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Reference : On 2-torsion in motivic coh. (V. Voevodsky)

$n \geq 1$, l prime $H90(n, l)$ $\forall k$ field $ch \neq l$ $H_L^{n+1,n}(k, \mathbb{Z}_{(l)}) = 0$

Aim : Prove $H90(n, 2)$

<Why "Norm Varieties"> l prime

Thm Assume $H90(n-1, l)$

If k: field ① No fin ext of deg prime to l, $ch \neq l$

② $K_n^M k/l = 0$

$\Rightarrow H_{et}^n(k, \mathbb{Z}/l) = 0$

Cor Assume $H90(n-1, l)$

If $\forall k$ field, $\forall (a_1, \cdots, a_n) \in (k^\times)^n$

$\exists E/k$ f.g. ext. s.t. $\begin{cases} \underline{a} = 0 & \text{in } K_n^M E/l \\ H_L^{n+1,n}(k, \mathbb{Z}_{(l)}) \hookrightarrow H_L^{n+1,n}(E, \mathbb{Z}_{(l)}) \end{cases}$

$\Rightarrow H90(n, l)$ holds

Thm $\Rightarrow$ Cor

k given By assumption + transfinite induction

$\exists k'/k$ s.t. $\begin{cases} K_n^M k/l \xrightarrow{0} K_n^M k'/l \\ H_L^{n+1,n}(k, \mathbb{Z}_{(l)}) \hookrightarrow H_L^{n+1,n}(k', \mathbb{Z}_{(l)}) \end{cases}$ $(*)$

$\exists \underbrace{k''/k'}_{\text{prime to } l}$: s.t. k'' has no ext. of deg. prime to l

($\Rightarrow (*)$ is still satisfied with k' replaced by k'')

Iterate this procedure ($k \rightsquigarrow k''$) : $\underset{k}{\underset{\parallel}{K_0}} \subset \underset{k''}{\underset{\parallel}{K_1}} \subset \underset{K_1''}{\underset{\parallel}{K_2}} \subset \cdots$

$K_\infty = \bigcup K_n$

$$\Rightarrow \begin{cases} K_n^M K_\infty/\ell = 0 \\ H_L^{n+1,n}(k, \mathbb{Z}_{(\ell)}) \hookrightarrow H_L^{n+1,n}(K_\infty, \mathbb{Z}_{(\ell)}) \\ K_\infty \text{ has no ext. of deg. prime to } \ell \end{cases}$$

$$\underset{\text{Thm}}{\Rightarrow} \quad H^n_{et}(K_\infty, \mathbb{Z}/\ell) = 0 \qquad\qquad 0 = H^{n+1,n}(K_\infty, \mathbb{Q}_\ell)$$

$$\Rightarrow \quad \to \underset{0}{\underset{\|}{H_L^{n,n}(K_\infty, \mathbb{Q}_\ell/\mathbb{Z}_\ell)}} \to H_L^{n+1,n}(K_\infty, \mathbb{Z}_{(\ell)}) \to H_L^{n+1,n}(K_\infty, \mathbb{Q}_\ell) \to$$

$$\Rightarrow \quad H_L^{n+1,n}(K_\infty, \mathbb{Z}_{(\ell)}) = H_L^{n+1,n}(k, \mathbb{Z}_{(\ell)}) = 0 \qquad \blacksquare$$

Rem To prove $H90(n, \ell)$, it is sufficient to prove

$$H_L^{n+1,n}(k, \mathbb{Z}_{(\ell)}) = 0 \quad \forall k : \text{perfect}$$

Def. $X \in Sm/k$ $\quad \check{C}(X) := ([n] \mapsto \overbrace{X \times \cdots \times X}^{n+1}) \in \Delta^{op} Sm/k$

Def. $K(n) := \mathrm{Cone}(\mathbb{Z}(n) \longrightarrow \tau_{\le n+1} R\varepsilon_* \mathbb{Z}(n)^{et}) \in D^-(Shv_{zar}(Sm/k))$

Rem $H90(n, \ell) \Leftrightarrow K(n) \otimes \mathbb{Z}_{(\ell)}$: acyclic

Then $\underset{k(X)}{\underset{\|}{\mathrm{Spec}\, K}} \longrightarrow X \longrightarrow \check{C}(X) \longrightarrow \mathrm{Spec}\, k$ induces

$$\begin{tikzcd}
 & & H^{n+1,n}(\check{C}(X), \mathbb{Z}_{(\ell)}) \arrow[d] & \text{yesterday} \arrow[d] \\
H_L^{n+1,n}(K, \mathbb{Z}_{(\ell)}) \arrow[d] & H_L^{n+1}(X, \mathbb{Z}_{(\ell)}) \arrow[l] \arrow[d] & H_L^{n+1,n}(\check{C}(X), \mathbb{Z}_{(\ell)}) \arrow[l] \arrow[d] & \simeq H_L^{n+1,n}(k, \mathbb{Z}_{(\ell)}) \\
H^{n+1}_{zar}(K, K(n)_{(\ell)}) & H^{n+1}_{zar}(X, K(n)_{(\ell)}) \arrow[l, "① \simeq"'] & H^{n+1}_{zar}(\check{C}(X), K(n)_{(\ell)}) \arrow[l, "⊛"'] & \\
 & H90(n-1, \ell) & (\text{exact}) &
\end{tikzcd}$$

① birational invariant of $K(n)_{(\ell)}$

$\lceil U \subset X \underset{\text{codim } d}{\supset} Z = X \setminus U$: smooth

$$\Rightarrow \quad \to H^{*-2d}(Z, K(*'-d)_{(l)}) \to H^*(X, K(*')_{(l)}) \to H^*(U, K(*')_{(l)}) \to \cdots$$

Thus to prove $H90(n,l)$ (under $H90(n-1,l)$)

For $\underline{a} = (a_1, \cdots, a_n) \in (k^\times)^n$, it suffices to find a variety $X_{\underline{a}}$ s.t.

A) $\underline{a} = 0$ in $K_n^M k(X_{\underline{a}})/l$ $\qquad$ C) $\circledast$ is injective

B) $H_B^{n+1,n}(\check{C}(X_{\underline{a}}), \mathbb{Z}_{(l)}) = 0$

< Rost's theorems and their corollaries >

(k perfect) $\qquad \underline{a} = (a_1, \cdots, a_n) \in (k^\times)^n$

$Q_{\underline{a}} \cdots$ (the quadric ass. to $V_{\underline{a}} = \langle\langle a_1, \cdots, a_{n-1} \rangle\rangle \perp \langle -a_n \rangle$) $\subset \mathbb{P}^{2^{n-1}}$

$\dim = 2^{n-1} - 1 =: d$

$K_{\underline{a}} = k(Q_{\underline{a}}) \qquad \mathfrak{X}_{\underline{a}} = \check{C}(Q_{\underline{a}})$

<u>Prop</u> $\underline{a} = 0$ in $K_n^M K_{\underline{a}}/2$ (calculation)

$\mathbb{Z}(1)[2]$

<u>Thm</u> (Rost)

(A) $\exists$ a direct summand $M_{\underline{a}}$ of $M(Q_{\underline{a}})$ $\exists\ \mathbb{L}^d \xrightarrow{\psi^*} M_{\underline{a}} \xrightarrow{\psi_*} \mathbb{Z}$ $\circledast$

s.t. (1) $M(Q_{\underline{a}}) \to M_{\underline{a}} \xrightarrow{\psi_*} \mathbb{Z} = M(\mathrm{Spec}\, k)$

natural hom

(2) If F/k ext $Q_{\underline{a}}(F) \neq \phi \Rightarrow \circledast \otimes_k F$ is split exact

(B) $H^{2d+1,d+1}(Q_{\underline{a}}, \mathbb{Z}) \xhookrightarrow{N} k^\times$

<u>Rem</u> $H^{2d+1,d+1}(Q_{\underline{a}}, \mathbb{Z}) \simeq CH^{d+1}(Q_{\underline{a}}, 1)$

$$\simeq \mathrm{Cok}\Big(\bigoplus_{x \in (Q_{\underline{a}})_{(1)}} K_2 k(x) \xrightarrow{\partial} \bigoplus_{x \in (Q_{\underline{a}})_{(0)}} k(x)^\times \Big) \xrightarrow{N} k^\times$$

$$\uparrow \qquad\qquad \nearrow \oplus N_{k(x)/k}$$

$$\oplus\, k(x)^\times$$

Formally follows

Cor1 $\exists$ dist. triangle in $DM^{eff}_{-}(k)$

$$M(\mathcal{X}_{\underline{a}})(d)[2d] \longrightarrow M_{\underline{a}} \to M(\mathcal{X}_{\underline{a}}) \to$$

Cor2. $H^{2d+1, d+1}(\mathcal{X}_{\underline{a}}, \mathbb{Z}) = 0$

Rost's construction $(M_{\underline{a}})$ - inductively

$\underline{a}' = (a_1, \dots, a_{n-1})$ $\quad\quad \mathbb{P}^{2^{n-1}-2} \supset$

$R_{\underline{a}'} = (\text{the quadric ass. to } W_{\underline{a}'} \text{ s.t. } \langle 1\rangle \perp W_{\underline{a}'} = \langle\langle a_1, \dots a_{n-1}\rangle\rangle)$

He inductively construct $M_{\underline{a}}$ and prove

$$M(Q_{\underline{a}}) = M_{\underline{a}} \oplus M(R_{\underline{a}'}) \otimes \mathbb{L}$$

$$M(R_{\underline{a}'}) = \bigoplus_{i=0}^{d-3/2} M_{\underline{a}'} \otimes \mathbb{L}^i$$

ex. $S = (\text{the quadric ass. to } \overset{\langle 1,-1\rangle}{\overset{\shortparallel}{\mathbb{H}}}{}^m \perp \langle 1\rangle)$ $\leftarrow$ dim $= 2m-1$

$$\Rightarrow M(S) = \bigoplus_{i=0}^{2m-1} \mathbb{L}^i$$

If F/k splits $Q_{\underline{a}}, R_{\underline{a}'}$ $\left(\begin{array}{l}\text{i.e. } V_{\underline{a}} \otimes_k F \simeq \mathbb{H}^{d+1/2} \perp \langle 1\rangle \\ W_{\underline{a}'} \otimes_k F \simeq \mathbb{H}^{d-1/2} \perp \langle 1\rangle\end{array}\right)$

$$M(Q_{\underline{a}}) \simeq M_{\underline{a}} \oplus \underbrace{M(R_{\underline{a}'})}_{(1)} \otimes \mathbb{L} \quad (2)$$

$$\simeq M_{\underline{a}} \oplus (M_{\underline{a}'} \otimes \mathbb{L}) \oplus \dots \oplus (M_{\underline{a}'} \otimes \mathbb{L}^{d-1/2})$$

$\times F \Big\downarrow$

$$M((Q_{\underline{a}})_F) \simeq \mathbb{Z} \oplus \underbrace{\mathbb{L} \oplus \mathbb{L}^2 \oplus \dots \oplus \mathbb{L}^{d-1/2}}_{(1)} \oplus \underbrace{\mathbb{L}^{d+1/2} \oplus \dots \oplus \mathbb{L}^{d-1} \oplus \mathbb{L}^d}_{(2)}$$

Cor (Milnor conj for $n=2$)

(:) Suffices to show (B), (C)

C): from Cor1, $M(\mathcal{X}_{\underline{a}})(1)[2] \to M_{\underline{a}} \to M(\mathcal{X}_{\underline{a}})(1)[3]$

$$\rightsquigarrow H^0(\mathcal{X}_{\underline{a}}, K(1)_{(\ell)}) \to H^3(\mathcal{X}_{\underline{a}}, K(2)_{(\ell)}) \to H^3(M_{\underline{a}}, K(2)_{(\ell)}) \to \cdots$$

$\|$ (easy) 0 ; inj $\to H^3(Q_{\underline{a}}, K(2)_{(\ell)})$; $\cap$ ---- in fact isom $(n=2)$

(B) $H^{2+1,2}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) = H^{2^2-1,2^{2-1}}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) \overset{\text{Cor } 2}{=} 0$ $\boxtimes$

The case for $n>2$

C) is O.K. $\Leftarrow$ $\mathrm{Hom}(M(\mathcal{X}_{\underline{a}})(2d+1)[d], K(n)[n+1]) \underset{\text{easy}}{\overset{\text{for } n>2}{=}} 0$

Remains to show $\boxed{H^{n+1,n}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) = 0}$

Want to relate $H^{n+1,n} \longleftrightarrow H^{2^n-1,2^{n-1}}$ ⇖ Known (Rost)

$$\underset{CH^n(\mathcal{X}, n-1)}{H^{n+1,n}} \qquad \underset{CH^{2^{n-1}}(\mathcal{X}, 1)}{H^{2^n-1,2^{n-1}}}$$

↗

<u>motivic cohomology operation</u>