彩 厚 望

On the original Milnor conj.

§ 1937 Witt

<u>Def.</u> k field $ch \neq 2$

quadratic space $= (V, \mu)$ fin. dim. k-v. sp., symm. bilin form □

(V, μ), (V', μ') quad sp.

$V \underset{\text{isom}}{\simeq} V' \underset{\text{def}}{\iff} \exists\, V \xrightarrow[\sim]{\tau} V'$ isom of v sp

st. $\mu' \circ \tau = \mu$

Classify (V, μ)'s

<u>ex.</u> a. $a \in k$ $\langle a \rangle \cdots$ 1-dim k-v sp $k e$ base

$\mu(e, e) = a$ □

$V = (V, \mu)$, $V' = (V', \mu')$

$\rightsquigarrow$ $V \perp V'$: v. sp. $V \oplus V'$ $\mu \oplus \mu'(v \oplus v', w \oplus w')$

$= \mu(v, w) + \mu'(v', w')$

$V \otimes V'$: v sp. $V \otimes V'$ $\mu \otimes \mu'(v \otimes v', w \otimes w')$

$= \mu(v, w) \cdot \mu'(v', w')$

$\langle a_1, \cdots, a_n \rangle := \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$

$\rightsquigarrow$ every (V, μ) is isomorphic to some $\langle a_1, \cdots, a_n \rangle$

$\langle a_1, \cdots, a_n \rangle \otimes \langle b_1, \cdots, b_n \rangle = \langle \cdots a_i b_j \cdots \rangle$

Def $v \in V$: isotropic $\underset{\text{def}}{\Leftrightarrow}$ $\mu(v, v) = 0$

V. isotropic $\underset{\text{def}}{\Leftrightarrow}$ $\exists \underset{\neq 0}{v} \in V$, v: isotropic

V: an " $\Leftrightarrow$ $\forall \underset{\neq 0}{v} \in V$ v not —

Lem $V \simeq \mathbb{H} = \langle 1, -1 \rangle$

$\Leftrightarrow$ $\exists \{x, y\}$ basis of V s.t. $\begin{cases} x, y : \text{isot.} \\ \mu(x, y) \neq 0 \end{cases}$

$\Leftrightarrow$ $V \simeq \langle a, -a \rangle$ $(\exists a \in k^\times)$

$\square$

Witt's decomp. thm (1937)

$\forall (V, \mu)$

$(V, \mu) \simeq (V_{an}, \mu_{an}) \perp \mathbb{H}^m \perp (V_t, \mu_t)$

isom class of anisot.

totally isot. (i.e. $\forall v \in V$ $\mu(v, v) = 0$)

$\left.\begin{matrix} (V_{an}, \mu_{an}) \\ m \\ (V_t, \mu_t) \end{matrix}\right\}$ uniquely det. by (V, μ)

$\square$

Witt's cancellation thm

$(V_1, \mu_1) \perp (V, \mu) \simeq (V_2, \mu_2) \perp (V, \mu)$

$\Rightarrow (V_1, \mu_1) \simeq (V_2, \mu_2)$

Def. (Witt ring)

$M(k) = \{\text{non deg quad sp.}\}/_{\simeq} \hookrightarrow \widehat{W}(k)$ = the Groth. gp. of $M(k)$

$\perp, \otimes$: semiring — cancellation — ring

Lem $\mathbb{Z}\cdot\mathbb{H} \subset \widehat{W}(k)$: ideal

$(\langle a, -a\rangle \simeq \langle 1, -1\rangle)$

Def. $W(k) := \widehat{W}(k)/\mathbb{Z}\cdot\mathbb{H}$: Witt ring

$\updownarrow$ 1:1

$\{\text{anisot.}\}/_{\simeq}$

classical invariant

$(V, \mu) \rightsquigarrow m := \dim V \in \mathbb{Z}$

non-deg. $\rightsquigarrow (-1)^{\frac{m(m-1)}{2}} \det\mu \in k^\times/(k^\times)^2$

$\quad = d(V,\mu)$

$\rightsquigarrow C(V,\mu) := \begin{cases} [C(V)] & (\dim V : \text{even}) \\ [C_0(V)] & (\dim V : \text{odd}) \end{cases}$

Clifford alg $T(V)/(v\otimes v - \mu(v,v))$

C_0: even part

$\in {}_2Br(k)$

induces a map $W(F) \xrightarrow{e_0} \mathbb{Z}/2$, $\xrightarrow{e_1} k^\times/(k^\times)^2$, $\xrightarrow{e_2} {}_2Br(k)$

(Rem: not a hom)

Def. $I := \mathrm{Ker}(e_0)$, $Gr_I W := \bigoplus_{n \geq 0} I^n / I^{n+1}$ $\square$

$\rightsquigarrow$ $e_1|_I$, $e_2|_{I^2}$ grp hom

$\bar{e}_0 : W/I \xrightarrow{\simeq} \mathbb{Z}/2$

$\bar{e}_1 : I/I^2 \xrightarrow{\simeq} k^\times/(k^\times)^2$

§ 1970 Milnor

$$\begin{tikzcd} & K_n^M k/2 \arrow[dl, "s_n"'] \arrow[dr, "h_n"] & \\ I^n/I^{n+1} \arrow[rr, dashed, "\bar{e}_n"'] & & H^n_{Gal}(k, \mathbb{Z}/2) \end{tikzcd}$$

Milnor conj : s_n, h_n isom

$$s_n(\{a_1, \cdots, a_n\}) := \langle\langle a_1, \cdots, a_n \rangle\rangle := \bigotimes_{i=1}^n \langle 1, -a_i \rangle$$

2^n-dim

(Pfister form).

Rem $s_n, h_n, \bar{e}_n$ are compatible $(n = 0, 1, 2)$

- Milnor also conjectured that $\bigcap_{n \geq 0} I^n = 0$

This was proven by Arason - Pfister (1971) $\square$

s_2 : isom ⋯ Milnor (1970)

h_2 : isom ⋯ Merkurjev (1981)

h_n : Voevodsky (1996)

s_n : Orlov - Vishik - Voevodsky (2001), Morel (2004)

§ 2001 Voevodsky

Idea of the proof by OVV

Elman - Lam (1971): $\langle\langle a_1, \cdots, a_n \rangle\rangle = \langle\langle b_1, \cdots b_n \rangle\rangle \mod I^{n+1}$

$\Leftrightarrow$ —— = ...

$\Leftrightarrow \{a_1, \cdots, a_n\} = \{b_1, \cdots, b_n\}$ in $K_n^M k/2$

In particular

$s_n(\{a_1, \cdots, a_n\}) = 0 \Rightarrow \{a_1, \cdots, a_n\} = 0$

It suffices to prove k: field, $x \in K_n^M k/2$, $x \neq 0$

$\Rightarrow \exists E/k$ s.t. $x_E \in K_n^M F/2$ is of the form $\{a_1, \cdots, a_n\}$

& $x_E \neq 0$

Def. $\underline{a} = (a_1, \cdots, a_n) \in (k^\times)^n$

$Q_{\underline{a}} := \left(\begin{array}{c} \text{the quadric ass. to the quad. sp.} \\ \langle\langle a_1, \cdots, a_{n-1} \rangle\rangle \perp \langle -a_n \rangle \end{array} \right)$ dim $2^{n-1}-1$

$K_{\underline{a}} = k(Q_{\underline{a}})$

ex $n=2$

$Q_{\underline{(a_1, a_2)}} = (x_0^2 - a_1 x_1^2 - a_2 x_2^2 = 0)$

$n=3$

$Q_{\underline{(a_1, a_2, a_3)}} = (x_0^2 - a_1 x_1^2 - a_2 x_2^2 + a_1 a_2 x_3^2 - a_3 x_4^2 = 0)$ □

Prop. $\{a_1, \cdots, a_n\} = 0$ in $K_n^M(K_{\underline{a}})/2$
(calculation)

Key Thm (OVV)

$\{a_1, \ldots, a_n\} \neq 0 \Rightarrow 0 \to \mathbb{Z}/2 \to K_n^M k/2 \to K_n^M K_{\underline{a}}/2$ (exact)

generated by $\{a_1, \ldots, a_n\}$

(calculated by the technique of $\mathbb{A}^1$-ho cat.
DM
+ Assume BK(n,2))

Assuming Key Thm

$x \neq 0$, $\in K_n^M k/2$ given, $x = y_1 + y_2 + \cdots + y_r$

y_i is of the form $\{\alpha_1, \cdots, \alpha_n\}$

$E_i := k(Q_{y_1} \times \cdots \times Q_{y_i})$

Then

$$\begin{array}{ccccccccc}
K_n^M k/2 & \to \cdots \to & K_n^M E_j/2 & \to & K_n^M E_{j+1}/2 & \to \cdots \to & K_n^M E_r/2 \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
x \neq 0 & \longmapsto & x' & \longmapsto & 0 & \longmapsto & 0 \\
& & \neq & & & & \\
& & 0 & & & &
\end{array}$$

$\underset{\text{Key Thm}}{\Longrightarrow}$ $x' = y_j$ $\boxtimes$

proof of Key thm

(sketch) $\mathbb{Z}(n) \in D^-(Shv_{zar}(Sm/k))$ A abel

$H^{p,q}(X, A) := H^p_{zar}(X, \mathbb{Z}(q) \overset{\mathbb{L}}{\otimes} A)$

$H_L^{p,q}(X, A) := H^p_{et}(X, \mathbb{Z}(q) \overset{\mathbb{L}}{\otimes} A)$

$$\rightsquigarrow \quad \mathbb{Z}/2(n-1) \xrightarrow{\tau} \mathbb{Z}/2(n) \longrightarrow H^n(\mathbb{Z}/2(n))[-n] \xrightarrow{+1}$$

$\tau : \mathbb{Z}/2 \to \mathbb{Z}/2(1)$ $\qquad R^n \varepsilon_* \mu_2^{\otimes n}$

Def $X \in Sm/k$

$$\check{C}(X) = ([n] \longmapsto \overbrace{X \times \cdots \times X}^{n+1}) \in \Delta^{op} Sm/k$$

(Čech s. sch)

Lem ① If $\mathrm{Hom}(Y, X) \neq \phi$

then $\check{C}(X) \times Y \xrightarrow{\simeq} Y$ in $\mathcal{H}^{\mathbb{A}^1}$

In particular $H^{*,*}(Y) \simeq H^{*,*}(\check{C}(X) \times Y)$

② $H_L^{*,*}(\mathrm{Spec}\, k) \simeq H_L^{*,*}(\check{C}(X))$ (Hochschild-Serre. If $k = k^{sep}$, use the above) □

Then (exact)

$$0 \to H^{n,n-1}(\check{C}(Q_a), \mathbb{Z}/2) \to H^{n,n}(\check{C}(Q_a), \mathbb{Z}/2) \to H^0(\check{C}(Q_a), H^n(\mathbb{Z}/2(n)))$$

vanishing of Margolis coh. $\downarrow$ $\qquad$ $\wr|$ BK(n, 2) $\qquad$ $\downarrow$ Gersten conj for presh

$H^{2^n-1, 2^{n-1}-1}(\check{C}(Q_a), \mathbb{Z}/2)$ $\qquad K_n^M k/2$ $\qquad H^n(\mathbb{Z}/2(n))(k(Q_a))$

$\uparrow$ Rost's result $\qquad\qquad$ || BK

$\mathbb{Z}/2$ $\qquad\qquad K_n^M(k(Q_a))/2$