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Construction

a diagram D set of objects $\mathcal{O}(D)$

$\forall p, \forall q \in \mathcal{O}(D)$ set of morphisms $M(p,q)$

(don't consider composition)

ex $\mathbb{C} \supset k$ fix $Sch/_k$: cat. of schemes of fin. type $/_k$

$H_* Sch_k$. Obj (X, Y, i) $X, Y \in Sch/_k$, $X \supset Y$ closed subsch, $i \in \mathbb{Z}$

for $f: \begin{matrix} X' & \longrightarrow & X \\ \cup & & \cup \\ Y' & \longrightarrow & Y \end{matrix}$ $\Rightarrow f_* \ (X', Y', i) \to (X, Y, i)$

$X \supset Y \supset Z$ (closed, closed) $X, Y, Z \in Sch_k$

$\Rightarrow \partial : (X, Y, i) \longrightarrow (Y, Z, i-1)$

if we reverse the arrow $\Rightarrow H^* Sch_k$

$\mathcal{C}$: cat a representation T of D in $\mathcal{C}$ is given

by a map $\begin{matrix} \mathcal{O}(D) \longrightarrow obj(\mathcal{C}) \\ p \longmapsto Tp \end{matrix}$ $\forall p, \forall q \in \mathcal{O}(D)$ $M(p,q) \to Mor_{\mathcal{C}}(Tp, Tq)$

ex. $H_* : H_* Sch_k \longrightarrow (Ab)$

$(X, Y, i) \longmapsto H_i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Z})$

$H^* : H^* Sch_k \longrightarrow (Ab)$

$(X, Y, i) \longmapsto H^i(\quad , \quad , \quad)$

Thm (Nori)

cat. of fin gen. R-mod

D : diag $\quad T : D \to (R\text{-mod})$ repr.

(1) $\exists\ \mathcal{C}(T)$: R-linear abelian cat.

$$H_T : \mathcal{C}(T) \longrightarrow (R\text{-mod})$$

(R-lin exact faithful functor)

$$\tilde{T} : D \longrightarrow \mathcal{C}(T) \text{ repr.} \qquad T = H_T \circ \tilde{T}$$

(2) $(\tilde{T}, \mathcal{C}(T))$ is universal for $\mathcal{A}$: R-lin abel. cat.

$f : \mathcal{A} \to (R\text{-mod})$ R-lin exact faithful functor

$F : D \longrightarrow \mathcal{A}$ repr. s.t. $T = f \circ F$

Then $\exists!\ L(F) : \mathcal{C}(T) \to \mathcal{A}$ R-lin. functor s.t.

$$\begin{tikzcd} & \mathcal{C}(T) \arrow[dr, "H_T"] \arrow[d, "L(F)"] & \\ D \arrow[ur, "\tilde{T}"] \arrow[dr, "F"'] & \mathcal{A} \arrow[r, "f"'] & R\text{-mod} \end{tikzcd}$$

construction of $\mathcal{C}(T)$:

first assume $\mathcal{O}(D)$ is finite

$$\mathrm{End}(T) := \left\{ \prod_{p \in \mathcal{O}(D)} e_p \in \prod_p \mathrm{End}(Tp) \;\middle|\; \begin{array}{ccc} Tp & \xrightarrow{Tm} & Tq \\ {\scriptstyle e_p}\downarrow & & \downarrow{\scriptstyle e_q} \\ Tp & \xrightarrow[Tm]{} & Tq \end{array} \quad \begin{array}{l} \forall p, \forall q \in \mathcal{O}(D) \\ \forall m \in M(p, q) \end{array} \right\}$$

$\mathcal{C}(T)$: cat. of f.g. End(T)-modules.

$\hat{T}p = Tp$ as End(T)-mod.

$\exists\ R \longrightarrow \mathrm{End}(T)$ $\quad$ ff_T: forget $\quad$ $\mathrm{End}(T)$-mod str

In general $\quad \mathcal{C}(T) = \varinjlim_{\substack{F \subset D \\ \text{finite}}} \mathcal{C}(T|_F)$ $\quad \blacksquare$

$(H_* Sch_k, H_*) \rightsquigarrow EHM(k)$ cat. of effective homological motives

$(H^* Sch_k, H^*) \rightsquigarrow ECM(k)$

universality

$\Rightarrow ECM(k) \longrightarrow \{ \mathbb{Q}_\ell\text{-}Gal(\bar{k}/k) \text{ repr.} \}$

$\qquad \searrow MHS$

$$Gal(\bar{k}/k) \longrightarrow \mathrm{End}(H^*) \otimes \mathbb{Q}$$

Construction of $\quad Sch_k \longrightarrow D^b(EHM(k))$

$\qquad\qquad\qquad\qquad \searrow D^b(ECM(k))^{op}$

<u>Basic Lemma</u> (Beilinson - Nori)

X/k. affine scheme f.t. $/k$ $\quad n = \dim X$

$X \supset Z$ closed subset, $\dim Z \leq n-1$.

Then $X \supset \exists Y$: closed subset $\dim Y \leq n-1$ s.t.

(1) $Y \supset Z$ $\quad$ (2) $H^j(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Z}) = 0$ for $j \neq n$ $\quad \square$

$\rightsquigarrow$ X has a "cellular decomposition" i.e.

a filtration by closed subsets

$\phi = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$ $\quad$ s.t. $H^j(X_i(\mathbb{C}), X_{i-1}(\mathbb{C}), \mathbb{Z}) =$

$\dim X_i \leq i$ $\qquad\qquad$ for $i \neq j$

$m^* : Sch/_k \longrightarrow D^b(ECM(k))^{op}$

X: affine $\longmapsto \{0 \to H^0(X_0) \xrightarrow{\partial} H^1(X_1, X_0) \to \quad \to H^n(X, X_{n-1}) \to 0\}$

X: separated $\Rightarrow$ Čech construction

Assume $X = U_1 \cup U_2$ U_i: affine open $U_1 \cap U_2 = U_{12}$

Take a cell decomp of U_{12}: $\phi = (U_{12})_{-1} \hookrightarrow (U_{12})_0 \hookrightarrow \cdots \hookrightarrow (U_{12})_n = U_{12}$

Then take cell decomp. of U_i s.t.

$(U_{12})_j \subset (U_i)_j \quad 0 \leq j \leq n \quad i = 1, 2$

$\exists$ restriction $m^*(U_i) \to m^*(U_{12})$

Čech cpx. $0 \longrightarrow m^*(U_1) \oplus m^*(U_2) \longrightarrow m^*(U_{12}) \longrightarrow 0$

$m^*(X) = \mathrm{Tot}\ (\text{——})$

$\rightsquigarrow m^* : Sch_k \longrightarrow D^b(FCM(k))^{op}$

$m_* : Sch_k \longrightarrow D^b(EHM(k))$

$X/_k$ variety $X \xrightarrow{f} \mathrm{Spec}\, k$

$\rightsquigarrow m^*(X) \otimes \mathbb{Q}_\ell \simeq Rf_* \mathbb{Q}_\ell$ in $D^b(Sh(\mathrm{Spec}\, k_{et})^{\mathbb{Q}_\ell})$

Nori extends

$m_*|_{Aff_k \cap Sm_k}$ to $\Pi : DM^{eff}_{gm}(k) \longrightarrow D^b(EHM(k))$

<u>Conj.</u> (someone) $\Pi \otimes \mathbb{Q}$ is fully faithful.

An application : second ℓ-adic Abel-Jacobi map

X/k : sm proj var $\dim X = n$

$z \in CH^i(X)$ $\exists$ a class $[z] \in H^{2i}_{cont}(X, \mathbb{Z}_\ell(i))$

usual class $cl(z)$ is the restriction of $[z]$ under

$H^{2i}_{cont}(X, \mathbb{Z}(i)) \to H^{2i}(\bar{X}, \mathbb{Z}_\ell(i))$

Hochschild - Serre spec. seq : $E_2^{p,q} = H^p(G_k, H^q(\bar{X}, \mathbb{Z}_\ell(i))) \Rightarrow H^{p+q}_{cont}(X, \mathbb{Z}_\ell(i))$

induces ℓ-adic Abel-Jacobi map

$$cl^1 : CH^i(X)_{hom} \to H^1(G_k, H^{2i-1}(\bar{X}, \mathbb{Z}_\ell(i))) = \mathrm{Ext}^1_{G_k}(\mathbb{Z}_\ell, H^{2i-1}(\bar{X}, \mathbb{Z}_\ell(i)))$$

description $Y = X - |z|$

$$0 \to H^{2i-1}(\bar{X}, \mathbb{Z}_\ell(i)) \to H^{2i-1}(\bar{Y}, \mathbb{Z}_\ell(i)) \to H^{2i}_{|z|}(\bar{X}, \mathbb{Z}_\ell(i))$$

$$\Downarrow\ cl^1(z)$$

$cl^1(z)$ is pull back of that by $\mathbb{Z}_\ell\, cl(z)$

second A-J map

$$\mathrm{Ker}(cl^1) \to H^2(G_k, H^{2i-2}(\bar{X}, \mathbb{Z}_\ell(i)))$$

a description : $g : Y \to \mathrm{Spec}\, k$ $Y = X - |z|$ $Rg_* \mathbb{Q}_\ell(i)$

truncation $\rightsquigarrow$

$$X_{2i-2}(Y) : 0 \to H^{2i-2}(\bar{Y}, \mathbb{Q}_\ell(i)) \to \frac{(Rg_*\mathbb{Q}_\ell(i))^{2i-2}}{\mathrm{Im}\, \partial^{2i-3}} \to \mathrm{Ker}\, \partial^{2i-1} \to H^{2i-1}(\bar{Y}, \mathbb{Q}_\ell(i)) \to 0$$

If $cl^1(z) = 0 \Rightarrow \exists$ a splitting $\mathbb{Q}_\ell \to H^{2i-1}(\bar{Y}, \mathbb{Q}_\ell(i))$

Thm (Jannsen)

$cl^2(z)$ is the pull-back of $X_{2i-2}(Y)$ by the splitting $\square$

Want more explicit description

Assume $i = n$ $\quad z = CH_0(X)_{\deg 0}$

$\rightsquigarrow$ $\exists H \ (\subset X)$ proj sm curve, intersection of

$n-1$ hypersurfaces $i_H : H \hookrightarrow X$ s.t. $z \in i_* CH_0(H)$

Then $U := X - H$, $Y = X - |z|$ $(U \subset Y)$

$$0 \to \frac{H^{2n-2}(\bar{Y}, \mathbb{Q}_\ell(n))}{H^{2n-2}_{Y\cap H}(\bar{Y}, \mathbb{Q}_\ell(n))} \to H^{2n-2}(\bar{U}, \mathbb{Q}_\ell(n))$$

$$\to H^{2n-1}_{Y\cap H}(Y, \mathbb{Q}_\ell(n)) \to Q \text{ (quotient)} \to 0$$

$(cl^1(z) = 0 \Rightarrow \exists$ a splitting $\mathbb{Q}_\ell \to Q$ $)$

Thm The push out of $cl^2(z)$ by the quotient

$H^{2n-2}(\bar{Y}) \to \dfrac{H^{2n-2}(\bar{Y})}{H^{2n-2}_{Y\cap H}(\bar{Y})}$ is given by the

pull-back by the splitting.