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X : proj smooth var / $\mathbb{C}$

$$Z_i(X) = \bigoplus_{\substack{Z \subset X \\ \text{sub var} \\ \dim i}} \mathbb{Q}\, Z$$

$$\cup$$

$$Z_i(X)_{rat} \qquad\qquad CH_i(X) = Z_i(X)/Z_i(X)_{rat}$$

properties of alg. cycles

- characterized by differential forms
- influenced by definition field

1. topological cycles and alg. cycles

$$\varphi : Z_i(X) \longrightarrow H_{2i}(X, \mathbb{Q})$$

(cycle map)

γ : top. $2i$-cycle

$$[\gamma] = 0 \in H_{2i}(X, \mathbb{Q}) \Longleftrightarrow \int_\gamma \omega = 0 \quad \forall \omega \in F^i H^{2i}_{dR}(X)$$

Hodge filt.

$$[\gamma] : \text{algebraic} \Longrightarrow \int_\gamma \omega = 0 \quad \forall \omega \in F^{i+1} H^{2i}_{dR}(X)$$

$$([\gamma] \in \mathrm{Im}\,\varphi) \qquad (\Longleftarrow)$$

Hodge conj.

$$Z_i(X)_{hom} = \mathrm{Ker}(\varphi)$$

$$\supset$$

$$Z_i(X)_{rat}$$

2. Abel-Jacobi map

$$\rho : \underset{\substack{\Psi \\ \alpha \\ \| \\ \partial\Gamma}}{Z_i(X)_{hom}} \longrightarrow \mathrm{Hom}(F^i H_{dR}^{2i+1}(X), \mathbb{C}) / H_{2i+1}(X, \mathbb{Q})$$

$$\left[\omega \longmapsto \int_\Gamma \omega \right]$$

$$\alpha \in Z_i(X)_{rat} \quad \Rightarrow \quad \rho(\alpha) = 0$$

$\not\Leftarrow$ (e.g. Mumford: $\dim X = 2$ $H^0(\Omega_X^2) \neq 0$ $\Rightarrow Z_0(X)_{hom} / Z_0(X)_{rat}$: big)

Conj (Bloch Beilinson)

If X and α are defined $/ \overline{\mathbb{Q}}$ then $\rho(\alpha) = 0 \Rightarrow \alpha \in Z_i(X)_{rat}$

3. Differential forms $/ \mathbb{Q}$

$\alpha \in Z_i(X)$ $\quad \mathbb{Q} \subset^{\exists} K \subset \mathbb{C}$ $\quad \exists X_K$: proj smooth $/K$

$\exists \alpha_K \in Z_i(X_K)$ $\quad$ s.t. $\dim_K \Omega^1_{K/\mathbb{Q}} =: m < \infty$ $\quad X_K \otimes_K \mathbb{C} = X$

$K = \overline{K}$ $\quad \alpha_K \otimes_K \mathbb{C} = \alpha$

<u>natural pairing</u>

$$\langle\ ,\ \rangle : \underset{\substack{Y_K \subset X_K \\ \text{smooth} \\ i\text{-dim}}}{Z_i(X_K)} \times H^i(X_K, \Omega^{i+m}_{X_K/\mathbb{Q}}) \longrightarrow \Omega^m_{K/\mathbb{Q}} \ (\simeq K)$$

$$\langle Y_K, * \rangle \quad H^i(\Omega^{i+m}_{X_K/\mathbb{Q}}) \longrightarrow H^i(\Omega^{i+m}_{Y_K/\mathbb{Q}})$$
$$\simeq H^i(\Omega^i_{Y_K/K}) \otimes \Omega^m_{K/\mathbb{Q}}$$
$$\simeq \Omega^m_{K/\mathbb{Q}}$$

$$Z_i(X)_\Omega := \{ \alpha \in Z_i(X) \mid \langle \alpha_K, \omega \rangle = 0 \quad \forall \omega \in H^i(\Omega^{i+m}_{X_K}) \}$$

$$Z_i(X)_{rat} \subset Z_i(X)_\Omega \subset Z_i(X)_{hom}$$

$$0 \to \Omega^1_{K/\mathbb{Q}} \otimes \mathcal{O}_{X_K} \to \Omega^1_{X_K/\mathbb{Q}} \to \Omega^1_{X_K/K} \to 0$$

$$\mathrm{Fil}^p \Omega^j_{X_K/\mathbb{Q}} := \mathrm{Im}(\Omega^p_{K/\mathbb{Q}} \otimes \Omega^{j-p}_{X_K/\mathbb{Q}} \to \Omega^j_{X_K/\mathbb{Q}})$$

$$\rightsquigarrow F^p_\Omega Z_i(X) = \{ \alpha \in Z_i(X) \mid \langle \alpha, \omega \rangle = 0 \ \forall \omega \in \mathrm{Fil}^{m+1-p} H^i(\Omega^{i+m}_{X_K/\mathbb{Q}}) \}$$

- $F^0_\Omega Z_i(X) = Z_i(X)$
- $F^1_\Omega Z_i(X) = Z_i(X)_{hom}$
- $p > \dim X - i \quad F^p_\Omega Z_i(X) = Z_i(X)_\Omega$

Infinitesial inv.

$$\mathrm{Gr}^p_{F_\Omega} Z_i(X_K) \times \underbrace{\mathrm{Gr}^{m-p}_{\mathrm{Fil}} H^i(\Omega^{i+m}_{X_K/\mathbb{C}})} \to \Omega^m_{K/\mathbb{Q}}$$

$$\| $$

$$H(\cdot \to \Omega^{m-p}_{\mathbb{Q}} \otimes H^i(\Omega^{i+p}_{X_K/K}) \to \Omega^{m-p+1}_{K/\mathbb{Q}} \otimes H^{i+1}(\Omega^{i+p+1}_{X_K/K}))$$

non trivial $\alpha \in \mathrm{Gr}^*_{F_\Omega} Z_i(X)$ trdeg of $\alpha \geq p$

4. Motive and filtr. on Chow grp

Conj. (Bloch - Beilinson)

$\exists F_M$ filt. on Chow gp s.t.

$CH_i(X) = CH^r(Y) = F^0_M CH^r(X) \qquad r+i = \dim X$

$CH^r(X)_{hom} = F^1_M CH^r(X)$

$\mathrm{Gr}^p_{F_M} CH^r(X) \simeq \mathrm{Ext}^p(h(\mathrm{Spec}\, \mathbb{C}), h^{2r-p}(X)(r))$

5. Algebraic equivalence

$$CH_i(X)_{alg} := \sum_{\substack{Y:\text{proj sm}/\mathbb{C} \\ T\in CH_{i+\dim Y}(Y\times X)}} \mathrm{Im}\left(CH_0(Y)_{hom} \xrightarrow{T_*} CH_i(X)\right)$$

$$\cap \quad \neq$$

$$CH_i(X)_{hom}$$

(e.g. (Griffiths)
$X\subset \mathbb{P}^4$: generic quartic hypersurf.
$\mathrm{Griff}(X) := CH_1(X)/CH_1(X)_{alg} \neq 0$)

$$F^p_\Omega CH_i(X)_{alg} := \sum_{Y,T} \mathrm{Im}\left(F^p_\Omega CH_0(Y) \longrightarrow CH_i(X)\right)$$

$$\mathrm{Griff}^p_i(X) = F^p_\Omega CH_i(X)/F^p_\Omega CH_i(X)_{alg}$$

6. Examples

C : curve $\quad y^e = x^d + a_1x^{d-1} + \cdots + a_d \quad (e > d)$

$a_1, \cdots, a_d \in \mathbb{C}$ alg. indep. $/\mathbb{Q}$

J : jacobian of C $\Rightarrow$ $i \geq 1,\ p\geq 1,\ d \geq i+p+1$

$\Rightarrow 0 \neq \mathrm{Griff}^p_i(J)$