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$CH_* X = \{\text{alg cycles}\} \ni \sum n_i [V_i]$

$\sim$ deformation over $\mathbb{P}^1$ $\updownarrow$ $[0,1]$

X smooth proj $/\mathbb{C}$

C alg. curve $\quad Z_0 C \ni [P_1] + \cdots + [P_n]$

$\underbrace{C \times \cdots \times C}_{n} / \mathfrak{S}_n = \mathrm{Sym}^n C$

$\mathrm{Sym}^n C \longrightarrow J(C)$ fibers are proj. sp.

$\Rightarrow \quad CH_0(C) \simeq \mathbb{Z} \oplus J(C)$

finite dimensional

$X = C \times D,\ g(C) > 0,\ g(D) > 0$ universal abel. var.

$X \longrightarrow \mathrm{Alb}(X) = J(C) \times J(D)$

$\mathrm{Sym}^n X \nearrow$

$$
\begin{array}{ccc}
CH_0 C \otimes_{\mathbb{Z}} CH_0 D & \longrightarrow\!\!\!\!\rightarrow & CH_0(C \times D) \\
[P] \otimes [Q] & \longmapsto & [(P,Q)] \\
\mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} \\
 & & \oplus \\
J(C) \oplus J(D) & \longrightarrow & \mathrm{Alb}(C \times D) \\
 & & \oplus \\
J(C) \otimes_{\mathbb{Z}} J(D) & \longrightarrow\!\!\!\!\rightarrow & ?
\end{array}
$$

Albanese Kernel

$\mathrm{Ker}(CH_0(C \times D) \to \mathbb{Z} \oplus \mathrm{Alb}(C \times D))$

Thm (Mumford 1969) $\quad (p_g(C \times D) = p_g(C) \cdot p_g(D) > 0)$

$CH_0(C \times D)$ is "infinite dimensional" in the following sense.

$\forall N > 0,\ \exists U \underset{\text{dense open}}{\subset} \mathrm{Sym}^N(C \times D)$ s.t. $U \longrightarrow CH_0(C \times D)$

$[P_1] + \cdots + [P_n] \in \mathrm{Sym}^N(C \times D)$

sub var $W \subset U$, $\quad W \longrightarrow \{*\}$

$\Rightarrow \dim W \leq N \quad (\Leftrightarrow \operatorname{cod}_{\mathrm{Sym}^N(C\times D)} W \geq N)$ $\square$

If $\dim T = d \Rightarrow {}^\forall P \in T$ is defined by d equasions (locally, set-theoretically)

$\forall S \xrightarrow{\varphi} T$, $\varphi^{-1}(P)$ is defined by d equations

$\Rightarrow \operatorname{cod}(\varphi^{-1}(P)) \leq d$

If $CH_0(C\times D)$ has a "scheme structure"

$\Rightarrow \dim CH_0(C\times D) \geq N \quad ({}^\forall N) \Rightarrow \dim CH_0(C\times D) = \text{"}\infty\text{"}$

Because $J(C)$ and $J(D)$ are finite dimensional, should Alb. kernel behave finite dimensionally?

Goal 1. $N \gg 0 \qquad \beta_1, \cdots, \beta_N \in \mathrm{Alb\ Ker} \subset CH_0(C\times D)$

$(N = p_g(C\times D)+1)$ $\quad \beta_1 \wedge \cdots \wedge \beta_N := \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma)(\beta_{\sigma(1)} \times \cdots \times \beta_{\sigma(N)}) \in CH_0((C\times D)^N)$

$\Downarrow$

$\beta_1 \wedge \cdots \wedge \beta_N = 0$

Tool: Pontrjagin Product of Chow group of Abelian var.

A. abelian var $\quad \mu : A \times A \to A$ multiplication morphism

$$(P, Q) \longmapsto P+Q$$

$$CH_*A \otimes CH_*A \longrightarrow CH_*A$$

$$\alpha \otimes \beta \longmapsto \mu_*(\alpha \times \beta) =: \alpha * \beta$$ Pontrjagin product

$P, Q \in A \Rightarrow [P] * [Q] = [P+Q] \quad CH_0A$ is a subring

$CH_0A \xrightarrow{\deg} \mathbb{Z}$, $\quad \sum n_i[P_i] \longmapsto \sum n_i \quad$ is a ring hom.

$\mathrm{Ker}(\deg) = I$ is an ideal
$\parallel$
$CH_0(A)_{hom}$ generated by $[P]-[0]$

$CH_0A \supset I \supset I^{*2} \supset \cdots$ (Rem $I/I^{*2} \simeq A$)

<u>Thm</u> (Bloch) $I^{*(\dim A+1)} = 0$

(Pf) outline : Mukai Beauville Fourier transform

$\hat{A}$: dual Abelian var. $\mathcal{P}/A\times\hat{A}$: Poincaré line bundle

$\mathcal{F} := \exp(c_1(\mathcal{P})) = \sum_{i=0}^{\infty} c_1(\mathcal{P})^i/i!$ ← considered as a correspondence $A \vdash \hat{A}$
$\in CH_*(A\times\hat{A})$

① $\mathcal{F}_* : CH_*A \xrightarrow{\sim} CH_*\hat{A}$ $\left((\because) (\hat{\mathcal{F}}\circ\mathcal{F})_* = (-1)^{\dim A}(-1_A)_* \right)$

② $\mathcal{F}(\alpha * \beta) = (-1)^{\dim A}(-1_{\hat{A}})_*(\mathcal{F}(\alpha)\cdot\mathcal{F}(\beta))$
↑ intersection product

③ $\alpha \in I \subset CH_0(A) \Rightarrow \mathcal{F}(\alpha) \in \bigoplus_{i=0}^{\dim A-1} CH_i(\hat{A})$

$\left(\mathcal{F}([P]) = [\hat{A}] + c_1(L_P) + c_1(L_P)^2/2! + \cdots \right)$
$\quad L_P/\hat{A}$

$N = \dim A+1$. $\alpha_1, \cdots, \alpha_N \in I$

$$\mathcal{F}(\alpha_1 * \cdots * \alpha_N) \underset{②}{=} (-1)^0(-1_{\hat{A}})_* \mathcal{F}(\alpha_1)\cdots\mathcal{F}(\alpha_N)$$

$\Downarrow$ ① $\qquad\qquad$ $\cap$ ③

$\alpha_1 * \cdots * \alpha_N = 0 \qquad \bigoplus_{i<0} CH_i\hat{A} = 0$

∎

<u>Cor</u> C : curve $g(C)=g$, $N>g$ $\alpha_1, \cdots, \alpha_N \in CH_0C$, $\deg\alpha_i = 0$

$$\mathrm{Sym}(\alpha_1, \cdots, \alpha_N) := \frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N} \alpha_{\sigma(1)}\times\cdots\times\alpha_{\sigma(N)} \in CH_0(C^N)$$

If $N > g \Rightarrow \mathrm{Sym}(\alpha_1, \cdots, \alpha_N) = 0$

(∵) $C^N \xrightarrow{\pi} \mathrm{Sym}^N C \xrightarrow{\varphi} J(C)$

$\mathrm{Sym}^N C = C^N / \mathfrak{S}_N$

$$\mathrm{Sym}(\alpha_1, \ldots, \alpha_N) = \frac{1}{N!} \pi^* \underbrace{\pi_*(\alpha_1 \times \cdots \times \alpha_N)}_{\text{want } = 0}$$

fibers of φ are proj. sp $\Rightarrow \varphi_* \; CH_0(\mathrm{Sym}^N C) \simeq CH_0 \, J(C)$

It is sufficient to prove $\varphi_* \pi_* (\alpha_1 \times \cdots \times \alpha_N)$

$= \underbrace{(\varphi \circ \pi)_* \alpha_1}_{I} * \cdots * \underbrace{(\varphi \circ \pi) \alpha_N}_{I} = 0$

▨

Claim Because $CH_0 C_{hom}$ and $CH_0 D_{hom}$ are "finite dimensional" as above, $(CH_0 C_{hom}) \otimes_{\mathbb{Z}} (CH_0 D_{hom})$ is also finite dim.

(∵) Step 1. Prove that $V \otimes W$ is fin. dim v. sp when V and W are fin. gen using rep. theory

Step 2. Mimic it ▨

Can lift to motives $h^1(C) := (C, [\Delta_C] - [P \times C] - [C \times P])$

$\mathrm{Sym}^{2g+1} h^1(C) = 0 \Rightarrow \mathrm{Sym}(\alpha_1, \cdots, \alpha_d) \in CH_*(\underset{\neq 0}{\mathrm{Sym}^N h^1(C)})$

$\Downarrow$

$\Lambda^{4g(C)g(D)+1} \, h^1(C) \otimes h^1(D) = 0$

<u>Def</u> M is evenly fin dim $\Leftrightarrow$ for $N \gg 0$ $\bigwedge^N M = 0$

" oddly " $\Leftrightarrow$ " $\mathrm{Sym}^N M = 0$

M is fin. dim $\Leftrightarrow$ $M = \underset{\text{even}}{M^+} \oplus \underset{\text{odd}}{M^-}$

<u>Thm</u> ① $h(C)$ is fin. dim

② fin. dim. is stable under $\otimes, \oplus$, direct summand $\square$

<u>Rem</u> All known examples are obtained by above

<u>Thm</u> If M is fin dim. motive, then $M=0 \Leftrightarrow H^*(M)=0$