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S: noether scheme finite Krull dim

$Sp(S) := \Delta^{op} Shv_{Nis}(Sm/S)$

(simplicial Nisnevich sheaf over Sm/S)

$Sp(S)_\bullet := \Delta^{op} Shv_{Nis}(Sm/S)_\bullet$

$$\begin{array}{ccc} Sm/S & \longrightarrow & Sp(S). \\ X & \longmapsto & X_+ \end{array}$$

$\Delta^{op}Set_* \longrightarrow Sp(S)$

$E \to X$: vector bundle

$Th\,E = E/E - s(X)$

zero section

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y/X \end{array}$$

$Th(E \boxtimes F) = Th\,E \wedge Th\,F$ (一般の presheaf では X)

$E \xrightarrow{f} X$ $\qquad (X, x), (Y, y)$

$F \xrightarrow{g} Y$ $\qquad (X, x) \wedge (Y, y) = (X \times Y / x \times Y \cup X \times y, \text{can})$

Define model str. on $Sp(S)$

Step 1 (W_s, C, F_s)

$W_s = \{ f \in Mor\, Sp(S) \mid x^*(f)$ is w.e. for every points $x \}$

$C = \{$monomorphism$\}$

F_s : defined by left lifting property

$Sp(S)$ has an internal hom. obj

$\mathcal{X}, \mathcal{Y} \in Sp(S)$

$S(\mathcal{X}, \mathcal{Y}) := \mathrm{Hom}_{Sp(S)}(\mathcal{X} \times \Delta^{\cdot}, \mathcal{Y})$

$\mathcal{H}_s(S) := W_s^{-1} Sp(S)$

<u>Step 2.</u> $\mathcal{X} \in Sp(S)$ is called $\mathbb{A}^1$-local

$\mathrm{Hom}_{\mathcal{H}_s(S)}(\mathcal{Y}, \mathcal{X}) \to \mathrm{Hom}_{\mathcal{H}_s(S)}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$

is isom $\forall \mathcal{Y}$

$W_{\mathbb{A}^1} := \left\{ \begin{array}{l} f \in \mathrm{Mor}\, Sp(S) \\ f : \mathcal{X} \to \mathcal{Y} \end{array} \middle| \begin{array}{l} S(\mathcal{Y}, \mathcal{Z}) \to S(\mathcal{X}, \mathcal{Z}) \text{ w.e. for} \\ \forall \mathcal{Z} \text{ fibrant } \mathbb{A}^1\text{-local obj} \end{array} \right\}$

$\mathcal{Z} \to *$ fibration

$C := \{\text{monomorphism}\}$

$F_{\mathbb{A}^1}$ defined by left lifting property

<u>Thm</u> (Morel - Voevodsky)

$(Sp(S), W_{\mathbb{A}^1}, C, F_{\mathbb{A}^1})$ proper model category $\square$

$\mathcal{H}^{\mathbb{A}^1}(S) := W_{\mathbb{A}^1}^{-1} Sp(S)$

$\mathcal{H}^{\mathbb{A}^1}(S)_\bullet := W_{\mathbb{A}^1}^{-1} Sp(S)_\bullet$

S^1_s -- simplicial circle

$\cap$

$\Delta^{op}Set_* \longrightarrow Sp(S)$

$S^1_t = (\mathbb{A}^1 - \{0\}, 1)$

[illegible] case

$M(\mathbb{A}^1 - \{0\}, 1) = \mathbb{Z}(1)[1]$

$M(\mathbb{P}^1, \infty) = \mathbb{Z}(1)[2]$

$M(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$

$C^*(\mathbb{Z}_{tr}(\mathbb{P}^1) / \mathbb{Z}_{tr}(\mathrm{Spec}\, k))$

$(\mathbb{P}^1, \infty) = S^1_s \wedge S^1_t$ $\quad M(\mathbb{P}^1, \infty) = \mathbb{Z}[1] \otimes \mathbb{Z}(1)[1] = \mathbb{Z}(1)[2]$

in $\mathcal{H}^{\mathbb{A}^1}(S)$.

$S^{p,q} := S^{p-q}_s \wedge S^q_t \quad \mathbb{Z}(q)[p]$

(X, Z) smooth pair

Thm (homotopy purity theorem)

$X/X-Z \xrightarrow{\simeq} Th\, N_{X,Z}$

(considering Nisnevich local we may assume
$(X, Z) = (\mathbb{A}^n, \mathbb{A}^d)$)

$M_Z X \simeq M\, Th\, N_{X,Z} \xrightarrow{\simeq} M(Z)(r)[2r] \quad (Z \hookrightarrow X \text{ cod } r)$

$\|$

$C^*(\mathbb{Z}_{tr}(X) / \mathbb{Z}_{tr}(X-Z))$

G : sheaf of group $\in Sp(S)$

we can define $BG*$

(similar to simplicial construction)

$U \in Sm/S$

$$\mathrm{Hom}_{\mathcal{H}_s(S)}(S_s^i \wedge U_+, BG) = \begin{cases} H^1_{Nis}(U, G) & i = 0 \\ G(U) & i = 1 \\ 0 & i \geq 1 \end{cases}$$

If G : $\mathbb{A}^1$-local $\longrightarrow\ \|$

$$\mathrm{Hom}_{\mathcal{H}(\mathbb{A}^1(S))}(S_s^i \wedge U_+, BG)$$

$G = \mathbb{G}_m$ or abelian var $\Rightarrow$ $\mathbb{A}^1$-local

$$\mathrm{Hom}_{\mathcal{H}\mathbb{A}^1(S).}(U_+, B\mathbb{G}_m) = H^1_{Nis}(U, \mathbb{G}_m) = \mathrm{Pic}\, U$$

$BGL \longleftarrow$ Grassman

$B\mathbb{G}_m = (\mathbb{P}^\infty, \infty)$ in $\mathcal{H}\mathbb{A}^1(S)$.

Over $\mathcal{H}\mathbb{A}^1(S)_\bullet$ $- \wedge S_t^1$ is not an isom.

Dold - Puppe :

$$\begin{array}{ccc} \Delta^{op}(Ab) & \xrightarrow{\sim} & C_{\geq 0}(Ab) \\ & & \cap \\ & & C(Ab) \end{array}$$

$(\mathbb{P}^1, \infty)$ - spectra

$$\{X_i,\ X_i \wedge (\mathbb{P}^1, \infty) \xrightarrow{e_i} X_{i+1}\}_i$$

$\Rightarrow$ the category of $(\mathbb{P}^1, \infty)$ - spectra $Spc(S)$.

$\downarrow$ localizing

$\mathcal{SH}(S) \leftarrow$ symmetric monoidal category
triangulated cat.

$$\Sigma^\infty : Sm/S \longrightarrow \mathcal{SH}(S)$$

$E \in \mathcal{SH}(S) \qquad (X, x) \in (Sm/S)_*$

$$\begin{cases} E^{p,q}(X, x) := \mathrm{Hom}(\Sigma^\infty(X, x), E \wedge S^{p,q}) \\ E_{p,q}(X, x) := \mathrm{Hom}(S^{p,q}, \Sigma^\infty(X, x) \wedge E) \end{cases}$$

$H\mathbb{Z} \in \mathcal{SH}((X)$

$S = \mathrm{Spec}\, k$

$H_{\mathbb{Z}}^{p,q}(X_+) = H_{\mathcal{M}}^p(X, \mathbb{Z}(q))$

<u>Ex.</u> X pointed simplicial set

$$\mathrm{Symm}^\infty(X) := \varinjlim \mathrm{Symm}^n(X)$$

$$\left(\mathrm{Symm}^n(X) = \underbrace{X \times \cdots \times X}_{n} / \mathfrak{S}_n \right.$$

abelian monoid

In topological homotopy category

$$\mathrm{Symm}^\infty(X)^+ = \prod K(\tilde{H}_n(X, \mathbb{Z}), n)$$

$$\mathrm{Symm}^\infty(S^n)^+ = K(\mathbb{Z}, n)$$

In the context of motivic homotopy theory,

$\mathrm{ch}\, k = 0$ $\quad U$: smooth connected

$$\mathrm{Hom}(U, \mathrm{Symm}^{\infty}(X)) = \bigoplus_{\substack{Z \subset U \times X \\ \downarrow \text{ finite surj} \\ U}} \mathbb{N} Z$$

$$(\mathrm{Symm}^{\infty})^{+} \doteqdot \mathbb{Z}_{tr}(\ )$$

$$K(\mathbb{Z}(n), 2n) = \mathbb{Z}_{tr}((\mathbb{P}^1, \infty)^{\wedge n})$$

$$\Downarrow$$

$$H\mathbb{Z} = \{ K(\mathbb{Z}(n), 2n) \}$$