Reduced Power Operation :

topology stable homotopy operation

$$
\begin{array}{ccc}
\widetilde{H}^{*}(X_+, \mathbb{Z}/\ell) & \longrightarrow & \widetilde{H}^{*+k}(X_+, \mathbb{Z}/\ell) \\
\downarrow s & \circlearrowleft & \downarrow s \\
\widetilde{H}^{*+1}(\Sigma X_+, \mathbb{Z}/\ell) & \longrightarrow & \widetilde{H}^{*+k+1}(\Sigma X_+, \mathbb{Z}/\ell)
\end{array}
$$

Steenrod alg

$$\mathcal{A}^* = \left(\begin{array}{c}\text{stable}\\ \text{coh. operation}\end{array}\right)$$

$$= \mathrm{Hom}(H_{\mathbb{Z}}, \Sigma^* H_{\mathbb{Z}})$$

$$\underset{\text{Serre}}{\overset{?}{=}} \langle \beta, P^i \rangle_{\mathbb{Z}/\ell\text{-alg}}$$

Milnor

H^* : cup prod $\rightsquigarrow$ $\mathcal{A}^*$: coprod.

$\exists!\ \psi^*$ ← graded co-commutative

$\mathcal{A}_*$ dual Steenrod alg.

← graded commutative

Th (Milnor)

$$\ell > 2 \Rightarrow \mathcal{A}_* = \mathbb{Z}/\ell[\xi_1, \xi_2, \cdots] \otimes_{\mathbb{Z}/\ell} \Lambda^*\left(\bigoplus_{i=0}^{\infty} \mathbb{Z}/\ell\cdot\tau_i\right)$$

$$\deg \tau_i = 2\ell^i - 1$$

$$\deg \xi_i = 2(\ell^i - 1)$$

$$\ell = 2 \Rightarrow \mathcal{A}_* = \mathbb{Z}/\ell[\tau_0, \tau_1, \cdots] \quad (\ell = 2 \Rightarrow \tau_i^2 = \xi_{i+1})$$

$\Rightarrow \{ \prod_{i\geq 0} \tau_i^{\varepsilon_i} \prod_{j\geq 1} \xi_j^{r_j} \}_{\varepsilon_i = 0,1,\ r_j \geq 0}$ formes a basis of $\mathcal{A}_*$ $\quad \xi_i \quad \tau_i$

$\updownarrow$ dual basis of $\mathcal{A}^*$ $\qquad\qquad \updownarrow \quad \updownarrow$

$\{ P(E, R) \}$ $\qquad\qquad \mathcal{P}_i \quad Q_i$

Milnor's primitive operation

$\deg Q_i = 2\ell^i - 1 \qquad \deg \mathcal{P}_i = 2\ell^i - 2$

$Q_0 = \beta \qquad Q_{i+1} = [Q_i, P^{\ell^i}]$

($\uparrow$ NOT holds in the motivic case)

$Q_i^2 = 0 \Rightarrow \{ H^*(X, \mathbb{Z}/\ell), Q_i \}$: complex

$\Rightarrow$ Margolis coh. $\widetilde{MH}^*{}_i(X)$

Want to construct a motivic analogue

How to use in the last stage of the proof of Milnor conj.?

(a glance)

<u>H90(n,2)</u> $H^{n+1,n}(\check{C}(Q_{\underline{a}}), \mathbb{Z}/2) = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow$ want to prove

$\cong$

$\tilde{H}^{n+2,n}(\check{C}(Q_a))$

<u>3 differences</u>

① bigraded $\rightsquigarrow$ harmless

② $H^{*,*}(\mathrm{Spec}\, k, \mathbb{Z}/\ell)$: non-trivial

$a = (\cup a) \in \mathcal{A}^{*,(1)}$

$\to$ $\ell > 2$, harmless

$\ell = 2 \Rightarrow$ a little complicated

$\tau \in H^{0,1}(k)$ $\qquad$ $\rho \in H^{1,1}(k)$

$\uparrow \qquad \cong \qquad\qquad \uparrow \qquad \cong$

$-1 \in \mu_2(k) = \{\pm 1\}$ $\qquad$ $-1 \in k^\times/(k^\times)^2$

② We don't know

$\mathcal{A}^{*,*} \overset{?}{=} \mathrm{Hom}(H\mathbb{Z}/\ell, \Sigma_s^* \Sigma_t^* H\mathbb{Z}/\ell)$

$\to$ $\ell = 2$ harmless

$\ell > 2 \Rightarrow$ harm!

<u>strategy of construction of P^i</u>

<u>Rem</u> bistable coh. op.

$\tilde{H}^{*,*} \longrightarrow \tilde{H}^{*+i, *+j}$ functorial

$S_s^1 \wedge$, $S_t^1 \wedge$ compatible with suspension isom

$\Leftrightarrow \tilde{H}^{2n,n} \longrightarrow \tilde{H}^{2n+i, n+j}$ compatible with $\wedge T = \mathbb{A}^1/\mathbb{A}^1 - \{0\}$

$\downarrow$

$\mathbb{Z}(1)[2]$

$T = S_s^1 \wedge S_t^1$

$\left(\tilde{H}^{p,q}(X_+) = \tilde{H}^{p+a+\ell, q+\ell}(S_s^a \wedge S_t^\ell \wedge X_+) \right.$

$\updownarrow \qquad \uparrow$

$(1,0) \qquad (1,1)$

を用いる $\left. \right)$

$\tilde{H}^{2n,n}(F_\bullet, \mathbb{Z}/\ell) \xrightarrow{P} \tilde{H}^{2n\ell, n\ell}(F_\bullet \wedge (B\mathfrak{S}_\ell)_+, \mathbb{Z}/\ell)$

pointed simplicial sheaf

classifying space of ℓ-th sym. grp.

$\tilde{H}^{*,*}(F_\cdot \wedge (B\mathfrak{S}_\ell)_+, \mathbb{Z}/\ell)$

$\tilde{H}^{*,*}(F_\cdot)[[c, d]] / (c^2)$ $\quad \deg(c) = (2\ell-3, \ell-1)$, $\deg(d) = (2\ell-2, \ell-1)$ $\quad (\ell > 2)$

$\tilde{H}^{*,*}(F_\cdot)[[c, d]] / (c^2 - \tau d - \rho c)$ $\quad \ell = 2$

$\hat{H}^{2n,n}(F_\cdot) \ni \alpha$ $\qquad \hat{H}^{2n\ell, n\ell}(F_\cdot \wedge (B\mathfrak{S}_\ell)_+) \ni$

$P(\alpha) = \sum B^i(\alpha)\, c d^i + \sum P^i(\alpha)\, d^i$

property of P $\rightsquigarrow$ property of P^i, B^i

$\rightsquigarrow$ $\mathcal{A}^{*,*}$: left $H^{*,*}$-mod

$H^{*,*}$: cup prod $\longrightarrow$ $\mathcal{A}^{*,*}$: coprod

$\mathcal{A}_{*,*} := \mathrm{Hom}_{H^{*,*}(k)}(\mathcal{A}^{*,*}, H^{*,*}(k))$ $\curvearrowleft$ prod

<u>calculate</u> $\tilde{H}^{**}(F_\cdot \wedge (BG_\ell)_+, \mathbb{Z}/\ell)$

$k \ni \zeta_\ell \neq 1$ $\qquad \mu_\ell \overset{\rho_\zeta}{\cong} \mathbb{Z}/\ell \hookrightarrow \mathfrak{S}_\ell$

$$\tilde{H}^*(F_\cdot \wedge (B\mathfrak{S}_\ell)_+) \xrightarrow{\rho_\zeta^*} \tilde{H}(F_\cdot \wedge (B\mu_\ell)_+)$$

$\uparrow$ splitting injection

<u>Lem</u> $B\mu_\ell = \mathcal{O}(-\ell)_{\mathbb{P}^\infty} - \mathbb{P}^\infty$

$\uparrow$ zero section

($\because$ $\mathbb{A}^{n+1} - \{0\} \twoheadrightarrow \mathbb{P}^n$ $\qquad \mathcal{O}(-\ell)_{\mathbb{P}^n} - \mathbb{P}^n$

$\searrow (\mathbb{A}^{n+1} - \{0\})/\mu_\ell \nearrow$ $\qquad n \to \infty$

$$B\mu_\ell \hookrightarrow (\mathcal{O}(-\ell))_{\mathbb{P}^\infty} \twoheadrightarrow Th_{\mathbb{P}^\infty}(\mathcal{O}(-\ell)_{\mathbb{P}^\infty})$$

cofibration seq.

$$c = [\mathcal{O}(-\ell)] \in H^{2,1}(\mathbb{P}^\infty)$$

$$\begin{array}{l} \to H^{*,*}(Th(\mathcal{O}(-\ell))) \to (H(k)[\![c]\!])^{*,*} \to H^{*,*}(B\mu_\ell) \to H^{*+1,*}(Th) \\ \quad \uparrow x\, t_{\mathcal{O}(-\ell)} \quad \cong \qquad\qquad\qquad , x\ell c \\ \quad x \quad (H(k)[\![c]\!])^{*-2,*-1} \qquad \ni x \end{array}$$

$t_{\mathcal{O}(-\ell)}$ Thom class of $\mathcal{O}(-\ell)$

$$0 \to (H(k)[\![c]\!])^{*,*} \to H^{*,*}(B\mu_\ell) \to (H(k)[\![c]\!])^{*-1,*-1} \to 0$$

(exact)