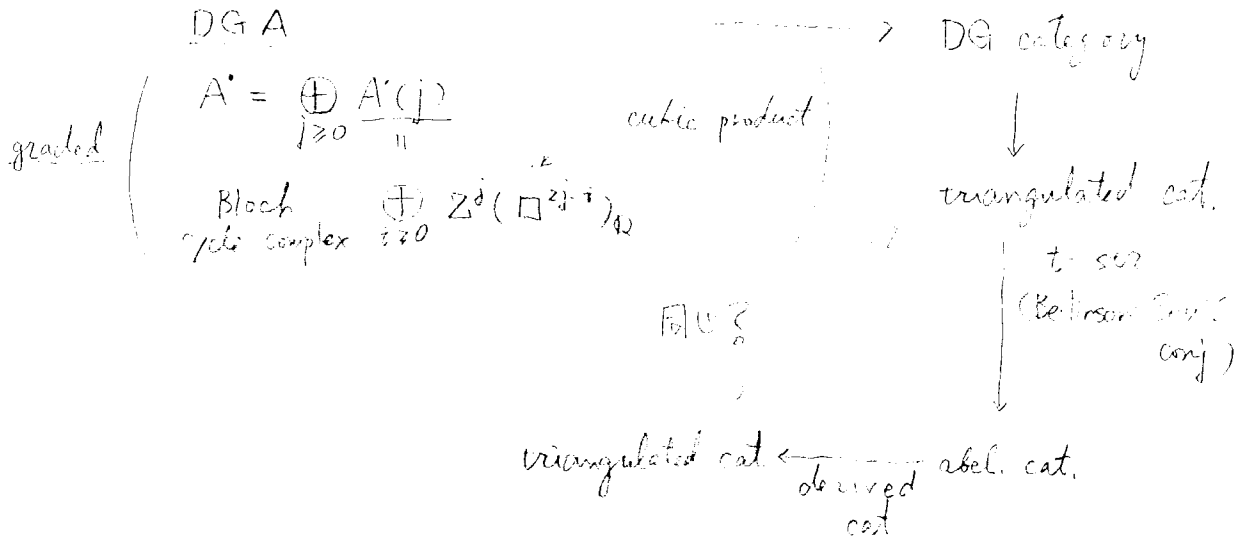


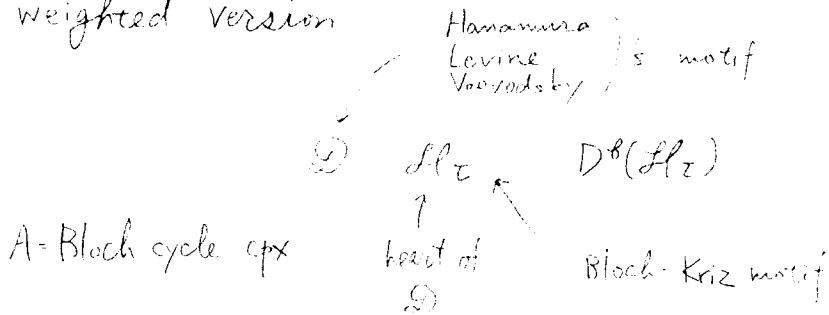
寺山友秀

Plan



$K\pi_1$ -conj \Rightarrow $2 \Rightarrow$ triangulated cat. is eg.

② weighted version



Reference } Levine
 } Kriz - May : Bock

$S = \text{Spec } \mathbb{Z}$

(Goncharov's modification)

③ Application

① motivic construct $(BGL)^+ = \Omega BQM$

(we can construct) Quillen's constr.

② What kind of object can generate $DM(\text{Spec } \mathbb{Z})$

DG category

\mathcal{C} $\text{obj}(\mathcal{C}) \ni x, y$ $\underline{\text{Hom}}^*(x, y) : \text{complex} / \mathbb{Q}$
 composite $x, y, z \in \text{obj}(\mathcal{C})$ $\text{shift op. } [+1] \text{ (auto)}$

$$\underline{\text{Hom}}^*(x, y) \otimes \underline{\text{Hom}}^*(y, z) \rightarrow \underline{\text{Hom}}^*(x, z)$$

hom of complex

Assume \circ associativity

$\circ \exists \text{ shift } [+1]$

$$\left. \begin{array}{l} \underline{\text{Hom}}^i(x, y[1]) \\ \underline{\text{Hom}}^i(x[1], y) \end{array} \right\} = \underline{\text{Hom}}_{i-1}^{i+1}(x, y)$$

$\mathcal{C} : \text{DG-category}$

$\Rightarrow H^0 \mathcal{C} : \text{category}$

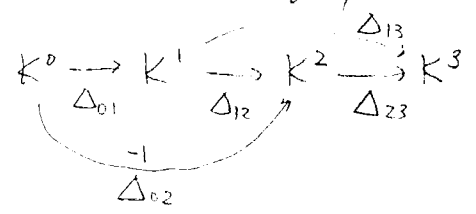
$$\left\{ \begin{array}{l} \text{obj}(\mathcal{C}) = \text{obj}(H^0 \mathcal{C}) \\ \underline{\text{Hom}}_{H^0 \mathcal{C}}(\) = H^0 \underline{\text{Hom}}_{\mathcal{C}}^*(\) \end{array} \right.$$

$\circ \mathcal{C} : \text{DG-category}$

$\Rightarrow K\mathcal{C} : \text{the category of DG-complex}$

e.g.

$$\begin{aligned} \Delta_{23} \circ \Delta_{02} + \Delta_{13} \circ \Delta_{01} &= d\Delta_{03} \\ \Delta_{12} \circ \Delta_{01} &= d\Delta_{02} \end{aligned}$$



$$\left(\begin{array}{l} \Delta_{01} \in \underline{\text{Hom}}^0(K^0, K^1) \\ \Delta_{02} \in \underline{\text{Hom}}^{-1}(K^0, K^2) \\ \dots \\ \text{condition } \Delta^2 = d\Delta \\ \text{(Frobenius integrality)} \end{array} \right.$$

Fact: $K\mathcal{C}$ is also
 DG-cat

Prop (1) $H^0 K\mathcal{C}$ is a triangulated cat.

(2) $KK\mathcal{C} \xrightarrow{\sigma} K\mathcal{C}$ ass. simpl

$H^0 KK\mathcal{C} \xrightarrow{H^0(\sigma)} H^0 K\mathcal{C}$ is equiv of cat.

Rem $(K, \Delta), (L, \Delta) : DG\text{-complex}$

$$\text{Hom}^\bullet(K, L) = \bigoplus_{i,j} \text{Hom}^{i-j}(K^i, L^j)$$

\cap
differential

$$D(F) = \underbrace{\Delta F - F \Delta}_{\text{outer diff}} + \underbrace{dF}_{\text{inner diff}}$$

□

DGA \rightarrow DA-category

A : DGA, associative

generated by
v.sp., $[+]$, \oplus

\mathcal{S} : DG-cat obj: fin dim v.sp. / \mathbb{Q} V

$$\text{morph: } \text{Hom}_{\mathcal{S}}^j(V, W[\mathbb{Z}]) = A^{i+j} \otimes \text{Hom}_{\mathbb{Q}}(V, W)$$

composite: multiplication of $A \otimes$ composite of $(\text{Vec}_{\mathbb{Q}})$

$K\mathcal{S} : DG\text{-cat}$

"

$$\mathcal{D} = \mathcal{D}(A)$$

Def: $\text{SEC} \subset \mathcal{D}$
full sub.
cat.

$$V_0 \rightarrow V_1[1] \rightarrow V_2[2] \rightarrow V_3[3]$$

V_i v.sp. / \mathbb{Q}

Condition for A^*

(1) $H^0(A^*) = \mathbb{Q}$ connected
 \exists augmentation $\epsilon: A \rightarrow \mathbb{Q}$ which induce isom on H^0

(2) Beilinson - Soulé cond. (BS cond)
 $H^i(A^*) = 0$ for $i < 0$
 (B-S conj. for Bloch's cycle complex)

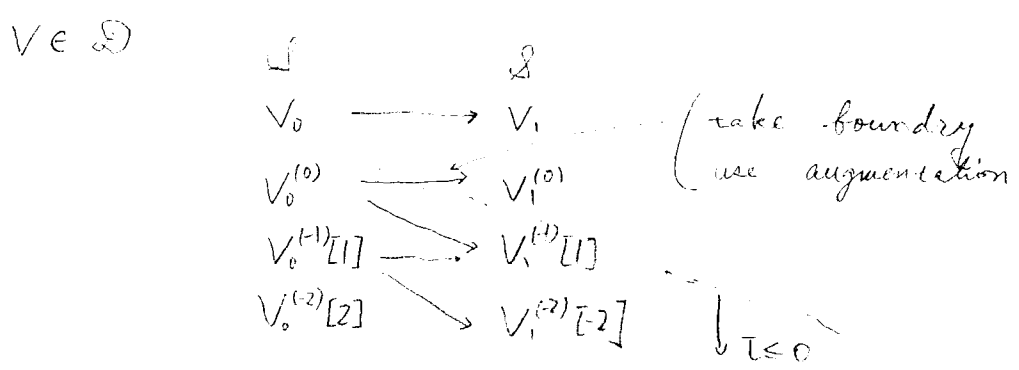
Prop $H^0(\text{SEC})$ is an abelian full subcat. $\subset H^0\mathcal{D}$ if A is connected (i.e. \exists augmentation)

Prop $\mathcal{D} = \mathcal{D}(A^*) \rightsquigarrow H^0\mathcal{D}$ triangulated

If A^* satisfies BS-conditions

$\Rightarrow \exists$ t-structure on $H^0\mathcal{D}$
 s.t. $H^0\mathcal{D}^{\tau \leq 0} \cap H^0\mathcal{D}^{\tau > 0} = \mathcal{H}_\tau = H^0(\text{SEC})$

outline of proof



□

Comparison between $D^{\mathbb{Z}}(\mathbb{A}_t)$ and $H^0 \mathcal{D}$

Def. $K\pi_1$ -condition

$$A: \text{DGA} \rightsquigarrow \text{Bar}(\varepsilon_1 | A | \varepsilon_2)$$

\uparrow $\varepsilon_1, \varepsilon_2$: augmentation

$$\left(\begin{array}{l} H^i(A) = 0 \quad (i < 0) \\ H^0(A) = \mathbb{Q} \end{array} \right)$$

$$\begin{array}{c} A \xrightarrow{\varepsilon_1} \mathbb{Q} \\ \varepsilon_2 \searrow \mathbb{Q} \end{array}$$

Bar complex

$$\Delta_n := \{0 < x_1 < \dots < x_n < 1\}$$

$\sigma(\Delta_n)$ = the set of faces in Δ_n

\downarrow

$$\mathcal{P}(E) = \{E\}$$

$$E = \{e_1, e_2, \dots, e_{n+1}\}$$

$$e_0: \{0 = x_1\}, e_2: \{x_1 = x_2\}$$

$$\dots e_{n+1}: \{x_n = 1\}$$

$$\tau: (0 | 2 3 4 5)$$

\circlearrowleft

$$\circlearrowright$$

$$0 = x_1, x_2 = x_3, x_4 = 1$$

\swarrow

copy of A

$$A_\tau: \mathbb{Q}_{01} \otimes A_{23} \otimes \mathbb{Q}_{45}$$

\nearrow
via ε_1

\nearrow
this A -alg via ε_2

$$0 = x_1 \quad x_3 = x_4 \quad x_6 = 1$$

$$(0 | 2 | 3 4 5 | 6 7)$$

$$\mathbb{Q} \otimes A \otimes A \otimes A \otimes \mathbb{Q}$$

confluence

\downarrow
multiplication or
augmentation

$\tau < \sigma$: τ is a face of σ

$$\dim \tau < \dim \sigma$$

$$\rightsquigarrow A_\sigma \longrightarrow A_\tau$$

Bar($\mathcal{E}_1 | A^\bullet | \mathcal{E}_2$) Beilinson's Bar complex

$$= \bigoplus_{\substack{d_i(\tau)=n \\ \tau \in \mathcal{S}(\Delta_n)}} A_i^\bullet \rightarrow \bigoplus_{d_i(\tau)=n-1} A_i^\bullet \rightarrow \dots$$

↖ has shuffle alg. str.

Rem. • Chen reduced Bar complex

$$\mathcal{E}: A \rightarrow \mathbb{Q} \quad \text{Ker}(\mathcal{E}) = I$$

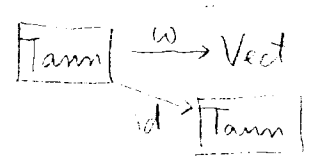
$$\text{Bar}(\) : \dots \rightarrow I^{\otimes 3} \xrightarrow{\text{inn. diff}} I^{\otimes 2} \xrightarrow{\text{out diff}} I^{\otimes 1} \xrightarrow{\text{0-map}} \mathbb{Q}$$

($\mathcal{E}_1 = \mathcal{E}_2$)

Chen's \cong Beilinson's bar complex

Advantage

We can generalize "augmentation"
 ↑
 fiber functor



$$\mathbb{Q} \in \mathcal{D} = \mathcal{D}(A^\bullet)$$

$$\mathbb{Q}_S \quad (\text{Hom}^i(\mathbb{Q}_S, \mathbb{Q}_S) = A^i)_{A, \text{univ}}$$

$$(0 | 23 | 45) \rightsquigarrow \mathbb{Q}_{01} \otimes A_{23}^\bullet \otimes \mathbb{Q}_{S,45} \in \mathcal{D}$$

$$\mathbb{Q}_{01} \otimes A_{23}^\bullet \otimes A_4 \otimes \mathbb{Q}_{S,5}$$

Take action
 in the sense of
 DG cat.

Universality $A \xrightarrow{\varepsilon_1} \mathbb{Q}$

$$\text{Bar}(\varepsilon_1 | A \cdot | \text{univ}) = \bigoplus_{\text{dim}=n} A_{\mathbb{Z}, \text{univ}} \in \mathcal{D}$$

$$H^0(\text{Bar}(\varepsilon_2 | A \cdot | \text{univ})) \in H^0(\mathcal{D})$$

Def. $A \cdot$ is $K\pi_1 \Leftrightarrow H^i(\text{Bar}(\varepsilon_2 | A \cdot | \varepsilon_1)) = 0$
 $\varepsilon_1, \varepsilon_2: A \cdot \rightarrow \mathbb{Q}$ for $i \neq 0$

□

Thm. Assume $A \cdot$ is $K\pi_1$

$$(1) H^0(\text{Bar}(\varepsilon | A \cdot | \text{univ})) \in H^0(\text{SEC})$$

$\stackrel{(\ast)}{\cong} \mathcal{U}^*$

(2) $\mathcal{U} = H^0(\text{Bar}(\varepsilon | A | \varepsilon))^*$ is a Hopf alg.
 \mathcal{U}^* has right \mathcal{U} -mod. str.

(3) $H^0(\text{SEC})$ is a Tannakian category

$\uparrow \hookrightarrow$ fiber functor is obtained by ε
 \mathcal{U} (Lie Tannaka fund grp)
 \uparrow univ. envelope

$$H^0(\text{SEC}) \xrightarrow{\sim} (\text{left Mod } \mathcal{U})$$

$$\downarrow \quad \downarrow$$

$$\mathcal{M} \xrightarrow{\quad} \text{Hom}_{H^0(\text{SEC})}(\mathcal{U}^*, \mathcal{M})$$

$$\mathcal{U}^* \otimes_{\mathcal{U}} \mathcal{M} \xrightarrow{\quad} \mathcal{M}$$

left action of \mathcal{U}
 inverse to each other

(4) $D^b(\mathbb{Z}) \xrightarrow{\sim} H^0(\mathcal{D})$
 is a category equiv □

Applications

* $\begin{matrix} \mathbb{Z} \\ \downarrow \\ X \rightarrow Y \end{matrix}$ simplicial set

$C^*(X)$: singular cochain

DGA ser. = Alexander-Whitney rule

$$\text{Bar}_n(C^*(X) | C^*(Y) | C^*(Z)) = \bigoplus_{d_1+d_2=n} \dots$$

$$\left(\begin{matrix} (0 | 1 | 2 | 3 | 4 | 5) \\ C^*(X) \otimes C^*(Y) \otimes C^*(Z) \\ \begin{matrix} 01 & 23 & 45 \end{matrix} \end{matrix} \right)$$

$C^*(X)$ is algebra / $C^*(Y)$

$$\text{Bar} = \varinjlim \text{Bar}_n$$

Thm (Eilenberg-Moore)

X : simply connected

$$\{ (x, z, \gamma) \in X \times Z \times \text{Path}(Y) \mid \gamma(0) = f(x), \gamma(1) = g(z) \}$$

$$H^i(\text{Bar}(C^*(X) | C^*(Y) | C^*(Z))) \cong H^i(X \times_Y Z, \mathbb{Q}) \quad \square$$

$$\begin{matrix} \text{BGL}/\text{Spec } \mathbb{Z} & \rightarrow & \text{LBGL}/\text{Spec } \mathbb{Z} & \rightarrow & \text{LBGL} \times_{\text{Spec } \mathbb{Z}} \text{LBGL} & \rightarrow & \text{LBGL}/\text{Spec } \mathbb{Z} \\ & & \downarrow \text{is defined} & & \downarrow \text{Loday product in} & & \\ & & & & \text{chain class} & & \text{K-theory} \end{matrix}$$

$$\begin{matrix} X/\text{Spec } \mathbb{Z} & \rightarrow & C^*(X(\mathbb{Z})) & \rightarrow & \text{K-theory} \\ & & \downarrow & & \downarrow \\ & & \text{Hom}^*(\mathbb{Q}_X, \mathbb{Q}_Z) & \rightarrow & \text{Bock-Kriz's bar} \end{matrix}$$

$\Rightarrow \text{motif}/\text{Spec } \mathbb{Z}$ is gen. by $\text{OGL} \times \text{GL} \times \text{GL} \dots \Rightarrow \text{universal}$