[illegible]

< The vanishing of Margolis cohomology >

X : smooth $\rightsquigarrow$ $\check{C}(X)$ $\rightsquigarrow$ $\tilde{C}(X) = \mathrm{Cone}(\check{C}(X)_+ \to (\mathrm{Spec}\, k)_+)$

$\tilde{\mathcal{X}}_{\underline{a}} := \tilde{C}(Q_a)$

Thm $\mathrm{ch}\, k \neq l$ $m \geq 0$

$$\begin{cases} X : \text{sm. proj. var. of dim } l^m - 1, \ \deg s_{l^m-1}(X) \not\equiv 0 \pmod{l^2} \\ Y : \text{sm. proj.} \end{cases}$$

$\exists X \to Y \Rightarrow \widetilde{MH}^{*,*}_m(\tilde{C}(Y), \mathbb{Z}/l) = 0$ $\square$

s_j : characteristic class s.t. $\begin{cases} s_j(L) = c_1(L)^j & (L : \text{l.b.}) \\ s_j(E \oplus F) = s_j(E) \oplus s_j(F) \end{cases}$

$$s_{l^m-1}(X) := s_{l^m-1}(TX) \in H^{2(l^m-1), l^m-1}(X, \mathbb{Z}) \ (= CH_0(X)) \xrightarrow[\deg]{} \mathbb{Z}$$

Cor Q : smooth quadric in $\mathbb{P}^{2^n}_k$ ($\mathrm{ch}\, k \neq 2$)

$$\Rightarrow \widetilde{MH}^{*,*'}_i(\tilde{C}(Q), \mathbb{Z}/2) = 0 \quad (\forall i \leq n, \ \forall *, *')$$

End of the proof

$$\begin{array}{ccccc}
H^{n+1,n}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) \simeq \tilde{H}^{n+2,n}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbb{Z}_{(2)}) & \xrightarrow[]{\overset{(1)}{\subset}} & \tilde{H}^{n+2,n}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbb{Z}/2) & \boxed{1} \\
\vdots & & \downarrow Q_1 & \boxed{2} \\
(2) & (3) & \tilde{H}^{n+5,n+1}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbb{Z}/2) & \\
\vdots & & \downarrow Q_2 & \\
\vdots & & \vdots & \\
\downarrow & & \downarrow Q_{n-2} & \boxed{n-1} \\
\tilde{H}^{2^n-1, 2^{n-1}}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) \simeq \tilde{H}^{2^n, 2^{n-1}}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbb{Z}_{(2)}) & \xrightarrow[(1)]{\subset} & \tilde{H}^{2^n, 2^{n-1}}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbb{Z}/2) & \\
\| \, (\text{Rost}) & & & \\
0 & & &
\end{array}$$

① $\exists E/k$ deg 2 $\quad Q_S(E) \neq \phi$

$$\Rightarrow \hat{H}^{*,*}(\hat{\mathcal{X}}_{\underline{a}}, \mathbb{Z}_{(2)}) \xrightarrow{2} \hat{H}^{*,*'}(\hat{\mathcal{X}}_{\underline{a}}, \mathbb{Z}_{(2)}) \to \hat{H}^{*,*'}(\hat{\mathcal{X}}_{\underline{a}}, \mathbb{Z}/2)$$

$$\begin{tikzcd} \hat{H}^{*,*}(\hat{\mathcal{X}}_{\underline{a}}, \mathbb{Z}_{(2)}) \arrow[dr, "i"] & & \hat{H}^{*,*'}(\hat{\mathcal{X}}_{\underline{a}}, \mathbb{Z}_{(2)}) \\ & \hat{H}^{*,*}((\hat{\mathcal{X}}_{\underline{a}})_E, \mathbb{Z}_{(2)}) \arrow[ur, "N"] \arrow[d, equal] & \\ & 0 & \end{tikzcd}$$

② $\quad Q_i = \beta \mathcal{P}_i - \mathcal{P}_i \beta$

$\rightsquigarrow$ 「 $\beta x = 0 \Rightarrow \beta Q_i x = 0$ 」

③ $\widetilde{MH}^{*,*'}_i = 0 \quad$ (Cor)

$$\tilde{H}^{n-i,n-i} \to \tilde{H}^{2^{i+1}+n-i-1,\, 2^i+n-i-1} \xrightarrow{Q_i} \tilde{H}^{2^{i+2}+n-i-2,\, 2^{i+1}+n-i-2}$$

$\cap$ H90$(n-1,2)$ $\quad \boxed{i} \quad\quad \boxed{i+1}$

$\tilde{H}_L^{n-i,n-i} = 0 \quad\quad \Rightarrow Q_i$ inj

<u>proof of vanishing</u>

$m = 0$ omitt $\quad\quad m > 0$

<u>Key Lemma</u> $\quad X$: sm. proj of dim d / k

$\Rightarrow \begin{cases} \exists V : \text{v.b. of rk } n \\ \exists f_V : T^{n+d} \to Th_X V \text{ in } \mathcal{H}.A^1 \end{cases}$ s.t

Tate obj.

① $V + TX = \mathcal{O}^{n+d}$ in $K_0(X)$

②
$$\begin{array}{ccc} H^{2d,d}(X, \mathbb{Z}) & \simeq & H^{2(n+d),n+d}(Th_X V, \mathbb{Z}) \\ \text{deg} \downarrow & \circlearrowleft & \downarrow f_V^* \\ \mathbb{Z} & = & H^{2(n+d),n+d}(T^{n+d}, \mathbb{Z}) \end{array}$$

<u>Rem</u>. A formal corollary to Spanier - Whitehead duality

· Voevodsky constructs the above in the category $\mathcal{H}^{\mathbb{A}^1}$ □

All coh. is $\mathbb{Z}/\ell$ - coeff unless otherwise stated.

$$d = \ell^n - 1 > 0 \qquad T^{n+d} \xrightarrow{t_V} Th_X V \longrightarrow \underset{\substack{\parallel \\ \mathrm{Cone}(t_V)}}{\mathrm{Cone}} \longrightarrow \Sigma_s^1$$

$$\rightsquigarrow \quad \begin{array}{ccc} \tilde{H}^{2n,n}(Th_X V) & \xleftarrow{\sim} & \tilde{H}^{2n,n}(\mathrm{Cone}) \\ \rotatebox{90}{$\in$} & & \\ t_V & \longmapsto & \exists ! \, \alpha \end{array}$$

$$\begin{array}{ccc} \tilde{H}^{2n+2d+1,n+d}(\mathrm{Cone}) & \longleftarrow & \tilde{H}^{2n+2d+1,n+d}(\Sigma_s^1 T^{n+d}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \gamma & \longleftarrow\!\shortmid & id \end{array}$$

$$\rightsquigarrow \quad \begin{tikzcd} \tilde{H}^{p,q}(\hat{C}(Y)) \arrow[r, "\sim", "\alpha\wedge -"'] \arrow[rdd, dotted, "\phi"'] \arrow[rd, phantom, "\circlearrowright"] & \tilde{H}^{p+2n,q+n}(\mathrm{Cone}\wedge \hat{C}(Y)) \\ & \tilde{H}^{p+2n,q+n}(\Sigma_s^1 T^{n+d}\wedge \hat{C}(Y)) \arrow[u, "s\ \circledast"'] \\ & \tilde{H}^{p-2d-1,q-d}(\hat{C}(Y)) \arrow[u, "s"'] \arrow[uu, bend right=60, "\gamma\wedge -"'] \end{tikzcd}$$

$$\left(\begin{array}{l} \circledast \quad \begin{tikzcd} Th_X V \wedge \hat{C}(Y) \arrow[r] & \mathrm{Cone}\wedge \hat{C}(Y) \arrow[r] & \Sigma_s^1 T^{n+d}\wedge \hat{C}(Y) \\ V_+ \wedge \hat{C}(Y) \simeq * \arrow[u] & & \exists\, V \to Y \\ (V - (0\text{-sect}))_+ \wedge \hat{C}(Y) \simeq * \arrow[u] & & \exists\, (V - (0\text{-sect})) \to Y \end{tikzcd} \end{array} \right)$$

<u>Lem</u>

$$\begin{tikzcd}
\arrow[r, "Q_m"] & \tilde{H}^{p,q} \arrow[r, "Q_m"] \arrow[dl, "\phi"'] \arrow[d, "\exists c"] & \tilde{H}^{p+2d+1,\, q+d} \arrow[dl, "\phi"] \\
\tilde{H}^{p-2d-1,\, q-d} \arrow[r, "Q_m"'] & \hat{H}^{p,q} \arrow[r, "Q_m"'] & {}
\end{tikzcd}$$

$$\exists\, c \in (\mathbb{Z}/\ell)^{\times}, \quad c x = \phi Q_m(x) - Q_m \phi(x)$$
$$(\forall p, q)$$

(∵) Taking $\alpha \wedge$, suffices to prove

$$\exists\, c \in (\mathbb{Z}/\ell)^{\times} \qquad c\gamma \wedge x = \alpha \wedge Q_m(x) - Q_m(\alpha \wedge x)$$
$$\left(\in \hat{H}^{p+2n+2d+1,\, q+n+d}(\mathrm{Cone} \wedge \hat{C}(\gamma)) \right)$$

$$\begin{tikzcd}
& \alpha \in \hat{H}^{2n,n}(\mathrm{Cone}) \arrow[r, "\sim"] \arrow[d, "Q_i"] \arrow[dr, phantom, "\circlearrowleft"] & \hat{H}^{2n,n}(\mathrm{Th}_X V) \ni t_V \arrow[d, "Q_i"] \arrow[ll, bend right, "\alpha"'] \\
\tilde{H}^{2n+2\ell^i-1,\, n+\ell^i-1}(\Sigma^1 T^{n+d}) \arrow[r] & \tilde{H}^{\cdot\cdot}(\mathrm{Cone}) \arrow[r] & \hat{H}^{\cdot\cdot}(\mathrm{Th}_X V) \\
& Q_i(\alpha) \arrow[u, phantom, "\in"] \arrow[urr, mapsto, bend right] & 0 \arrow[u, phantom, "\in"]
\end{tikzcd}$$

$$\underbrace{\begin{matrix} 0 & \mathbb{Z}/\ell \ni 1 \\ (i<m) & (i=m) \end{matrix}}_{=\ \tilde{H}^{2n+2\ell^i-1,\, n+\ell^i-1}(\Sigma^1 T^{n+d})} \qquad 1 \longmapsto \gamma \ (i=m)$$

$$Q_i(\alpha) = \begin{cases} 0 & (i<m) \\ \beta q_m(\alpha) - q_m \beta(\alpha) = \beta q_m(\alpha) & (i=m) \end{cases}$$

$$\rightsquigarrow \quad Q_m(\alpha \wedge x) = \alpha \wedge Q_m(x) + \beta q_m(\alpha) \wedge x$$

$$\rightsquigarrow \quad -\beta q_m(\alpha) \wedge x = \alpha \wedge Q_m(x) - Q_m(\alpha \wedge x)$$

Thus, suffices to show $Q_m(\alpha) \neq 0$ ($\Leftrightarrow \beta q_m(\alpha) \neq 0$)

May assume $\forall E/k$: deg. prime to ℓ

$Y(E) = \phi$ (otherwise $H^{*,*}(\tilde{\mathcal{C}}(Y)) = 0$)

$\leadsto H^{2d,d}(X, \mathbb{Z}) \xrightarrow{\deg} \mathbb{Z}$: not surj $\Rightarrow$ 0-map

$$\begin{tikzcd}
\text{space} \backslash \text{coeff} & 0 \arrow[r] & \mathbb{Z}/\ell \arrow[r] & \mathbb{Z}/\ell^2 \arrow[r] & \mathbb{Z}/\ell \arrow[r] & 0 \\
\text{Cone} & & \hat{H}^{\cdot\cdot} \arrow[r] \arrow[d] & \tilde{H}^{\cdot\cdot} \arrow[r] \arrow[d] & \tilde{H}^{\cdot\cdot} \arrow[r] \arrow[d] & \tilde{H}^{2n+2d+1,n+d} \\
\text{Th}_X V \arrow[u] & & \hat{H}^{\cdot\cdot} \arrow[r] & \tilde{H}^{\cdot\cdot} \arrow[r, dashed] & \tilde{H}^{\cdot\cdot} & \\
& & CH_0(X)/\ell \arrow[u, phantom, "\cong"] \arrow[r] \arrow[d] \arrow[d, bend right, "0"'] & CH_0(X)/\ell^2 \arrow[u, phantom, "\cong"] \arrow[r, dashed] \arrow[d] \arrow[d, bend left, "\deg"] & CH_0(X)/\ell \arrow[u, phantom, "\cong"] \arrow[r] \arrow[d] \arrow[d, bend right, "0"'] & 0 \\
T^{n+d} \arrow[uu] & & \mathbb{Z}/\ell \arrow[r] & \mathbb{Z}/\ell^2 \arrow[r] & \mathbb{Z}/\ell &
\end{tikzcd}$$

$q_m(\alpha) \longmapsto Q_m(\alpha)$

$q_m(\alpha) \mapsto q_m(t_V) \mapsto \bullet$

$s_d(X) \longmapsto \bullet$

(non-zero)

$(\cdot\cdot = 2n+2d, n+d)$

This diagram implies

「 $\deg s_d(X) \not\equiv 0 \mod \ell^2 \Rightarrow Q_m(\alpha) \neq 0$ 」