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Spanier - Whitehead duality in alg. geom.

hand written notes available at URL:

http://www.math.jussieu.fr/~riou/notes/

0 - Lemma on vector spaces

Lem Let A be a commutative ring

M, N be two A-modules

$I : A \to M \otimes N$ $\quad I(1) = \sum_i m_i \otimes n_i$

$E : N \otimes M \to A$

such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "I \otimes id_M"'] \arrow[rr, bend left, "id_M"] & M \otimes N \otimes M \arrow[r, "id_M \otimes E"'] & M \end{tikzcd}$$

$$\begin{tikzcd} N \arrow[r, "id_N \otimes I"'] \arrow[rr, bend left, "id_N"] & N \otimes M \otimes N \arrow[r, "E \otimes id_N"'] & N \end{tikzcd}$$

Then M, N are f.g. projective A-modules

(·) $\forall$ A-mod T, we have a can. isom

$$\mathrm{Hom}_A(M, T) \xrightarrow{\sim} T \otimes N$$

$$f \mapsto f \otimes id_N : M \otimes N \to T \otimes N$$

$$\begin{tikzcd} M \otimes N \arrow[r] & T \otimes N \\ A \arrow[u, "I"] \arrow[ur, dashed] & \end{tikzcd}$$

$\Rightarrow \mathrm{Hom}_A(M, A) \xrightarrow{\sim} N$

$\boxtimes$

Def. (Dold-Puppe) Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal cat.

M, N : obj on $\mathcal{C}$
$I : \mathbb{1} \to M \otimes N$
$\varepsilon : M \otimes N \to \mathbb{1}$

it is strong duality if the previous diag commutes.

$\forall T \in \mathcal{C}$ $\quad \underline{\mathrm{Hom}}(M, T) \xrightarrow{\sim} T \otimes N$

$N \simeq \underline{\mathrm{Hom}}(M, \mathbb{1})$

$\rightsquigarrow$ notion of strong dualizable obj

$\varepsilon : \underline{\mathrm{Hom}}(M, \mathbb{1}) \otimes M \longrightarrow \mathbb{1}$ evaluation map

$I : \mathbb{1} \xrightarrow[\mathrm{id}_M]{} \underline{\mathrm{Hom}}(M, M) \xleftarrow{\sim} M \otimes \underline{\mathrm{Hom}}(M, \mathbb{1})$

A: commutative ring In $(D(A\text{-mod}), \otimes, A)$ strong dualizable objects are perfect complexes

I SW in alg. top.

Thm (Poincaré duality)

Let X be an oriented connected compact smooth mfd. of dim d.

$\forall$ field k $\forall i \in \mathbb{Z}$ $\quad H^i(X, k) \times H^{d-i}(X, k) \longrightarrow H^d(X, k) \xrightarrow{\sim} k$

is a perfect pairing

Idea divide this into two statements

(a) orientation
(b) duality

(a) Thom isom

X : smooth mfd E : vector bundle of rk r

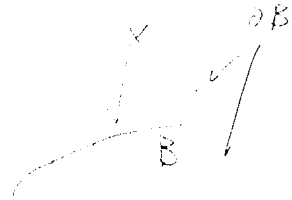
$\|\cdot\|$: Euclidean metric on E

$B = \{e \in E, \|e\| \le 1\}$ $\quad B/\partial B = Th_X E$

$\cup$ trivial bundle

∂B $\quad Th_X(\varepsilon^n) = S^n \wedge X_+$

Thm: If E is oriented, there is a class $\mu \in H^r(Th_X E)$ inducing isom. $H^*(X) \xrightarrow{\simeq} \tilde{H}^{*+r}(Th_X E)$



(b) X : smooth compact mfd

$i: X \hookrightarrow \mathbb{R}^n$ normal bundle ν

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow & \\ \partial B \subset B & \hookrightarrow & \mathbb{R}^n \end{array}$$

tubular nbd.

$Th_X \nu = B/\partial B$

$B \times X \longrightarrow \mathbb{R}^n$ $\quad$ if $b \in \partial B$, then $b - x \neq 0$

$(b, x) \longmapsto b - x$

$B \times X / \partial B \times X \longrightarrow \mathbb{R}^n / \mathbb{R}^n - \left(\begin{array}{l}\text{small open ball} \\ \text{centered on } 0\end{array}\right)$

$Th_X \nu \wedge X_+ \longrightarrow S^n$

$\downarrow$ $\tilde{C}_*$ singular chain complex functor

$\tilde{C}_*(Th_X \nu) \otimes C_* X \longrightarrow \mathbb{Z}[n]$ in $D(Ab)$

$\tilde{C}_*(Th_X \nu)[-n] \otimes C_* X \longrightarrow \mathbb{Z}$

<u>Thm</u>: This map induces a strong duality in $D(Ab)$ between $C_* X$ and $\tilde{C}_*(Th_X \nu)[-n]$

part (a) of 5| qis Thom isom

$C_* X[-d]$

Refinement $\mathcal{H}^{f}_{\bullet}$ the pointed homotopy cat. of finite CW-complexes

$\wedge : \mathcal{H}^{f}_{\bullet} \times \mathcal{H}^{f}_{\bullet} \to \mathcal{H}^{f}_{\bullet}$

$S : \mathcal{H}^{f}_{\bullet} \to \mathcal{H}^{f}_{\bullet}$
$\;\; = - \wedge S^1$

Def. The Spanier-Whitehead cat SW is the category

objects (X, n) $X \in \mathcal{H}^{f}_{\bullet}$ $n \in \mathbb{Z}$

hom $\mathrm{Hom}_{SW}((X,n),(Y,m)) = \varinjlim_{r \gg 0} (S^{r+n} \wedge X, S^{r+m} \wedge Y)$ $\square$

$\mathcal{H}^{f}_{\bullet} \to SW \; ; \; X \mapsto (X, 0)$

SW is a symmetric monoidal triangulated cat

$(X,n)[1] = (X, n+1)$

Prop. All objects in SW are strong dualizable
(i.e. SW is a rigid $\otimes$-cat)
$SW = \langle S^0 \rangle$

the proof uses Ayoub's four functors

Thm X: smooth compact mfd $X \hookrightarrow \mathbb{R}^n$
ν: normal bundle

There is a strong duality in SW between

X_+ and $\mathrm{Th}_X \nu[-n]$ $\square$

part (b) of Poincaré duality

$0 \to TX \to \mathcal{E}^n \to \nu \to 0$
$\quad\quad\quad\quad \| $
$\quad\quad\quad T\mathbb{R}^n|_X$

$\mathrm{Th}_X \nu[-n] = \mathrm{Th}_X(-TX)$

$SW \xrightarrow{\tilde{C}_*} D(Ab)$

§ 2 SW in alg. geom.

Lefschetz motive
$h(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{L}$

- k: field CHM(k) : Chow motives

X proj. smooth var $h(X)$ has a strong dual $h(X) \otimes \mathbb{L}^{-d_X}$

i.e. corresponds to elements in $CH^{d_X}(X\times X)$, they are given by $[\Delta_X]$.

· S: noetherian scheme $\rightsquigarrow$ $(\mathcal{SH}(S), \wedge, S^0)$ stable $\mathbb{A}$-ho cat

<u>Def</u>: X: scheme / S, E/X: vector bundle

$Th_X E = E/(E - \text{zero section}) \in \mathcal{SH}(S)$

<u>Rem</u> a short exact seq $0 \to E' \to E \to E'' \to 0$ gives a can. isom. $Th_X E \xrightarrow{\sim} Th_X(E' \oplus E'')$

(Deligne's virtual cat) $\longrightarrow \mathcal{SH}(S)$ $\quad E \mapsto Th_X E$

<u>Thm</u> If X is proj smooth / S, there is a strong duality between X_+ and $Th_X(-TX)$ $\square$

several sub-triangulated cat of $\mathcal{SH}(S)$

$\mathcal{SH}(X)^{pf} = \{ X \in \mathcal{SH}(S) \mid Hom_{\mathcal{SH}(S)}(X,-) : \mathcal{SH}(S) \to Ab$ commutes with $\oplus \}$

$= \langle X_+ \wedge (\mathbb{P}^1)^{\wedge n},\ Y : \text{smooth}/S,\ n \in \mathbb{Z} \rangle^{\natural}$ ← pseudo ab. hull

$\mathcal{SH}(X)^{proj} = \langle \quad " \quad X \text{ proj smooth} \quad " \rangle^{\natural}$

$\mathcal{SH}(X)^{rig} = \{ X \in \mathcal{SH}(S) \mid X : \text{strong dual} \}$

$$\mathcal{SH}(S)^{proj} \underset{\text{Thm}}{\subset} \mathcal{SH}(S)^{rig} \underset{\text{triv}}{\subset} \mathcal{SH}(S)^{pf}$$

<u>Thm</u> Let k be a field admitting res of sing. (weak RS(k)) then $\mathcal{SH}(k)^{proj} = \mathcal{SH}(k)^{rig} = \mathcal{SH}(k)^{pf} \underset{\text{def}}{=} \mathcal{SW}(k)$ $\square$

If k is perfect, replace $\mathcal{SH}(k)$ by $\mathcal{SH}(k)_{\mathbb{Q}}$ (use de Jong)

$k = \mathbb{C}$ $\quad \mathbb{H}_{dR.diff} \xrightarrow{\sim} \mathbb{H}_{dR.alg}$

$\mathbb{H}_{dR.diff} \in \mathcal{SH}(\mathbb{C})$ representing de Rham coh of $X(\mathbb{C})$

thm by Grothendieck

$\mathbb{H}_{dR.alg}$: rep. alg de Rham coh.

reduce to proj smooth and use GAGA.