三藤 周友

ℓ: prime number $\neq ch(k)$

Thm A″ (S-V)

Assume $RS(k)$ and $wBK(k, g', \ell)$ for ${}^\forall g' \leq g$. Then

$$\alpha^g_{\ell^r} : \mathbb{Z}(g)^{\mathrm{Nis}} \otimes^{\mathbb{L}} \mathbb{Z}/\ell^r \longrightarrow \underbrace{\tau_{\leq g} R\gamma_* \left(\mathbb{Z}(g)^{\acute{e}t} \otimes^{\mathbb{L}} \mathbb{Z}/\ell^r\right)}_{\simeq\ \mu_{\ell^r}^{\otimes g}}$$

is an isom. in $DM^{\mathrm{eff}}_{-}(k)$ for ${}^\forall r \geq 1$ $\square$

Thm B (S-V)

Assume $RS(k)$ and $BV(k, g', \ell)$ for ${}^\forall g' \leq g$.

Then $\alpha^g_{\ell^r}$ is an isom for ${}^\forall r \geq 1$ $\square$

Rem

$$H^i_{\acute{e}t}(X, \mu_{\ell^r}^{\otimes g}) \xrightarrow{\sim} H^i_{\acute{e}t}(X \times \mathbb{A}^1, \mu_{\ell^r}^{\otimes g}) \quad \begin{pmatrix} {}^\forall X \in Sm/k \\ {}^\forall i \in \mathbb{Z}_{\geq 0} \end{pmatrix}$$

Rem α^g_ℓ is an isom $\Leftrightarrow$ $\alpha^g_{\ell^r}$: isom for ${}^\forall r \geq 1$

$\Leftrightarrow$ $\alpha^g_{\ell^\infty} : \mathbb{Z}(g) \otimes^{\mathbb{L}} \mathbb{Q}_\ell/\mathbb{Z}_\ell \xrightarrow{\sim} \tau_{\leq g} R\gamma_* \mathbb{Z}(g)^{\acute{e}t} \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$

Thm A″

Thm B

Def. $RS(k)$: the following two conditions

(1) ${}^\forall X \in Sch/k$ integral ${}^\exists Y \longrightarrow X$ proper birational, Y smooth$/k$

(2) ${}^\forall X \in Sm/k$ integral ${}^\forall Y \longrightarrow X$ proper birational

$${}^\exists X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X \quad \text{a blow up at smooth centers}$$

$$X_n \longrightarrow Y \longrightarrow X$$

§5 Proof of Thm A″ (outline)

$\mathbb{Z}/\ell(q) := \mathbb{Z}(q)^{Nis} \overset{\mathbb{L}}{\otimes} \mathbb{Z}/\ell$, $B_\ell(q) := \tau_{\leq q} R\gamma_* \mu_\ell^{\otimes q}$ $\in DM^{eff}_-(k)$

Key fact (Voevodsky) For $f : F_1 \to F_2$ homomorphism of homotopy inv PST

f is an isom $\Longleftrightarrow$ $f(E) : F_1(E) \to F_2(E)$ is bij for any f.g. fields E/k □

Suffices to show:

$\alpha_\ell^{i,q}(F) : H^i(F, \mathbb{Z}/\ell(q)) \longrightarrow H^i(F, B_\ell(q))$ is bij for $\forall i \leq q$, $\forall F/k$ f.g. field

Nis is omitted

Surjectivity of $\alpha_\ell^{i,q}(F)$: easy

Reduced the case $\mu_\ell \in F$, Then $0 \leq i \leq q$

$$\begin{array}{ccc} H^i(F, \mathbb{Z}/\ell(i)) \otimes H^0(F, \mathbb{Z}/\ell(q-i)) & \xrightarrow{cup} & H^i(F, \mathbb{Z}/\ell(q)) \\ \downarrow{\scriptstyle \cong BK(k,i,\ell)} & & \downarrow \\ H^i(F, B_\ell(i)) \otimes H^0(F, B_\ell(q-i)) & \xrightarrow{\sim} & H^i(F, B_\ell(q)) \end{array}$$

Injectivity of $\alpha_\ell^{i,q}(F)$: hard

Use induction on $q > 0$. Necessary staffs :

(1) For $X \in Sch/k$ define $M(X) \in DM^{eff}_-(k)$
$\qquad\qquad \| $
$\qquad C^*(\mathbb{Z}_{tr}(X))$

(2) For $\mathcal{K} \in DM^{eff}_-(k)$

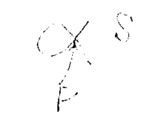
$H^i(X, \mathcal{K}) := \mathrm{Hom}_{DM^{eff}_-(k)}(M(X), \mathcal{K})$

$X \in Sm/k$ $\cong$ $H^i_{Nis}(X, \mathcal{K})$

(3) $\Delta^\bullet$: standard cosimplicial scheme.

$\partial\Delta^n$ = union of faces $\mathrm{Im}(\partial^i : \Delta^{n-1} \to \Delta^n)$ $(i = 0, \dots, n)$

$S := \mathbb{A}^1_k$ with 0 and 1 identified



$P \in S$: singular pt

(4) $Z \hookrightarrow X$: closed immersion of schemes $/k$

$$M_Z(X) = C^*(\mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X \setminus Z)) \in DM^{eff}_-(k)$$

Step 1 $\mathcal{K} \in DM^{eff}_-(k)$, $q \geq 0$ $\quad H^i(F, \mathcal{K}) \hookrightarrow H^{q+1}(\partial\Delta_F^{q-i+1} \times S, \mathcal{K})$ (functorial in $\mathcal{K}$)

Step 2. $U :=$ semi-localization of $\partial\Delta_F^{q-i+1} \times S$ at $v_1 \times P, v_2 \times P, \dots, v_{q-i+1} \times P$

($v_1, \dots, v_n \in \partial\Delta^n_F$ vertices of $\partial\Delta^n_F$ intersecting pt of $(n-1)$ components)

Show

$$H^i(F, \mathbb{Z}/\ell(q)) \hookrightarrow H^{q+1}(\partial\Delta_F^{q-i+1} \times S, \mathbb{Z}/\ell(q)) \xrightarrow{(*)} H^{q+1}(U, \mathbb{Z}/\ell(q))$$

is zero map.

Step 3. $H^{q+1}(\partial\Delta_F^{q-i+1} \times S, \mathbb{Z}/\ell(q)) \xrightarrow{\alpha^q_\ell} H^{q+1}(\partial\Delta_F^{q-i+1} \times S, B_\ell(q))$ is injective on Ker$(*)$

- induction on q · cancellation of $\mathbb{Z}(1)$
- Gabber's base change for henselian pairs

§ Proof of Thm B

BV(k, q, ℓ) $\quad \beta_{q,j} : H^q_{Gal}(F, \mu_\ell^{\otimes q}) \longrightarrow H^{q+1}_{Gal}(F, \mu_{\ell^j}^{\otimes q})$

$(0 \to \mu_{\ell^j}^{\otimes q} \to \mu_{\ell^{j+1}}^{\otimes q} \to \mu_\ell^{\otimes q} \to 0)$

is zero for $\forall j > 0$ and $\forall F/k$ f.g field

Lem BV(k, q, ℓ) + (BL(q', ℓ) for $\forall q' < q$) + RS(k)

$\Rightarrow \chi^q_{F, \ell^\infty} : K^M_q(F) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow H^q_{Gal}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(q))$ is surj.

(Lem $\Rightarrow$ Thm B)