An Introduction to Lawson Homology, II

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Review of Lawson homology

For X quasi-projective over $\mathbb{C}$:

$$\mathcal{Z}_r(X) = \text{topological abelian group of } r\text{-cycles on } X$$

$$L_r H_m(X) := \pi_{m-2r} \mathcal{Z}_r(X)$$

There are maps:

$$H_m^{\mathcal{M}}(X, \mathbb{Z}(r)) \to L_r H_m(X) \to H_m^{\mathsf{BM}}(X(\mathbb{C}), \mathbb{Z}(r))$$

The left-hand map is an isomorphism with $\mathbb{Z}/n$-coefficients.

The right-hand map is the topic of Suslin's Conjecture (see below).

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The right-hand map is the topic of Suslin's Conjecture (see below).

Additional comment: These are maps of (non-finitely generated) MHS's, where $H_m^{\mathcal{M}}(X, \mathbb{Z}(r))$ has the trivial MHS. The MHS of $L_r H_m(X)$ is induced by MHS on $H_*^{\mathsf{sing}}(\mathcal{C}_{r,e}(X))$.

Related theories

I mentioned (but did not define) morphic cohomology, $L^*H^*(X)$, yesterday.

There is also a version of K-theory that uses algebraic equivalence, called semi-topological K-theory: Let $\mathrm{Grass} = \coprod_n \mathrm{Grass}(\mathbb{C}^n)$. For X projective,

$$K_q^{\text{semi}}(X) := \pi_q \left(\mathrm{Maps}(X, \mathrm{Grass})^{h+} \right)$$

where $h+$ denotes "homotopy theoretic group completion" of the homotopy-commutative H-space $\mathrm{Maps}(X, \mathrm{Grass})$.

For example, $K_0^{semi}(X) = K_0(X)/(\text{alg. equiv.})$.

"Every formal property one might expect involving L_*H_*, L^*H^* and K_*^{semi} does indeed hold."

Some of the properties of K_*^{semi}

- There are natural maps

$$K_n(X) \to K_n^{\text{semi}}(X) \to ku^{-n}(X(\mathbb{C})).$$

- $K_n(X, \mathbb{Z}/m) \xrightarrow{\cong} K_n^{\text{semi}}(X, \mathbb{Z}/m)$ for $m > 0$.
- There is a Chern character isomorphism

$$ch : K_n^{\text{semi}}(X)_{\mathbb{Q}} \xrightarrow{\cong} \oplus\, L^q H^{2q-n}(X, \mathbb{Q}).$$

- For X smooth, there is an Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = L^{-q} H^{p-q} \Rightarrow K_{-p-q}^{\text{semi}}(X),$$

which degenerates upon tensoring with $\mathbb{Q}$.

Conjecture (Suslin's Conjecture — Lawson form)

For a smooth, quasi-projective variety X, the map

$$L_r H_m(X) \to H_m^{sing}(X(\mathbb{C}), \mathbb{Z}(r))$$

is an isomorphism for $m \geq d + r$ and a monomorphism for $m = d + r - 1$.

"Suslin's Conjecture = Bloch-Kato (really, Beilinson-Lichtenbaum) with $\mathbb{Z}$-coefficients (over $\mathbb{C}$)":

$$L^t H^n(X) \overset{?}{\cong} \mathbb{H}^n_{Zar}(X, tr^{\leq t}\mathbb{R}\pi_* \mathbb{Z}),$$

where $\pi : (Var/\mathbb{C})_{analytic} \to (Var/\mathbb{C})_{Zar}$.

(Thin) Evidence: Cases where Suslin's Conjecture is known

- For codimension one cycles. Proof is by explicit calculation of $L_{\dim(X)-1}H_*(X)$.
- For L_0H_* (by Dold-Thom Theorem) and $L_{\dim(X)}H_*$ (trivially). In particular, it's known for all surfaces.
- For special varieties, such as toric varieties, cellular varieties, linear varieties (that are smooth).
- Certain hyper-surfaces of dim. 3 [Voineagu]
- With finite coefficients — i.e., Bloch-Kato is known [Voevodsky].
- The cohomological form holds for π_0 [Bloch-Ogus]

$$L_{d-t}H_{2(d-t)}(X) = L^tH^{2t}(X) \cong \mathbb{H}^{2t}_{Zar}(X, tr^{\leq t}\mathbb{R}\pi_*\mathbb{Z})$$

Finite generation of Lawson homology

Suslin's Conjecture predicts, in particular, that

$L_rH_m(X)$ is finitely generated for $m \geq \dim(X) + r - 1$.

The converse is also known: let $C_rH_*(X)$ be the "cone" of the map from L_rH_* to H_*^{BM}, so that

$$\cdots \to C_rH_m(X) \to L_rH_m(X) \to H_m^{\mathrm{BM}}(X) \to C_rH_{m-1}(X) \to \cdots.$$

Voevodsky's Bloch-Kato $\Rightarrow C_rH_m(X, \mathbb{Z}/n) = 0$ for $m \geq \dim(X) + r - 1$, and hence $C_rH_m(X, \mathbb{Z})$ is a divisible group in this range.

Thus, if $L_rH_m(X)$ is finitely generated for $m \geq \dim(X) + r - 1$, then $C_rH_m(X) = 0$ in this range, and hence Suslin's Conjecture holds.

A map $Y \to \mathcal{C}_r(X)$ (for example, an inclusion) determines a cycle

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[dr, "p", "\text{rel. dim. } r"'] & Y \times X \arrow[d] \\ & Y \end{tikzcd}$$

and hence a map on cycles spaces

$$\Gamma_* : \mathcal{Z}_0(Y) \to \mathcal{Z}_r(X)$$

determined by

$$y \mapsto \Gamma_y = p^{-1}(y) \in \mathcal{Z}_r(X).$$

Applying π_{m-2r} gives

$$\Gamma_* : H^{\text{sing}}_{m-2r}(Y(\mathbb{C})) = L_0 H_{m-2r}(Y) \to L_r H_m(X).$$

Lifting elements to singular cohomology

Given an element

$$\alpha \in L_r H_m(X) = \pi_{m-2r} \mathcal{Z}_r(X)$$

it lifts to $\tilde{\alpha} \in H^{\mathsf{sing}}_{m-2r}(\mathcal{C}_{r,e}(X)(\mathbb{C}))$ along the maps

$$H^{\mathsf{sing}}_{m-2r}(\mathcal{C}_{r,e}(X)(\mathbb{C})) \to H^{\mathsf{sing}}_{m-2r}(\mathcal{Z}_r(X)) \twoheadrightarrow \pi_{m-2r} \mathcal{Z}_r(X).$$

Using singular Lefschetz, $\tilde{\alpha}$ lifts to

$$\tilde{\tilde{\alpha}} \in H^{\mathsf{sing}}_{m-2r}(Y(\mathbb{C})) \to L_r H_m(X)$$

for some $Y \subset \mathcal{C}_{r,e}(X)$ with $\dim(Y) \leq m - 2r$.

Lifting along correspondences

Thus we have a surjection

$$\bigoplus_{\dim(Y)\leq m-2r} H^{\mathsf{sing}}_{m-2r}(Y) \twoheadrightarrow L_r H_m(X)$$

where each map $H^{\mathsf{sing}}_{m-2r}(Y) \to L_r H_m(X)$ is the map associated to an equi-dimensional correspondence $\Gamma : Y \dashrightarrow X$ of rel. dim. r:

$$H^{\mathsf{sing}}_{m-2r}(Y) \cong L_0 H_{m-2r}(Y) \xrightarrow{\Gamma_*} L_r H_m(X).$$

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Remark: This surjection can be used to understand MHS on $L_r H_m(X)$.

Lifting to smooth varieties, using Hodge theory

For such a correspondence Γ, the composition

$$H^{\mathsf{sing}}_{m-2r}(Y) \xrightarrow{\Gamma_*} L_r H_m(X) \to H^{\mathsf{sing}}_m(X)$$

coincides with the map on singular cohomology induced by Γ:

$$\Gamma_* : H^{\mathsf{sing}}_{m-2r}(Y) \to H^{\mathsf{sing}}_m(X)$$

When X is smooth, letting $\tilde{Y} \to Y$ be a resolution of singularities, Hodge theory gives:

$$\operatorname{im}\left(H^{\mathsf{sing}}_{m-2r}(Y) \to H^{\mathsf{sing}}_m(X)\right) =$$

$$\operatorname{im}\left(H^{\mathsf{sing}}_{m-2r}(\tilde{Y}) \to H^{\mathsf{sing}}_{m-2r}(Y) \to H^{\mathsf{sing}}_m(X)\right)$$

Characterizing image in singular homology

Proposition (Friedlander-Mazur)

For X smooth and projective, the topological filtration is contained in the "correspondence" filtration: Every element of

$$F_r^{top} H_m^{sing}(X) := \operatorname{im}\left(L_r H_m(X) \to H_m^{sing}(X(\mathbb{C}))\right)$$

is contained in the image of

$$\Gamma_* : H_{m-2r}^{sing}(W(\mathbb{C})) \to H_m^{sing}(X(\mathbb{C}))$$

where W is smooth with $\dim(W) \leq m - 2r$ and Γ is a correspondence.
(Since $H_m^{sing}(X(\mathbb{C}))$ is f.g., a single pair W, Γ suffices.)

Weak form of Suslin's Conjecture

Recall Suslin's conjecture predicts

$$L_r H_m(X) \xrightarrow{\cong} H_m^{\text{sing}}(X(\mathbb{C})), \text{ for } m \geq \dim(X) + r.$$

Conjecture (Weak form of Suslin's Conjecture (or Friedlander-Mazur Conjecture))

For a smooth, projective variety X, the map

$$L_r H_m(X) \to H_m^{\text{sing}}(X(\mathbb{C}))$$

is onto for $m \geq \dim(X) + r$.

In particular, the weak Suslin conjecture predicts:

$$L_{m-\dim(X)} H_m(X) \twoheadrightarrow H_m^{\text{sing}}(X)$$

is onto, for all m. (If $m < \dim(X)$, let $L_{m-\dim(X)} := L_0$.)

Consequence of weak Suslin conjecture

Using the Proposition concerning the lifting of elements of $\mathrm{im}(L_r H_m \to H_m^{\text{sing}})$ along correspondences:

Proposition

Assume the weak form of Suslin's Conjecture holds for X. Let $d = \dim(X)$.
Then for each integer m, there is a smooth, projective variety Y of dim $2d - m$ and a correspondence $\Gamma : Y \dashrightarrow X$ of rel. dim. $m - d$ such that

$$\Gamma_* : H_{2d-m}^{\textit{sing}}(Y(\mathbb{C})) \twoheadrightarrow H_m^{\textit{sing}}(X(\mathbb{C}))$$

is onto.

The existence of such a Y, Γ turns out to be a very strong condition on X. In fact....

Beilinson's Theorem

Theorem (Beilinson)

The validity of all Grothendieck's standard conjectures over $\mathbb{C}$ is equivalent to the following property: For each smooth, projective X, there is a Y and Γ as above such that

$$\Gamma_* : H^{sing}_{2d-m}(Y(\mathbb{C})) \twoheadrightarrow H^{sing}_m(X(\mathbb{C}))$$

is onto.

Corollary (Beilinson)

The weak form of Suslin's Conjecture is equivalent to the validity of all of Grothendieck's standard conjectures over $\mathbb{C}$.

Note: The $\Leftarrow$ direction of the Corollary was originally shown by Friedlander-Mazur.

Should we believe Suslin's Conjecture?

Beilinson's result makes it clear Suslin's Conjecture is a very strong conjecture. Its weak form is equivalent to Grothendieck's standard conjectures.

Perhaps the strong form of Suslin's conjecture is simply false.

The first unknown case occurs for 1-cycles on a smooth projective 3-dimensional variety X:

Question

For a smooth, projective 3-dimensional variety X, is

$$\pi_{m-2}\mathcal{Z}_1(X) =: L_1H_m(X) \to H_m^{sing}(X(\mathbb{C}))$$

one-to-one for $m \geq 3$?

Mixed Hodge structures for Lawson homology

Assume (for simplicity) X is projective. Then we have a surjection

$$\bigoplus_{Y, \dim(Y) \leq m-2r} H^{\text{sing}}_{m-2r}(Y(\mathbb{C})) \twoheadrightarrow L_r H_m(X)$$

of (non f.g.) MHS's (and where the maps are given by correspondences).

Thus, $L_r H_m(X)$ has same Hodge type as H^{sing}_{m-2r} of a union of (highly singular) varieties of dimension $m - 2r$.

MHS for $L_1H_3(X))$

For example,

$$\bigoplus_{Y, \dim(Y)=1} H_1^{\text{sing}}(Y(\mathbb{C})) \twoheadrightarrow L_1H_3(X)$$

and so $L_1H_3(X)$ has Hodge type: $(0,0), (-1,0), (0,-1)$.

If we assume $\dim(X) = 3$, then Suslin's conjecture predicts

$$L_1H_3(X) \rightarrowtail H_3^{\text{sing}}(X(\mathbb{C}), \mathbb{Z}(1)) \cong H^3_{\text{sing}}(X, \mathbb{Z}(2))$$

and the target has Hodge type $(-1,0), (0,-1)$.

A Conjecture concerning Hodge type

Conjecture

For X smooth, projective of dimension 3, the Lawson group $L_1H_3(X)$ has Hodge type $(-1,0),(0,-1)$.

A proof (or counter-example) of just this conjecture would represent highly significant progress.

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Note that the validity of this conjecture implies:

Conjecture

For X smooth, projective of dimension 3, the map

$$H_3^{\mathcal{M}}(X,\mathbb{Z}(1)) \to L_1H_3(X) \quad \textit{is a torsion map.}$$

$\dim(X) = 3$, $Y \subset X$, $\dim(Y) = 2$, $U = X \setminus Y$

$$\begin{tikzcd}
L_1H_4(Y) \arrow[r, "\cong", "\dim(Y)=2"'] \arrow[d] & H_4^{\mathsf{sing}}(Y) \arrow[d] \\
L_1H_4(X) \arrow[r, "\mathsf{SuslinConj} \Rightarrow \cong", "\mathsf{GSConj} \Rightarrow \mathsf{onto}"'] \arrow[d] & H_4^{\mathsf{sing}}(X) \arrow[d] \\
L_1H_4(U) \arrow[r, "\mathsf{SuslinConj} \Rightarrow \cong"] \arrow[d] & H_4^{\mathsf{sing}}(U) \arrow[d] \\
L_1H_3(Y) \arrow[r, "\cong", "\dim(Y)=2"'] \arrow[d] & H_3^{\mathsf{sing}}(Y) \arrow[d] \\
L_1H_3(X) \arrow[r, "\mathsf{SuslinConj} \Rightarrow \text{1-1}"] \arrow[d] & H_3^{\mathsf{sing}}(X) \arrow[d] \\
L_1H_3(U) \arrow[r] & H_3^{\mathsf{sing}}(U)
\end{tikzcd}$$

Passing to function fields

Let $\mathbb{C}(X) := \varinjlim_{U \subset X} U$.

Proposition

$$L_r H_m(\mathbb{C}(X)) = 0 \text{ if } m < d + r.$$

For example, $L_1 H_3(\operatorname{Spec} \mathbb{C}) = 0$ *if* $\dim(X) = 3$.

Assuming Grothendieck Standard Conjectures:

$$L_1 H_4(\mathbb{C}(X)) \to H_4^{\mathrm{sing}}(\mathbb{C}(X)) \xrightarrow{\text{SuslinC} \Rightarrow 0} L_1 H_3(X) \to H_3^{\mathrm{sing}}(X).$$

Challenge

Find a good method of describing/constructing elements of

$$H_m^{\mathrm{sing}}(\mathbb{C}(X)) = H^{2d-m}_{\mathrm{sing}}(\mathbb{C}(X)).$$

Toy examples

- With finite coefficients:

$$\begin{aligned} H_m^{\text{sing}}(\mathbb{C}(X), \mathbb{Z}/n) &= L_{m-\dim(X)} H_m(\mathbb{C}(X), \mathbb{Z}/n) \\ &= H_{\mathcal{M}}^{2d-m}(\mathbb{C}(X), \mathbb{Z}/n(2d-m)) \\ &= K_{2d-m}^{\text{Milnor}}(\mathbb{C}(X))/n \end{aligned}$$

- For H^1_{sing}:

$$\varinjlim_Y H^1_{\text{sing},Y}(X) \to H^1_{\text{sing}}(X) \to H^1_{\text{sing}}(\mathbb{C}(X)) \to \varinjlim_Y H^2_{\text{sing},Y}(X) \to H^2_{\text{sing}}(X)$$

Since $H^1_{\text{sing},Y}(X) = 0$ and $H^2_{\text{sing},Y}(X) \cong H^{\text{sing}}_{2\dim(Y)}(Y) =$ free abelian group on integral components of Y:

$$0 \to H^1_{\text{sing}}(X) \to H^1_{\text{sing}}(\mathbb{C}(X)) \to Z^1(X)_{\text{hom}\sim 0} \to 0$$

The End

Thanks for your attention!