s.t

A) $\underline{a} = 0$ in $K_n^M k(X_{\underline{a}})/\ell$ C) ✪ is injective

B) $H_B^{n+1,n}(\check{C}(X_{\underline{a}}), \mathbb{Z}_{(\ell)}) = 0$

< Rost's theorems and their corollaries >

(k perfect) $\underline{a} = (a_1, \cdots, a_n) \in (k^\times)^n$

$Q_{\underline{a}}$: (the quadric ass. to $V_{\underline{a}} = \langle\langle a_1, \cdots, a_{n-1} \rangle\rangle \perp \langle -a_n \rangle$)

$\uparrow$ dim $= 2^{n-1} - 1 =: d$ $\cap \; \mathbb{P}^{2^{n-1}}$

$K_{\underline{a}} = k(Q_{\underline{a}})$ $\mathcal{X}_{\underline{a}} = \check{C}(Q_{\underline{a}})$

Prop $\underline{a} = 0$ in $K_n^M K_{\underline{a}}/2$ (calculation)

$\mathbb{Z}(1)[2]$

Thm (Rost)

(A) ∃ a direct summand $M_{\underline{a}}$ of $M(Q_{\underline{a}})$ ∃ $\mathbb{L}^d \xrightarrow{\psi^*} M_{\underline{a}} \xrightarrow{\psi_*} \mathbb{Z}$ ✪

s.t. (1)

$$\begin{tikzcd}M(Q_{\underline{a}}) \arrow[r] \arrow[rr, bend right, "\text{natural hom}"'] & M_{\underline{a}} \arrow[r, "\psi_*"] & \mathbb{Z} = M(\operatorname{Spec} k)\end{tikzcd}$$

(2) If F/k ext $Q_{\underline{a}}(F) \neq \emptyset \Rightarrow$ ✪$\otimes_k F$ is split exact

(B) $H^{2d+1,d+1}(Q_{\underline{a}}, \mathbb{Z}) \xhookrightarrow{N} k^\times$

Rem $H^{2d+1,d+1}(Q_{\underline{a}}, \mathbb{Z}) \simeq CH^{d+1}(Q_{\underline{a}}, 1)$

$$\simeq \begin{tikzcd}\mathrm{Cok}\Big(\bigoplus_{x \in (Q_{\underline{a}})_{(1)}} K_2 k(x) \arrow[r, "\partial"] & \bigoplus_{x \in (Q_{\underline{a}})_{(0)}} k(x)^\times \Big) \arrow[r, dashed, "N"] & k^\times \\ & \bigoplus k(x)^\times \arrow[u] \arrow[ur, "\oplus N_{k(x)/k}"'] & \end{tikzcd}$$

Formally follows $\rightsquigarrow$ **Cor 1** $\exists$ dist. triangle in $DM^{eff}_{-}(k)$

$$M(\mathcal{X}_{\underline{a}})(d)[2d] \longrightarrow M_{a} \to M(\mathcal{X}_{\underline{a}}) \to$$

Cor 2 $H^{2d+1,d+1}(\mathcal{X}_a, \mathbb{Z}) = 0$

Proof: construction (M_a) - inductively

$\underline{a}' = (a_1, \dots, a_{n-1})$

$R_{\underline{a}'} = (\text{the quadric ass. to } W_{\underline{a}'} \text{ s.t. } \langle 1\rangle \perp W_{\underline{a}'} = \langle\langle a_1, \dots, a_{n-1}\rangle\rangle)$ $\quad \mathbb{P}^{2^{n-1}-2} \cup$

He inductively construct $M_{\underline{a}}$ and prove

$$M(Q_{\underline{a}}) = M_{\underline{a}} \oplus M(R_{\underline{a}'}) \otimes \mathbb{L}$$

$$M(R_{\underline{a}'}) = \bigoplus_{i=0}^{d-3/2} M_{\underline{a}'} \otimes \mathbb{L}^i$$

ex. $S = (\text{the quadric ass. to } \overset{\langle 1,-1\rangle}{\overset{\shortparallel}{\mathbb{H}}}{}^m \perp \langle 1\rangle)$ $\leftarrow$ dim. $2m-1$

$$\Rightarrow M(S) = \bigoplus_{i=0}^{2m-1} \mathbb{L}^i$$

If F/k splits $Q_{\underline{a}}, R_{\underline{a}'}$ $\left(\begin{array}{l} \text{i.e. } V_{\underline{a}} \otimes_k F \simeq \mathbb{H}^{d+1/2} \perp \langle 1\rangle \\ W_{\underline{a}'} \otimes_k F \simeq \mathbb{H}^{d-1/2} \perp \langle 1\rangle \end{array}\right)$

$$M(Q_{\underline{a}}) \simeq M_{\underline{a}} \oplus \underset{①}{M(R_{\underline{a}'})} \otimes \mathbb{L} \quad ②$$

$$\simeq M_{\underline{a}} \oplus (M_{\underline{a}'} \otimes \mathbb{L}) \oplus \cdots \oplus (M_{\underline{a}'} \otimes \mathbb{L}^{d-1/2})$$

$\times F \rightsquigarrow$

$$M((Q_{\underline{a}})_F) \simeq \mathbb{Z} \oplus \underbrace{\mathbb{L} \oplus \mathbb{L}^2 \oplus \cdots \oplus \mathbb{L}^{d-1/2}}_{①} \oplus \underbrace{\mathbb{L}^{d+1/2} \oplus \cdots \oplus \mathbb{L}^{d-1}}_{②} \oplus \mathbb{L}^d$$

Cor (Milnor conj for $n=2$)

$\because$ Suffices to show (B), (C)

C): from Cor 1, $M(\mathcal{X}_{\underline{a}})(1)[2] \to M_{\underline{a}} \to M(\mathcal{X}_{\underline{a}})(1)[3]$

$$\begin{tikzcd}
{} \arrow[r, rightsquigarrow] & H^0(\mathcal{X}_{\underline{a}}, K(1)_{(2)}) \arrow[r] \arrow[d, equal, "\text{(easy)}"] & H^3(\mathcal{X}_{\underline{a}}, K(2)_{(2)}) \arrow[r] \arrow[dr, "\text{inj}"'] & H^3(M_{\underline{a}}, K(2)_{(2)}) \arrow[r] & \cdots \\
 & 0 & & H^3(Q_{\underline{a}}, K(2)_{(2)}) \arrow[u, hook, "\text{in fact isom } (n=2)"'] &
\end{tikzcd}$$

(B) $H^{2+1,2}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) = H^{2^2-1, 2^{2-1}}(\mathcal{X}_a, \mathbb{Z}_{(2)}) \overset{\text{Cor 2}}{=} 0$

▨

The case for $n>2$

C) is O.K. $\Leftarrow$ $\mathrm{Hom}(M(\mathcal{X}_a)(2d+1)[2d], K(n)[n+1]) \overset{\text{for } n>2}{\underset{\text{easy}}{=}} 0$

Remains to show $\boxed{H^{n+1,n}(\mathcal{X}_a, \mathbb{Z}_{(2)}) = 0}$

Want to relate $H^{n+1,n} \longleftrightarrow H^{2^n-1, 2^{n-1}}$

$ \overset{\shortparallel}{CH^n(*, n-1)} \qquad \overset{\shortparallel}{CH^{2^{n-1}}(*\ 1)} \Leftarrow$ Known (Rost)

$\nearrow$

<u>motivic cohomology operation</u>

[illegible] 宗 啓

< The vanishing of Margolis cohomology >

X : smooth $\rightsquigarrow$ $\check{C}(X)$ $\rightsquigarrow$ $\tilde{C}(X) = \mathrm{Cone}(\check{C}(X)_+ \to (\mathrm{Spec}\, k)_+)$

$\hat{\mathcal{X}}_{\underline{a}} := \tilde{C}(Q_a)$

Thm $\mathrm{ch}\, k \neq l$ $m \geq 0$

$$\begin{cases} X : \text{sm. proj. var. of dim } l^m-1, \ \deg s_{l^m-1}(X) \neq 0 \pmod{l^2} \\ Y : \text{sm. proj.} \end{cases}$$

$\exists X \to Y \Rightarrow \widetilde{MH}^{*,*}_m(\tilde{C}(Y), \mathbb{Z}/l) = 0$ $\square$

s_j : characteristic class s.t. $\begin{cases} s_j(L) = c_1(L)^j & (L : \text{l.b.}) \\ s_j(E \oplus F) = s_j(E) \oplus s_j(F) \end{cases}$

$s_{l^m-1}(X) := s_{l^m-1}(TX) \in H^{2(l^m-1), l^m-1}(X, \mathbb{Z}) \ (= CH_0(X)) \xrightarrow[\deg]{} \mathbb{Z}$

Cor Q : smooth quadric in $\mathbb{P}^{2^n}_k$ $(\mathrm{ch}\, k \neq 2)$

$\Rightarrow \widetilde{MH}^{*,*'}_i(\tilde{C}(Q), \mathbb{Z}/2) = 0$ $(\forall i \leq n, \ \forall *, *')$

End of the proof

$$\begin{tikzcd}
H^{n+1,n}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) \arrow[r, phantom, "\simeq"] & \hat{H}^{n+2,n}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbb{Z}_{(2)}) \arrow[r, dotted, "\overset{①}{c}"] \arrow[ddd, dotted, "(2)"] & \tilde{H}^{n+2,n}(\hat{\mathcal{X}}_{\underline{a}}, \mathbb{Z}/2) \ \boxed{1} \arrow[d, "Q_1"] \\
 & (3) & \tilde{H}^{n+5,n+1}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbb{Z}/2) \ \boxed{2} \arrow[d, "Q_2"] \\
 & & \vdots \arrow[d, "Q_{n-2} \quad \boxed{n-1}"] \\
\tilde{H}^{2^n-1,2^{n-1}}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) \arrow[r, phantom, "\simeq"] \arrow[d, equal, "(\mathrm{Rost})"] & \tilde{H}^{2^n,2^{n-1}}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbb{Z}_{(2)}) \arrow[r, "c", "①"'] & \tilde{H}^{2^n,2^{n-1}}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbb{Z}/2) \\
0 & &
\end{tikzcd}$$

① $\exists\, E/k$ deg 2 $\quad Q_S(E) \neq \phi$

$$\begin{tikzcd}
\Rightarrow \hat{H}^{*,*}(\hat{\mathfrak{X}}_{\underline{a}}, \mathbb{Z}_{(2)}) \arrow[r, "2"] \arrow[dr, "i"'] & \hat{H}^{*,*'}(\hat{\mathfrak{X}}_{\underline{a}}, \mathbb{Z}_{(2)}) \arrow[r] & \hat{H}^{*,*'}(\hat{\mathfrak{X}}_{\underline{a}}, \mathbb{Z}/2) \\
& \hat{H}^{*,*}((\hat{\mathfrak{X}}_{\underline{a}})_E, \mathbb{Z}_{(2)}) \arrow[u, "N"'] \arrow[d, equal] & \\
& 0 &
\end{tikzcd}$$

② $\quad Q_i = \beta q_i - q_i \beta$

$\rightsquigarrow$ 「$\beta x = 0 \Rightarrow \beta Q_i x = 0$」

③ $\widetilde{MH}_i^{*,*'} = 0$ (Cor)

$$\tilde{H}^{n-i,n-i} \to \tilde{H}^{2^{i+1}+n-i-1,\, 2^i+n-i-1} \xrightarrow{Q_i} \tilde{H}^{2^{i+2}+n-i-2,\, 2^{i+1}+n-i-2}$$

$\cap$ H90$(n-1,2)$ $\quad \boxed{i} \quad\quad \boxed{i+1}$

$\tilde{H}_L^{n-i,n-i} = 0 \quad\quad \Rightarrow Q_i$ inj

proof of vanishing

$m = 0$ omitt $\quad\quad m > 0$

Key Lemma $\quad X$: sm. proj of dim $d\,/k$

$\Rightarrow \begin{cases} \exists\, V : \text{v.b. of rk } n \\ \exists\, f_V : T^{n+d} \to Th_X V \text{ in } \mathcal{H}.\mathbb{A}^1 \end{cases}$ s.t

Tate obj.

① $V + TX = \mathcal{O}^{n+d}$ in $K_0(X)$

②
$$\begin{tikzcd}
H^{2d,d}(X, \mathbb{Z}) \arrow[r, phantom, "\simeq"] \arrow[d, "\deg"'] \arrow[dr, phantom, "\circlearrowleft"] & H^{2(n+d),n+d}(Th_X V, \mathbb{Z}) \arrow[d, "f_V^*"] \\
\mathbb{Z} \arrow[r, phantom, "="] & H^{2(n+d),n+d}(T^{n+d}, \mathbb{Z})
\end{tikzcd}$$

Rem. A formal corollary to Spanier-Whitehead duality

- Voevodsky constructs the above in the category $\mathcal{H}.^{\mathbb{A}^1}$ $\square$

All coh. is $\mathbb{Z}/\ell$-coeff unless otherwise stated.

$$d = \ell^{n} - 1 > 0 \qquad \begin{tikzcd} T^{n+d} \arrow[r, "t_V"] & Th_x V \arrow[r] & \underset{\underset{\textstyle Cone(t_V)}{\shortparallel}}{Cone} \arrow[r] & \Sigma_s^1 \end{tikzcd}$$

$$\rightsquigarrow \begin{tikzcd} \tilde{H}^{2n,n}(Th_x V) & \tilde{H}^{2n,n}(Cone) \arrow[l, "\simeq"'] \\ \overset{\Psi}{t_V} \arrow[r, mapsto] & \exists!\,\alpha \end{tikzcd}$$

$$\begin{tikzcd} \tilde{H}^{2n+2d+1,n+d}(Cone) & \tilde{H}^{2n+2d+1,n+d}(\Sigma_s^1 T^{n+d}) \arrow[l] \\ \overset{\Psi}{\gamma} & \overset{\Psi}{id} \arrow[l, mapsto] \end{tikzcd}$$

$$\rightsquigarrow \begin{tikzcd} \tilde{H}^{p,q}(\hat{C}(Y)) \arrow[r, "\sim", "\alpha\wedge -"'] \arrow[ddr, dotted, "\phi"'] & \tilde{H}^{p+2n,q+r}(Cone \wedge \hat{C}(Y)) \\ \circlearrowright & \tilde{H}^{p+2n,q+n}(\Sigma_s^1 T^{n+d} \wedge \hat{C}(Y)) \arrow[u, "s\ \circledast"'] \\ & \tilde{H}^{p-2d-1,q-d}(\hat{C}(Y)) \arrow[u, "s"'] \arrow[uu, bend right=60, "\gamma\wedge -"'] \end{tikzcd}$$

$$\left(\begin{tikzcd} \circledast\ Th_x V \wedge \hat{C}(Y) \arrow[r] & Cone \wedge \hat{C}(Y) \arrow[r] & \Sigma_s^1 T^{n+d} \wedge \hat{C}(Y) \\ V_+ \wedge \hat{C}(Y) \simeq * \arrow[u] & & \exists\, V \to Y \arrow[ll, bend right] \arrow[dll] \\ (V - (0\text{-sect}))_+ \wedge \hat{C}(Y) \simeq * \arrow[u] & & \exists\, (V - (0\text{-sect})) \to Y \end{tikzcd} \right)$$

Lem

$$\begin{tikzcd}
{} \arrow[r, dashed, "Q_m"] & \tilde{H}^{p.q} \arrow[r, "Q_m"] \arrow[dl, "\phi"] \arrow[d, "{}^{\exists}c"] & \tilde{H}^{p+2d+1,q+d} \arrow[dl, dashed, "\phi"] \\
\tilde{H}^{p-2d-1,q-d} \arrow[r, dashed, "Q_m"'] & \hat{H}^{p.q} \arrow[r, "Q_m"'] & {}
\end{tikzcd}$$

$$\exists c \in (\mathbb{Z}/\ell)^{\times}, \quad c x = \phi Q_m(x) - Q_m \phi(x)$$

$$(\forall p.q)$$

(∵) Taking $\alpha\wedge$, suffices to prove

$$\exists c \in (\mathbb{Z}/\ell)^{\times} \qquad c\gamma \wedge x = \alpha \wedge Q_m(x) - Q_m(\alpha \wedge x)$$

$$(\in \hat{H}^{p+2n+2d+1,q+n+d}(\mathrm{Cone} \wedge \hat{C}(\gamma)))$$

$$\begin{tikzcd}
\alpha \arrow[ddr, dashed, mapsto, bend right] & \tilde{H}^{2n,n}(\mathrm{Cone}) \arrow[r, "\sim"] \arrow[d, "Q_i"'] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{H}^{2n,n}(\mathrm{Th}_x V) \ni t_v \arrow[d, "Q_i"] \arrow[ll, dashed, bend right] \arrow[dd, dashed, mapsto, bend left] \\
\tilde{H}^{2n+2\ell^i-1,n+\ell^i-1}(\Sigma^1 T^{n+d}) \arrow[r] \arrow[dd, phantom, "\shortparallel"] & \tilde{H}^{\cdot\cdot}(\mathrm{Cone}) \arrow[r] \arrow[d, phantom, sloped, "\ni"] & \tilde{H}^{\cdot\cdot}(\mathrm{Th}_x V) \arrow[d, phantom, sloped, "\ni"] \\
 & Q_i(\alpha) \arrow[d, phantom, sloped, "\ni"] & 0 \\
\underbrace{0}_{(i<m)} \quad \underbrace{\mathbb{Z}/\ell}_{(i=m)} \ni 1 \arrow[r, mapsto] & \gamma \ (i=m) &
\end{tikzcd}$$

$$Q_i(\alpha) = \begin{cases} 0 & (i<m) \\ \beta q_m(\alpha) - q_m \beta(\alpha) = \beta q_m(\alpha) & (i=m) \end{cases}$$

$$\rightsquigarrow \quad Q_m(\alpha \wedge x) = \alpha \wedge Q_m(x) + \beta q_m(\alpha) \wedge x$$

$$\rightsquigarrow \quad -\beta q_m(\alpha) \wedge x = \alpha \wedge Q_m(x) - Q_m(\alpha \wedge x)$$

Thus, suffices to show $Q_m(\alpha) \neq 0 \ (\Leftrightarrow \beta q_m(\alpha) \neq 0)$

May assume $\forall E/k$: deg. prime to ℓ

$Y(E) = \emptyset$ (otherwise $H^{*,*}(\check{C}(Y)) = 0$)

$\leadsto H^{2d,d}(X, \mathbb{Z}) \xrightarrow{\deg} \mathbb{Z}$: not surj $\Rightarrow$ 0-map

$$\begin{tikzcd}
\text{space}\backslash\text{coeff} & 0 \arrow[r] & \mathbb{Z}/\ell \arrow[rr] & & \mathbb{Z}/\ell^2 \arrow[rr] & & \mathbb{Z}/\ell \arrow[r] & 0 \\
\text{Cone} & & \hat{H}^{\cdot\cdot} \arrow[rr] \arrow[dd] & & \tilde{H}^{\cdot\cdot} \arrow[rr] \arrow[dd] & & \tilde{H}^{\cdot\cdot} \arrow[r] \arrow[dd] & \tilde{H}^{2n+2d+1,n+d} \\
& & & & & q_m(\alpha) \arrow[ur, phantom, "\in" sloped] \arrow[rrr, mapsto] \arrow[d, mapsto] & & & Q_m(\alpha) \arrow[ul, phantom, "\in" sloped] \\
\text{Th}_X V \arrow[uu] & & \hat{H}^{\cdot\cdot} \arrow[rr] \arrow[ddd] & & \hat{H}^{\cdot\cdot} \arrow[rr] \arrow[ddd] & q_m(t_V) \arrow[dd, mapsto] & \hat{H}^{\cdot\cdot} \arrow[ddd] & \\
& \mathrm{CH}_0(X)/\ell \arrow[ur, phantom, "\cong" sloped] \arrow[rrr] \arrow[ddr, "0"'] & & & \mathrm{CH}_0(X)/\ell^2 \arrow[u, phantom, "\cong" sloped] \arrow[rr, dashed] \arrow[dd, "\deg"'] & & \mathrm{CH}_0(X)/\ell \arrow[u, phantom, "\cong" sloped] \arrow[r] \arrow[dd, "0"'] & 0 \\
& & & s_d(X) \arrow[ur, phantom, "\in" sloped] \arrow[rr, dashed, bend right] \arrow[dd, dotted, mapsto] & & \bullet \arrow[ur, phantom, "\in" sloped] & & \\
\text{T}^{n+d} \arrow[uuu] & & \mathbb{Z}/\ell \arrow[rr] & & \mathbb{Z}/\ell^2 \arrow[rr] & & \mathbb{Z}/\ell & \\
& & & (\text{non-zero}) \arrow[ur, phantom, "\in" sloped] & & & &
\end{tikzcd}$$

$(\cdot\cdot = 2n+2d, n+d)$

This diagram implies

「$\deg s_d(X) \not\equiv 0 \mod \ell^2 \Rightarrow Q_m(\alpha) \neq 0$」

寺杣友秀

<u>Plan</u>

$$\begin{tikzcd}
\text{DG A} \arrow[rr, dashed] & & \text{DG category} \arrow[d] \\
& & \text{triangulated cat.} \arrow[d, "\begin{matrix}\text{t-str} \\ \text{(Beilinson Soulé conj)}\end{matrix}"] \arrow[dl, dashed, "\text{同じ？}"'] \\
& \text{triangulated cat} & \text{abel. cat.} \arrow[l, dashed, "\text{derived cat}"]
\end{tikzcd}$$

graded ($A^{\cdot} = \bigoplus_{j \geq 0} \underline{A^{\cdot}(j)}$ cubic product

$\qquad\qquad\qquad\qquad\quad \shortparallel$

Bloch cycle complex $\bigoplus_{i \geq 0} Z^j(\square^{2j-i})_{\mathbb{Q}}$)

$K\pi_1$-conj $\Rightarrow$ 2つの triangulated cat. は eq.

⊙ weighted version

Hanamura, Levine, Voevodsky 's motif $\rightarrow$ $\mathcal{D}$

$\mathcal{H}_\tau$ $\quad D^b(\mathcal{H}_\tau)$

A = Bloch cycle cpx $\quad$ heart of $\mathcal{D}$ $\uparrow$ $\quad$ Bloch-Kriz motif $\nwarrow$

<u>Reference</u> { Levine
Kriz-May : Book

⊙ <u>Application</u>

$S = \operatorname{Spec} \mathbb{Z}$
$\downarrow$ (Goncharov's modification

⊙ motivic construct $(BGL)^+ = \Omega BQM$
(we can construct) $\quad$ Quillen's constr.

⊙ What kind of "object" can generate $DM(\operatorname{Spec}\mathbb{Z})$

DG category

$\mathcal{C}$ $\quad obj(\mathcal{C}) \ni x, y \quad \underline{Hom}^*(x\ y)$: complex / $\mathbb{Q}$

shift op. $[+1]$ (auto)

composite $\quad x, y, z \in obj(\mathcal{C})$

$$\underline{Hom}^*(x, y) \otimes \underline{Hom}^*(y\ z) \longrightarrow \underline{Hom}^*(x\ z)$$

hom of complex

Assume ∘ associativity

∘ $\exists$ shift $[+1]$ $\quad \left\{ \begin{array}{l} \underline{Hom}^i(x, y[1]) \\ \qquad (x[1], y) \end{array} \right. = \underline{Hom}^{i+1}_{i-1}(x, y)$

$\mathcal{C}$: DG-category

$\Rightarrow$ $H^0\mathcal{C}$: category

$$\left\{ \begin{array}{l} obj(\mathcal{C}) = obj(H^0(\mathcal{C})) \\ Hom_{H^0\mathcal{C}}(\ ) = H^0\, \underline{Hom}^*_{\mathcal{C}}(\ ) \end{array} \right.$$

(*) $\mathcal{C}$: DG-category

$\Rightarrow$ $K\mathcal{C}$: the category of DG-complex

$$\begin{tikzcd} K^0 \arrow[r, "\Delta_{01}"'] \arrow[rr, bend right, "-1", "\Delta_{02}"'] & K^1 \arrow[r, "\Delta_{12}"'] \arrow[rr, bend left, "\Delta_{13}"] & K^2 \arrow[r, "\Delta_{23}"'] & K^3 \end{tikzcd}$$

e.g.
$$\left(\begin{array}{l} \Delta_{23} \circ \Delta_{02} + \Delta_{13} \circ \Delta_{01} \\ \qquad = d\Delta_{03} \\ \Delta_{12} \circ \Delta_{01} = d\Delta_{02} \end{array} \right.$$

$$\left(\begin{array}{l} \Delta_{01} \in \underline{Hom}^0(K^0, K^1) \\ \Delta_{02} \in \underline{Hom}^{-1}(K^0, K^2) \\ \quad \cdots \end{array} \right.$$

<u>condition</u> $\Delta^2 = d\Delta$
(Frobenius integrality)

Fact: $K\mathcal{C}$ is also
DG-cat

$\underline{\text{Prop}}$ (1) $H^0 K\mathcal{C}$ is a triangulated cat.

(2) $KK\mathcal{C} \xrightarrow{s} K\mathcal{C}$ ass. simpl

$H^0KK\mathcal{C} \xrightarrow[H^0(s)]{} H^0K\mathcal{C}$ is equiv of cat.

$\underline{\text{Rem}}$ $(K^\cdot, \Delta), (L^\cdot, \Delta)$: DG-complex

$$\underset{\substack{\circlearrowleft \\ \text{differential}}}{\underline{Hom}^\cdot(K^\cdot, L^\cdot)} = \bigoplus_{i,j} Hom^{i-j}(K^i, L^j)$$

$$\overset{\cup}{F}$$

$$D(F) = \underbrace{\Delta F - F\Delta}_{\text{outer diff}} + \underbrace{dF}_{\text{inner diff}}$$

□

$\underline{\text{DGA}} \longrightarrow \underline{\text{DA-category}}$

$A^\cdot$: DGA, associative

$\mathcal{S}$: DG-cat obj : fin dim v.sp. / $\mathbb{Q}$ V ↙ generated by v.sp, $[+1]$, $\oplus$

morph : $\underline{Hom}^j_{\mathcal{S}}(V, W[i]) = A^{i+j} \otimes Hom_{\mathbb{Q}}(V, W)$

composite : multiplication of $A \otimes$ composite of $(Vec_{\mathbb{Q}})$

$K\mathcal{S}$: DG-cat

$\|$

$\mathcal{D} = \mathcal{D}(A^\cdot)$

$\underline{\text{Def}}$: $\text{SEC} \underset{\substack{\text{full sub.} \\ \text{cat}}}{\subset} \mathcal{D}$

$$\begin{tikzcd} V_0 \arrow[r] \arrow[rr, bend right, "-1"'] & V_1[1] \arrow[r] \arrow[rr, bend left, "-1"] & V_2[2] \arrow[r] & V_3[3] \end{tikzcd}$$

V_i : v. sp. / $\mathbb{Q}$

Condition for $A^{\cdot}$

(1) $H^0(A^{\cdot}) - \mathbb{Q}$ connected

$\exists$ augmentation $\varepsilon : A \to \mathbb{Q}$ which induce isom on H^0

(2) Beilinson - Soulé cond. (BS cond)

$H^i(A^{\cdot}) = 0$ for $i < 0$

(B-S conj. for Bloch's cycle complex)

<u>Prop</u> $H^0(SEC)$ is an abelian full subcat $\subset H^0\mathcal{D}$ if A is connected (i.e. $\exists$ augmentation)

<u>Prop</u> $\mathcal{D} = \mathcal{D}(A^{\cdot}) \rightsquigarrow H^0\mathcal{D}$ triangulated

If $A^{\cdot}$ satisfies BS-condition

$\Longrightarrow \exists$ t-structure on $H^0\mathcal{D}$

s.t. $H^0\mathcal{D}^{\tau \le 0} \cap H^0\mathcal{D}^{\tau \ge 0} = \mathcal{H}_{\tau} = H^0(SEC)$

<u>outline of proof</u>

$V \in \mathcal{D}$

$$\begin{tikzcd}
\sqcup & \delta \\
V_0 \arrow[r] & V_1 \\
V_0^{(0)} \arrow[r] \arrow[dr] & V_1^{(0)} \\
V_0^{(-1)}[1] \arrow[r] \arrow[dr] & V_1^{(-1)}[1] \\
V_0^{(-2)}[2] & V_1^{(-2)}[-2]
\end{tikzcd}$$

(take boundry
use augmentation

$\downarrow \tau \le 0$

$\square$

Comparison between $D^b(\mathcal{H}_t)$ and $H^0\mathcal{D}$

Def. $K\pi_1$- condition

Bar complex

$A^\cdot$: DGA $\rightsquigarrow \mathrm{Bar}(\varepsilon_1 | A | \varepsilon_2)$

$\varepsilon_1, \varepsilon_2$: augmentation

$$\begin{tikzcd} A \arrow[r, "\varepsilon_1"] \arrow[dr, "\varepsilon_2"'] & \mathbb{Q} \\ & \mathbb{Q} \end{tikzcd}$$

$\begin{pmatrix} H^i(A^\cdot) = 0 \quad (i<0) \\ H^0(A^\cdot) = \mathbb{Q} \end{pmatrix}$

$\Delta_n := \{0 < x_1 < \cdots < x_n < 1\}$

$\sigma(\Delta_n)$ = the set of faces in Δ_n

$\updownarrow$

$\mathcal{P}(E) - \{E\}$

$E = \{e_1, e_2, \cdots e_{n+1}\}$ $\quad e_0 : \{0 = x_1\}, \; e_2 : \{x_1 = x_2\}$
$\quad -- \quad e_{n+1} : \{x_n = 1\}$

$\tau : (0 \; 1 \,|\, 2 \; 3 \; 4 \; 5)$
$\quad ⓪ \quad \quad \quad \quad ①$

$0 = x_1, \; x_2 = x_3, \; x_4 = 1$

copy of $A^\cdot$

$A_\tau : \mathbb{Q}_{01} \otimes A_{23} \otimes \mathbb{Q}_{45}$

via ε_1 $\qquad$ this A-alg via ε_2

$0 = x_1 \quad x_3 = x_4 \quad x_6 = 1$
$(0 \; 1 \,|\, 2 \,|\, 3 \; 4 \; 5 \; 6 \; 7)$
$\mathbb{Q} \otimes A \otimes A \otimes A \otimes \mathbb{Q}$

$\tau < \sigma$: τ is a face of σ
$\dim \tau < \dim \sigma$

$\rightsquigarrow A_\sigma \longrightarrow A_\tau$

confluence
$\rightsquigarrow$
multiplication or
augmentation

$\mathrm{Bar}(\varepsilon_1 | A^\cdot | \varepsilon_2)$ Beilinson's Bar complex

$$= \bigoplus_{\substack{\dim(\tau)=n \\ \tau\in\sigma(\Delta_n)}} A_\tau^\cdot \longrightarrow \bigoplus_{\dim(\tau)=n-1} A_\tau \longrightarrow \cdots$$

↖ has shuffle alg. str.

Rem ∘ Chen reduced Bar complex

$\varepsilon : A \to \mathbb{Q}$ $\mathrm{Ker}(\varepsilon) = I$

$$\begin{tikzcd} \mathrm{Bar}(\ ): \cdots \arrow[r] & I^{\otimes 3} \arrow[r] \arrow[loop below, "\text{inn. diff}"] & I^{\otimes 2} \arrow[r, "\text{out diff}"'] & I^{\otimes 1} \arrow[r, "\text{0-map}"'] & \mathbb{Q} \end{tikzcd}$$

$(\varepsilon_1 = \varepsilon_2)$

Chen's $\underset{\text{qis}}{\simeq}$ Beilinson's bar complex

Advantage

We can generalize "augmentation"
↑
fiber functor

$$\begin{tikzcd} \boxed{\mathrm{Tann}} \arrow[r, "\omega"] \arrow[dr, "id"'] & \mathrm{Vect} \\ & \boxed{\mathrm{Tann}} \end{tikzcd}$$

$\mathbb{Q} \in \mathcal{D} = \mathcal{D}(A^\cdot)$
$\|$
$\mathbb{Q}_S$ $(\ \underline{\mathrm{Hom}}^i(\mathbb{Q}_S, \mathbb{Q}_S) = A^i\ )$

$$(01|23|45) \rightsquigarrow \mathbb{Q}_{01} \otimes A_{23}^\cdot \otimes \overset{A_{\tau,\mathrm{univ}}}{\overset{\|}{\mathbb{Q}_{S,45}}} \in \mathcal{D}$$

Take action in the sense of DG cat.

$$\mathbb{Q}_{01} \otimes A_{23}^\cdot \otimes A_4^\cdot \otimes \mathbb{Q}_{S,5}$$

Universality $A \xrightarrow{\varepsilon_1} \mathbb{Q}$

$$\mathrm{Bar}(\varepsilon_1 | A^\cdot | \mathrm{univ}) = \bigoplus_{\dim \tau = n} A_{\tau,\mathrm{univ}} \in \mathcal{D}$$

$$H^0(\mathrm{Bar}(\varepsilon_2 | A^\cdot | \mathrm{univ})) \in H^0 \mathcal{D}$$

Def. $A^\cdot$ is $K\pi_1$ $\Leftrightarrow$ $H^i(\mathrm{Bar}(\varepsilon_2 | A^\cdot | \varepsilon_1)) = 0$ for $i \neq 0$

$\varepsilon_1, \varepsilon_2 : A^\cdot \to \mathbb{Q}$

□

Thm. Assume $A^\cdot$ is $K\pi_1$

(1) $$H^0(\underset{\overset{\cap}{\mathcal{D}}}{\mathrm{Bar}(\varepsilon | A^\cdot | \mathrm{univ})}) \in \underset{\overset{\shortparallel :}{\mathcal{U}^*}}{H^0(\mathrm{SEC})}$$

(2) $\mathcal{U} = H^0(\mathrm{Bar}(\varepsilon | A | \varepsilon))^*$ is a Hopf alg.

$\mathcal{U}^*$ has right $\mathcal{U}$-mod str.

(3) $H^0(\mathrm{SEC})$ is a Tannakian category

↳ fiber functor is obtained by ε

↗ $\mathcal{U}$(Lie Tannaka fund grp)

↖ univ. envelope

$$\begin{tikzcd} H^0(\mathrm{SEC}) \arrow[r, "\simeq"] & (\text{left } \mathrm{Mod}\,\mathcal{U}) \end{tikzcd}$$

← left action of $\mathcal{U}$

$$\begin{tikzcd} \overset{\Psi}{M} \arrow[r, mapsto] & \mathrm{Hom}_{H^0(\mathrm{SEC})}(\mathcal{U}^*, M) \\ \mathcal{U}^* \otimes_{\mathcal{U}} M & M \arrow[l, mapsto] \end{tikzcd}$$

inverse to each other

(4) $D^b(\mathcal{S}\ell_\tau) \xrightarrow{\sim} H^0\mathcal{D}$

is a category equiv

$\square$

Application

$$\begin{tikzcd} & Z. \arrow[d] \\ X. \arrow[r] & Y. \end{tikzcd}$$ simplicial set

$C^*(X)$ singular cochain

DGA str. = Alexander - Whitney rule

$\mathrm{Bar}_n(C^\cdot(X) \mid C^\cdot(Y) \mid C^\cdot(Z)) = \bigoplus_{\dim I = n}$

$(0\,1 \mid 2\,3 \mid 4\,5)$

$\underset{01}{C^\cdot(X)} \otimes \underset{23}{C^\cdot(Y)} \otimes \underset{45}{C^\cdot(Z)}$

$C^\cdot(X)$ is algebra $/ C^\cdot(Y)$

$\mathrm{Bar} = \varinjlim \mathrm{Bar}_n$

Thm (Eilemburg - Moore)

X. simply connected

$\{(x, z, \gamma) \in X \times Z \times \mathrm{Path}(Y) \mid \gamma(0) = f(x), \gamma(1) = g(z)\}$

$\shortparallel$

$H^i(\mathrm{Bar}(C^\cdot(X) \mid C^\cdot(Y) \mid C^\cdot(Z))) \simeq H^i(X \overset{h}{\underset{Y}{\times}} Z, \mathbb{Q})$ $\square$

$\rightsquigarrow$ $BGL/_{\mathrm{Spec}\,\mathbb{Z}} \rightsquigarrow LBGL/_{\mathrm{Spec}\,\mathbb{Z}} \rightsquigarrow LBGL \underset{\mathrm{Spec}\,\mathbb{Z}}{\times} LBGL/_{\mathrm{Spec}\,\mathbb{Z}} \to LBGL/_{\mathrm{Spec}\,\mathbb{Z}}$

$\ominus$ is defined

Loday product in K-theory.

Chern class

$X./_{\mathrm{Spec}\,\mathbb{Z}} \longrightarrow C^\cdot(X(\mathbb{Z})) \rightsquigarrow$ K-theory

$\downarrow$

$\longrightarrow \underline{\mathrm{Hom}}^*(\mathbb{Q}_{X.}, \mathbb{Q}_S) \rightsquigarrow$ Bloch-Kriz's bar

$\Rightarrow$ motif $/_{\mathrm{Spec}\,\mathbb{Z}}$ is gen. by $\circ GL \times GL \to GL$ = universal