

• morning session

goal:

Voevodsky's proof of  
Milnor Conj.

Sato, Mochizuki, Yamashita  
Hagihara

• afternoon session

• Review from (classical)  
algebraic topology

• topics

Review and Overview

Voevodsky's idea

transport concepts and techniques from alg. top.

• spectrum • coh. operation • Margolis coh.

last year : motivic (co)homology

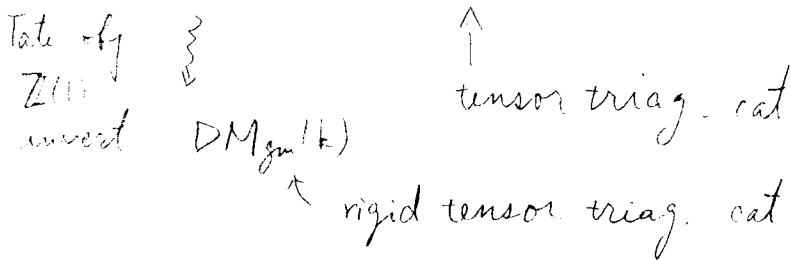
this year : motivic homotopy

{	homological alg.	→	homotopical alg.
	abel. cat	→	model cat
	derived cat	→	homotopy cat.
	inj. obj	→	fibrant obj
	[+1]	→	$S_0^1 \wedge$ " sphere

Review  $k$ : perfect field

$S_m/k$  cat. of smooth sch. /  $k$

$$DM_{gm}^{eff}(k) \subset DM_{-}^{eff}(k) \quad \leftarrow \text{last year.}$$



$DM_{-}^{eff}(k)$  = the derived cat of bounded above complex of Nisnevich sheaves with transfers with homotopy invariant wh sheaves

Def.  $\{X_i \xrightarrow{f_i} X\}_{i \in I} \subset S_m/k$  Nisnevich cov.

$$\stackrel{\text{def}}{\left\{ \begin{array}{l} \cdot \text{ étale cov.} \\ \cdot X \ni x, \exists i \in I, \exists y \in X_i \\ \text{ s.t. } f_i(y) = x, \quad k(x) \xrightarrow{\sim} k(y) \\ \quad \quad \quad \uparrow \\ \quad \quad \quad \text{residue field} \end{array} \right.} \quad \square$$

a coarse line :

$$Zar < Nis < et$$

$$\forall x \in X \subset S_m/k$$

$$\text{Nis. loc. } X \simeq A^n$$

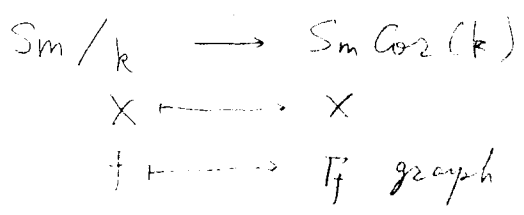
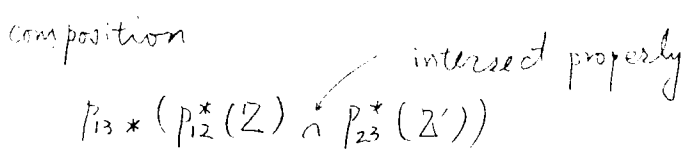
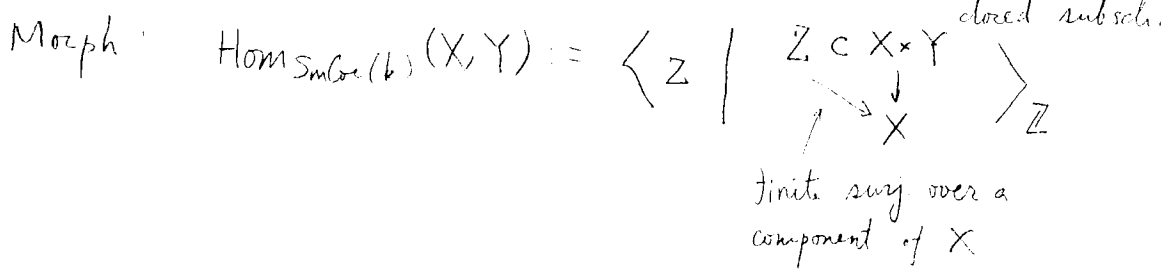
$\rightsquigarrow$  Nis. top, Nis. sheaf.

Def.  $F$ : presheaf of abel. gp.

homotopy inv  $\stackrel{\text{def}}{\iff} \forall X \in \text{Sm}/k \quad F(X) \xrightarrow[\text{pr}]{\cong} F(X \times A^1)$

□

$\text{SmCos}(k)$  obj.  $X \in \text{Sm}/k$



Def.  $F$ : presheaf with transfers

$\stackrel{\text{def}}{\iff} F: \text{SmCos}(k) \longrightarrow (A-b)$

$F \left( \begin{smallmatrix} Nis \\ Zar \\ \text{et} \\ \text{cdh} \end{smallmatrix} \right)$  sheaf with trans  $\iff F$ : presheaf with trans +  $F|_{\text{Sm}/k} : \left( \begin{smallmatrix} Nis \\ \vdots \end{smallmatrix} \right)$  sheaf

motivic  $\subset DM_{-}^{\text{eff}}(k)$   
 spx.  $\mathbb{Z}(n) \quad (n \geq 0)$   
 $M(X) \quad (X \in \text{Sm}/k)$

Def.  $\mathbb{Z}_{\text{tr}}(X)(-) := \text{Hom}_{\text{Sm}/k}(-, X)$  : presheaf with trans  
 $\rightsquigarrow$  Zar Nis sheaf  $\square$

$$\Delta^n = \text{Spec } k[T_0, \dots, T_n] / \left( \sum_i T_i - 1 \right)$$

Def. (Suslin complex)

$F$  presheaf  $C_n(F)(-) = F(\Delta^n \times -)$

$\rightsquigarrow C_*(F)$  associated complex to the simplicial presheaf  $n \mapsto C_n(F)$

Fact  $C_*(F)$ : homologies are homotopy inv. presheaf

Def.  $X \in \text{Sm}/k$

$M(X) \in \text{DM}_{\text{eff}}(k)$  motive of  $X$

"  $C_*(\mathbb{Z}_{\text{tr}}(X)) \leftrightarrow$  alg. analogue of singular chain complex

$$C^m(F) = C_{-m}(F)$$

Def.  $\mathbb{Z}_{\text{tr}}(\widehat{G}_m^n) := \text{Cok} \left( \bigoplus_{i=0}^{n-1} \mathbb{Z}_{\text{tr}}(\widehat{G}_m^{i(n-1)}) \xrightarrow{\gamma} \mathbb{Z}_{\text{tr}}(\widehat{G}_m^{i(n)}) \right)$   
 $n \geq 0$  direct summand

Def.  $Z(n) := C_*(\mathbb{Z}_{\text{tr}}(\widehat{G}_m^n))[-n] \in \text{DM}_{\text{eff}}(k)$

$\left( \begin{array}{l} H_{\text{Nis}}^p(X, \mathbb{Z}(q)) = \text{Hom}_{\text{DM}_{\text{eff}}(k)}(M(X), \mathbb{Z}(q)) \\ \text{"} \\ H_{\text{Zar}}^p(X, \mathbb{Z}(q)) \end{array} \right)$  motivic complex

properties

Milnor's  
↓  
K-grp

- ①  $\mathbb{Z}(0) = \mathbb{Z}$       ②  $\mathbb{Z}(1) = \mathcal{O}_X^* / [1]$
- ③  $F$  f.g. field /  $k \Rightarrow H^n(\text{Spec } F, \mathbb{Z}(n)) \cong K_n^M(F)$
- ④  $X \in \text{Sm}/k \Rightarrow H_{\text{Zar}}^{2n}(X, \mathbb{Z}(n)) \cong CH^n(X)$   
(Nis) chow grp.
- (  $H_{\text{Zar}}^p(X, \mathbb{Z}(q)) \cong CH^q(X, 2q-p)$  )

⑤  $X \in \text{Sm}/k$

$\exists$  spec seq      Atiyah-Hirzebruch

$$E_2^{p,q} = H_{\text{Zar}}^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p,q}(X)$$

(motivic homotopy)

- Bloch-Lichtenbaum, Friedlander-Suslin
- Levin-Voevodsky
- Grayson-Suslin

Beilinson Suslin vanishing conjecture

$$X \in \text{Sm}/k \quad n \geq 0$$

$$H_{\text{Zar}}^i(X, \mathbb{Z}(n)) = 0 \quad \text{for } i > 0$$

Beilinson Lichtenbaum Conj.  $l \neq \text{char } k$

$$\mathbb{Z}/l(q) := \mathbb{Z}(q) \otimes \mathbb{Z}/l$$

BL.  $(q, \mathbb{Z}/l)$

$\mu_l^{\otimes q}$

$\alpha: (Sm/k)_{\text{ét}} \rightarrow (Sm/k)_{\text{zar}}$

$$\mathbb{Z}/l(q) \xrightarrow{q!S} \Gamma_{\text{ét}} R\alpha_* \sqrt{\alpha^* \mathbb{Z}/l(q)}$$

BL( $\mathfrak{g}, \mathbb{Z}/\ell$ )  $\quad \ell \geq 2$  Milnor conj

$\hookrightarrow$  Bloch-Kato conj  $\quad$  coincides with the classical map by symbol

$$K_0^M(F)/\ell \xrightarrow{\sim} H_{\text{et}}^0(\text{Spec } F, \mu_{\ell}^{\otimes \mathfrak{g}})$$

(only surj.  $\rightsquigarrow$  weak Bloch-Kato)

weak BK( $\mathfrak{g}, \ell$ )

$\Rightarrow$  vanishing of Blochstein  $BV(\mathfrak{g}, \ell)$

$$F: \text{f.g.}/k \quad \forall j \quad \beta_{\mathfrak{g}, j}: H^j(F, \mu_{\ell}^{\otimes \mathfrak{g}}) \rightarrow H^{j+1}(F, \mu_{\ell}^{\otimes \mathfrak{g}})$$

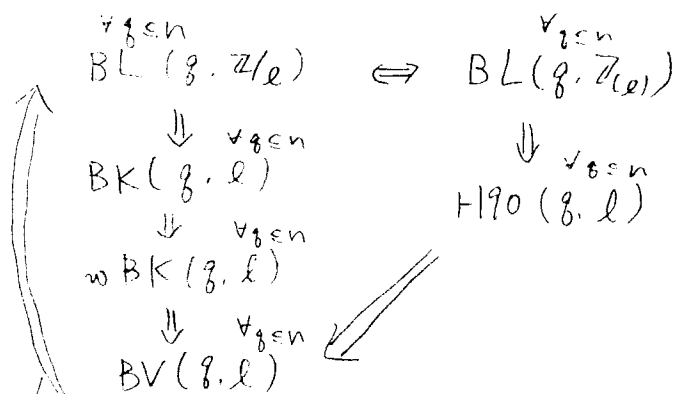
trivial

$$BL(\mathfrak{g}, \mathbb{Z}(\ell)) : \mathbb{Z}(\ell)(\mathfrak{g}) := \mathbb{Z}(\mathfrak{g}) \otimes_{\mathbb{Z}} \mathbb{Z}(\ell)$$

$$\mathbb{Z}(\ell)(\mathfrak{g}) \xrightarrow{\tau_{\mathfrak{g}, \ell}} \tau_{\leq \mathfrak{g}+1} R\alpha_* \alpha^* \mathbb{Z}(\ell)(\mathfrak{g})$$

$\Rightarrow$  generalised Hilbert  $\mathfrak{g}_0$   $H^0(\mathfrak{g}, \ell)$

$$\forall F: \text{f.g.}/k \quad \Rightarrow H_{\text{et}}^{\mathfrak{g}+1}(F, \mathbb{Z}(\ell)(\mathfrak{g})) = 0$$



} Suslin-Voevodsky  
Geisser-Lavine