山下剛

- morning session

<u>goal</u>:
Voevodsky's proof of
Milnor Conj.
Sato, Mochizuki, Yamashita
Hagihara

- afternoon session
  - Review from (classical) algebraic topology
  - topics

Review and Overview

Voevodsky's idea

transport concepts and techniques from alg. top.
- spectrum
- coh. operation
- Margolis coh.

last year : motivic (co)homology

this year : motivic homotopy

homological alg.	$\rightsquigarrow$	homotopical alg.
abel. cat	$\rightsquigarrow$	model cat
derived cat	$\rightsquigarrow$	homotopy cat
inj. obj	$\rightsquigarrow$	fibrant obj
$[+1]$	$\rightsquigarrow$	$S_s^1 \wedge$
		" sphere

Review k: perfect field

Sm/k cat of smooth sch. $/k$

$$DM^{eff}_{gm}(k) \subset DM^{eff}_{-}(k) \quad \leftarrow \text{last year}$$

Tate obj $\mathbb{Z}(1)$ invert $\Downarrow$ $DM_{gm}(k)$ $\uparrow$ tensor triag. cat

$\uparrow$ rigid tensor triag. cat

$DM^{eff}_{-}(k)$ = the derived cat of bounded above complex of Nisnevich sheaves with transfers with homotopy invariant coh sheaves

Def. $\{X_i \xrightarrow{f_i} X\}_{i\in I} \subset Sm/k$: Nisnevich cov

$\overset{\Longleftrightarrow}{def}$ $\begin{cases} \cdot \text{ étale cov.} \\ \cdot\ X \ni x,\ \exists i \in I,\ \exists y \in X_i \\ \quad \text{s.t } f_i(y) = x,\ \ k(x) \xrightarrow{\sim} k(y) \end{cause}$

$\uparrow$ residue field $\square$

· coarse line

$Zar < Nis < et$

$\forall x \in X \subset Sm/k$

Nis. loc. $X \simeq \mathbb{A}^n$

$\rightsquigarrow$ Nis. top. Nis. sheaf

Def. F: presheaf of abel. gp.

homotopy inv $\overset{\text{def}}{\iff}$ $\forall X \in Sm/k$ $\quad F(X) \xrightarrow[pr]{\sim} F(X \times \mathbb{A}^1)$ $\quad\square$

$SmCor(k)$ $\quad$ obj. $X \in Sm/k$

Morph: $\mathrm{Hom}_{SmCor(k)}(X,Y) := \left\langle Z \;\middle|\; \begin{array}{c} Z \subset X \times Y \\ \downarrow \\ X \end{array} \right\rangle_{\mathbb{Z}}$

reduced irr. closed subsch.

finite surj over a component of X

composition

$$p_{13*}\left(p_{12}^*(Z) \cap p_{23}^*(Z')\right)$$

intersect properly

$$\begin{array}{ccc} Sm/k & \longrightarrow & SmCor(k) \\ X & \longmapsto & X \\ f & \longmapsto & \Gamma_f \text{ graph} \end{array}$$

Def. F: presheaf with transfers

$\overset{\text{def}}{\iff}$ $F : SmCor(k) \longrightarrow (Ab)$

F $\begin{pmatrix} \text{Nis} \\ \text{Zar} \\ \text{ét} \\ \text{cdh} \end{pmatrix}$ sheaf with trans $\iff$ F: presheaf with trans $+$ $F|_{Sm/k}$: $\begin{pmatrix} \text{Nis} \\ \vdots \end{pmatrix}$ sheaf

motivic cpx. $\in DM^{\mathrm{eff}}_{-}(k)$

$\mathbb{Z}(n)$ $(n \geq 0)$

$M(X)$ $(X \in Sm/k)$

Def: $\mathbb{Z}_{tr}(X)(-) := \mathrm{Hom}_{SmCor(k)}(-, X)$ presheaf with trans

$\rightsquigarrow$ $\begin{matrix}\text{Zar}\\ \text{Nis}\end{matrix}$ sheaf $\square$

$$\Delta^n = \mathrm{Spec}\, k[T_0, \cdots, T_n]/(\sum_i^n T_i - 1)$$

Def. (Suslin complex)

F presheaf $\quad C_n(F)(-) = F(\Delta^n \times -)$

$\rightsquigarrow$ $C_*(F)$ associated complex to the simplicial presheaf $n \longmapsto C_n(F)$

Fact $C_*(F)$: homologies are homotopy inv. presheaf

Def. $X \in Sm/k$

$M(X) \in DM_-^{eff}(k)$ motive of X

$\shortparallel$

$C_*(\mathbb{Z}_{tr}(X)) \longleftrightarrow$ alg analogue of singular chain complex

$C^m(F) = C_{-m}(F)$

Def. $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}) := \mathrm{Cok}\left(\bigoplus_{i=0}^{n-1} \mathbb{Z}_{tr}(\mathbb{G}_m^{\times(n-1)}) \to \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})\right)$

$n \geq 0$ $\qquad$ direct summand

Def. $\mathbb{Z}(n) := C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))[-n] \in DM_-^{eff}(k)$ motivic complex

$$\left(\begin{matrix} H^p_{Nis}(X, \mathbb{Z}(q)) = \mathrm{Hom}_{DM_-^{eff}(k)}(M(X), \mathbb{Z}(q)) \\ \shortparallel \\ H^p_{Zar}(X, \mathbb{Z}(q)) \end{matrix}\right)$$

properties

Milnor's K-grp

① $\mathbb{Z}(0) = \mathbb{Z}$ ② $\mathbb{Z}(1) = \mathcal{O}_X^*[-1]$

③ F f.g. field $/k$ $\Rightarrow$ $H^n(\operatorname{Spec} F, \mathbb{Z}(n)) \simeq K_n^M(F)$

④ $X \in Sm/k$ $\Rightarrow$ $H^{2n}_{\substack{Zar \\ (Nis)}}(X, \mathbb{Z}(n)) \simeq CH^n(X)$ chow grp.

$\left(H^p_{Zar}(X, \mathbb{Z}(q)) \simeq CH^q(X, 2q-p) \right)$

⑤ $X \in Sm/k$

$\exists$ spec seq Atiyah-Hirzebruch

$$E_2^{p,q} = H^{p-q}_{\substack{Zar \\ (Nis)}}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

(motivic homotopy)

- Bloch-Lichtenbaum, Friedlander-Suslin
- Levin-Voevodsky
- Grayson-Suslin

Beilinson Suslin vanishing conjecture

$X \in Sm/k$ $n \geq 0$

$H^i_{Zar}(X, \mathbb{Z}(n)) = 0$ for $i > 0$

Beilinson Lichtenbaum Conj. $\ell \neq \operatorname{char} k$

$\mathbb{Z}/\ell(q) := \mathbb{Z}(q) \otimes \mathbb{Z}/\ell$

BL$(q, \mathbb{Z}/\ell)$ $\alpha : (Sm/k)_{et} \to (Sm/k)_{Zar}$

$$\mathbb{Z}/\ell(q) \xrightarrow{qis} \tau_{\leq q} R\alpha_* \underbrace{\alpha^* \mathbb{Z}/\ell(q)}_{= \mu_\ell^{\otimes q}}$$

$BL(q, \mathbb{Z}/\ell)$ — $\ell = 2$ Milnor conj

/ Bloch-Kato conj

F: f.g. field $/k$

$K^M_q(F)/\ell \xrightarrow{\sim} H^q_{et}(\mathrm{Spec}\, F, \mu_\ell^{\otimes q})$

coincides with the classical map by symbol

(only surj $\rightsquigarrow$ weak Bloch-Kato)

weak $BK(q, \ell)$

$\Rightarrow$ vanishing of Bochstein $BV(q, \ell)$

F: f.g. $/k$ $\quad \forall j \quad \beta_{q,j}: H^q(F, \mu_\ell^{\otimes q}) \to H^{q+1}(F, \mu_\ell^{\otimes q})$ trivial

$BL(q, \mathbb{Z}_{(\ell)})$: $\mathbb{Z}_{(\ell)}(q) := \mathbb{Z}(q) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)}$

$\mathbb{Z}_{(\ell)}(q) \xrightarrow[qis]{} \tau_{\leq q+1} R\alpha_* \alpha^* \mathbb{Z}_{(\ell)}(q)$

$\Rightarrow$ generalized Hilbert 90 $H90(q, \ell)$

$\forall F$: f.g. $/k$ $\Rightarrow H^{q+1}_{et}(F, \mathbb{Z}_{(\ell)}(q)) = 0$

$$\begin{array}{ccc}
BL(q, \mathbb{Z}/\ell)\ \forall q \leq n & \Leftrightarrow & BL(q, \mathbb{Z}_{(\ell)})\ \forall q \leq n \\
\Downarrow \forall q \leq n & & \Downarrow \forall q \leq n \\
BK(q, \ell) & & H90(q, \ell) \\
\Downarrow \forall q \leq n & & \\
wBK(q, \ell) & & \\
\Downarrow \forall q \leq n & & \\
BV(q, \ell) & &
\end{array}$$

Suslin-Voevodsky
Geisser Levine