No. 1

Date

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Plan

DGA $\dashrightarrow$ DG category

graded $\left(\begin{array}{l} A^{\cdot} = \bigoplus_{j \geq 0} A^{\cdot}(j) \quad \text{cubic product} \\ \qquad\qquad\quad \shortparallel \\ \text{Bloch cycle complex} \quad \bigoplus_{i \geq 0} Z^{j}(\square^{2j-i})_{\mathbb{Q}} \end{array} \right.$

$\downarrow$

triangulated cat.

$\downarrow$ t-str (Beilinson Soulé conj)

同じ？

triangulated cat $\xleftarrow[\text{derived cat}]{}$ abel. cat.

$K\pi_1$-conj $\Rightarrow$ 2つの triangulated cat. は eq.

◎ weighted version

Hanamura, Levine, Voevodsky's motif

$\mathcal{D}$ $\quad \mathcal{H}_{\tau} \quad D^b(\mathcal{H}_{\tau})$

A = Bloch cycle cpx $\quad$ heart of $\mathcal{D}$ $\quad$ Bloch-Kriz motif

Reference $\begin{cases} \text{Levine} \\ \text{Kriz - May : Book} \end{cases}$

$S = \operatorname{Spec} \mathbb{Z}$

(Goncharov's modification)

◎ Application

⊙ motivic construct $(BGL)^{+} = \Omega BQM$

(we can construct) $\quad$ Quillen's constr.

⊙ What kind of [object] can generate $DM(\operatorname{Spec} \mathbb{Z})$

DG category

$\mathcal{C}$ $\quad obj(\mathcal{C}) \ni x, y \quad \underline{Hom}^*(x, y)$: complex / $\mathbb{Q}$

shift op. $[+1]$ (auto)

composite $\quad x, y, z \in obj(\mathcal{C})$

$$\underline{Hom}^*(x, y) \otimes \underline{Hom}^*(y, z) \longrightarrow \underline{Hom}^*(x, z)$$

hom of complex

Assume ∘ associativity

∘ $\exists$ shift $[+1]$ $\quad \left\{ \begin{array}{l} \underline{Hom}^i(x, y[1]) \\ \qquad (x[1], y) \end{array} \right. = \underline{Hom}^{i+1}_{i-1}(x, y)$

$\mathcal{C}$: DG-category

$\Rightarrow H^0\mathcal{C}$: category

$$\left\{ \begin{array}{l} obj(\mathcal{C}) = obj(H^0(\mathcal{C})) \\ Hom_{H^0\mathcal{C}}(\ ) = H^0\, \underline{Hom}^*_{\mathcal{C}}(\ ) \end{array} \right.$$

(+) $\mathcal{C}$: DG-category

$\Rightarrow K\mathcal{C}$: the category of DG-complex

e.g.
$\Delta_{23} \circ \Delta_{02} + \Delta_{13} \circ \Delta_{01} = d\Delta_{03}$
$\Delta_{12} \circ \Delta_{01} = d\Delta_{02}$

$$\begin{tikzcd} K^0 \arrow[r, "\Delta_{01}"'] \arrow[rr, bend right, "\Delta_{02}"', "-1"] & K^1 \arrow[r, "\Delta_{12}"'] & K^2 \arrow[r, "\Delta_{23}"', "\Delta_{13}"] & K^3 \end{tikzcd}$$

$\Delta_{01} \in \underline{Hom}^0(K^0, K^1)$
$\Delta_{02} \in \underline{Hom}^{-1}(K^0, K^2)$
...

<u>condition</u> $\Delta^2 = d\Delta$
(Frobenius integrality)

<u>Fact</u>: $K\mathcal{C}$ is also DG-cat

<u>Prop</u> (1) $H^0 K\mathcal{C}$ is a triangulated cat.

(2) $KK\mathcal{C} \xrightarrow{s} K\mathcal{C}$ ass. simpl

$$H^0 KK\mathcal{C} \xrightarrow[H^0(s)]{} H^0 K\mathcal{C}$$ is equiv of cat.

<u>Rem</u> $(K^\cdot, \Delta), (L^\cdot, \Delta)$: DG-complex

$$\underline{Hom}^\cdot(K^\cdot, L^\cdot) = \bigoplus_{i,j} Hom^{i-j}(K^i, L^j) \ni F$$

$\circlearrowleft$ differential

$$D(F) = \underbrace{\Delta F - F\Delta}_{\text{outer diff}} + \underbrace{dF}_{\text{inner diff}}$$

$\square$

<u>DGA $\longrightarrow$ DA-category</u>

$A^\cdot$: DGA, associative

$\mathcal{S}$: DG-cat obj : fin dim v.sp. / $\mathbb{Q}$ V — generated by v.sp, $[+1]$, $\oplus$

morph : $\underline{Hom}^j_{\mathcal{S}}(V, W[i]) = A^{i+j} \otimes Hom_{\mathbb{Q}}(V, W)$

composite : multiplication of A $\otimes$ composite of $(Vec_{\mathbb{Q}})$

$K\mathcal{S}$: DG-cat

$\parallel$

$\mathcal{D} = \mathcal{D}(A^\cdot)$

<u>Def.</u> $SEC \subset \mathcal{D}$
full sub. cat.

$$V_0 \to V_1[1] \to V_2[2] \to V_3[3]$$

V_i : v. sp. / $\mathbb{Q}$

Condition for $A^\cdot$

(1) $H^0(A^\cdot) - \mathbb{Q}$ connected

$\exists$ augmentation $\varepsilon : A \to \mathbb{Q}$ which induce isom on H^0

(2) Beilinson - Soulé cond. (BS cond)

$H^i(A^\cdot) = 0$ for $i < 0$

(B-S conj. for Bloch's cycle complex)

Prop $H^0(SEC)$ is an abelian full subcat. $\subset H^0\mathcal{D}$ if A is connected (i.e. $\exists$ augmentation)

Prop $\mathcal{D} = \mathcal{D}(A^\cdot) \rightsquigarrow H^0\mathcal{D}$ triangulated

If $A^\cdot$ satisfies BS-condition

$\Rightarrow \exists$ t-structure on $H^0\mathcal{D}$

s.t. $H^0\mathcal{D}^{\tau \le 0} \cap H^0\mathcal{D}^{\tau \ge 0} = \mathcal{H}_\tau = H^0(SEC)$

outline of proof

$V \in \mathcal{D}$

$$\begin{array}{ccc} \cup & & \mathcal{S} \\ V_0 & \longrightarrow & V_1 \\ V_0^{(0)} & \longrightarrow & V_1^{(0)} \\ V_0^{(-1)}[1] & \longrightarrow & V_1^{(-1)}[1] \\ V_0^{(-2)}[2] & \longrightarrow & V_1^{(-2)}[-2] \end{array}$$

(take boundry
use augmentation

$\downarrow \tau \le 0$

$\square$

Comparison between $D^b(\mathcal{H}_t)$ and $H^0\mathcal{D}$

Def. $K\pi_1$- condition

Bar complex

$A^\cdot$: DGA $\rightsquigarrow$ $\mathrm{Bar}(\varepsilon_1 | A | \varepsilon_2)$

$\varepsilon_1, \varepsilon_2$: augmentation

$A \xrightarrow{\varepsilon_1} \mathbb{Q}$, $A \xrightarrow{\varepsilon_2} \mathbb{Q}$

$\left(\begin{array}{l} H^i(A^\cdot) = 0 \quad (i<0) \\ H^0(A^\cdot) = \mathbb{Q} \end{array} \right)$

$\Delta_n := \{ 0 < x_1 < \cdots < x_n < 1 \}$

$\sigma(\Delta_n)$ = the set of faces in Δ_n

$\updownarrow$

$\mathcal{P}(E) - \{E\}$

$E = \{ e_1, e_2, \cdots e_{n+1} \}$ $\quad e_0 : \{0 = x_1\}$, $e_2 : \{x_1 = x_2\}$
$\cdots$ $e_{n+1} : \{x_n = 1\}$

$\tau : (0\ 1 \mid 2\ 3\ 4\ 5)$ $\quad 0 = x_1,\ x_2 = x_3,\ x_4 = 1$

copy of $A^\cdot$

$A_\tau : \mathbb{Q}_{01} \otimes A_{23} \otimes \mathbb{Q}_{45}$

via ε_1 $\qquad$ this A-alg via ε_2

$0 = x_1 \quad x_3 = x_4 \quad x_6 = 1$
$(0\ 1 | 2 | 3\ 4\ 5 \mid 6\ 7)$
$\mathbb{Q} \otimes A \otimes A \otimes A \otimes \mathbb{Q}$

confluence
$\Downarrow$
multiplication or augmentation

$\tau < \sigma$: τ is a face of σ
$\dim \tau < \dim \sigma$

$\rightsquigarrow A_\sigma \longrightarrow A_\tau$

$\mathrm{Bar}(\varepsilon_1 | A^\cdot | \varepsilon_2)$ Beilinson's Bar complex

$$= \bigoplus_{\substack{\dim(\tau)=n \\ \tau \in \sigma(\Delta_n)}} A_\tau^\cdot \longrightarrow \bigoplus_{\dim(\tau)=n-1} A_\tau \longrightarrow \cdots$$

has shuffle alg. str.

Rem · Chen reduced Bar complex

$\varepsilon : A \to \mathbb{Q}$ $\quad \mathrm{Ker}(\varepsilon) = I$

$$\mathrm{Bar}(\ ) : \quad \cdots \to I^{\otimes 3} \to I^{\otimes 2} \to I^{\otimes 1} \xrightarrow{\text{0-map}} \mathbb{Q}$$

inn. diff

out diff

$(\varepsilon_1 = \varepsilon_2)$

Chen's $\underset{\text{qis}}{\simeq}$ Beilinson's bar complex

Advantage

We can generalize "augmentation"

fiber functor

$\boxed{\mathrm{Tann}} \xrightarrow{\omega} \mathrm{Vect}$

$\boxed{\mathrm{Tann}} \xrightarrow{\mathrm{id}} \boxed{\mathrm{Tann}}$

$\mathbb{Q} \in \mathcal{D} = \mathcal{D}(A^\cdot)$

$\shortparallel$

$\mathbb{Q}_S \qquad (\ \underline{\mathrm{Hom}}^i(\mathbb{Q}_{S}, \mathbb{Q}_S) = A^i\ )$

$$(01|23|45) \rightsquigarrow \mathbb{Q}_{01} \otimes A_{23}^\cdot \otimes \overset{A_{\tau,\mathrm{univ}}}{\overset{\shortparallel}{\mathbb{Q}_{S,45}}} \in \mathcal{D}$$

Take action in the sense of DG cat.

$\mathbb{Q}_{01} \otimes A_{23}^\cdot \otimes A_4^\cdot \otimes \mathbb{Q}_{S,5}$

Universality $A \xrightarrow{\varepsilon_1} \mathbb{Q}$

$$\mathrm{Bar}(\varepsilon_1 | A^\cdot | \mathrm{univ}) = \bigoplus_{\dim \tau = n} A_{\tau, \mathrm{univ}} \in \mathcal{D}$$

$$H^0(\mathrm{Bar}(\varepsilon_2 | A^\cdot | \mathrm{univ})) \in H^0 \mathcal{D}$$

Def. $A^\cdot$ is $K\pi_1 \iff H^i(\mathrm{Bar}(\varepsilon_2 | A^\cdot | \varepsilon_1)) = 0$

$\varepsilon_1, \varepsilon_2 : A^\cdot \to \mathbb{Q}$ for $i \neq 0$

$\square$

Thm. Assume $A^\cdot$ is $K\pi_1$

(1) $H^0(\underset{\underset{\mathcal{D}}{\cap}}{\mathrm{Bar}(\varepsilon | A^\cdot | \mathrm{univ})}) \in H^0(SEC)$

$=: \mathcal{U}^*$

(2) $\mathcal{U} = H^0(\mathrm{Bar}(\varepsilon | A | \varepsilon))^*$ is a Hopf alg.

$\mathcal{U}^*$ has right $\mathcal{U}$-mod. str.

(3) $H^0(SEC)$ is a Tannakian category

$\hookrightarrow$ fiber functor is obtained by ε

$\mathcal{U}$(Lie Tannaka fund grp)

univ. envelope

$H^0(SEC) \xrightarrow{\simeq} (\text{left Mod}\ \mathcal{U})$ left action of $\mathcal{U}$

$\mathbb{M} \longmapsto \mathrm{Hom}_{H^0(SEC)}(\mathcal{U}^*, \mathbb{M})$ inverse to each other

$\mathcal{U}^* \otimes_{\mathcal{U}} M \longleftarrow M$

(4) $D^b(\mathcal{L}l_\tau) \xrightarrow{\sim} H^0\mathcal{D}$

is a category equiv $\square$

Application

$$\begin{array}{ccc} & & Z. \\ & & \downarrow \\ X. & \longrightarrow & Y. \end{array}$$

simplicial set

$C^*(X)$ singular cochain

DGA str. = Alexander - Whitney rule

$\mathrm{Bar}_n(C^\cdot(X) \mid C^\cdot(Y) \mid C^\cdot(Z)) = \bigoplus_{\dim I = n}$

$\left(\begin{array}{c} (0\,1 \mid 2\,3 \mid 4\,5) \\ C^\cdot(X)_{01} \otimes C^\cdot(Y)_{23} \otimes C^\cdot(Z)_{45} \end{array} \right)$

$C^\cdot(X)$ is algebra $/ C^\cdot(Y)$

$\mathrm{Bar} = \varinjlim \mathrm{Bar}_n$

Thm (Eilemburg - Moore)

X. simply connected

$H^i(\mathrm{Bar}(C^\cdot(X) \mid C^\cdot(Y) \mid C^\cdot(Z))) \simeq H^i(X \overset{h}{\underset{Y}{\times}} Z, \mathbb{Q})$ $\square$

$\{ (x, z, \gamma) \in X \times Z \times \mathrm{Path}(Y) \mid \gamma(0) = f(x), \gamma(1) = g(z) \}$

$\rightsquigarrow$ $BGL/_{\mathrm{Spec}\,\mathbb{Z}} \rightsquigarrow LBGL/_{\mathrm{Spec}\,\mathbb{Z}} \rightsquigarrow LBGL \times_{\mathrm{Spec}\,\mathbb{Z}} LBGL /_{\mathrm{Spec}\,\mathbb{Z}} \to LBGL/_{\mathrm{Spec}\,\mathbb{Z}}$

$\ominus$ is defined

Chern class

Loday product in K-theory.

$X./_{\mathrm{Spec}\,\mathbb{Z}} \longrightarrow C^\cdot(X(\mathbb{Z})) \rightsquigarrow$ K-theory

$\downarrow$

$\longrightarrow \underline{\mathrm{Hom}}^*(\mathbb{Q}_X, \mathbb{Q}_S) \rightsquigarrow$ Bloch-Kriz's bar

$\Rightarrow$ motif $/_{\mathrm{Spec}\,\mathbb{Z}}$ is gen. by $\circ GL \times GL \to GL$ $\Rightarrow$ universal