望月 哲史 　　Reading Morel-Voevodsky

Contents

1. Raw material

Mumford : Lect. on curves on alg. surface

Ex. $\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1} \hookrightarrow \cdots \hookrightarrow \mathrm{Pic}$

$h_{\mathbb{P}^n} \to \mathrm{Pic}$ 　　 $h_{\mathbb{P}^n}(X) \longrightarrow \mathrm{Pic}X$

$f \longmapsto f^*\mathcal{O}(1)$

$\mathrm{Pic} \fallingdotseq \mathbb{P}^\infty$ up to homotopy

In algebraic topology.

Hot = " the homotopy cat. of CW-complexes"

$\mathrm{Hot} \longrightarrow \mathrm{Set}$

$X \longmapsto \{\text{top line bundle}/X\}/\text{isom}$ ($\mathbb{C}$)

is represented by $\mathbb{P}^\infty_{\mathbb{C}}$

This statement is true in the context of alg geom. using the theory of Morel-Voevodsky

2. Q1. What is the meaning of phrase

"to do homotopy theory"

Q2. What is the meaning of the phrase

"to analyse (co)homology theory"

2-1 Q Throughout the geom. realization functor, how approximated homotopical property of topological space by combiractorial str.?

$$\begin{tikzcd}
Cat \arrow[r, "Yoneda"] \arrow[dr, dashed, "N"] & (Cat)^{\wedge} \arrow[d] \\
\Delta \arrow[u, hook] \arrow[r, dashed, "Yoneda"] \arrow[d] \arrow[dr, dashed] & \Delta^{op} Set \arrow[dl, "||"'] \\
Top \arrow[r, "Sing"] & Top^{\wedge} \arrow[u]
\end{tikzcd}$$

Prop (Kan)

X, Y simplicial sets (Kan complex)

$f: X \to Y$ simplicial homotopic

$\Leftrightarrow$ weak homotopy equiv.

Homological alg analogue $\mathcal{A}$ abelian cat

$X., Y.$ bounded below complex consisting of proj obj

$f: X. \to Y.$ chain homotopic $\Leftrightarrow$ f qis

sketch of proof Considering $Cone(f)$. We reduce to the following statement

" $Z.$ bounded below complex $\Leftrightarrow$ consisting of proj. obj.

$Z.$: contractible (chain homotopic to 0 $\Leftrightarrow$ id_Z chain homotopic to 0)

$\Leftrightarrow$ acyclic

$\Leftarrow$) uniqueness of (lifting up to homotopy

$$\begin{array}{ccc} Z_\cdot & \longrightarrow & 0 \\ {\scriptstyle id_Z} \downarrow\downarrow {\scriptstyle 0} & & \downarrow {\scriptstyle id=0} \\ Z_\cdot & \longrightarrow & 0 \end{array} \qquad id_Z \sim 0$$

Axiomized !!

Weak answer for Q1

$S_{cat} = \{ f \in Mor\, Cat : |Nf| : \text{weak eq} \}$

$S_{top} = \{ f \in Mor\, Top : f \text{ weak eq.} \}$

$S_{\Delta^{op}Set} \{ f \in Mor\, \Delta^{op}Top : |f| : \text{weak eq} \}$

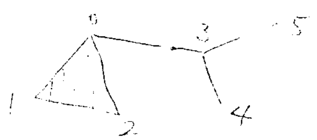
$$S_{cat}^{-1}\, Cat \xrightarrow[\sim]{N} S_{\Delta^{op}Set}^{-1}\, \Delta^{op} Set \xrightarrow[\sim]{|\;|} S_{Top}^{-1}\, Top \qquad S \subset Mor\, \mathcal{C}$$

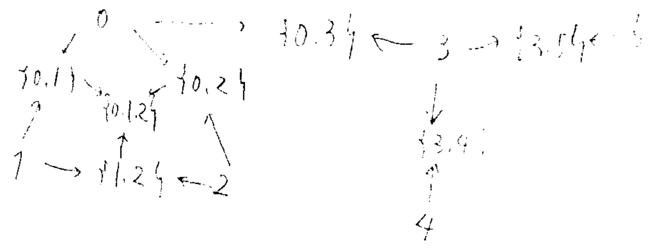
Provisional def. : (1) A "homotopy theory" is a pair $(\mathcal{C}, S)$

$\mathcal{H}o(\mathcal{C}) := S^{-1}\mathcal{C}$ $\quad$ $\mathcal{C}$ is called model category co
do homotopy theory

Ex. X simplicial complex

$Face(X) = \{$ ordered set of faces in $X \}$ is
beyantric decomposition of X





$(Cat, \mathcal{W}_{cat})$, $(\Delta^{op}Set, \mathcal{W}_{\Delta^{op}Set})$, $(Top, \mathcal{W}_{Top})$ have a same homotopy theory

Q. Is $S^{-1}C$ actually local small category?

we need more axioms !!

2.2

Thm (A. Neeman)

$\mathcal{T}$ compactly generated triangulated cat

$H: \mathcal{T}^{op} \to Ab$ cohomological functor

the following is equivalent

(1) H: representable

(2) H is commutative with small coproducts □

cohomology theory $\Leftrightarrow$ obj in $\mathcal{T}$

Weak answer for Q2

C the category of spaces

$\Downarrow$

C: (compactly generated) triangulated cat

Control higher homotopical str → Model str.

$\circlearrowleft$

(the cat. of (symmetric) spectra)

homotopy cat

lose many info

Toy's model V: fin. dim vector sp. / $\mathbb{C}$ with inner prod

$\cup$

W subspace $\boxed{V/W \xrightarrow{\sim} W^{\perp}}$

cf. $D^{-}(Shv_{Nis}(SmCor(k))) / \langle [X \times \mathbb{A}^1] - [X] \rangle$

$\xrightarrow{\sim} \langle [X \times \mathbb{A}^1] - [X] \rangle^{\perp} \simeq \mathbb{A}^1$ - local obj.

$\mathcal{C}$: cat $\hookrightarrow \mathcal{C}^\wedge$

Grothendieck $\rightsquigarrow J$

$\mathcal{C}^\wedge/J \xrightarrow{\sim} J^\perp$ $\qquad J^\perp \overset{\text{full subcat}}{\hookrightarrow} \mathcal{C}^\wedge$

$\shortparallel \qquad$ sheaf

$S_J^{-1}\mathcal{C}$ $\qquad S_J = \{ f \in \mathrm{Mor}\,\mathcal{C} \mid a(f) \text{ isom} \}$

<u>classical</u>

Topos

$\rightsquigarrow$

$\mathcal{O}_X$-Mod Abelian cat

$\rightsquigarrow$ lose info

$D(\mathcal{O}_X\text{-Mod})$, triag cat

<u>Morel-Voevodsky</u> (Bousfield localization

site with interval theory

the category of spaces

$\downarrow$

the category of spectrum $\supset$ Model str.

$\downarrow$

$S\mathcal{H}(\ )$

<u>Def.</u> (Model Category)

$\mathcal{M}$ category $\qquad \begin{matrix} W \\ F \\ Cof \end{matrix} \subset \mathcal{M}$: subcat

$(\mathcal{M}, W, F, Cof)$ $\qquad$ W cat. of weak eq.

F cat of fibrations

Cof : " cofibrations

(M0) $\mathcal{M}$ is closed under limits and colimits

(M1) (retraction) W, F, Cof are closed under retraction

$$(f \in \mathrm{Mor}W) \quad \begin{tikzcd} A \arrow[r, dashed] \arrow[rr, bend left, "id"] \arrow[d, "g"'] & X \arrow[r, dashed] \arrow[d, "f"] & A \arrow[d, "g"] \\ B \arrow[r, dashed] \arrow[rr, bend right, "id"'] & Y \arrow[r, dashed] & B \end{tikzcd} \Rightarrow g \in \mathrm{Mor}W$$

(M2) (2-out of 3) $X \xrightarrow{f} Y \xrightarrow{g} Z$

If 2 of f, g, gf are w.e then so the 3rd

(M3) (lifting property)

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "j"'] \arrow[ur, dashed, "\exists h"] & Y \end{tikzcd}$$

$f \in Cof$
$g \in Fib$

If $f \in W$ or $g \in W$

$\Rightarrow \exists h,\ B \to X$ s.t.
$g \cdot h = j,\ hf = i$

(M4) (factorisation) $X \xrightarrow{f} Y$

$$\underbrace{X \xrightarrow{g} Z \xrightarrow{h} Y}_{f}$$

$\left(\begin{array}{l} h : fib \ \& \text{ weak} \\ g : cof \end{array} \right.$ or

This factorisation is functorial

$\left(\begin{array}{l} h : fib \\ g : cof \ \& \text{ w.e.} \end{array} \right.$

$\square$

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "X \times \{0\}"'] & Y \arrow[d, shift left] \\ & X \times [0,1] \cup Y \arrow[u, shift left] \end{tikzcd}$$

$(x,1) \sim f(x)$

$$\begin{tikzcd} X & \subset & Y \\ & \{\gamma : [0,1] \to Y \mid \gamma(0) \in X\} \arrow[ul, "\gamma(0)"] \arrow[ur, "\gamma(1)"'] & \end{tikzcd}$$