佐藤周友

Motivic cohomology with finite coefficents

k: perfect field

Aim (1) Define $\mathbb{Z}(q)$ $(q \geq 0)$ in $D^-(\mathrm{Shv}_{zar}(Sm/k))$

(2) Study the canonical adjunction map

$$\mathbb{Z}(q) \longrightarrow R\varepsilon_* \varepsilon^* \mathbb{Z}(q) \qquad (Sm/k)_{et} \to (Sm/k)_{Nis} \xrightarrow{\beta} (Sm/k)_{zar}$$

ε

using Voevodsky's framework

Background

(1) $\mathbb{Z}(q)$ is a strong candidate for $\Gamma(q)$ conjectured by Beilinson - Lichtenbaum

(2) comparison between

$$H^i_{zar}(X, \mathbb{Z}(q)) \dashrightarrow H^i_{et}(X, \varepsilon^* \mathbb{Z}(q))$$

(should be the same for any $i \leq q+1$)

§1. $\mathbb{Z}(q)$

Presheaf with trans. $F : (SmCor(k))^{op} \longrightarrow (Ab)$

Zariski sheafification F_{zar}

Nisnevich sheafification F_{Nis}
(prserves transfer str.)

Thm (Voevodsky) $F \in PST(k)$

(1) $F_{zar} = F_{Nis}$ as presheaves over (Sm/k)

If F [illegible] inv.

(2) F_{Nis} has canonical str. of $PST(k)$

(3) $X \in Sm/k$ $\quad H^i_{zar}(X, F_{zar}) \xrightarrow{\sim} H^i_{Nis}(X, F_{Nis})$ for $\forall i \geq 0$

$\square$

Def. $\mathbb{Z}(q)^{Nis} \in DM^{eff}_{-}(k)$

$\mathbb{Z}(0)^{Nis} = \mathbb{Z}$ $\qquad \mathbb{Z}(1)^{Nis} := \mathcal{O}^{\times}[-1] \quad \left(\simeq C^*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge 1})[-1])\right)$

$q \geq 2$ $\quad \mathbb{Z}(q)^{Nis} = \underbrace{\mathbb{Z}(1)^{Nis} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathbb{Z}(1)^{Nis}}_{q\text{-times}} \quad \left(\simeq C^*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q])\right)$

$\mathbb{Z}(q) =$ image of $\mathbb{Z}(q)^{Nis}$ via $R\beta_* \circ$ (forgetful)

$\cap$

$D^-(Shv_{zar}(Sm/k))$

Rem $\mathbb{Z}(q) \underset{\substack{\uparrow \\ \text{Voevodsky}}}{\xrightarrow{\text{qis}}}$ Zariski sheafification of Bloch's cycle cpx.

Thm A (Suslin-Voevodsky / Geisser-Levine $\times 2$ / Voevodsky)

Assume $wBK(k, q, \ell)$ for all prime $\ell \neq ch(k)$

Then
$$\begin{cases} R^{q+1} \varepsilon_* \varepsilon^* \mathbb{Z}(q) = 0 \\ \mathbb{Z}(q) \xrightarrow[\alpha^q]{\sim} \tau_{\leq q} R\varepsilon_* \varepsilon^* \mathbb{Z}(q) \end{cases}$$

in $D^-(Shv_{zar}(Sm/k))$

Thm A' (essential part of Thm A)

Assume wBK(k, q, l) for all $l \neq ch(k)$

Then $\mathbb{Z}(q) \otimes^{L} \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \tau_{\leq q} R\varepsilon_{*} \varepsilon^{*}(\mathbb{Z}(q) \otimes^{L} \mathbb{Q}/\mathbb{Z})$

in $D^{-}(Shv_{Zar}(Sm/k))$

Rem on Thm A'

$ch(k) = p > 0 \Rightarrow$ p-primary part : Geisser-Levine (unconditional)

prime to ch(k) part. S-V : assuming res. of sing.

and wBK(k, m, l) for $m \leq q$

G-L : assuming only wBK(k, q, l)

§2 Galois symbol and wBK(k, q, l)

<Milnor K-grp>

k : field $q \geq 0$

$$K_q^M(F) = \begin{cases} \mathbb{Z} & (q=0) \\ F^\times & (q=1) \\ \underbrace{F^\times \otimes \cdots \otimes F^\times}_{q\text{-times}} / I \end{cases}$$

I = subgp of $(F^\times)^{\otimes q}$ generated by elements of the form $x_1 \otimes \cdots \otimes x_q$, $x_i \in F^\times$, $x_i + x_j = 1$ for some $i \neq j$

<Galois symbol> $n \in \mathbb{N}$, $ch(k) \nmid n$

μ_n := group of n-th roots of 1 in $(F^{sep})^\times$

$\circlearrowright$

$G_F = Gal(F^{sep}/F)$

$0 \to \mu_n \to (F^{sep})^\times \to (F^{sep})^\times \to 0$

(exact seq.)

$\rightsquigarrow \quad F^\times \xrightarrow{d_{F,n}} H^1_{Gal}(G_F, \mu_n) = H^1(F, \mu_n)$

$(F^\times)^{\otimes q} \xrightarrow{(d_{n,F})^{\otimes q}} H^1(F, \mu_n)^{\otimes q} \xrightarrow{\text{cup prod.}} H^q(F, \mu_n^{\otimes q})$

Lem (Tate) The above map factors through $K_q^M(F)$

($\because$) essential case : $q = 2$ $\quad \zeta_\ell \in F$

Have to show. $\forall a \in F^\times \setminus \{1\}$ $\quad d_{\ell^r}(a) \cup d_{\ell^r}(1-a) = 0$

in $H^2(F, \mu_{\ell^r}^{\otimes 2})$

We show : For $\forall r' \leq r$, $\exists A \in H^2(F, \mu_{\ell^r}^{\otimes 2})$

s.t. $d_{\ell^r}(a) \cup d_{\ell^r}(1-a) = \ell^{r'} A$

Induction on $r' \geq 0$

$a \in (F^\times)^\ell \Rightarrow \exists b \in F^\times$ s.t. $a = b^\ell$

$$d_{\ell^r}(a) \cup d_{\ell^r}(1-a) = \ell \cdot \sum_{i=0}^{\ell-1} d_{\ell^r}(\zeta_\ell^i b) \cup d_{\ell^r}(1 - \zeta_\ell^i b)$$

$a \notin (F^\times)^\ell \Rightarrow$ Put $E = F(\sqrt[\ell]{a})$, $N_{E/F}(1 - \sqrt[\ell]{a}) = 1-a$

$$d_{\ell^r}(a) \cup d_{\ell^r}(1-a) = d_{\ell^r}(a) \cup \mathrm{Cor}_{E/F}(d_{\ell^r,E}(1-\sqrt[\ell]{a}))$$
$$= \ell \cdot \mathrm{Cor}_{E/F}(d_{\ell^r,E}(\sqrt[\ell]{a}) \cup d_{\ell^r,E}(1-\sqrt[\ell]{a}))$$

$\blacksquare$

$\chi^q_{n,F} : K_q^M(F)/n \longrightarrow H^q(F, \mu_n^{\otimes q})$ (Galois symbol)

Known to be bij in the following cases

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- $q = 0$ (clear)
- $q = 1$ (Hilbert 90, $H^1(F, (F^{sep})^\times) = 0$)
- $q = 2$ (Merkurjev - Suslin, 1983)
- $n = 2^r$, $ch(F) \neq 2$ ($q = 3$: M-S, $q \geq 4$ Voevodsky)

<u>Conj</u> (B-K) $\chi^q_{n,F}$ is always bijective

$\Leftarrow BK(k, q, \ell) = \chi^q_{\ell, F}$ is surjective for all finitely generated fields F/k

§3 Relation between $\chi^q_{n,F}$ and α^q

<u>Lem</u>: $\mathbb{Z}(q)^{\acute{e}t} := \varepsilon^* \mathbb{Z}(q) \in D^-(Shv_{\acute{e}t}(Sm/k))$

$ch(k) \nmid n \Rightarrow \mathbb{Z}(q)^{\acute{e}t} \overset{\mathbb{L}}{\otimes} \mathbb{Z}/n \simeq \mu_n^{\otimes q}$

(:) $\mathbb{Z}(1)^{\acute{e}t} \simeq \mathcal{O}^\times[-1]$

$0 \to \mu_n \to \mathcal{O}^\times \xrightarrow{\times n} \mathcal{O}^\times \to 0$ exact in the étale top.

$q \geq 2$ $DM^{eff}_{-,\acute{e}t}(k, \mathbb{Z}[1/p])$ $p = ch(k) > 0$

$$\mathbb{Z}(q)^{\acute{e}t} \overset{\mathbb{L}}{\otimes} \mathbb{Z}/n = \mathbb{Z}(1)^{\acute{e}t} \overset{\mathbb{L}}{\otimes} \cdots \overset{\mathbb{L}}{\otimes} \underbrace{\mathbb{Z}(1)^{\acute{e}t} \overset{\mathbb{L}}{\otimes} \mathbb{Z}/n}_{\mu_n}$$

$$= \mu_n^{\otimes q}$$

$\boxtimes$

<u>Prop</u> $ch(k) \nmid n$ $\exists$ a commutative diagram
F/k f.g. field

adj. + lem

induced by $F^\times = H^1_{Zar}(F, \mathbb{Z}(1))$ and prod.

$$\begin{array}{ccc} K^M_q(F)/n & \xrightarrow{\chi^q_{n,F}} & H^q_{Gal}(G_F, \mu_n^{\otimes q}) \\ \downarrow \simeq (S\text{-}V) & & \| \\ H^q_{\mathcal{M}}(F, \mathbb{Z}(q) \overset{\mathbb{L}}{\otimes} \mathbb{Z}/n) & \longrightarrow & H^q_{\acute{e}t}(Spec(F), \mu_n^{\otimes q}) \end{array}$$

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Rem

Nesterenko-Suslin

$$\begin{tikzcd} CH^q(F, q) \arrow[d, "\wr"', "\text{Voevodsky}"] & & K_q^M(F) \arrow[ll, "\sim"'] \arrow[dll, "\sim", "\text{S-V}"'] \\ H^q_{zar}(F, \mathbb{Z}(q)) & & \end{tikzcd}$$

$$wBK(k, q, \ell) \underset{\text{Thm } A'}{\Longrightarrow} BK(k, q, \ell^r) \quad (\forall r > 0)$$