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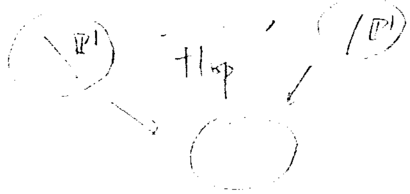
intro /$\mathbb{C}$

flop

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow f & & \searrow & \\ X : \text{3-dim} & & & & Y \end{array} \quad \Rightarrow \quad K_{Z/X} = K_{Z/Y}$$

$$K_Z - f^* K_X$$



flop

Def. X, Y birational smooth var.

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

X & Y are K-equivalent

$\underset{\text{def}}{\iff} K_{Z/X} = K_{Z/Y}$ $\square$

K-eq varieties have the same "invariant"

⇑ motivic integration

↓ generalize

K-eq. orbifold (smooth Deligne Mumford stack) have the same "orbifold inv" (Homological McKay)

⇑ motivic int. over DM stack

generalize

normal, $\mathbb{Q}$-div

(X, D) KLT (Kawamata log terminal)

K-eq. KLT pairs have the same "stringy inv."

<u>motivic int.</u>

the Grothendieck ring of MHS

X : var. Hodge char.

$$\chi_h(X) := \sum (-1)^i [H_c^i(X, \mathbb{Q})] \in K_0(HS)$$

$$\mathrm{Arcs}(X) = \mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[\![t]\!], X)$$

$\uparrow$ $\widehat{K}_0$-valued measure

$K_0(HS) =: K_0 \rightsquigarrow \widehat{K}_0$
weight completion

$F : \mathrm{Arcs}(X) \to \widehat{K}_0$
measurable function

$\downarrow$ $\int F \, d\mu_X$

X : sm $\quad \int 1 \, d\mu_X = \chi_h(X)$

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, bend right, leftrightarrow, "K\text{-eq}"'] & & Y \end{tikzcd}$$

$f_* : \mathrm{Arcs}(Z) \to \mathrm{Arcs}(X)$

<u>Key fact</u> : f_* is almost bij
(outside measure zero subsets)

$$\left(\begin{tikzcd} \mathrm{Spec}\, \mathbb{C}(\!(t)\!) \arrow[r, dashed] \arrow[d] & Z \arrow[d] \\ \mathrm{Spec}\, \mathbb{C}[\![t]\!] \arrow[r] \arrow[ur, dashed, "\exists !"] & X \end{tikzcd} \right)$$

$$\chi_h(X) = \int d\mu_X \underset{\substack{\uparrow \\ \text{change of variables}}}{=} \int \underbrace{\mathbb{L}^{F_{K_{Z/X}}}}_{} d\mu_Z = \int d\mu_Y = \chi_h(Y)$$

$\mathbb{L} = \chi_h(\mathbb{A}^1) = [\mathbb{Q}(-1)]$

$F_{K_{Z/X}} : \mathrm{Arcs}(Z) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$

depend only on
$K_{Z/X} - K_{Z/Y}$

Furthermore if X, Y : proj, then $H^*(X) \underset{\text{Hodge str.}}{\simeq} H^*(Y)$

Homological McKay correspondence

$$\begin{array}{ccc} & \mathbb{C}^d & G \subset SL_d \text{ finite subgroup} \\ & \downarrow & \\ \text{smooth } Y & \xrightarrow[\text{crepant resol}]{\pi} & \mathbb{C}^d/G = X \end{array}$$

$\pi^* K_X = K_Y$

<u>Th</u> (Batyrev)

$$H^i(Y, \mathbb{Q}) = \begin{cases} 0 & (i : \text{odd}) \\ \mathbb{Q}(-i/2)^{\oplus n_i} & (i : \text{even}) \end{cases}$$

$$\left(\begin{array}{l} g \in G \quad \text{for suitable basis}, \ g = \operatorname{diag}(\zeta_\ell^{a_1}, \ldots, \zeta_\ell^{a_d}) \quad 0 \le a_i \le \ell \\ \operatorname{age}(g) := \frac{1}{\ell} \sum_i a_i \in \mathbb{Z} \quad (\Leftarrow G \subset SL_d) \\ n_i = \# \{ (g) \in \operatorname{Conj}(G) \mid \operatorname{age}(g) = i/2 \} \end{array} \right)$$

$\square$

$$\begin{array}{ccccc} \mathbb{A}^d & \to & [\mathbb{A}^d/G] & \to & \mathbb{A}^d/G \\ \circlearrowleft & & \text{quotient} & & \text{quotient} \\ G & & \text{stack} & & \text{var.} \end{array}$$

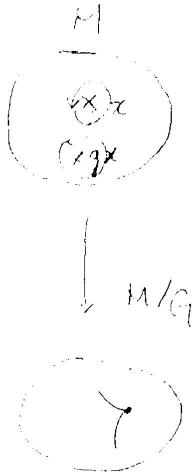
smooth DM stack — have quotient singularity

Points have automorphism groups

manifold $\rightsquigarrow$ orbifold (Satake's V-mfd)

scheme (alg. space) $\rightsquigarrow$ DM stack

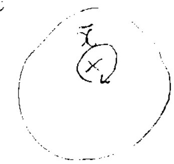
$\forall$ DM stack is locally isom. to a quotient stack



scheme (mfd) $\circlearrowleft$ G : finite gp.

$\mathrm{stab}(x) \subset G$

$\mathrm{stab}(gx) = g\,\mathrm{stab}(x)\,g^{-1}$ $\rightsquigarrow$ $[M/G]$



quotient singularity

$\mathrm{Aut}(\bar{x}) \simeq \mathrm{stab}(x) \simeq \mathrm{stab}(gx)$

$$\begin{array}{ccc} [\mathbb{A}^d/G] & \xrightarrow{\pi} & \mathbb{A}^d/G \\ \uparrow & \underline{\text{birational}} & \uparrow \\ [U/G] & = & U/G \end{array}$$

$\mathbb{A}^d \supset U := \{x \mid \mathrm{stab}(x) = 1\}$

$G \subset SL$

$\Rightarrow$ no reflection

$\mathrm{cod}\ \mathbb{A}^d \setminus U \geq 2$

π has no exceptional div.

$\Rightarrow$ crepant

$$\begin{array}{ccc} & & [\mathbb{A}^d/G] \\ & \overset{K\text{-eq}}{\nearrow} & \downarrow \\ Y & \longrightarrow & X \end{array}$$

motivic integration over DM stacks

$\Downarrow$

$\chi_h(Y) = \chi_h^{orb}([\mathbb{A}^d/G])$

<u>Th</u> $\mathcal{X}, \mathcal{Y}$: smooth DM-stack K-eq.

$\Rightarrow \chi_h^{orb}(\mathcal{X}) = \chi_h^{orb}(\mathcal{Y})$ $\square$

DM stack

$\mathcal{X} \rightsquigarrow I\mathcal{X}$ inertia stack $\quad \mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X}$

$I\mathcal{X} = \mathcal{X} \times_{\Delta, \mathcal{X}\times\mathcal{X}, \Delta} \mathcal{X} = \{(x, [g]) \mid x \in \mathcal{X}, [g] \in \mathrm{Conj}(\mathrm{Aut}(x))\}$

$$\chi_{\hbar}^{orb}(\mathcal{X}) = \sum_{\substack{D \subset I\mathcal{X} \\ \text{conn comp}}} \chi_{\hbar}(\bar{D})\, \mathbb{L}^{sht(D)}$$

$\bar{D}$: coarse moduli space

$sht(D) = age(g) \in \mathbb{Q}$

$\operatorname{Spec} \mathbb{C}[[t]] \longrightarrow \mathcal{X}$

$$\begin{array}{ccccc} & & \operatorname{Spec}\mathbb{C}((t)) & \dashrightarrow & \mathcal{Y} \\ & & \downarrow & & \downarrow \text{ proper fin} \\ \operatorname{Spec}\mathbb{C}[[t^{1/n}]] & \longrightarrow & \operatorname{Spec}\mathbb{C}[[t]] & \longrightarrow & \mathcal{X} \end{array}$$

γ induces a hom of fin grp. $\mu_\ell \to G$

$\ell \in \mathbb{N}$, $\mu_\ell \subset \mathbb{C}^\times$ $\quad \mathcal{D}_\ell = [\operatorname{Spec}\mathbb{C}[[t]]/\mu_\ell] \xrightarrow{\gamma} \mathcal{X}$

representable twisted stack

injective

$\operatorname{TwArcs}(\mathcal{X})/\text{isom}$

$\hookleftarrow \mu_{\mathcal{X}} \quad \int F\, d\mu_{\mathcal{X}}$

$$\begin{array}{ccc} \operatorname{TwArcs}(\mathcal{X}) & \ni & \gamma \\ \downarrow & & \rotatebox{90}{$\mapsto$} \\ I\mathcal{X} & \ni & (\gamma(\text{special pt}), \ ) \end{array}$$

Ass. lift

$$\chi_{\hbar}^{orb}(\mathcal{X}) = \int \mathbb{L}^{sht(\ )}\, d\mu_{\mathcal{X}}$$