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Operations on algebraic K-theory and regulators via the homotopy theory of schemes

K_0 SGA 6

Let X be a scheme

Def: $K_0(X)$ is the abelian group

$$\begin{cases} \text{generators} & [M] \quad M: \text{isom class of vector bundles on } X \\ \text{relations} & [M']+[M'']=M \text{ if } 0\to M'\to M\to M''\to 0 \text{ is exact} \end{cases}$$

G: any ab. group

$$K_0(X) \longrightarrow G$$
$$\Updownarrow$$
$$\text{maps } \left\{ \begin{matrix} \text{isom. classes} \\ \text{of vector bundles}/X \end{matrix} \right\} \longrightarrow G$$

which are additive on short exact seq

$K_0(X)$ is a commutative ring

$$[M]\cdot[N] = [M\otimes N]$$

Def M: vector bundle $/X$, $\lambda_t M = \sum_{n\geq 0} [\Lambda^n M] t^n$

$$\lambda_t M \in K_0(X)[[t]]$$

Lem If $0\to M'\to M\to M''\to 0$ is exact seq. of bundles then $\lambda_t M = \lambda_t M' \cdot \lambda_t M''$

$$\Rightarrow \text{morphism } K_0(X) \xrightarrow{\lambda_t} 1 + K_0(X)[[t]]$$

on $\Lambda^* M$ there is a filtration such that

$$Gr^p \Lambda^n M \simeq \Lambda^p M' \otimes \Lambda^{n-p} M''$$

$\rightsquigarrow$ $K_0(X)$ is a λ-ring
(pré-λ-anneau)

<u>Thm</u> (SGA 6) $K_0(X)$ is a special λ-ring
(λ-anneau)

there are formulas for $\lambda^n(t)$, $\lambda^n(uv)$, $\lambda^n(\lambda^m(u))$

$\square$

$-X$: compact space $\rightsquigarrow K_0^{top}(X)$ use top. complex v.b.

<u>Grassman varieties</u>

$(d,r) \in \mathbb{N}^2$

$$Gr_{d,r}(\mathbb{C}) = \{ V \subset \mathbb{C}^{d+r} \mid V : \text{sub } \mathbb{C}\text{-v sp. of dim } d \}$$

$\mathcal{M}'_{d,r} \to Gr_{d,r}$ tauto. vector bundle of rk d

$$\begin{array}{ccccccc} Gr_{d,r}(\mathbb{C}) & \hookrightarrow & Gr_{d,r+1}(\mathbb{C}) & \hookrightarrow \cdots & \hookrightarrow & Gr_{d,\infty}(\mathbb{C}) \\ \downarrow & & & & & \vdots \\ Gr_{d+1,r}(\mathbb{C}) & \hookrightarrow & Gr_{d+1,r+1}(\mathbb{C}) & \hookrightarrow & & \downarrow \\ & & & & & Gr(\mathbb{C}) \end{array}$$

$$\mathcal{M}_{d,r} = [\mathcal{M}'_{d,r}] - d \in K_0^{top}(Gr_{d,r}(\mathbb{C}))$$

$$(\mathcal{M}_{d,r})_{(d,r)\in\mathbb{N}^2} \in \varprojlim_{(d,r)} (Gr_{d,r}(\mathbb{C}))$$

<u>Thm</u> X : compact sp.

$$[X, \mathbb{Z} \times Gr(\mathbb{C})] \simeq K_0^{top}(\mathbb{C})$$

$\square$

def. $\Rightarrow$ $K_n^{top}(X) = [S^n \wedge X_+, \mathbb{Z} \times Gr(\mathbb{C})]$
X: space

Thm (Morel Voevodsky) S: regular scheme, X/S: smooth

$\mathrm{Hom}_{\mathcal{H}(S)}(X, \mathbb{Z}\times Gr) \simeq K_0(X)$

$\mathrm{Hom}_{\mathcal{H}(S)}(S^n \wedge X_+, \mathbb{Z}\times Gr) \simeq K_n(X)$ (defined by Quillen)
$\forall n \geq 0$ $\square$

more precisely, any pointed endomorphism of $\mathbb{Z}\times Gr$ in $\mathcal{H}_\cdot(S)$ gives map $K_n(X) \to K_n(X)$ $\forall n$, $\forall X$: smooth

Thm S: regular scheme, $K_0(-): (Sm/S)^{opp} \to (Sets)$

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{H}(S)}(\mathbb{Z}\times Gr, \mathbb{Z}\times Gr) & \xrightarrow[\alpha]{\sim} & \mathrm{Hom}_{(Sm/S)^{opp} \to (Sets)}(K_0(-), K_0(-)) \\
 & & \beta \downarrow \wr \\
 & \overset{\gamma}{\longrightarrow} & [\varprojlim_{d,r} K_0(Gr_{d,r})]^{\mathbb{Z}} \\
 & & \wr\! \mid \\
 & & (K_0(S)[[c_1, c_2, \cdots]])^{\mathbb{Z}}
\end{array}$$

$\xi: K_0(-) \to K_0(-)$
$\downarrow$
(evaluate on "$d_r + n$", $n \in \mathbb{Z}$)

proof γ: bijection. Milnor's exact seq. $\rightsquigarrow$ γ: surj.
$\rightsquigarrow$ $\mathrm{Ker}\, \gamma = [R^1 \varprojlim_{(d,r)} K_1(Gr_{d,r})]^{\mathbb{Z}}$

$d \leq d'$
$r \leq r'$ $K_1(Gr_{d',r'}) \twoheadrightarrow K_1(Gr_{d,r})$
$\wr\!\mid$
$K_1(S) \otimes_{K_0(S)} K_0(Gr_{d',r'})$

K-theory of Grassmannian (SGA6)

B. surjective $\quad \tau, \tau' : K_0(-) \longrightarrow K_0(-)$ s.t.

$$\tau(\mathcal{M}_{d,r} + n) = \tau'(\mathcal{M}_{d,r}+n)$$

$X \in Sm/S \qquad x \in K_0(X)$

- X : affine and connected

$$x = [M] - \underset{\overset{\parallel}{d}}{\mathrm{rk}\, M} + n \qquad n \in \mathbb{Z}$$

M : subbundle of $\mathcal{E}^{d+r}$, $r \gg 0$

$\exists f : X \longrightarrow Gr_{d,r} \quad$ s.t. $\quad f^*\mathcal{M}_{d,r} = M$

$$x = f^*(\mathcal{M}_{d,r}+n) \quad \Rightarrow \quad \tau(x) = \tau'(x)$$

- X : general $\quad$ use Jouanolou's trick

$$\begin{array}{ll} \exists \ T & \text{torsor under some} \\ \ \ \downarrow & \text{vector bundle} \\ \ \ X & \text{s.t. } T \text{ affine} \end{array}$$

$\simeq$ in $\mathcal{H}(S) \qquad K_0(X) \simeq K_0(T)$ $\qquad \square$

Variant with several variables

maps $\quad (\mathbb{Z} \times G)^n \longrightarrow \mathbb{Z} \times G \quad$ in $\mathcal{H}(S)$

$$\updownarrow 1:1$$

$(K_0(-))^n \longrightarrow K_0(-)$ in $(Sm/S)^{opp} \to Sets$

$\lambda^n : K_0(-) \longrightarrow K_0(-) \quad \rightsquigarrow \quad \lambda^n : \mathbb{Z} \times Gr \longrightarrow \mathbb{Z} \times Gr$

$\cdot : K_0(-) \times K_0(-) \longrightarrow K_0(-) \quad \rightsquigarrow \quad \times : (\mathbb{Z}\times G)^2 \longrightarrow \mathbb{Z} \times G$

$$\begin{tikzcd} (\mathbb{Z}\times G)^2 \arrow[d] \arrow[r] & \mathbb{Z}\times G \\ (\mathbb{Z}\times G) \wedge (\mathbb{Z}\times G) \arrow[ur] & \end{tikzcd}$$

$K_i(X) \times K_j(X) \longrightarrow K_{i+j}(X)$ (the same as those of Quillen, Loday ~~Waldhausen~~)

<u>Thm</u> $\mathbb{Z}\times Gr$ is a special λ-Ring in $\mathcal{H}(S)$ $\square$

Soulé defined operation for elements of $\begin{array}{c} R_{\mathbb{Z}}GL \\ \downarrow \\ \mathrm{End}(\mathbb{Z}\times Gr)\end{array}$

<u>Additive operations</u>.

understand $\tau : K_0(-) \longrightarrow K_0(-)$ that are additive

<u>example</u> Adam's operation $\psi^{*} : K_0(-) \longrightarrow K_0(-)$

$$\forall x \in K_0(X) \qquad \frac{1}{\lambda_t(x)} \frac{d\lambda_t(x)}{dt} = \sum_{k=1}^{\infty} (-1)^{k-1} \psi^k(x)\, t^{k-1}$$

L line bundle $\quad \psi^k([L]) = [L^{\otimes k}]$

<u>Thm</u> S : regular sch.

$$\begin{array}{ccc} \mathrm{Hom}_{(Sm/S)^{opp}\to(Ab)}(K_0(-), K_0(-)) & \longrightarrow & \mathrm{Hom}_{(Sm/S)^{opp}\to(Sets)}(\mathrm{Pic}(-), K_0(-)) \\ & \searrow^{c^*}_{\wr} & \wr\!\wr \\ & & \varprojlim_n K_0(\mathbb{P}^n) \\ & & \| \\ & K_0(S)[\![U]\!] = & \varprojlim_n K_0(S)[U]/(U^{n+1}) \\ & & U = [\mathcal{O}(1)] - 1 \end{array}$$

<u>proof</u>: injectivity. "splitting principle"

M: v. bundle on X

$$\begin{array}{ccc} \exists\ \mathrm{Flag}(M) & & [f^*M] = \sum [\text{line bundles}] \\ \downarrow f & \nearrow & \\ X & M & K_0(X) \hookrightarrow K_0(\mathrm{Flag}\, M) \end{array}$$

<u>surjectivity</u> : $c^*(\psi^k) = (1+U)^k$

$x \in K_0(S)$

$x\psi^k : K_0(-) \longrightarrow K_0(-)$ | $c^*(x\psi^k) = x(1+U)^k$

$y \longmapsto x\cdot\psi^k(y)$

$\boxtimes$

<u>Rem</u> : there is a composition law

$* : K_0(S)[[U]] \times K_0(S)[[U]] \longrightarrow K_0(S)[[U]]$ $\quad$ $\mathbb{Z}$-bilinear continuous

$x(1+U)^k * y(1+U)^{k'} = (x\,\psi^k(y))(1+U)^{kk'}$ $\square$

- <u>Regulators</u>

k : perfect field $\quad K(\mathbb{Z}(n), 2n) \in \mathcal{H}_\bullet(k)$

$$\mathrm{Hom}_{\mathcal{H}(k)}(X, K(\mathbb{Z}(n), 2n)) \simeq CH^n(X)$$

<u>Prop</u> : $\mathrm{Hom}_{\mathcal{H}_\bullet(k)}(\mathbb{Z}\times Gr, K(\mathbb{Z}(n), 2n))$

$$\xrightarrow{\sim} \underbrace{\mathrm{Hom}_{(Sm/S)^{opp}\to(Sets)}(K_0(-), CH^n(-))}_{\text{polynomials in Chern classes}}$$

<u>Prop</u> : $\mathrm{Hom}_{(Sm/S)^{opp}\to(Ab)}(K_0(-), CH^n(-)) \simeq \mathbb{Z}\chi_n$

$\chi_n : K_0(-) \dashrightarrow CH^n(-)$ additive

$[L] \longmapsto [D]^n$

line bundle

divisor D

$\rightsquigarrow$ computes maps

$$\mathbb{Z} \times G \longrightarrow \mathbb{H}_A(q)[p]$$

$A \cdot \mathbb{Q}$ Chern character $K_0(-) \to \bigoplus CH^*(-)_{\mathbb{Q}}$

$$BGL \longrightarrow \bigoplus_p \mathbb{H}_{\mathbb{Q}}(p)[2p]$$

higher Chern character $K_i \longrightarrow H_{\mathcal{M}}^{2n+i}(X, \mathbb{Q}(n))$