No. 1

Date

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Higher Chern maps

$$c_{ij} : K_i(X) \longrightarrow H^{2j-i}(X, \Gamma(j)) \qquad j \geq 0,\ i \geq 1$$

$c_{3,2}$ $K_3(\mathbb{C}) \to \mathbb{R}(1)$ is written by $D_2$

Bloch Wigner function

$S$ : a base scheme

$V \subset \mathrm{Sch}/S$ full sub cat

s.t.

(i) $X \in ob(V)$ $U \underset{open}{\subset} X \Rightarrow U \in ob(V)$

(ii) $E \longrightarrow X$ : vector bundle $\Rightarrow \mathbb{P}(E) \in ob(V)$

$V_{ZAR}$ the big Zariski site

$\Gamma = \bigoplus_{j \in \mathbb{Z}} \Gamma(j)$ complex of abel sheaves on $V_{ZAR}$

$$H(X, \Gamma(j)) \underset{def}{=} H(X_{Zar}, \Gamma(j))$$

(i) $\Gamma$ is a unitary graded commutative

$e : \mathbb{Z} \to \Gamma(0)$ $\qquad \cup : \Gamma(j) \overset{L}{\otimes} \Gamma(j') \longrightarrow \Gamma(j+j')$

(ii) $i : X \hookrightarrow Y$ closed imm. $\Rightarrow i_* : H^*(X, \Gamma) \longrightarrow H^*(Y, \Gamma)$

s.t. • if $X$ is pure codim $r$ $\quad i_* H^k(X, \Gamma(j)) \subset H^{k+2r}(Y, \Gamma(j+r))$

• $i_*(x \cdot i^* y) = i_* x \cdot y$

cup product

(iii) $\mathbb{G}_m[-1] \longrightarrow \Gamma(1)$ in $D(V_{ZAR})$

s.t.

(1) $X \overset{i}{\hookrightarrow} Y$ codim 1 $Y$ regular

$$\Rightarrow \quad H^0(X, \Gamma(0)) \xrightarrow{i_*} H^2(Y, \Gamma(1))$$

$$\uparrow c$$

$$H^1(Y, \mathbb{G}_m) = \mathrm{Pic}\, Y$$

$$i_*(1_X) = c(\mathcal{O}_Y(X))$$

(2) projective bundle formula

$E \longrightarrow X$ vector bundle of rk $r$

$\mathbb{P}(E) \xrightarrow[\pi]{} X \qquad \xi_E := c(\mathcal{O}_{\mathbb{P}(E)}(1))$

$$\Rightarrow \quad \bigoplus_{k=0}^{r-1} H^{*-2k}(X, \Gamma(j-k)) \xrightarrow[\oplus \pi^*(\ )\cdot \xi_E^k]{\sim} H^*(\mathbb{P}(E), \Gamma(j))$$

<u>Examples</u>

(de Rham)

$V =$ (smooth scheme$/k$) ch $k=0$ $\Gamma(j) = \Omega^{\cdot}_{/k}$

(Betti)

$V =$ (sep. schemes of finite type $/\mathbb{C}$)

$\Gamma(j) := R u^{an}_* \underbrace{\mathbb{Z}(j)}_{= (2\pi i)^j \mathbb{Z}} \qquad u^{an} \quad V_{an} \to V_{ZAR}$

(Etale) $n \geq 2$ integer

$V = (\text{sep. schemes} / \mathbb{Z}[1/n]) \qquad \Gamma(j) := R u^{et}_* \mathbb{Z}/n(j)$

$u^{et} : V_{et} \longrightarrow V_{ZAR}$

(Deligne - Beilinson)

$V$ = (smooth schemes / $\mathbb{C}$)  cone $(Rj_*\mathbb{Z}_X(j) \to \Omega_{\bar{X}}^{\leq j-1}(\log D))[-1]$

$\Gamma(j) = \mathbb{Z}(j)_D \quad \mathbb{R}(j)_D$

$$H_D(X, \mathbb{Z}(j)) = H(X_{Zar}, \mathbb{Z}(j)_D) \simeq H(\bar{X}, \ '' \ )$$

$\bar{X} \underset{j}{\supset} X$ smooth compactification

$D = \bar{X} - X$ NCD

$X.$ simplicial scheme (in $V$)

$\longrightarrow H(X., \Gamma(j))$ cohomology gp of simp. sch.

e.g. projective bundle formula O.K.

$E. \longrightarrow X.$ simp. vector bundle $rk = r$

$\Rightarrow c_j(E.) \in H^{2j}(X., \Gamma(j))$ Chern class

defined by

$$\xi_E^r + \pi^* c_1(E.) \cdot \xi_E^{r-1} + \cdots + c_r(E.) = 0 \quad \text{in } H^{2r}(\mathbb{P}(E.), \Gamma(r))$$

$G/S$ group sch. $\qquad X$: sch.

$G \times_S X \longrightarrow X$ left action

$[X/G]_n = \underbrace{G \times \cdots \times G}_{n} \times X \qquad \Rightarrow [X/G].$

$\partial_\cdot : [X/G]_n \longrightarrow [X/G]_{n-1} \qquad s_i (g_1, \cdots, g_n, x)$

$(g_1, \cdots g_n, x) \xmapsto{\partial_0} (g_2, \cdots, g_n, x) \qquad \xmapsto{s_0} (1, g_1, \cdots g_n, x)$

$\xmapsto{\partial_1} (g_1 g_2, \cdots, g_n, x)$

$\qquad \xmapsto{s_n} (g_1, \cdots g_n, 1, x)$

$\xmapsto{\partial_n} (g_1, \cdots, g_{n-1}, g_n x)$

$$[\mathbb{A}^n_S / GL_{n,S}] \longrightarrow [S / GL_{n,S}]$$

$\| \qquad\qquad \|$

$E_n^{univ}$ the univ vector bundle

$B.GL_{n,S}$ the classifying space

$$c_j(E_n^{univ}) \in H^{2j}(B.GL_{n,S}, \Gamma(j)) \qquad c_0 = 1$$

$$GL_{n,S} \hookrightarrow GL_{n+1,S} \qquad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

$$E_{n+1}^{univ}|_{B.GL_n} = E_n^{univ} + [\mathcal{O}] \quad \text{in } K_0(B.GL_n)$$

<u>Def.</u> $c_j^{univ} = \varprojlim_n c_j(E_n^{univ}) \in \varprojlim_n H^{2j}(B.GL_n, \Gamma(j))$ the univ Chern class

$\mathbb{Z}X$ : sheaf ass. to $U \mapsto \mathbb{Z}\mathrm{Hom}_S(U, X)$

$$\begin{array}{l}
c_j^{univ} \in \varprojlim H^{2j}(B.GL_n, \Gamma(j)) \\
\qquad\qquad \| \\
\qquad \varprojlim \mathrm{Hom}_{D(V_{ZAR})}(\mathbb{Z}\,\underline{B.GL_n}, \Gamma(j)[2j]) \\
\qquad\qquad \downarrow \\
\qquad \mathrm{Hom}_{D(V_{ZAR})}(\mathbb{Z}\,\underline{B.GL}, \Gamma(j)[2j]) \\
\qquad\qquad \downarrow \\
\qquad \bigoplus \mathrm{Hom}(H^{-i}(X, \mathbb{Z}\,\underline{B.GL}), H^{2j-i}(X, \Gamma(j))) \\
\qquad\qquad \downarrow \\
c_{i,j} \in \bigoplus \mathrm{Hom}(K_i(X), H^{2j-i}(X, \Gamma(j)))
\end{array}$$

$C_{3,2} : K_3(\mathbb{C}) \longrightarrow \mathbb{R}(1)$

Bloch group of $\underset{\text{infinite field}}{F}$

$$P(F) := \bigoplus_{x \in F^\times - \{1\}} \mathbb{Z}[x] \Big/ \left\langle [x]-[y]+\left[\frac{y}{x}\right]-\left[\frac{1-x^{-1}}{1-y^{-1}}\right]+\left[\frac{1-x}{1-y}\right] \right\rangle$$

scissors congruence

$$P(F) \xrightarrow{\lambda} F^\times \wedge F^\times$$
$$[x] \longmapsto x \wedge (1-x)$$

Def. $B(F) := \operatorname{Ker} \lambda$ Bloch grp.

Thm (Suslin)

$K_3^{ind}(F) = K_3(F) / K_3^M(F)$ indecomp $K_3$

$K_3^{ind}(F)_{\mathbb{Q}} \simeq B(F)_{\mathbb{Q}}$

$K_3^{ind}(F)_{\mathbb{Q}} \xrightarrow{\simeq} H_3(SL_2(F), \mathbb{Q}) \longrightarrow B(F)_{\mathbb{Q}}$

induced by Hurewicz

$\infty \in \mathbb{P}^1(F)$

$$\sum [g_0 : g_1 : g_2 : g_3] \longmapsto \text{“}\sum [g_0\infty : g_1\infty : g_2\infty : g_3\infty]\text{”}$$

$g_i \in SL_2(F)$

$$[a_0 : a_1 : a_2 : a_3] \underset{\text{def}}{=} \left[ \frac{a_0 - a_2}{a_0 - a_3} \cdot \frac{a_1 - a_3}{a_1 - a_2} \right]$$

Thm (Bloch-Wigner)

$$D_2(z) = \arg(1-z) \log|z| - \operatorname{Im} \int_0^z \log(1-t) \frac{dt}{t} \qquad z \in \mathbb{C} - \{0,1\}$$

$(D_2(0) = D_2(1) = D_2(\infty) = 0)$ $C^\infty$-func. on $\mathbb{P}^1(\mathbb{C})$

$$
\begin{array}{ccc}
K_3^{ind}(\mathbb{C})_{\mathbb{Q}} & \xrightarrow{\sim} & B(\mathbb{C})_{\mathbb{Q}} \ni [z] \\
{\scriptstyle c_{3,2}} \searrow & \swarrow & \swarrow \\
& \mathbb{R}(1) \ni iD_2(-z) & \\
& \| & \\
& H^1_D(\mathbb{C}, \mathbb{R}(2)) &
\end{array}
$$

proof (Sketch)

$$
\begin{array}{l}
c_2^{univ} \in H^4_D(B.GL_2(\mathbb{C}), \mathbb{R}(2)) \\
\qquad \downarrow \\
H^4_D(B.SL_2(\mathbb{C}), \mathbb{R}(2)) \\
\qquad \downarrow \star \\
H^3_{cont}(SL_2(\mathbb{C}), \mathbb{R}(1)) \\
\qquad \| \text{ def} \\
H^3(B.SL_2(\mathbb{C}), A^0_{\mathbb{R}} \otimes \mathbb{R}(1)) \\
\qquad \downarrow \\
c_{3,2} \in \mathrm{Hom}(H_3(SL_2(\mathbb{C}), \mathbb{Z}), \mathbb{R}(1))
\end{array}
$$

in simp. top. sp.

$\star$ is induced from

$$
\begin{array}{c}
\mathbb{R}(2) \to \mathcal{O} \to \Omega^1 \\
\downarrow \mathrm{Im} \\
A^0_{\mathbb{R}} \otimes \mathbb{R}(1)[-1]
\end{array}
$$

sheaf of $C^\infty$-func

$H^3 = SL_2(\mathbb{C})/SO(2)$ hyperbolic 3-space

$$0 \to \mathbb{R} \to A^0(H^3) \to A^1(H^3) \to A^2(H^3) \to A^3(H^3) \to 0$$

1-form

acyclic res. of cont $SL_2(\mathbb{C})$-mod

$$H^3_{cont}(SL_2(\mathbb{C}), \mathbb{R}) = A^3(H^3)^{SL_2(\mathbb{C})} / dA^2$$

$= \mathbb{R} \cdot \omega$ $\quad \omega$: hyperbolic volume

$$\omega \longmapsto \left[ [g_0 : g_1 : g_2 : g_3] \longmapsto \int_{\Delta(g_0\omega, \dots, g_3\omega)} \omega \right]$$

$$\omega \longmapsto \left[ [z] \longmapsto \int_{\Delta(z,0,1,\infty)} \omega \right] \underset{\exists \lambda}{=} \lambda D_2(z) \quad \text{(Lobachevsky)}$$