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A PROOF OF THE ABC CONJECTURE AFTER MOCHIZUKI

By

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Abstract

We give a survey of S. Mochizuki's ingenious inter-universal Teichmüller theory and explain how it gives rise to Diophantine inequalities. The exposition was designed to be as self-contained as possible.

Contents

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§ 0. Introduction.

The author once heard the following observation, which was attributed to Grothendieck: There are two ways to crack a nut — one is to crack the nut in a single stroke by using a nutcracker; the other is to soak it in water for an extended period of time until its shell *dissolves naturally.* Grothendieck's mathematics may be regarded as an example of the latter approach.

In a similar vein, the author once heard a story about a mathematician who asked an expert on étale cohomology what the *main point* was in the ℓ-adic (not the p-adic) proof of the rationality of the congruence zeta function. The expert was able to recall, on the one hand, that the Lefschetz trace formula was proved by checking various commutative diagrams and applying various base change theorems (e.g., for proper or smooth morphisms). On the other hand, *neither* the commutativity of various diagrams *nor* the various base change theorems could be described as the *main point* of the proof. Ultimately, the expert was not able to point out precisely what the *main point* in the proof was. From the point of view of the author, the main point of the proof seems to lie in the *establishment of a suitable framework* (i.e., scheme theory and étale cohomology theory) in which the Lefschetz trace formula, which was already well known in the field of algebraic topology, could be formulated and proved even over fields of positive characteristic.

A similar statement can be made concerning S. Mochizuki's proof of the *abc* Conjecture. Indeed, once the reader admits the main results of the preparatory papers (especially [AbsTopIII], [EtTh]), the numerous constructions in the series of papers [IUTchI], [IUTchII], [IUTchIII], [IUTchIV] on inter-universal Teichmüller theory are likely to strike the reader as being somewhat trivial. On the other hand, the way in which the main results of the preparatory papers are *interpreted* and *combined* in order to perform these numerous constructions is *highly nontrivial* and based on very delicate considerations (cf. Remark 9.6.2 and Remark 12.8.1) concerning, for instance, the notions of multiradiality and uniradiality (cf. Section 11.1). Moreover, when taken together, these numerous trivial constructions, whose exposition occupies literally hundreds of pages, allow one to conclude a *highly nontrivial consequence* (i.e., the desired Diophantine inequality) practically effortlessly! Again, from the point of view of the author, the point of the proof seems to lie in the *establishment of a suitable framework* in which one may deform the structure of a number field by abandoning the framework of conventional scheme theory and working instead in the framework furnished by inter-universal Teichmüller theory (cf. also Remark 1.15.3).

In fact, the main results of the preparatory papers [AbsTopIII], [EtTh], etc. are also obtained, to a substantial degree, as consequences of numerous constructions that are not so difficult. On the other hand, the *discovery of the ideas and insights* that underlie these constructions may be regarded as highly nontrivial in content. Examples of such ideas and insights include the "hidden endomorphisms" that play a central role in the mono-anabelian reconstruction algorithms of Section 3.2, the notions of arithmetically holomorphic structure and mono-analytic structure (cf. Section 3.5), and the distinction between étale-like and Frobenius-like objects (cf. Section 4.3). Thus, in summary, it seems to the author that, if one *ignores* the delicate considerations that occur in the course of interpreting and combining the main results of the preparatory papers, together with the ideas and insights that underlie the theory of these preparatory papers, then, in some sense, the only nontrivial mathematical ingredient in inter-universal Teichmüller theory is the classical result [pGC], which was already known in the last century!

A more technical introduction to the mathematical content of the main ideas of inter-universal Teichmüller theory may be found in Appendix A and the discussion at the beginning of Section 13.

The following results are consequences of inter-universal Teichmüller theory (cf. Section 1.1 for more details on the notation):

Theorem 0.1. (Vojta's Conjecture [Voj] for Curves, [IUTchIV, Corollary 2.3]) *Let X be a proper, smooth, geometrically connected curve over a number field; $D \subset X$ a reduced divisor; $U_X := X \setminus D$. Write ω_X for the canonical sheaf on X. Suppose that U_X is a hyperbolic curve, i.e.,* $\deg(\omega_X(D)) > 0$. *Then for any $d \in \mathbb{Z}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$, we have*

$$\mathrm{ht}_{\omega_X(D)} \lesssim (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)$$

on $U_X(\overline{\mathbb{Q}})^{\leq d}$.

Corollary 0.2. (The *abc* Conjecture of Masser and Oesterlé [Mass1], [Oes]) *For any $\epsilon \in \mathbb{R}_{>0}$, we have*

$$\max\{|a|,|b|,|c|\} \leq \left(\prod_{p|abc} p\right)^{1+\epsilon}$$

for all but finitely many coprime $a, b, c \in \mathbb{Z}$ with $a + b = c$.

Proof. We apply Theorem 0.1 in the case where $X = \mathbb{P}^1_{\mathbb{Q}} \supset D = \{0, 1, \infty\}$, and $d = 1$. Thus, we have $\omega_{\mathbb{P}^1}(D) = \mathcal{O}_{\mathbb{P}^1}(1)$, $\text{log-diff}_{\mathbb{P}^1}(-a/b) = 0$, $\text{log-cond}_{\{0,1,\infty\}}(-a/b) = \sum_{p|a,b,a+b} \log p$, and $\mathrm{ht}_{\mathcal{O}_{\mathbb{P}^1}(1)}(-a/b) \approx \log\max\{|a|,|b|\} \approx \log\max\{|a|,|b|,|a+b|\}$ for coprime $a, b \in \mathbb{Z}$ with $b \neq 0$, where the first "$\approx$" follows from [Silv1, Proposition 7.2], and we apply the inequality $|a+b| \leq 2\max\{|a|,|b|\}$. Now let $\epsilon, \epsilon' \in \mathbb{R}_{>0}$ be such that $\epsilon > \epsilon'$. According to Theorem 0.1, there exists $C \in \mathbb{R}$ such that $\log\max\{|a|,|b|,|c|\} \leq (1+\epsilon')\sum_{p|abc}\log p + C$ for any coprime $a, b, c \in \mathbb{Z}$ with $a+b = c$. Observe that there are only finitely many triples $a, b, c \in \mathbb{Z}$ with $a+b = c$ such that $\log\max\{|a|,|b|,|c|\} \leq \frac{1+\epsilon}{\epsilon-\epsilon'}C$. Thus, we have $\log\max\{|a|,|b|,|c|\} \leq (1+\epsilon')\sum_{p|abc}\log p + \frac{\epsilon-\epsilon'}{1+\epsilon}\log\max\{|a|,|b|,|c|\}$ for all but finitely many coprime triples $a, b, c \in \mathbb{Z}$ with $a + b = c$. This completes the proof of Corollary 0.2. $\square$

§ 0.1. Un Fil d'Ariane.

By combining a relative anabelian result (a relative version of the Grothendieck Conjecture over sub-p-adic fields (Theorem B.1)) and the "hidden endomorphism" diagram (EllCusp) (resp. the "hidden endomorphism" diagram (BelyiCusp)), one obtains a(n) (absolute) mono-anabelian result, i.e., the *elliptic cuspidalisation* (Theorem 3.7) (resp. the *Belyi cuspidalisation* (Theorem 3.8)). Then, by applying Belyi cuspidalisations, one obtains a mono-anabelian reconstruction algorithm for the NF-portion of the base field and function field of a hyperbolic curve of strictly Belyi type over a sub-p-adic field (Theorem 3.17), as well as a mono-anabelian reconstruction algorithm for the base field of a hyperbolic curve of strictly Belyi type over a mixed characteristic local field (Corollary 3.19). This motivates the philosophy of mono-analyticity and arithmetic holomorphicity (Section 3.5), as well as the theory of Kummer isomorphisms from Frobenius-like objects to étale-like objects (cf. Remark 9.6.1).

The theory of Aut-holomorphic (orbi)spaces and related reconstruction algorithms (Section 4) is an Archimedean analogue of the mono-anabelian reconstruction algorithms discussed above and yields another application of the technique of elliptic cuspidalisation. On the other hand, the Archimedean theory does not play a very central role in inter-universal Teichmüller theory.

The theory of the étale theta function centers around the establishment of various rigidity properties of mono-theta environments. One applies the technique of elliptic cuspidalisation to show the *constant multiple rigidity* of a mono-theta environment (Theorem 7.23 (3)). The *cyclotomic rigidity* of a mono-theta environment is obtained as a consequence of the ("precisely") quadratic structure of a Heisenberg group (Theorem 7.23 (1)). Finally, by applying the "at most" quadratic structure of a Heisenberg group (and excluding the algebraic section in the definition of a mono-theta environment), one shows the *discrete rigidity* of a mono-theta environment (Theorem 7.23 (2)).

The notions of *étale-like and Frobenius-like objects* play a very important role in inter-universal Teichmüller theory (cf. Section 4.3). The significance of Frobenius-like objects (cf. the theory of Frobenioids, as discussed in Section 8) lies in the fact they allow one to construct links, or "walls", such as the Θ-link and $\mathfrak{log}$-link (cf. Definition 10.8; Corollary 11.24 (3); Definition 13.9 (2); Definition 12.1 (1), (2); and Definition 12.3). (The main theorems of the theory of Frobenioids concern category-theoretic reconstruction algorithms; however, these algorithms do not play a very central role in inter-universal Teichmüller theory (cf. [IUTchI, Remark 3.2.1 (ii)]).) By contrast, the significance of étale-like objects lies in the fact that they allow one to penetrate these walls (cf. Remark 9.6.1).

The notion of *multiradiality* plays a central role in inter-universal Teichmüller theory (cf. Section 11.1). The significance of the multiradial algorithms that are ultimately established lies in the fact that they allow one to

> "*permute*" (up to mild indeterminacies) the theta values in the source of the Θ-link and the theta values in the target of the Θ-link.

In other words, multiradiality makes it possible to "*see*" (up to mild indeterminacies) the "alien" ring structure on one side of Θ-link from the point of view of the ring structure on the other side (cf. the discussion at the beginning of Section 13). This multiradiality, together with the compatibility of the algorithms under consideration with the Θ-link, will, ultimately, lead to the desired height estimate (cf. Remark 11.1.1).

The multiradial algorithm that we ultimately wish to establish consists, roughly speaking, of three main objects (cf. the column labelled "(1)" of the table before Corollary 13.13): (mono-analytic étale-like) *log-shells* (which are related to the *local units* of the number fields under consideration) equipped with log-volume functions (cf. Section 5), *theta values* (which are related to the *local value groups* of the number fields under consideration) acting on these log-shells, and (*global*) *number fields* acting on these log-shells. In this context, the theta function (resp. κ-coric functions (Definition 9.2)) serve(s) as a geometric container for the theta values (resp. the number fields) just mentioned and allow(s) one to establish the multiradiality of the reconstruction algorithms under consideration. Here, suitable versions of Kummer theory for the theta function and κ-coric functions allow one to relate the respective étale-like objects and Frobenius-like objects under consideration. These versions of Kummer theory depend on suitable versions of cyclotomic rigidity.

The cyclotomic rigidity of mono-theta environments discussed above allows one to perform Kummer theory for the theta function in a multiradial manner (Proposition 11.4, Theorem 12.7, Corollary 12.8). Similarly, a certain version of cyclotomic rigidity that is deduced from the elementary fact $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\}$ (Definition 9.6) allows one to perform Kummer theory for κ-coric functions in a multiradial manner. At a more concrete level, the cyclotomic rigidity of mono-theta environments and κ-coric functions plays the role of protecting the Kummer theory surrounding the theta function and κ-coric functions from the $\widehat{\mathbb{Z}}^\times$-indeterminacies that act on the local units and hence ensures the compatibility of the Θ-link with the portion of the final multiradial algorithm that involves the Kummer theory surrounding the theta function and κ-coric functions (cf. the column labelled "(3)" of the table before Corollary 13.13). By contrast, the most classical version of cyclotomic rigidity, which is deduced from local class field theory for MLF's (cf. Section 0.2), does not yield a multiradial algorithm (cf. Remark 11.4.1, Proposition 11.15 (2), and Remark 11.17.2 (2)).

The Kummer theory discussed above for mono-theta environments and theta functions (resp. for κ-coric functions) leads naturally to the theory of *Hodge-Arakelov-theoretic evaluation* (resp. the NF-counterpart (cf. Section 0.2) of the theory of Hodge-

Arakelov-theoretic evaluation) and the construction of Gaussian monoids, i.e., in essence, monoids generated by theta values (Section 11.2) (resp. the construction of elements of number fields (Section 9.2, Section 11.4)). In the course of performing Hodge-Arakelov-theoretic evaluation at the bad primes, one applies a certain consequence of the theory of semi-graphs of anabelioids ("*profinite conjugates vs. tempered conjugates*" Theorem 6.11). The reconstruction of mono-theta environments from (suitable types of) topological groups (Corollary 7.22 (2) "$\Pi \mapsto \mathbb{M}$") and tempered Frobenioids (Theorem 8.14 "$\mathcal{F} \mapsto \mathbb{M}$"), together with the discrete rigidity of mono-theta environments, allows one to derive Frobenioid-theoretic versions of the group-theoretic versions of Hodge-Arakelov evaluation and the construction of Gaussian monoids just described (Corollary 11.17). In the course of performing Hodge-Arakelov-theoretic evaluation, one applies the $\mathbb{F}_l^{\rtimes\pm}$-symmetry in the Hodge theatres under consideration (Section 10.5) to *synchronise the conjugacy indeterminacies* that occur (Corollary 11.16 (1)). The theory of synchronisation of conjugacy indeterminacies makes it possible to construct "good diagonals", which give rise, in the context of the log-theta-lattice, to horizontally coric objects.

By combining the construction of Gaussian monoids just discussed with the theory of $\mathfrak{log}$-links, one obtains LGP-monoids (Proposition 13.6). Here, it is of interest to observe that this construction of LGP-monoids makes use of the compatibility of the cyclotomic rigidity of mono-theta environments with the profinite topology, which is closely related to the isomorphism class compatibility of mono-theta environments (cf. Remark 9.6.2 (5)). LGP-monoids are equipped with natural canonical splittings, which arise, via canonical splittings of theta monoids (i.e., in essence, monoids generated by theta functions), from the constant multiple rigidity of mono-theta environments (Proposition 11.7, Proposition 13.6).

The theory of $\mathfrak{log}$-*links* and *log-shells*, both of which are closely related to the *local units* of number fields under consideration (Section 5, Section 12), together with the Kummer theory that relates corresponding Frobenius-like and étale-like versions of objects, gives rise to the $\mathfrak{log}$-Kummer correspondences for the theta values (which are related to the *local value groups* of the number fields under consideration) and (global) *number fields* under consideration (Proposition 13.7 and Proposition 13.11). The canonical splittings of LGP-monoids discussed above may be interpreted, in the context of the $\mathfrak{log}$-Kummer correspondence, as a non-interference property (Proposition 13.7 (2c)) of the LGP-monoids, while the classical fact $F_{\mathrm{mod}}^{\times} \cap \prod_{v \leq \infty} O_v = \mu(F_{\mathrm{mod}}^{\times})$ may be interpreted, in the context of the $\mathfrak{log}$-Kummer correspondence, as a non-interference property (Proposition 13.11 (2)) of the number fields involved (cf. the column labelled "(2)" of the table before Corollary 13.13).

By passing from arithmetically holomorphic structures to the underlying mono-analytic structures, admitting three kinds of mild indeterminacies, and applying the non-interefence properties of $\mathfrak{log}$-Kummer correspondences, one obtains the *multiradiality* of the final multiradial algorithm (Theorem 13.12). In the final multiradial algorithm, we use the *theta values* (which are related to the *local value groups*) to construct the Θ-pilot objects (Definition 13.9 (1)), (global) *number fields* to relate (global Frobenioids arising from) $\boxtimes$-line bundles to (global Frobenioids arising from) $\boxplus$-line bundles (cf. Section 9.3), and *log-shells* (which arise from the *local units*) as mono-analytic containers for theta values and (global) number fields.

Since the *labels* attached to the theta values depend on the arithmetically holomorphic structure, one cannot, *a priori*, transport these labels from one side of a Θ-link to the other side of the Θ-link. On the other hand, by using *processions*, one can reduce the indeterminacy that arises from forgetting these labels (cf. Remark 13.1.1).

Finally, by combining the multiradiality of the final multiradial algorithm with the compatibility of this algorithm with the Θ-link, the compatibility of the log-volumes with the $\mathfrak{log}$-links (Section 5), and various properties concerning global Frobenioids, we obtain an upper bound for the height of the given elliptic curve (Corollary 13.13, cf. Remark 13.13.2). The fact that the leading term of the upper bound is of the expected form may be regarded as a consequence of a certain calculation in Hodge-Arakelov theory (Remark 1.15.3 (the *"miracle identity"*)).

Leitfaden

$$
\begin{tikzcd}
\text{§2. Prel. Anab.} \arrow[r] \arrow[d] \arrow[rrrd, bend left] & \text{§6. Prel. Temp.} \arrow[d] \arrow[dd, bend left] & & \\
\text{§3. Mono-anab.} \arrow[r] \arrow[d] \arrow[dr] \arrow[ddr] \arrow[rrr, bend left] & \text{§7. Ét. Theta} \arrow[d] \arrow[ddl] & \text{§4. Aut-hol.} \arrow[r] \arrow[dll] & \text{§5. Log-vol./-sh.} \arrow[dl] \\
\text{§10. Hodge Th.} \arrow[r] & \text{§11. H-A. Eval.} \arrow[r] & \text{§12. Log-link} \arrow[r] & \text{§13. Mlt. Alg'm} \arrow[d] \\
\text{§8. Fr'ds} \arrow[r] \arrow[u] & \text{§9. Prel. NF-Eval.} \arrow[u] & \text{§1. Gen. Arith.} \arrow[r] & \text{Thm. 0.1}
\end{tikzcd}
$$

The above dependences are rough (or conceptual) relations. For example, we use some portions of §7 and §9 in the constructions in §10; however, conceptually, §7 and §9 are mainly used in §11, and so on.

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§ 0.2. Notation.

General Notation:

For a finite set A, we write $\#A$ for the cardinality of A. For a group G and a subgroup $H \subset G$ of finite index, we write $[G : H]$ for $\#(G/H)$. (For a finite extension of fields $K \supset F$, we also write $[K : F]$ for $\dim_F K$. This will not result in any confusion between the notations "$[G : H]$" and "$[K : F]$".) For a function f on a set X and a subset $Y \subset X$, we write $f|_Y$ for the restriction of f to Y.

[1]Ivan Fesenko wrote, in the published version of his survey "Arithmetic deformation theory via arithmetic fundamental groups and nonarchimedean theta-functions, notes on the work of Shinichi Mochizuki", that he encouraged the author to learn and scrutinise arithmetic deformation theory subsequent to his meeting with Mochizuki in mid-September 2012. In fact, the author had already sent an email to Mochizuki on the 1st of September 2012, in which the author expressed his interest in studying inter-universal Teichmüller theory.

[2]In particular, the author began his study of inter-universal Teichmüller theory *of his own will*. In the latest version of Fesenko's survey (posted on Fesenko's web site subsequent to the publication of the published version of the survey), Fesenko replaced the expression "encouraged Yamashita" by the expression "supported his interest to study the theory".

For a prime number $l > 2$, we write $\mathbb{F}_l^{\circledast} := \mathbb{F}_l^{\times}/\{\pm 1\}$, $\mathbb{F}_l^{\rtimes\pm} := \mathbb{F}_l \rtimes \{\pm 1\}$, where $\{\pm 1\}$ acts on $\mathbb{F}_l$ by multiplication, and $|\mathbb{F}_l| := \mathbb{F}_l/\{\pm 1\} = \mathbb{F}_l^{\circledast} \coprod \{0\}$. We also write $l^{\circledast} := \frac{l-1}{2} = \#\mathbb{F}_l^{\circledast}$ and $l^{\pm} := l^{\circledast} + 1 = \frac{l+1}{2} = \#|\mathbb{F}_l|$.

Categories:

For an object A in a category, we shall call an object isomorphic to A an **isomorph** of A.

For a category $\mathcal{C}$ and a filtered ordered set $I \neq \emptyset$, we write pro-$\mathcal{C}_I$(= pro-$\mathcal{C}$) for the category of pro-objects of $\mathcal{C}$ indexed by I, i.e., whose objects are of the form $((A_i)_{i\in I}, (f_{i,j})_{i<j\in I})$ (= $(A_i)_{i\in I}$), where A_i is an object in $\mathcal{C}$, and $f_{i,j}$ is a morphism $A_j \to A_i$ satisfying $f_{i,j}f_{j,k} = f_{i,k}$ for any $i < j < k \in I$, and whose morphisms are given by $\mathrm{Hom}_{\text{pro-}\mathcal{C}}((A_i)_{i\in I}, (B_j)_{j\in I}) := \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathcal{C}}(A_i, B_j)$. We also regard objects of $\mathcal{C}$ as objects of pro-$\mathcal{C}$ (by setting every transition morphism to be identity). Thus, relative to this convention, we have $\mathrm{Hom}_{\text{pro-}\mathcal{C}}((A_i)_{i\in I}, B) = \varinjlim_i \mathrm{Hom}_{\mathcal{C}}(A_i, B)$.

For a category $\mathcal{C}$, we write $\mathcal{C}^0$ for the full subcategory of connected objects, i.e., the non-initial objects which are not isomorphic to a coproduct of two non-initial objects of $\mathcal{C}$. We write $\mathcal{C}^{\top}$ (resp. $\mathcal{C}^{\perp}$) for the category whose objects are formal (possibly empty) countable (resp. finite) coproducts of objects in $\mathcal{C}$, and whose morphisms are given by $\mathrm{Hom}_{\mathcal{C}^{\top}\text{ (resp. }\mathcal{C}^{\perp})}(\coprod_i A_i, \coprod_j B_j) := \prod_i \coprod_j \mathrm{Hom}_{\mathcal{C}}(A_i, B_j)$ (cf. [SemiAnbd, §0]).

Let $\mathcal{C}_1, \mathcal{C}_2$ be categories. We say that two isomorphism classes of functors $f : \mathcal{C}_1 \to \mathcal{C}_2$, $f' : \mathcal{C}_1' \to \mathcal{C}_2'$ are **abstractly equivalent** if there exist isomorphisms $\alpha_1 : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_1'$, $\alpha_2 : \mathcal{C}_2 \xrightarrow{\sim} \mathcal{C}_2'$ such that $f' \circ \alpha_1 = \alpha_2 \circ f$.

Let $\mathcal{C}$ be a category. We define a **poly-morphism** $A \overset{\text{poly}}{\to} B$ for $A, B \in \mathrm{Ob}(\mathcal{C})$ to be a (possibly empty) set of morphisms $A \to B$ in $\mathcal{C}$. A poly-morphism for which each constituent morphism is an isomorphism will be called a **poly-isomorphism**. If $A = B$, then a poly-isomorphism $A \overset{\text{poly}}{\xrightarrow{\sim}} B$ will be called a **poly-automorphism**. We define the **full poly-isomorphism** $A \overset{\text{full poly}}{\xrightarrow{\sim}} B$ to be the set of all isomorphisms $A \xrightarrow{\sim} B$. We define the composite of poly-morphisms $\{f_i : A \to B\}_{i\in I}$ and $\{g_j : B \to C\}_{j\in J}$ to be $\{g_j \circ f_i : A \to C\}_{(i,j)\in I\times J}$. We define a **poly-action** to be an action via poly-automorphisms.

Let $\mathcal{C}$ be a category. We define a **capsule** of objects of $\mathcal{C}$ to be a finite collection $\{A_j\}_{j\in J}$ of objects of $\mathcal{C}$. We shall also refer to $\{A_j\}_{j\in J}$ as a **$\#J$-capsule**. We define a **morphism** $\{A_j\}_{j\in J} \to \{A'_{j'}\}_{j'\in J'}$ between **capsules** of objects of $\mathcal{C}$ to be a collection of data $(\iota, (f_j)_{j\in J})$ consisting of an injection $\iota : J \hookrightarrow J'$ and a morphism $f_j : A_j \to A'_{\iota(j)}$ in $\mathcal{C}$ for each $j \in J$. (Thus, the capsules of objects of $\mathcal{C}$ and the morphisms between capsules of objects of $\mathcal{C}$ form a category.) We define a **capsule-full poly-morphism** to be a poly-morphism

$$\Big\{\{f_j : A_j \xrightarrow{\sim} A'_{\iota(j)}\}_{j\in J}\Big\}_{(f_j)_{j\in J}\in\prod_{j\in J}\mathrm{Isom}_{\mathcal{C}}(A_j,A'_{\iota(j)})} \ (=\prod_{j\in J}\mathrm{Isom}_{\mathcal{C}}(A_j,A'_{\iota(j)}))$$

in the category of the capsules of objects of $\mathcal{C}$, associated with a fixed injection $\iota : J \hookrightarrow J'$. If the fixed ι is a bijection, then we shall refer to the capsule-full poly-morphism as a **capsule-full poly-isomorphism**.

Number Fields and Local Fields:

In this survey, we define a **number field** to be a finite extension of $\mathbb{Q}$ (i.e., we exclude infinite extensions). We define a mixed characteristic (or non-Archimedean) **local field** to be a finite extension of $\mathbb{Q}_p$ for some p. We use the abbreviations NF for "number field", MLF for "mixed characteristic local field", and CAF for "complex Archimedean field" (i.e., a topological field isomorphic to $\mathbb{C}$). For a topological field k which is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, we write $|-|_k : k \to \mathbb{R}_{\geq 0}$ for the absolute value associated to k, i.e., the unique continuous map such that the restriction of $|-|_k$ to $k^\times$ determines a homomorphism $k^\times \to \mathbb{R}_{>0}$ with respect to the multiplicative structures of $k^\times$ and $\mathbb{R}_{>0}$, and $|n|_k = n$ for $n \in \mathbb{Z}_{\geq 0}$. We write $\pi \in \mathbb{R}$ for the mathematical constant pi (i.e., $\pi = 3.14159\cdots$).

For a number field F, we write $\mathbb{V}(F)$ for the set of equivalence classes of valuations of F and $\mathbb{V}(F)^{\mathrm{arc}} \subset \mathbb{V}(F)$ (resp. $\mathbb{V}(F)^{\mathrm{non}} \subset \mathbb{V}(F)$) for the subset of Archimedean (resp. non-Archimedean) equivalence classes of valuations. For number fields $F \subset L$ and $v \in \mathbb{V}(F)$, we write $\mathbb{V}(L)_v := \mathbb{V}(L) \times_{\mathbb{V}(F)} \{v\}$ $(\subset \mathbb{V}(L))$, where $\mathbb{V}(L) \twoheadrightarrow \mathbb{V}(F)$ is the natural surjection. For $v \in \mathbb{V}(F)$, we write F_v for the completion of F with respect to v. For $v \in \mathbb{V}(F)^{\mathrm{non}}$ (resp. $v \in \mathbb{V}(F)^{\mathrm{arc}}$), we write p_v for the characteristic of the residue field (resp. $p_v := e = 2.71828\cdots$) and f_v for the extension degree of the residue field at v over $\mathbb{F}_{p_v}$ (resp. $f_v := 1$). We write $\mathfrak{m}_v$ for the maximal ideal and ord_v for the valuation normalised by $\mathrm{ord}_v(p_v) = 1$ for $v \in \mathbb{V}(F)^{\mathrm{non}}$. We normalise $v \in \mathbb{V}(F)^{\mathrm{non}}$ by $v(\text{uniformiser of } F_v) = 1$. (Thus, $v(-) = e_v \cdot \mathrm{ord}_v(-)$, where we write e_v for the ramification index of F_v over $\mathbb{Q}_{p_v}$.) We shall write ord for ord_v when there is no fear of confusion. For $v \in \mathbb{V}^{\mathrm{arc}}$, we write $|-|_v := |-|_{F_v}$.

For a non-Archimedean (resp. complex Archimedean) local field k, we write O_k for the valuation ring (resp. the *subset* of elements of absolute value ≤ 1) of k, $O_k^\times \subset O_k$ for the subgroup of units (resp. the subgroup of units, i.e., elements of absolute value $= 1$), and $O_k^\triangleright := O_k \setminus \{0\} \subset O_k$ for the multiplicative topological monoid of non-zero elements of O_k. We shall also refer to O_k as the subset of **integral elements** of k. When k is a non-Archimedean local field, we shall write $\mathfrak{m}_k$ for the maximal ideal of

O_k.

For a non-Archimedean local field K with residue field k, and an algebraic closure $\overline{k}$ of k, we write $\mathrm{Frob}_K \in \mathrm{Gal}(\overline{k}/k)$ or $\mathrm{Frob}_k \in \mathrm{Gal}(\overline{k}/k)$ for the (arithmetic) Frobenius element, i.e., the map $\overline{k} \ni x \mapsto x^{\#k} \in \overline{k}$. (Note that in this survey, neither the term "Frobenius element", the notation Frob_K, nor the notation Frob_k will be used to refer to the geometric Frobenius morphism, i.e., the map $\overline{k} \ni x \mapsto x^{1/\#k} \in \overline{k}$.)

Topological Groups and Topological Monoids:

For a Hausdorff topological group G, we write $(G \twoheadrightarrow)\, G^{\mathrm{ab}}$ for the abelianisation of G as a Hausdorff topological group, i.e., the quotient of G by the *closure of* the commutator subgroup of G, and $G_{\mathrm{tors}}\, (\subset G)$ for the subset of torsion elements in G.

For a commutative topological monoid M, we write $(M \to)M^{\mathrm{gp}}$ for the groupification of M (i.e., the coequaliser of the diagonal homomorphism $M \to M \times M$ and the zero-homomorphism), $M_{\mathrm{tors}}(\subset M)$ for the subgroup of torsion elements of M, $M^{\times}(\subset M)$ for the subgroup of invertible elements of M, and $(M \to)M^{\mathrm{pf}}$ for the perfection of M (i.e., the inductive limit $\varinjlim_{n \in \mathbb{N}_{\geq 1}} M$, where the index set $\mathbb{N}_{\geq 1}$ is equipped with the order structure determined by divisibility, and the transition map from the copy of M at n to the copy of M at m is given by multiplication by m/n).

For a Hausdorff topological group G, and a closed subgroup $H \subset G$, we write

$$
\begin{aligned}
&Z_G(H) := \{g \in G \mid gh = hg, \forall h \in H\}\,, \\
&\subset N_G(H) := \left\{g \in G \mid gHg^{-1} = H\right\}, \text{ and} \\
&\subset C_G(H) := \left\{g \in G \mid gHg^{-1} \cap H \text{ has finite index in } H,\ gHg^{-1}\right\}
\end{aligned}
$$

for the centraliser, normaliser, and commensurator of H in G, respectively. (Note that $Z_G(H)$ and $N_G(H)$ are always closed in G; however, $C_G(H)$ is not necessarily closed in G (cf. [AbsAnab, Section 0], [Anbd, Section 0]).) If $H = N_G(H)$ (resp. $H = C_G(H)$), then we shall say that H is **normally terminal** (resp. **commensurably terminal**) in G. (Thus, if H is commensurably terminal in G, then H is normally terminal in G.)

For a group G, we write $\mathrm{Inn}(G)$ $(\subset \mathrm{Aut}(G))$ for the group of inner automorphisms of G and $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$. We call $\mathrm{Out}(G)$ the group of outer automorphisms of G. Let G be a group with $Z_G(G) = \{1\}$. Then $G \to \mathrm{Inn}(G)$ $(\subset \mathrm{Aut}(G))$ is injective, and we have an exact sequence $1 \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$. If $f : H \to \mathrm{Out}(G)$ is a homomorphism of groups, we write $G \overset{\mathrm{out}}{\rtimes} H \twoheadrightarrow H$ for the pull-back of $\mathrm{Aut}(G) \twoheadrightarrow \mathrm{Out}(G)$

with respect to f:

$$\begin{tikzcd} 1 \arrow[r] & G \arrow[r] & \mathrm{Aut}(G) \arrow[r] & \mathrm{Out}(G) \arrow[r] & 1 \\ 1 \arrow[r] & G \arrow[r] \arrow[u, "="'] & G \overset{\mathrm{out}}{\rtimes} H \arrow[r] \arrow[u] & H \arrow[r] \arrow[u, "f"] & 1. \end{tikzcd}$$

We shall call $G \overset{\mathrm{out}}{\rtimes} H$ the **outer semi-direct product** of H with G with respect to f. (Note that $G \overset{\mathrm{out}}{\rtimes} H$ is *not necessarily* naturally isomorphic to a semi-direct product.) When G is a compact Hausdorff topological group, then we equip $\mathrm{Aut}(G)$ with the compact open topology and $\mathrm{Inn}(G)$, $\mathrm{Out}(G)$ with the induced topology. If, moreover, H is a topological group, and f is a continuous homomorphism, then we equip with $G \overset{\mathrm{out}}{\rtimes} H$ the induced topology.

Curves:

For a field K, we write $U_{\mathbb{P}^1} = U_{\mathbb{P}^1_K} := \mathbb{P}^1_K \setminus \{0, 1, \infty\}$. We shall call an algebraic curve over K that is isomorphic to $U_{\mathbb{P}^1_K}$ over K a **tripod** over K. We write $\mathcal{M}_{\mathrm{ell}} \subset \overline{\mathcal{M}_{\mathrm{ell}}}$ for the fine moduli stack of elliptic curves and its canonical compactification.

If X is a generically scheme-like algebraic stack over a field k which has a finite étale Galois covering $Y \to X$, where Y is a hyperbolic curve over a finite extension of k, then we call X a **hyperbolic orbicurve** over k ([AbsTopI, §0]).

Cyclotomes:

For a field K of characteristic 0 and a separable closure $\overline{K}$ of K, we write $\mu_{\widehat{\mathbb{Z}}}(\overline{K}) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \overline{K}^{\times})$ and $\mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K}) := \mu_{\widehat{\mathbb{Z}}}(\overline{K}) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}/\mathbb{Z}$. Note that $\mathrm{Gal}(\overline{K}/K)$ acts naturally on both. We shall use the term **cyclotome (associated to $\overline{K}$)** to refer to any of the following objects: $\mu_{\widehat{\mathbb{Z}}}(\overline{K})$, $\mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K})$, $\mu_{\mathbb{Z}_l}(\overline{K}) := \mu_{\widehat{\mathbb{Z}}}(\overline{K}) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_l$ (for some prime number l), $\mu_{\mathbb{Z}/n\mathbb{Z}}(\overline{K}) := \mu_{\widehat{\mathbb{Z}}}(\overline{K}) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/n\mathbb{Z}$ (for some positive integer n). We shall refer to an isomorph (in the category of topological abelian groups equipped with a continuous $\mathrm{Gal}(\overline{K}/K)$-action) of any of the above cyclotomes associated to $\overline{K}$ (we mainly use the case of $\mu_{\widehat{\mathbb{Z}}}(\overline{K})$) as a **cyclotome**. We write $\chi_{\mathrm{cyc}} = \chi_{\mathrm{cyc},K}$ (resp. $\chi_{\mathrm{cyc},l} = \chi_{\mathrm{cyc},l,K}$) for the (full) cyclotomic character (resp. the l-adic cyclotomic character) of $\mathrm{Gal}(\overline{K}/K)$ (i.e., the character determined by the action of $\mathrm{Gal}(\overline{K}/K)$ on $\mu_{\widehat{\mathbb{Z}}}(\overline{K})$ (resp. $\mu_{\mathbb{Z}_l}(\overline{K})$)).

§ 1. Reduction Steps via General Arithmetic Geometry.

In this section, we apply arguments in elementary arithmetic geometry to reduce Theorem 0.1 to a certain inequality $-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})|$, which will ultimately be proved by applying the main theorem concerning the final multiradial algorithm (Section 13).

§ 1.1. Height Functions.

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$, X a normal, $\mathbb{Z}$-proper, and $\mathbb{Z}$-flat scheme. For $d \in \mathbb{Z}_{\geq 1}$, we write

$$X(\overline{\mathbb{Q}}) \supset X(\overline{\mathbb{Q}})^{\leq d} := \bigcup_{[F:\mathbb{Q}] \leq d} X(F).$$

Write X^{arc} for the complex analytic space determined by $X(\mathbb{C})$. An **arithmetic line bundle** on X is defined to be a pair $\overline{\mathcal{L}} = (\mathcal{L}, || - ||_{\mathcal{L}})$, where $\mathcal{L}$ is a line bundle on X, and $|| - ||_{\mathcal{L}}$ is a hermitian metric on the line bundle $\mathcal{L}^{\mathrm{arc}} := \mathcal{L}|_{X^{\mathrm{arc}}}$ (i.e., the line bundle determined by $\mathcal{L}$ on X^{arc}) which is compatible with complex conjugation on X^{arc}. A morphism of arithmetic line bundles $\overline{\mathcal{L}_1} \to \overline{\mathcal{L}_2}$ is defined to be a morphism of line bundles $\mathcal{L}_1 \to \mathcal{L}_2$ such that, locally on X^{arc}, sections of $\mathcal{L}_1$ that satisfy $|| - ||_{\mathcal{L}_1} \leq 1$ map to sections of $\mathcal{L}_2$ that satisfy $|| - ||_{\mathcal{L}_2} \leq 1$. We define the set of global sections $\Gamma(\overline{\mathcal{L}})$ to be $\mathrm{Hom}(\overline{\mathcal{O}_X}, \overline{\mathcal{L}})$, where $\overline{\mathcal{O}_X}$ is the arithmetic line bundle on X determined by the trivial line bundle equipped with the trivial hermitian metric. We write $\mathrm{APic}(X)$ for the set of isomorphism classes of arithmetic line bundles on X; thus, $\mathrm{APic}(X)$ is equipped with the group structure determined by forming tensor products of arithmetic line bundles. If $f : X \to Y$ is a morphism of normal, $\mathbb{Z}$-proper, $\mathbb{Z}$-flat schemes, then we have a natural pull-back map $f^* : \mathrm{APic}(Y) \to \mathrm{APic}(X)$.

Let F be a number field. An **arithmetic divisor** (resp. $\mathbb{R}$-arithmetic divisor) on F is defined to be a finite formal sum $\mathfrak{a} = \sum_{v \in \mathbb{V}(F)} c_v v$, where $c_v \in \mathbb{Z}$ (resp. $c_v \in \mathbb{R}$) for $v \in \mathbb{V}(F)^{\mathrm{non}}$ and $c_v \in \mathbb{R}$ for $v \in \mathbb{V}(F)^{\mathrm{arc}}$. We shall call $\mathrm{Supp}(\mathfrak{a}) := \{v \in \mathbb{V}(F) \mid c_v \neq 0\}$ the **support** of $\mathfrak{a}$ and say that $\mathfrak{a}$ is **effective** if $c_v \geq 0$ for all $v \in \mathbb{V}(F)$. We write $\mathrm{ADiv}(F)$ (resp. $\mathrm{ADiv}_{\mathbb{R}}(F)$) for the group of arithmetic divisors (resp. $\mathbb{R}$-arithmetic divisors) on F. A **principal arithmetic divisor** is defined to be an arithmetic divisor of the form $\sum_{v \in \mathbb{V}(F)^{\mathrm{non}}} v(f)v - \sum_{v \in \mathbb{V}(F)^{\mathrm{arc}}} [F_v : \mathbb{R}] \log(|f|_v) v$ for some $f \in F^{\times}$. We have a natural isomorphism of groups $\mathrm{ADiv}(F)/(\text{principal elements}) \cong \mathrm{APic}(\mathrm{Spec}\, O_F)$ sending $\sum_{v \in \mathbb{V}(F)} c_v v$ to the line bundle determined by the rank one projective O_F-module $M = (\prod_{v \in \mathbb{V}(F)^{\mathrm{non}}} \mathfrak{m}_v^{c_v})^{-1} O_F$ equipped with the hermitian metric on $M \otimes_{\mathbb{Z}} \mathbb{C} = \prod_{v \in \mathbb{V}(F)^{\mathrm{arc}}} \overline{F_v}$ determined by $\prod_{v \in \mathbb{V}(F)^{\mathrm{arc}}} e^{-\frac{c_v}{[F_v:\mathbb{R}]}} | - |_{\overline{F_v}}$, where we write $\mathfrak{m}_v$ for the maximal ideal of O_F determined by v, and $\overline{F_v}$ denotes an algebraic closure of F_v. We have a (non-normalised) degree map

$$\deg_F : \mathrm{APic}(\mathrm{Spec}\, O_F) \cong \mathrm{ADiv}(F)/(\text{principal divisors}) \to \mathbb{R}$$

that sends $v \in \mathbb{V}(F)$ to $f_v \log(p_v)$. We also define (non-normalised) degree maps $\deg_F : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ in the same way. For any finite extension $K \supset F$ and any arithmetic line bundle $\overline{\mathcal{L}}$ on $\mathrm{Spec}\, O_F$, we have $\frac{1}{[F:\mathbb{Q}]}\deg_F(\overline{\mathcal{L}}) = \frac{1}{[K:\mathbb{Q}]}\deg_K(\overline{\mathcal{L}}|_{\mathrm{Spec}\, O_K})$; that is to say, the normalised degree $\frac{1}{[F:\mathbb{Q}]}\deg_F$ is *unaffected by passage to finite extensions of* F. Any non-zero element $0 \neq s \in \mathcal{L}$ of an arithmetic line bundle $\overline{\mathcal{L}} = (\mathcal{L}, ||-||_{\mathcal{L}})$ on $\mathrm{Spec}\, O_F$ determines a non-zero morphism $O_F \to \mathcal{L}$ and hence an isomorphism of $\mathcal{L}^{-1}$ with some fractional ideal $\mathfrak{a}_s$ of F. Thus, $\deg_F(\overline{\mathcal{L}})$ can be computed as the degree $\deg_F$ of the arithmetic divisor $\sum_{v\in\mathbb{V}(F)^{\mathrm{non}}} v(\mathfrak{a}_s)v - \sum_{v\in\mathbb{V}(F)^{\mathrm{arc}}}([F_v : \mathbb{R}]\log||s||_v)v$ for any $0 \neq s \in \mathcal{L}$, where $v(\mathfrak{a}_s) := \min_{a\in\mathfrak{a}_s} v(a)$, and $||-||_v$ is the v-component of $||-||_{\mathcal{L}}$ in the decomposition $\mathcal{L}^{\mathrm{arc}} \cong \coprod_{v\in\mathbb{V}(F)^{\mathrm{arc}}} \mathcal{L}_v$ over $(\mathrm{Spec}\, O_F)^{\mathrm{arc}} \cong \coprod_{v\in\mathbb{V}(F)^{\mathrm{arc}}} F_v \otimes_{\mathbb{R}} \mathbb{C}$.

For any arithmetic line bundle $\overline{\mathcal{L}}$ on X, we define the (logarithmic) **height function**

$$\mathrm{ht}_{\overline{\mathcal{L}}} : X(\overline{\mathbb{Q}}) \left(= \bigcup_{[F:\mathbb{Q}]<\infty} X(F)\right) \to \mathbb{R}$$

associated to $\overline{\mathcal{L}}$ by setting $\mathrm{ht}_{\overline{\mathcal{L}}}(x) := \frac{1}{[F:\mathbb{Q}]}\deg_F x_F^*(\overline{\mathcal{L}})$ for $x \in X(F)$, where $x_F \in X(O_F)$ is the element $\in X(F) = X(O_F)$ corresponding to x (recall that X is proper over $\mathbb{Z}$!), and $x_F^* : \mathrm{APic}(X) \to \mathrm{APic}(\mathrm{Spec}\, O_F)$ is the pull-back map. By definition, we have $\mathrm{ht}_{\overline{\mathcal{L}_1}\otimes\overline{\mathcal{L}_2}} = \mathrm{ht}_{\overline{\mathcal{L}_1}} + \mathrm{ht}_{\overline{\mathcal{L}_2}}$ for arbitrary arithmetic line bundles $\overline{\mathcal{L}_1}$, $\overline{\mathcal{L}_2}$ on X ([GenEll, Proposition 1.4 (i)]). For an arithmetic line bundle $(\overline{\mathcal{L}}, ||-||_{\mathcal{L}})$ with ample generic fiber $\mathcal{L}_{\mathbb{Q}}$ on $X_{\mathbb{Q}}$, it is well-known that $\#\{x \in X(\overline{\mathbb{Q}})^{\leq d} \mid \mathrm{ht}_{\overline{\mathcal{L}}}(x) \leq C\} < \infty$ for any $d \in \mathbb{Z}_{\geq 1}$ and $C \in \mathbb{R}$ (cf. [GenEll, Proposition 1.4 (iv)], Proposition C.1).

For functions $\alpha, \beta : X(\overline{\mathbb{Q}}) \to \mathbb{R}$, we write $\alpha \gtrsim \beta$ (resp. $\alpha \lesssim \beta$, $\alpha \approx \beta$) if there exists a constant $C \in \mathbb{R}$ such that $\alpha(x) > \beta(x) + C$ (resp. $\alpha(x) < \beta(x) + C$, $|\alpha(x) - \beta(x)| < C$) for all $x \in X(\overline{\mathbb{Q}})$. We call an equivalence class of functions relative to $\approx$ a **bounded discrepancy class**. Note that $\mathrm{ht}_{\overline{\mathcal{L}}} \gtrsim 0$ ([GenEll, Proposition 1.4 (ii)]) for any arithmetic line bunde $\overline{\mathcal{L}} = (\mathcal{L}, ||-||_{\mathcal{L}})$ such that the n-th tensor product $\mathcal{L}_{\mathbb{Q}}^{\otimes n}$ of the generic fiber $\mathcal{L}_{\mathbb{Q}}$ on $X_{\mathbb{Q}}$ is generated by global sections for some integer $n > 0$ (a condition that holds if, for instance, $\mathcal{L}_{\mathbb{Q}}$ is ample). (Indeed, suppose that $s_1, \ldots, s_m \in \Gamma(X_{\mathbb{Q}}, \mathcal{L}_{\mathbb{Q}}^{\otimes n})$ generate $\mathcal{L}_{\mathbb{Q}}^{\otimes n}$. Write $A_i := \{s_i \neq 0\} (\subset X(\overline{\mathbb{Q}}))$ for $i = 1, \ldots, m$ (so $A_1 \cup \cdots \cup A_m = X(\overline{\mathbb{Q}})$). After tensoring $\overline{\mathcal{L}}^{\otimes n}$ with the pull-back to X of an arithmetic line bundle on $\mathrm{Spec}\, \mathbb{Z}$ (cf. the property $\mathrm{ht}_{\overline{\mathcal{L}_1}\otimes\overline{\mathcal{L}_2}} = \mathrm{ht}_{\overline{\mathcal{L}_1}} + \mathrm{ht}_{\overline{\mathcal{L}_2}}$ mentioned above), we may assume (since X^{arc} is compact) that the section s_i extends to a section of $\mathcal{L}^{\otimes n}$ such that $||s_i||_{\mathcal{L}^{\otimes n}} \leq 1$ on X^{arc}. Then, for each $i = 1, \ldots, m$, the non-negativity of the height $\mathrm{ht}_{\overline{\mathcal{L}}}$ of points $\in A_i \subset X(\overline{\mathbb{Q}})$ may be verified by computing the height of such points by means of s_i and observing that both the Archimedean and non-Archimedean contributions to the height are ≥ 0.) We also note that the bounded discrepancy class of the height $\mathrm{ht}_{\overline{\mathcal{L}}}$ of an arithmetic line bundle $\overline{\mathcal{L}} = (\mathcal{L}, ||-||_{\mathcal{L}})$ depends only on the isomorphism class of the line bundle $\mathcal{L}_{\mathbb{Q}}$ on $X_{\mathbb{Q}}$ ([GenEll, Proposition 1.4 (iii)]). (Indeed, for $\overline{\mathcal{L}_1}$ and $\overline{\mathcal{L}_2}$ with $(\mathcal{L}_1)_{\mathbb{Q}} \cong (\mathcal{L}_2)_{\mathbb{Q}}$, since both the line bundle $(\mathcal{L}_1)_{\mathbb{Q}} \otimes (\mathcal{L}_2)_{\mathbb{Q}}^{\otimes(-1)} \cong \mathcal{O}_{X_{\mathbb{Q}}}$ and its inverse are generated by global sections, we have $\mathrm{ht}_{\overline{\mathcal{L}_1}} - \mathrm{ht}_{\overline{\mathcal{L}_2}} = \mathrm{ht}_{\overline{\mathcal{L}_1}\otimes\overline{\mathcal{L}_2}^{\otimes(-1)}} \gtrsim 0$ and

$\mathrm{ht}_{\overline{\mathcal{L}_2}} - \mathrm{ht}_{\overline{\mathcal{L}_1}} \gtrsim 0$.) When we are only interested in bounded discrepancy classes (and there is no fear of confusion), we shall write $\mathrm{ht}_{\mathcal{L}_{\mathbb{Q}}}$ for $\mathrm{ht}_{\overline{\mathcal{L}}}$.

For $x \in X(F) \subset X(\overline{\mathbb{Q}})$, where F denotes the minimal field of definition of x, the different ideal of F determines an effective arithmetic divisor $\mathfrak{d}_x \in \mathrm{ADiv}(F)$ supported in $\mathbb{V}(F)^{\mathrm{non}}$. We define the **log-different function** log-diff_X on $X(\overline{\mathbb{Q}})$ as follows:

$$X(\overline{\mathbb{Q}}) \ni x \mapsto \text{log-diff}_X(x) := \tfrac{1}{[F:\mathbb{Q}]}\deg_F(\mathfrak{d}_x) \in \mathbb{R}.$$

Let $D \subset X$ be an effective Cartier divisor. Write $U_X := X \setminus D$. For $x \in U_X(F) \subset U_X(\overline{\mathbb{Q}})$, where F denotes the minimal field of definition of x, write $x_F \in X(O_F)$ for the element in $X(O_F)$ corresponding to $x \in U_X(F) \subset X(F)$ via the equality $X(F) = X(O_F)$. (Recall that X is proper over $\mathbb{Z}$.) Write D_x for the pull-back of the Cartier divisor D on X to $\mathrm{Spec}\, O_F$ via $x_F : \mathrm{Spec}\, O_F \to X$. Thus, D_x may be regarded as an effective arithmetic divisor on F supported in $\mathbb{V}(F)^{\mathrm{non}}$. We shall refer to $\mathfrak{f}_x^D := (D_x)_{\mathrm{red}} \in \mathrm{ADiv}(F)$ as the **conductor** of x. We define the **log-conductor function** log-cond_D on $U_X(\overline{\mathbb{Q}})$ as follows:

$$U_X(\overline{\mathbb{Q}}) \ni x \mapsto \text{log-cond}_D(x) := \tfrac{1}{[F:\mathbb{Q}]}\deg_F(\mathfrak{f}_x^D) \in \mathbb{R}.$$

Note that the function log-diff_X on $X(\overline{\mathbb{Q}})$ depends only on the scheme $X_{\mathbb{Q}}$ ([GenEll, Remark 1.5.1]). By contrast, the function log-cond_D on $U_X(\overline{\mathbb{Q}})$ depends on the pair of $\mathbb{Z}$-schemes (X, D). Nevertheless, the bounded discrepancy class of log-cond_D on $U_X(\overline{\mathbb{Q}})$ depends only on the pair of $\mathbb{Q}$-schemes $(X_{\mathbb{Q}}, D_{\mathbb{Q}})$. (Indeed, this may be verified easily by applying the fact that any isomorphism $X_{\mathbb{Q}} \xrightarrow{\sim} X'_{\mathbb{Q}}$ that induces an isomorphism $D_{\mathbb{Q}} \xrightarrow{\sim} D'_{\mathbb{Q}}$ extends to an isomorphism between the respective restrictions of X, X' to a suitable open dense subset of $\mathrm{Spec}\, \mathbb{Z}$ ([GenEll, Remark 1.5.1]).)

§ 1.2. First Reduction.

In this subsection, we show that, to prove Theorem 0.1, it suffices to prove it in a situation subject to certain restrictions.

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. We shall say that a non-empty compact subset of a topological space is a **compact domain** if it is the closure of its interior. Let X be a normal, $\mathbb{Z}$-proper, and $\mathbb{Z}$-flat scheme and U_X an open dense subscheme of X. Let $V \subset \mathbb{V}_{\mathbb{Q}} := \mathbb{V}(\mathbb{Q})$ be a finite subset which contains $\mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$. For each $v \in V \cap \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$ (resp. $v \in V \cap \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$), let $\overline{\mathbb{Q}}_v$ be an algebraic closure of $\mathbb{Q}_v$, $\emptyset \neq \mathcal{K}_v \subsetneq U_X(\overline{\mathbb{Q}}_v)$ (resp. $\emptyset \neq \mathcal{K}_v \subsetneq U_X(\overline{\mathbb{Q}}_v)$) a $\mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$-stable compact domain (resp. a $\mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$-stable subset whose intersection with each $U_X(K) \subset U_X(\overline{\mathbb{Q}}_v)$, where K ranges over the finite subextensions of $\overline{\mathbb{Q}}_v/\mathbb{Q}_v$, is a compact domain in $U_X(K)$). (Thus, there is a natural $\mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$-orbit of bijections $X^{\mathrm{arc}} \xrightarrow{\sim} X(\overline{\mathbb{Q}}_v)$.) Then we write $\mathcal{K}_V \subset U_X(\overline{\mathbb{Q}})$ for the subset of points $x \in U_X(F) \subset U_X(\overline{\mathbb{Q}})$ where $[F : \mathbb{Q}] < \infty$ such that for each $v \in V \cap \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$

(resp. $v \in V \cap \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$) the set of $[F : \mathbb{Q}]$ points of $X(\overline{\mathbb{Q}}_v)$ (resp. $X(\overline{\mathbb{Q}}_v)$) determined by x is contained in $\mathcal{K}_v$. We shall refer to a subset $\mathcal{K}_V \subset U_X(\overline{\mathbb{Q}})$ obtained in this way as a **compactly bounded subset** and to V as its **support**. Note that it follows from the approximation theorem in elementary number theory that the $\mathcal{K}_v$'s and V are completely determined by $\mathcal{K}_V$.

Lemma 1.1. ([GenEll, Proposition 1.7 (i)]) *Let $f : Y \to X$ be a generically finite morphism of normal, $\mathbb{Z}$-proper, $\mathbb{Z}$-flat schemes of dimension two. Let e be a positive integer, $D \subset X$, $E \subset Y$ effective, $\mathbb{Z}$-flat Cartier divisors such that the generic fibers $D_{\mathbb{Q}}, E_{\mathbb{Q}}$ satisfy the following conditions: (a) $D_{\mathbb{Q}}, E_{\mathbb{Q}}$ are reduced, (b) $E_{\mathbb{Q}} = f_{\mathbb{Q}}^{-1}(D_{\mathbb{Q}})_{\mathrm{red}}$, and (c) $f_{\mathbb{Q}}$ restricts a finite étale morphism $(U_Y)_{\mathbb{Q}} \to (U_X)_{\mathbb{Q}}$, where $U_X := X \setminus D$ and $U_Y := Y \setminus E$.*

(1) We have $\text{log-diff}_X|_Y + \text{log-cond}_D|_Y \lesssim \text{log-diff}_Y + \text{log-cond}_E$ *on* $U_Y(\overline{\mathbb{Q}})$.

(2) Consider the following condition: (d) the ramification index of $f_{\mathbb{Q}}$ at each point of $E_{\mathbb{Q}}$ divides e. If, moreover, this condition (d) is satisfied, then we have

$$\text{log-diff}_Y \lesssim \text{log-diff}_X|_Y + \left(1 - \tfrac{1}{e}\right) \text{log-cond}_D|_Y$$

on $U_Y(\overline{\mathbb{Q}})$.

Proof. First, let us observe that there exists an open dense subscheme $\operatorname{Spec} \mathbb{Z}[1/S] \subset \operatorname{Spec} \mathbb{Z}$ such that the restriction of $Y \to X$ to $\operatorname{Spec} \mathbb{Z}[1/S]$ is a finite tamely ramified morphism of proper smooth families of curves. Then it follows from an elementary property of differents that the prime-to-S portion of the equality $\text{log-diff}_X|_Y + \text{log-cond}_D|_Y = \text{log-diff}_Y + \text{log-cond}_E$ and, if one assumes condition (d) to be in force, the prime-to-S portion of the inequality $\text{log-diff}_Y \leq \text{log-diff}_X|_Y + \left(1 - \frac{1}{e}\right) \text{log-cond}_D|_Y$ hold. (If the ramification index of $f_{\mathbb{Q}}$ at each point of $E_{\mathbb{Q}}$ is equal to e, then this inequality is an equality.) On the other hand, the S-portion of log-cond_E and $\text{log-cond}_D|_Y$ is ≈ 0, while the S-portion of $\text{log-diff}_Y - \text{log-diff}_X|_Y$ is ≥ 0. Thus, it suffices to show that the S-portion of $\text{log-diff}_Y - \text{log-diff}_X|_Y$ is bounded above on $U_Y(\overline{\mathbb{Q}})$. Such a bound may be obtained as a consequence of the following claim:

> Fix a prime number p and a positive integer d. Then there exists a positive integer n such that for any Galois extension L/K of finite extensions of $\mathbb{Q}_p$ with $[L : K] \leq d$, the different ideal of L/K contains $p^n O_L$.

This claim may be verified as follows. First, we observe that when the extension L/K is tamely ramified, we may take $n = 1$. Thus, by considering the maximal tamely ramified subextension of $L(\mu_p)/K$, we reduce immediately to the case where L/K is totally ramified p-power extension, and K contains μ_p. Since p-groups are solvable, we reduce further to the case where $[L : K] = p$. Since $K \supset \mu_p$, it follows from Kummer theory that we have $L = K(a^{1/p})$ for some $a \in K$. Here, $a^{1/p}$ denotes a p-th root of a in L. By multiplying by a suitable element of $(K^\times)^p$, we may assume that $a \in O_K$ and $a \notin \mathfrak{m}_K^p (\supset p^p O_K)$. Then we have $O_L \supset a^{1/p} O_L \supset p O_L$. We also have an inclusion of O_K-algebras $O_K[X]/(X^p - a) \hookrightarrow O_L$ given by sending $X \mapsto a^{1/p}$. Thus, the different ideal of L/K contains $p(a^{1/p})^{p-1} O_L \supset p^{1+(p-1)} O_L = p^p O_L$. This completes the proof of the claim and hence of Lemma 1.1. □

Proposition 1.2. ([GenEll, Theorem 2.1]) *Fix a finite subset V^{non} of $\mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$. To prove Theorem 0.1, it suffices to show the following: Write $U_{\mathbb{P}^1} := \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. Let $\mathcal{K}_V \subset U_{\mathbb{P}^1}(\overline{\mathbb{Q}})$ be a compactly bounded subset with support $V = V^{\mathrm{non}} \cup \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$. Then for any $d \in \mathbb{Z}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$, we have*

$$\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})} \lesssim (1+\epsilon)(\text{log-diff}_{\mathbb{P}^1} + \text{log-cond}_{\{0,1,\infty\}})$$

on $\mathcal{K}_V \cap U_{\mathbb{P}^1}(\overline{\mathbb{Q}})^{\leq d}$.

Proof. Let X, D, d, ϵ be as in Theorem 0.1. For any $e \in \mathbb{Z}_{>0}$, there exists (by the well-known structure of étale fundamental groups of hyperbolic curves over algebraically closed fields of characteristic zero) a connected étale Galois covering $U_Y \to U_X$ such that the normalisation Y of X in U_Y is hyperbolic and the ramification index of $Y \to X$ at each point in $E := (D \times_X Y)_{\mathrm{red}}$ is equal to e. (Later, we will take e sufficiently large.) First, we claim that

> it suffices to show that, for any $\epsilon' \in \mathbb{R}_{>0}$, we have $\mathrm{ht}_{\omega_Y} \lesssim (1+\epsilon')\text{log-diff}_Y$ on $U_Y(\overline{\mathbb{Q}})^{\leq d \cdot \deg(Y/X)}$.

This claim may be shown as follows: Let $\epsilon' \in \mathbb{R}_{>0}$ be such that $(1+\epsilon')^2 < 1+\epsilon$. Then we have

$$\begin{aligned}\mathrm{ht}_{\omega_X(D)}|_Y &\lesssim (1+\epsilon')\mathrm{ht}_{\omega_Y} \lesssim (1+\epsilon')^2 \text{log-diff}_Y \\ &\lesssim (1+\epsilon')^2(\text{log-diff}_X + \text{log-cond}_D)|_Y < (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)|_Y\end{aligned}$$

for $e > \frac{\deg(D)}{\deg(\omega_X(D))}\left(1 - \frac{1}{1+\epsilon'}\right)^{-1}$ on $U_Y(\overline{\mathbb{Q}})^{d \cdot \deg(Y/X)}$. Here, the first "$\lesssim$" follows from the computation

$$\begin{aligned}\deg(\omega_Y) &= \deg(\omega_Y(E)) - \deg(E) = \deg(\omega_Y(E))\left(1 - \tfrac{\deg(E)}{\deg(Y/X)\deg(\omega_X(D))}\right) \\ &= \deg(\omega_Y(E))\left(1 - \tfrac{\deg(D)}{e \cdot \deg(\omega_X(D))}\right) > \tfrac{1}{1+\epsilon'}\deg(\omega_Y(E)) = \tfrac{1}{1+\epsilon'}\deg(\omega_X(D)|_Y),\end{aligned}$$

together with the basic properties of height functions reviewed in Section 1.1; the second "$\lesssim$" is the hypothesis of the claim; the third "$\lesssim$" follows from Lemma 1.1 (2); the final inequality "$<$" follows from the choice of $\epsilon' \in \mathbb{R}_{>0}$. Thus, the claim follows from the fact that the fiber at any point in $U_X(\overline{\mathbb{Q}})^{\leq d}$ of the natural map $U_Y(\overline{\mathbb{Q}})^{\leq d \cdot \deg(Y/X)} \to U_X(\overline{\mathbb{Q}})$ is non-empty. This completes the proof of the claim.

Thus, it suffices to show Theorem 0.1 in the case where $D = \emptyset$. We assume that $\mathrm{ht}_{\omega_X} \lesssim (1+\epsilon)\text{log-diff}_X$ is false on $X(\overline{\mathbb{Q}})^{=d}$. It follows from the compactness of $X(K)$, where $K/\mathbb{Q}_v$ ($v \in V$) is a finite extension, that there exists a subset $\Xi \subset X(\overline{\mathbb{Q}})^{=d}$ and an unordered d-tuple of points $\Xi_v \subset X(\overline{\mathbb{Q}}_v)$ for each $v \in V$ such that $\mathrm{ht}_{\omega_X} \lesssim (1+\epsilon)\text{log-diff}_X$ is false on Ξ, and the unordered d-tuples of $\mathbb{Q}$-conjugates of points in Ξ converge to Ξ_v in $X(\overline{\mathbb{Q}}_v)$ for each $v \in V$. By Theorem C.2 (the existence of non-critical Belyi maps), there exists a morphism $f : X \to \mathbb{P}^1$ such that f is unramified over $U_{\mathbb{P}^1}$, and $f(\Xi_v) \subset U_{\mathbb{P}^1}(\overline{\mathbb{Q}}_v)$ for each $v \in V$. In particular, it follows that, after possibly eliminating finitely many elements from Ξ, there exists a compactly bounded subset $\mathcal{K}_V \subset U_{\mathbb{P}^1}(\overline{\mathbb{Q}})$ such that $f(\Xi) \subset \mathcal{K}_V$. Write $X \supset E := f^{-1}(\{0,1,\infty\})_{\mathrm{red}}$. Let $\epsilon'' \in \mathbb{R}_{>0}$ be such that $1+\epsilon'' \leq (1+\epsilon)(1-2\epsilon''\deg(E)/\deg(\omega_X))$. Then we have

$$\begin{aligned}
\mathrm{ht}_{\omega_X} &\approx \mathrm{ht}_{\omega_X(E)} - \mathrm{ht}_{\mathcal{O}_X(E)} \approx \mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}|_X - \mathrm{ht}_{\mathcal{O}_X(E)} \\
&\lesssim (1+\epsilon'')(\text{log-diff}_{\mathbb{P}^1}|_X + \text{log-cond}_{\{0,1,\infty\}}|_X) - \mathrm{ht}_{\mathcal{O}_X(E)} \\
&\lesssim (1+\epsilon'')(\text{log-diff}_X + \text{log-cond}_E) - \mathrm{ht}_{\mathcal{O}_X(E)} \\
&\lesssim (1+\epsilon'')(\text{log-diff}_X + \mathrm{ht}_{\mathcal{O}_X(E)}) - \mathrm{ht}_{\mathcal{O}_X(E)} = (1+\epsilon'')\text{log-diff}_X + \epsilon''\mathrm{ht}_{\mathcal{O}_X(E)} \\
&\lesssim (1+\epsilon'')\text{log-diff}_X + 2\epsilon''(\deg(E)/\deg(\omega_X))\mathrm{ht}_{\omega_X}
\end{aligned}$$

on Ξ. Here, the second "$\approx$" follows by applying the natural isomorphism $\omega_{\mathbb{P}^1}(\{0,1,\infty\})|_X \xrightarrow{\sim} \omega_X(E)$; the first "$\lesssim$" follows (since $f(\Xi) \subset \mathcal{K}_V$) from the displayed inequality in the statement of Proposition 1.2; the second "$\lesssim$" follows from Lemma 1.1 (1); the third "$\lesssim$" follows from the inequality $\text{log-cond}_E \lesssim \mathrm{ht}_{\mathcal{O}_X(E)}$ (discussed in more detail below); the fourth "$\lesssim$" follows from the fact that $\omega_X^{\otimes(2\deg(E))} \otimes \mathcal{O}_X(-E)^{\otimes(\deg(\omega_X))}$ is ample since its degree is equal to $2\deg(E)\deg(\omega_X) - \deg(E)\deg(\omega_X) = \deg(E)\deg(\omega_X) > 0$. (The inequality $\text{log-cond}_E \lesssim \mathrm{ht}_{\mathcal{O}_X(E)}$ may be proved by computing the height $\mathrm{ht}_{\mathcal{O}_X(E)}$ by means of the section $s \in \Gamma(X, \mathcal{O}_X(E))$ corresponding to the natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(E)$: the Archimedean contribution to $\mathrm{ht}_{\mathcal{O}_X(E)}$ is bounded from below in light of the compactness of the space X^{arc}, while the non-Archimedean contribution to $\mathrm{ht}_{\mathcal{O}_X(E)}$ is bounded from below by the corresponding non-Archimedean contribution to log-cond_E, as a consequence of the "$(-)_{\mathrm{red}}$" in the definition of log-cond_E.)

The inequalities of the above display imply that $(1-2\epsilon''\deg(E)/\deg(\omega_X))\mathrm{ht}_{\omega_X} \lesssim (1+\epsilon'')\text{log-diff}_X$ on Ξ. Thus, the choice of $\epsilon'' \in \mathbb{R}_{>0}$ implies that $\mathrm{ht}_{\omega_X} \lesssim (1+\epsilon)\text{log-diff}_X$ on Ξ. This contradicts the hypothesis on Ξ. $\square$

§ 1.3. Second Reduction — Log-volume Computations.

In the present and following subsections, we further reduce Theorem 0.1 to a certain relation "$-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})|$" (which will be treated in later sections) that arises naturally from the main results of inter-universal Teichmüller theory. It might seem to some readers that it is unnatural and bizzare to consider such objects as the image "$\phi(q_{\underline{v}_j}^{\frac{j^2}{2l}} O_{K_{\underline{v}_j}} \otimes_{O_{K_{\underline{v}_j}}} (\otimes_{0\leq i\leq j} O_{K_{\underline{v}_i}})^{\sim})$ via an arbitrary automorphism ϕ of $\mathbb{Q}\otimes\bigotimes_{0\leq i\leq j} \frac{1}{2p_{\underline{v}_i}} \log_p(O^{\times}_{K_{\underline{v}_i}})$ that induces an automorphism of $\bigotimes_{0\leq i\leq j} \frac{1}{2p_{\underline{v}_i}} \log_p(O^{\times}_{K_{\underline{v}_i}})$" (cf. Lemma 1.9). Moreover, since the reductions discussed in the present and following subsections consist of just elementary calculations and contain nothing deep, it may appear that the relation $-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})|$ is no less difficult than the inequality that we ultimately wish to show. However, we would like to give a complete treatment of these elementary calculations in these subsections before proceeding to our discussion of anabelian geometry and inter-universal Teichmüller theory.

For $a \in \mathbb{R}$, we write $\lfloor a \rfloor$ (resp. $\lceil a \rceil$) for the largest integer $\leq a$ (resp. the smallest integer $\geq a$).

Lemma 1.3. ([IUTchIV, Proposition 1.2 (i)]) *Let k be a finite extension of $\mathbb{Q}_p$. We write e for the ramification index of k over $\mathbb{Q}_p$. For $\lambda \in \frac{1}{e}\mathbb{Z}$, we write $p^{\lambda}O_k$ for the fractional ideal generated by any element $x \in k$ with* $\mathrm{ord}(x) = \lambda$. *Write*

$$a := \begin{cases} \frac{1}{e}\left\lceil \frac{e}{p-2} \right\rceil & p > 2, \\ 2 & p = 2, \end{cases} \qquad \textit{and} \qquad b := \left\lfloor \frac{\log\left(p\frac{e}{p-1}\right)}{\log p} \right\rfloor - \frac{1}{e}.$$

Then we have

$$p^a O_k \subset \log_p(O_k^{\times}) \subset p^{-b}O_k.$$

If, furthermore, $p > 2$ and $e \leq p-2$, then $p^a O_k = \log_p(O_k^{\times}) = p^{-b}O_k$.

Proof. We have $a > \frac{1}{p-1}$, since, for $p > 2$ (resp. $p = 2$), we have $a \geq \frac{1}{e}\frac{e}{p-2} = \frac{1}{p-2} > \frac{1}{p-1}$ (resp. $a = 2 > 1 = \frac{1}{p-1}$). We fix an embedding $k \hookrightarrow \mathbb{C}_p$. Then we have $p^a O_k \subset p^{\frac{1}{p-1}+\epsilon}O_{\mathbb{C}_p} \cap O_k \subset \log_p(O_k^{\times})$ for some $\epsilon > 0$, since the p-adic exponential map converges on $p^{\frac{1}{p-1}+\epsilon}O_{\mathbb{C}_p}$ and $x = \log_p(\exp_p(x))$ for any $x \in p^{\frac{1}{p-1}+\epsilon}O_{\mathbb{C}_p}$ for $\epsilon > 0$.

On the other hand, we have $p^{b+\frac{1}{e}} > \frac{e}{p-1}$, since $b + \frac{1}{e} > \frac{\log(p\frac{e}{p-1})}{\log p} - 1 = \frac{\log\frac{e}{p-1}}{\log p}$. Note that $b + \frac{1}{e} \in \mathbb{Z}_{\geq 0}$, and that $b + \frac{1}{e} \geq 1$ if and only if $e \geq p-1$. Thus, we have $(b + \frac{1}{e}) + \frac{1}{e} > \frac{1}{p-1}$, since, for $e \geq p-1$ (resp. for $e < p-1$), we have $(b + \frac{1}{e}) + \frac{1}{e} > b + \frac{1}{e} \geq 1 \geq \frac{1}{p-1}$ (resp. $(b+\frac{1}{e}) + \frac{1}{e} = \frac{1}{e} > \frac{1}{p-1}$). In summary, we have $\min\left\{(b+\frac{1}{e}) + \frac{1}{e}, \frac{1}{e}p^{b+\frac{1}{e}}\right\} > \frac{1}{p-1}$. For $b + \frac{1}{e} \in \mathbb{Z}_{\geq 0}$, we have $(1 + p^{\frac{1}{e}}O_{\mathbb{C}_p})^{p^{b+\frac{1}{e}}} \subsetneqq 1 + p^{\frac{1}{p-1}}O_{\mathbb{C}_p}$, since $\mathrm{ord}((1+p^{\frac{1}{e}}x)^{p^{b+\frac{1}{e}}} - 1) \geq \min\{(b+\frac{1}{e}) + \frac{1}{e}, \frac{p^{b+\frac{1}{e}}}{e}\} > \frac{1}{p-1}$ for $x \in O_{\mathbb{C}_p}$.

Thus, we obtain $p^{b+\frac{1}{e}}\log_p(O_k^{\times}) \subset O_k \cap \log_p(1+p^{\frac{1}{p-1}+\epsilon}O_{\mathbb{C}_p}) \subset O_k \cap p^{\frac{1}{p-1}+\epsilon}O_{\mathbb{C}_p} \subset p^{\frac{1}{e}}O_k$ for some $\epsilon > 0$, which gives us the second inclusion. The last claim follows from the definition of a and b. $\square$

For finite extensions $k \supset k_0$ of $\mathbb{Q}_p$, we write $\mathfrak{d}_{k/k_0}$ for $\mathrm{ord}(x)$, where x is any generator of the different ideal of k over k_0. For $a \in \mathbb{Q}$, we write $p^a \in \overline{\mathbb{Q}}_p$ for an element of $\overline{\mathbb{Q}}_p$ with $\mathrm{ord}(p^a) = a$.

Lemma 1.4. ([IUTchIV, Proposition 1.1]) *Let $\{k_i\}_{i\in I}$ be a finite set of finite extensions of $\mathbb{Q}_p$. Write $\mathfrak{d}_i := \mathfrak{d}_{k_i/\mathbb{Q}_p}$. Fix an element $* \in I$ and write $\mathfrak{d}_{I^*} := \sum_{i\in I\setminus\{*\}} \mathfrak{d}_i$. Then we have*

$$p^{\mathfrak{d}_{I^*}}(\otimes_{i\in I}O_{k_i})^{\sim} \subset \otimes_{i\in I}O_{k_i} \subset (\otimes_{i\in I}O_{k_i})^{\sim},$$

where $(\otimes_{i\in I}O_{k_i})^{\sim}$ denotes the normalisation of $\otimes_{i\in I}O_{k_i}$. (The tensor products are taken over $\mathbb{Z}_p$.) Note that $p^{\mathfrak{d}_{I^*}}(\otimes_{i\in I}O_{k_i})^{\sim}$ is well-defined for suitable $p^{\mathfrak{d}_{I^*}}$ (for example, products of $p^{\mathfrak{d}_i} \in O_{k_i}$ for $i \in I^*$).

Proof. It suffices to show that $p^{\mathfrak{d}_{I^*}}(O_{\overline{\mathbb{Q}}_p} \otimes_{O_{k_*}} \otimes_{i\in I}O_{k_i})^{\sim} \subset O_{\overline{\mathbb{Q}}_p} \otimes_{O_{k_*}} \otimes_{i\in I}O_{k_i}$, since $O_{\overline{\mathbb{Q}}_p}$ is faithfully flat over O_{k_*}. By applying induction on $\#I$, we reduce immediately to the case where $\#I = 2$. In this case, $O_{\overline{\mathbb{Q}}_p} \otimes_{O_{k_1}} (O_{k_1} \otimes_{\mathbb{Z}_p} O_{k_2}) \cong O_{\overline{\mathbb{Q}}_p} \otimes_{\mathbb{Z}_p} O_{k_2}$, and $p^{\mathfrak{d}_2}(O_{\overline{\mathbb{Q}}_p} \otimes_{\mathbb{Z}_p} O_{k_2})^{\sim} \subset O_{\overline{\mathbb{Q}}_p} \otimes_{\mathbb{Z}_p} O_{k_2}$ holds by the definition of the different ideal. $\square$

Lemma 1.5. ([IUTchIV, Proposition 1.3]) *Let $k \supset k_0$ be finite extensions of $\mathbb{Q}_p$. We write e and e_0 for the ramification indices of k and k_0 over $\mathbb{Q}_p$, respectively. Let m be the unique integer such that $p^m \mid [k : k_0]$ and $p^{m+1} \nmid [k : k_0]$. Write $\mathfrak{d}_k := \mathfrak{d}_{k/\mathbb{Q}_p}$ and $\mathfrak{d}_{k_0} := \mathfrak{d}_{k_0/\mathbb{Q}_p}$.*

(1) We have $\mathfrak{d}_{k_0} + 1/e_0 \le \mathfrak{d}_k + 1/e$. If, furthermore, k is tamely ramified over k_0, then we have $\mathfrak{d}_{k_0} + 1/e_0 = \mathfrak{d}_k + 1/e$.

(2) If k is a finite Galois extension of a tamely ramified extension of k_0, then we have $\mathfrak{d}_k \le \mathfrak{d}_{k_0} + m + 1/e_0$.

Remark 1.5.1. Note that it is the quantity "log-diff + log-cond", not the quantity "log-diff", that behaves well under passage to field extensions (cf. also the proof of Lemma 1.11 below). This is one of the reasons that the term log-cond appears in Diophantine inequalities (cf. Lemma 1.1 for the geometric case).

Proof. (1): We may replace k_0 by its maximal unramified subextension in $k \supset k_0$ and assume that k/k_0 is totally ramified. Choose uniformisers $\varpi_0 \in O_{k_0}$ and $\varpi \in O_k$ and let $f(x) \in O_{k_0}[x]$ be the minimal monic polynomial of ϖ over O_{k_0}. Then there exists an O_{k_0}-algebra isomorphism $O_{k_0}[x]/(f(x)) \xrightarrow{\sim} O_k$ which sends x to ϖ. Moreover,

$f(x) \equiv x^{e/e_0}$ modulo $\mathfrak{m}_{k_0} = (\varpi_0)$. Thus, $\mathfrak{d}_k - \mathfrak{d}_{k_0} \geq \min\{\mathrm{ord}(\varpi_0), \mathrm{ord}(\frac{e}{e_0}\varpi^{\frac{e}{e_0}-1})\} \geq \min\left\{\frac{1}{e_0}, \frac{1}{e}\left(\frac{e}{e_0}-1\right)\right\} = \frac{1}{e}\left(\frac{e}{e_0}-1\right)$, where the inequalities are equalities if k/k_0 is tamely ramified.

(2): We apply induction on m. If $m = 0$, then (2) follows from (1). Thus, we assume that $m > 0$. It suffices to show that $\mathfrak{d}_k \leq \mathfrak{d}_{k_0} + m + 1/e_0 + \epsilon$ for all $\epsilon > 0$. By assumption, k is a finite Galois extension of a tamely ramified extension k_1 of k_0. By replacing k_1 by its maximal tamely ramified subextension in $k \supset k_1$ and k_0 by its maximal unramified subextension in $k \supset k_0$, we may assume that $[k : k_1] = p^m$. Since any p-group is solvable, there exists a subextension $k \supset k_2 \supset k_1$ such that $[k : k_2] = p$ and $[k_2 : k_1] = p^{m-1}$. By the induction hypothesis, we have $\mathfrak{d}_{k_2} \leq \mathfrak{d}_{k_0} + (m-1) + 1/e_0$. By replacing k, k_2, and k_1 by suitable tamely ramified extensions, we may assume that $k_1 \supset \mu_p$ and $(e_2 \geq) e_1 \geq p/\epsilon$, where we write e_1 and e_2 for the ramification indices of k_1 and k_2 over $\mathbb{Q}_p$, respectively. By Kummer theory, there exists an inclusion of O_{k_2}-algebras $O_{k_2}[x]/(x^p - a) \hookrightarrow O_k$ for some $a \in O_{k_2}$. Write $a^{1/p} \in O_k$ for the image of x by this homomorphism. By replacing a by a suitable $(k_2^\times)^p$-multiple of a, we may assume that $\mathrm{ord}(a) \leq \frac{p-1}{e_2}$. Then we have $\mathfrak{d}_k \leq \mathrm{ord}(f'(a^{1/p})) + \mathfrak{d}_{k_2} \leq \mathrm{ord}(pa^{(p-1)/p}) + \mathfrak{d}_{k_0} + (m-1) + 1/e_0 \leq \frac{p-1}{p}\frac{p-1}{e_2} + \mathfrak{d}_{k_0} + m + 1/e_0 < p/e_2 + \mathfrak{d}_{k_0} + m + 1/e_0 \leq \mathfrak{d}_{k_0} + m + 1/e_0 + \epsilon$. This completes the proof of Lemma 1.5. $\square$

For a finite extension k of $\mathbb{Q}_p$, we write $\mu_k^{\log}$ for the (non-normalised) **log-volume function** (i.e., the logarithm of the usual p-adic measure on k) defined on compact open non-empty subsets of k and valued in $\mathbb{R}$. Thus, we have $\mu_k^{\log}(O_k) = 0$, $\mu^{\log}(pO_k) = -\log\#(O_k/pO_k) = -[k : \mathbb{Q}_p]\log p$. For a CAF k (cf. Section 0.2 for the definition of the notion of a CAF), we write $\mu_k^{\log}$ for the **radial log-volume function**. Thus, $\mu_k^{\log}$ is defined on compact domains of k (cf. Section 1.2 for the definition of the notion of a compact domain), is valued in $\mathbb{R}$, and satisfies $\mu_k^{\log}(O_k) = 0$, $\mu_k^{\log}(eO_k) = 1$. The non-normalised log-volume function $\mu_k^{\log}$ and twice the radial log-volume function, i.e., $2\mu_k^{\log}$, may be regarded as local versions of the non-normalised degree map $\deg_F$. (Indeed, note that if, for simplicity, F is a totally imaginary number field, and $\overline{\mathcal{L}}$ is an arithmetic line bundle over $\mathrm{Spec}\, O_F$ equipped with an embedding $\overline{\mathcal{L}} \hookrightarrow \overline{\mathcal{O}}$ into the trivial arithmetic line bundle over $\mathrm{Spec}\, O_F$, then we have $\deg_F(\overline{\mathcal{L}}) = \sum_{v \in \mathbb{V}(F)^{\mathrm{non}}} \mu_{F_v}^{\log}(\mathcal{L}_v) + \sum_{v \in \mathbb{V}(F)^{\mathrm{arc}}} 2\mu_{F_v}^{\log}(\mathcal{L}_v)$, where $\mathcal{L}_v$ denotes the ideal of O_{F_v} determined by $\overline{\mathcal{L}} \hookrightarrow \overline{\mathcal{O}}$.) Similarly, the normalised log-volume function $\frac{1}{[k:\mathbb{Q}_p]}\mu_k^{\log}$ and radial log-volume function $\mu_k^{\log}$ may be regarded as local versions of the normalised degree map $\frac{1}{[F:\mathbb{Q}]}\deg_F$. (Indeed, note that if, for simplicity, F, $\overline{\mathcal{L}}$ and $\mathcal{L}_v$ are as above, then the normalised degree of $\overline{\mathcal{L}}$

may be written as a sum of weighted averages

$$\frac{1}{[F:\mathbb{Q}]}\deg_F(\overline{\mathcal{L}}) = \sum_{v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}} \frac{1}{\sum_{\mathbb{V}(F)\ni v|v_{\mathbb{Q}}}[F_v:\mathbb{Q}_{v_{\mathbb{Q}}}]} \sum_{\mathbb{V}(F)\ni v|v_{\mathbb{Q}}} [F_v:\mathbb{Q}_{v_{\mathbb{Q}}}](\tfrac{1}{[F_v:\mathbb{Q}_{v_{\mathbb{Q}}}]}\mu^{\log}_{F_v}(\mathcal{L}_v))$$
$$+ \sum_{v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}} \frac{1}{\sum_{\mathbb{V}(F)\ni v|v_{\mathbb{Q}}}[F_v:\mathbb{Q}_{v_{\mathbb{Q}}}]} \sum_{\mathbb{V}(F)\ni v|v_{\mathbb{Q}}} [F_v:\mathbb{Q}_{v_{\mathbb{Q}}}]\mu^{\log}_{F_v}(\mathcal{L}_v)$$

with respect to the weights $\{[F_v:\mathbb{Q}_{v_{\mathbb{Q}}}]\}_{\mathbb{V}(F)\ni v|v_{\mathbb{Q}}}$.)

For finite extensions $\{k_i\}_{i\in I}$ of $\mathbb{Q}_p$, the sum of the normalised log-volume functions $\{\frac{1}{[k_i:\mathbb{Q}_p]}\mu^{\log}_{k_i}\}_{i\in I}$ determines (since, for any $i\in I$, we have $\frac{1}{[k_i:\mathbb{Q}_p]}\mu^{\log}_{k_i}(pO_{k_i}) = -\log p$) a log-volume function $\mu^{\log}_{k_I}$ on compact open non-empty subsets of $k_I := \otimes_{i\in I}k_i$ (where the tensor products are taken over $\mathbb{Q}_p$) valued in $\mathbb{R}$, normalised so that $\mu^{\log}_{k_I}(\otimes_{i\in I}O_{k_i}) = 0$. (Here, we note that $\mu^{\log}_{k_{\{i\}}} = \frac{1}{[k_i:\mathbb{Q}_p]}\mu^{\log}_{k_i}$.) For CAF's $\{k_i\}_{i\in I}$, we consider the tensor product $k_I := \otimes_{i\in I}k_i$ as a tensor product of topological rings. Then k_I decomposes as a direct sum (or, equivalently, as a direct product) of CAF's. Write $B_I \subset k_I$ for the subset determined by forming the direct product of the unit balls on the various direct summands. Then we define the log-volume function $\mu^{\log}_{k_I}$ on compact domains in k_I, valued in $\mathbb{R}$, by forming the sum of the radial log-volumes on the various direct summands. Thus, we have $\mu^{\log}_{k_I}(B_I) = 0$.

Lemma 1.6. ([IUTchIV, Proposition 1.2 (ii), (iv)], [IUTchIV, Proposition 1.4 (iii), (iv)] and [IUTchIV, "together with the fact..." in Steps (v) and (vi) in the proof of Theorem 1.10]) *Let $\{k_i\}_{i\in I}$ be a finite set of finite extensions of $\mathbb{Q}_p$. We write e_i for the ramification index of k_i over $\mathbb{Q}_p$; a_i and b_i for the quantities a and b defined in Lemma 1.3 for k_i, respectively; $\mathfrak{d}_i := \mathfrak{d}_{k_i/\mathbb{Q}_p}$, $a_I := \sum_{i\in I}a_i$, $b_I := \sum_{i\in I}b_i$, and $\mathfrak{d}_I := \sum_{i\in I}\mathfrak{d}_i$. For $\lambda\in\frac{1}{e_i}\mathbb{Z}$, we write $p^\lambda O_{k_i}$ for the fractional ideal generated by any element $x\in k_i$ with $\mathrm{ord}(x)=\lambda$. Let $\phi : \bigotimes_{i\in I}\log_p(O^\times_{k_i}) \xrightarrow{\sim} \bigotimes_{i\in I}\log_p(O^\times_{k_i})$ (where the tensor products are taken over $\mathbb{Z}_p$) be an automorphism of $\mathbb{Z}_p$-modules. Thus, ϕ extends uniquely to a $\mathbb{Q}_p$-linear automorphism of the $\mathbb{Q}_p$-vector space $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\bigotimes_{i\in I}\log_p(O^\times_{k_i})$. We consider $(\otimes_{i\in I}O_{k_i})^\sim$ as a submodule of $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\bigotimes_{i\in I}\log_p(O^\times_{k_i})$ via the natural isomorphisms $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}(\otimes_{i\in I}O_{k_i})^\sim \cong \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\otimes_{i\in I}O_{k_i} \cong \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\otimes_{i\in I}\log_p(O^\times_{k_i})$.*

(1) Write $I\supset I^* := \{i\in I \mid e_i > p-2\}$. For any $\lambda\in\frac{1}{e_{i_0}}\mathbb{Z}$, $i_0\in I$, we have

$$\phi\left(p^\lambda(\otimes_{i\in I}O_{k_i})^\sim\right) \bigcup p^{\lfloor\lambda\rfloor}\bigotimes_{i\in I}\frac{1}{2p}\log_p(O^\times_{k_i})$$
$$\subset p^{\lfloor\lambda-\mathfrak{d}_I-a_I\rfloor}\bigotimes_{i\in I}\log_p(O^\times_{k_i}) \subset p^{\lfloor\lambda-\mathfrak{d}_I-a_I\rfloor-b_I}(\otimes_{i\in I}O_{k_i})^\sim, \text{ and}$$

$$\mu^{\log}_{k_I}(p^{\lfloor\lambda-\mathfrak{d}_I-a_I\rfloor-b_I}(\otimes_{i\in I}O_{k_i})^\sim) \le (-\lambda+\mathfrak{d}_I+1)\log(p) + \sum_{i\in I^*}(3+\log(e_i)).$$

(2) If $p > 2$ and $e_i = 1$ for each $i \in I$, then we have

$$\phi((\otimes_{i\in I} O_{k_i})^{\sim}) \subset \bigotimes_{i\in I} \frac{1}{2p} \log_p(O_{k_i}^{\times}) = (\otimes_{i\in I} O_{k_i})^{\sim},$$

and $\mu_{k_I}^{\log}((\otimes_{i\in I} O_{k_i})^{\sim}) = 0$.

Remark 1.6.1. If, for simplicity, $e_i < p-2$, then we have $(O_{k_i} \subset) \frac{1}{2p} \log_p(O_{k_i}^{\times}) = \frac{1}{p}\mathfrak{m}_{k_i}$ (cf. Lemma 1.3), where we write $\mathfrak{m}_{k_i}$ for the maximal ideal of k_i. Then observe that

> if we consider $\frac{1}{2p} \log_p(O_{k_i}^{\times}) = \frac{1}{p}\mathfrak{m}_{k_i}$ as a $\mathbb{Z}_p$-module (i.e., if we regard its submodule O_{k_i} as *not being equipped with a ring structure*), then, at least from an *a priori* point of view, there is no evident intrinsic way (i.e., e.g., no intrinsically defined algorithm that allows one) to distinguish the elements $\frac{1}{p}\varpi, \frac{1}{p}\varpi^2, \ldots, \frac{1}{p}\varpi^e$,

where we write ϖ for a uniformiser of k_i. One can consider this phenomenon as a kind of "**differentiation over** $\mathbb{F}_1$" (cf. also the point of view of **Teichmüller dilations** discussed in Section 3.5).

Proof. (1): We have $p^{\mathfrak{d}_I + a_I}(\otimes_{i\in I} O_{k_i})^{\sim} \subset p^{a_I} \otimes_{i\in I} O_{k_i} \subset \bigotimes_{i\in I} \log_p(O_{k_i}^{\times})$, where the first (resp. second) inclusion follows from Lemma 1.4 (resp. Lemma 1.3). Then by Lemma 1.3, we have

$$\begin{aligned} p^{\lambda}(\otimes_{i\in I} O_{k_i})^{\sim} &= p^{\lambda - \mathfrak{d}_I - a_I} p^{\mathfrak{d}_I + a_I}(\otimes_{i\in I} O_{k_i})^{\sim} \subset p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} p^{\mathfrak{d}_I + a_I}(\otimes_{i\in I} O_{k_i})^{\sim} \\ &\subset p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \log_p(\otimes_{i\in I} O_{k_i}^{\times}) \subset p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I}(\otimes_{i\in I} O_{k_i})^{\sim}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \phi\left(p^{\lambda}(\otimes_{i\in I} O_{k_i})^{\sim}\right) &\subset \phi\left(p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \bigotimes_{i\in I} \log_p(O_{k_i}^{\times})\right) \\ &= p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \bigotimes_{i\in I} \log_p(O_{k_i}^{\times}) \subset p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I}(\otimes_{i\in I} O_{k_i})^{\sim}. \end{aligned}$$

If $p = 2$, then we have $\lceil \mathfrak{d}_I + a_I \rceil \geq \mathfrak{d}_I + a_I \geq a_I = 2 \cdot (\#I)$. If $p > 2$, then we have $a_i \geq \frac{1}{e_i}$, and $\mathfrak{d}_i \geq 1 - \frac{1}{e_i}$ by Lemma 1.5 (1), hence we have $\mathfrak{d}_I + a_I \geq \#I$. Thus, we obtain the remaining inclusion $p^{\lfloor \lambda \rfloor} \bigotimes_{i\in I} \frac{1}{2p} \log_p(O_{k_i}^{\times}) \subset p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \bigotimes_{i\in I} \log_p(O_{k_i}^{\times})$ for $p \geq 2$.

Next, we show the upper bound on the log-volume. We have $a_i - \frac{1}{e_i} < \frac{4}{p} < \frac{2}{\log(p)}$, where the first inequality for $p > 2$ (resp. $p = 2$) follows from the inequalities $a_i < \frac{1}{e_i}(\frac{e_i}{p-2} + 1) = \frac{1}{p-2} + \frac{1}{e_i}$ and $\frac{1}{p-2} < \frac{4}{p}$ for $p > 2$ (resp. $a_i - \frac{1}{e_i} = 2 - \frac{1}{e_i} < 2 = \frac{4}{p}$), and the second inequality follows from the fact that $x > 2\log x$ for $x \geq 2$. We also have $(b_i + \frac{1}{e_i})\log(p) \leq \log(\frac{pe_i}{p-1}) \leq \log(2e_i) < 1 + \log(e_i)$, where the first inequality follows from the definion of b_i, the second inequality follows from the fact that $\frac{p}{p-1} \leq 2$ for $p \geq 2$, and the third inequality follows from the fact that $\log(2) < 1$. Thus, by combining these inequalities, we obtain the inequality $(a_i + b_i)\log(p) \leq 3 + \log(e_i)$. For $i \in I \setminus I^*$, we have $a_i = -b_i$ $(= 1/e_i)$, hence we have $(a_i + b_i)\log(p) = 0$. Thus, we obtain $\mu^{\log}_{k_I}(p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I}(\otimes_{i\in I} O_{k_i})^{\sim}) \leq (-(\lambda - \mathfrak{d}_I - a_I - 1) + b_I)\log(p) = (-\lambda + \mathfrak{d}_I + a_I + b_I + 1)\log(p) \leq (-\lambda + \mathfrak{d}_I + 1)\log(p) + \sum_{i\in I^*}(3 + \log(e_i))$.

(2) follows from (1) and Lemma 1.3. □

For a non-Archimedean local field k, we write $\mathcal{I}_k := \frac{1}{2p_{v_{\mathbb{Q}}}} \log_p(O_k^{\times})$. For a CAF k, we write $\mathcal{I}_k := \{a \in k \mid |a|_k \leq \pi\}$. If k is a non-Archimedean local field or a CAF, then we shall refer to $\mathcal{I}_k$ as the **log-shell of k**. Let F be a totally complex number field and $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$. For $\mathbb{V}(F) \ni v_1, \ldots, v_n \mid v_{\mathbb{Q}}$, write

$$\mathcal{I}_{v_1,\cdots,v_n} := \otimes_{1\leq i\leq n}\mathcal{I}_{F_{v_i}} \subset \mathcal{I}^{\mathbb{Q}}_{v_1,\cdots,v_n} := \otimes_{1\leq i\leq n} F_{v_i}.$$

(Here, the tensor products are taken in the category of topological modules.) Let $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$. For $\mathbb{V}(F) \ni v_1, \ldots, v_n \mid v_{\mathbb{Q}}$, we write

$$\mathcal{I}_{v_1,\ldots,v_n} := \pi^n B_{\{1,\ldots,n\}} \subset \mathcal{I}^{\mathbb{Q}}_{v_1,\ldots,v_n} := \otimes_{1\leq i\leq n} F_{v_i},$$

where we take the "k_i" in the definition of B_I to be F_{v_i}. (Here, the tensor products are taken in the category of topological modules.) Let $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}$ (resp. $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}$). For a subset $A \subset \mathcal{I}^{\mathbb{Q}}_{v_1,\cdots,v_n}$, we define the **holomorphic hull** of A to be the smallest subset containing A of the form $\oplus_{i\in I} a_i O_{L_i}$ with $a_i \in L_i^{\times}$ in the natural direct sum decomposition of the topological fields $\otimes_{1\leq i\leq n} F_{v_i} \cong \oplus_{i\in I} L_i$.

For a CAF k, we define the group $\mathrm{Aut}(k)^{\mathrm{prim}}$ of **primitive automorphisms** of the underlying additive topological module of k to be the subgroup generated by the complex conjugation and the multiplication by $\sqrt{-1}$. (Thus, $\mathrm{Aut}(k)^{\mathrm{prim}} \cong \mathbb{Z}/4\mathbb{Z} \rtimes \{\pm 1\}$.) We also define the subgroup $\mathrm{Aut}(k)^{\boxplus\mathrm{prim}} \subset \mathrm{Aut}(k)^{\mathrm{prim}}$ of **$\boxplus$-primitive automorphisms** of the underlying additive topological module of k to be the subgroup $\{\alpha \in \mathrm{Aut}(k)^{\mathrm{prim}} \mid \alpha(1) = \pm 1\} \subset \mathrm{Aut}(k)^{\mathrm{prim}}$. (Thus, $\mathrm{Aut}(k)^{\boxplus\mathrm{prim}} \cong \{\pm 1\} \times \{\pm 1\}$.)

In the rest of this subsection, we consider a collection of data $(\overline{F}/F, E_F, \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}, l, \underline{\mathbb{V}})$ satisfying the following conditions:

(1) F is a number field such that $\sqrt{-1} \in F$, and $\overline{F}$ is an algebraic closure of F.

(2) E_F is an elliptic curve over F, whose origin we denote by 0_E, such that $\mathrm{Aut}_{\overline{F}}(E_{\overline{F}}) = \{\pm 1\}$, where $E_{\overline{F}} := E_F \times_F \overline{F}$, and $\mathrm{Aut}_{\overline{F}}(E_{\overline{F}})$ denotes the group of $\overline{F}$-automorphisms of the scheme $E_{\overline{F}}$ that preserve 0_E; the $2 \cdot 3 \cdot 5$ $(= 30)$-torsion points of E_F are rational over F; F is Galois over the field of moduli F_{mod} of E_F, i.e., the subfield of $\overline{F}$ consisting of elements fixed by the image of the natural homomorphism

$\mathrm{Aut}(E_{\overline{F}}) \to \mathrm{Aut}(\overline{F}) = \mathrm{Gal}(\overline{F}/\mathbb{Q})$ ($\supset \mathrm{Gal}(\overline{F}/F)$) (so we have a short exact sequence $1 \to \mathrm{Aut}_{\overline{F}}(E_{\overline{F}}) \to \mathrm{Aut}(E_{\overline{F}}) \to \mathrm{Gal}(\overline{F}/F_{\mathrm{mod}}) \to 1$), where we write $\mathrm{Aut}(E_{\overline{F}})$ for the group of automorphisms of the scheme $E_{\overline{F}}$ that preserve 0_E.

(3) $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$ is a non-empty finite subset $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \subset \mathbb{V}^{\mathrm{non}}_{\mathrm{mod}}$ ($\subset \mathbb{V}_{\mathrm{mod}} := \mathbb{V}(F_{\mathrm{mod}})$), such that $v \nmid 2$ holds for each $v \in \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$, and E_F has bad multiplicative reduction over $w \in \mathbb{V}(F)_v$.

(4) l is a prime number $l \geq 5$ such that l is prime to the elements of $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$, as well as to $w(-)$ of the q-parameters of E_F at $w \in \mathbb{V}(F)^{\mathrm{bad}} := \mathbb{V}(F) \times_{\mathbb{V}_{\mathrm{mod}}} \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$.

(5) $\underline{\mathbb{V}}$ is a subset $\underline{\mathbb{V}} \subset \mathbb{V}(K)$, where $K := F(E_F[l]) \subset \overline{F}$ denotes the extension field of F generated by the fields of definition of the l-torsion points of E_F, such that the restriction of the natural surjection $\mathbb{V}(K) \twoheadrightarrow \mathbb{V}_{\mathrm{mod}}$ to $\underline{\mathbb{V}}$ induces a bijection $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$.

(Note that this is *not* the definition of initial Θ-data, which includes more objects and conditions, as discussed in Definition 10.1.) Write

$$d_{\mathrm{mod}} := [F_{\mathrm{mod}} : \mathbb{Q}],$$

$$(\mathbb{V}^{\mathrm{arc}}_{\mathrm{mod}} \subset)\, \mathbb{V}^{\mathrm{good}}_{\mathrm{mod}} := \mathbb{V}_{\mathrm{mod}} \setminus \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}, \quad \text{and} \quad \mathbb{V}(F)^{\mathrm{good}} := \mathbb{V}(F) \times_{\mathbb{V}_{\mathrm{mod}}} \mathbb{V}^{\mathrm{good}}_{\mathrm{mod}}$$

Also, we shall write $\underline{v} \in \underline{\mathbb{V}}$ for the element corresponding to $v \in \mathbb{V}_{\mathrm{mod}}$ via the bijection of (5).

Lemma 1.7. ([IUTchIV, Lemma 1.8 (ii), (iii), (iv), (v)])

(1) Let $E_{F_{\mathrm{mod}}}$ be a model of E_F over F_{mod}. Write $F_{\mathrm{tpd}} := F_{\mathrm{mod}}(E_{F_{\mathrm{mod}}}[2]) \subset \overline{F}$ for the extension field of F_{mod} generated by the fields of definition of the 2-torsion points of $E_{F_{\mathrm{mod}}}$. Then F_{tpd} is independent of the choice of a model $E_{F_{\mathrm{mod}}}$ of E_F over F_{mod}. (Here, "tpd" stands for "tripod", i.e., the projective line minus three points.)

(2) The elliptic curve E_F has semistable reduction at all $w \in \mathbb{V}(F)^{\mathrm{non}}$.

(3) Any model $E^{\dagger}_F$ of $E_{\overline{F}}$ over F such that the 3-torsion points of $E^{\dagger}_F$ are defined over F is isomorphic to E_F over F. In particular, for any model $E_{F_{\mathrm{tpd}}}$ of $E_{\overline{F}}$ over F_{tpd} such that the 3-torsion points of $E_{F_{\mathrm{tpd}}}$ are defined over F, it holds that $E_{F_{\mathrm{tpd}}} \times_{F_{\mathrm{tpd}}} F$ is isomorphic to E_F over F.

(4) The extension $K \supset F_{\mathrm{mod}}$ is Galois.

Proof. (1): Consider the short exact sequence

$$1 \to \mathrm{Aut}_{\overline{F}}(E_{\overline{F}}) \to \mathrm{Aut}(E_{\overline{F}}) \to \mathrm{Gal}(\overline{F}/F_{\mathrm{mod}}) \to 1.$$

One verifies immediately that there exists a natural bijective correspondence between sections of the surjection $\mathrm{Aut}(E_{\overline{F}}) \twoheadrightarrow \mathrm{Gal}(\overline{F}/F_{\mathrm{mod}})$ and models $E_{F_{\mathrm{mod}}}$ of $E_{\overline{F}}$ over F_{mod}, and that the subfield $F_{\mathrm{mod}}(E_{F_{\mathrm{mod}}}[2]) \subset \overline{F}$ determined by such a model is the field of invariants with respect to the kernel of the composite of the corresponding section $\mathrm{Gal}(\overline{F}/F_{\mathrm{mod}}) \hookrightarrow \mathrm{Aut}(E_{\overline{F}})$ and the natural homomorphism $\mathrm{Aut}(E_{\overline{F}}) \to \mathrm{Aut}(E_{\overline{F}}[2])$. On the other hand, the assumption $\mathrm{Aut}_{\overline{F}}(E_{\overline{F}}) = \{\pm 1\}$ implies that the natural homomorphism $\mathrm{Aut}(E_{\overline{F}}) \to \mathrm{Aut}(E_{\overline{F}}[2])$ factors through the quotient $\mathrm{Aut}(E_{\overline{F}}) \twoheadrightarrow \mathrm{Gal}(\overline{F}/F_{\mathrm{mod}})$ (since the action of $\mathrm{Aut}_{\overline{F}}(E_{\overline{F}}) = \{\pm 1\}$ on $E_{\overline{F}}[2]$ is trivial, i.e., $-P = P$ for $P \in E_{\overline{F}}[2]$). This implies that the kernel of the composite $\mathrm{Gal}(\overline{F}/F_{\mathrm{mod}}) \hookrightarrow \mathrm{Aut}(E_{\overline{F}}) \to \mathrm{Aut}(E_{\overline{F}}[2])$ is independent of the choice of a section $\mathrm{Gal}(\overline{F}/F_{\mathrm{mod}}) \hookrightarrow \mathrm{Aut}(E_{\overline{F}})$, i.e., that $F_{\mathrm{mod}}(E_{F_{\mathrm{mod}}}[2])$ is independent of the choice of model $E_{F_{\mathrm{mod}}}$. This completes the proof of (1).

(2): For a prime $r \geq 3$, there exists a fine moduli scheme $X(r)_{\mathbb{Z}[1/r]}$ over $\mathbb{Z}[1/r]$ of elliptic curves with level r structure. (Note that it is a scheme since $r \geq 3$.) Moreover, since $X(r)_{\mathbb{Z}[1/r]}$ is proper over $\mathbb{Z}[1/r]$, any F_w-valued point with $w \nmid r$ arises from a unique O_{F_w}-valued point. We apply this to the F_w-valued point defined by E_F equipped with some level $r = 3$ (resp. $r = 5$) structure (which is defined over F by condition (2)). Thus, E_F has semistable reduction for $w \nmid 3$ (resp. $w \nmid 5$). This completes the proof of (2).

(3): First, observe that since the image of $-1 \in \mathrm{Aut}_{\overline{F}}(E_{\overline{F}}) = \{\pm 1\}$ in $\mathrm{Aut}(E_{\overline{F}}[3])\}$ is nontrivial, we have $\mathrm{Aut}_{\overline{F}}(E_{\overline{F}}) \cap \ker(\rho) = \{1\}$. (Note that if $-P = P \in E_{\overline{F}}[3]$ then $P \in E_{\overline{F}}[2] \cap E_{\overline{F}}[3] = \{O\}$.) Next, recall from the proof of (1) that models of $E_{\overline{F}}$ over F correspond bijectively to sections of $\mathrm{Aut}_F(E_{\overline{F}}) \twoheadrightarrow \mathrm{Gal}(\overline{F}/F)$. Relative to this correspondence, a model of $E_{\overline{F}}$ over F all of whose 3-torsion points are rational over F corresponds to a section of $\mathrm{Aut}_F(E_{\overline{F}}) \twoheadrightarrow \mathrm{Gal}(\overline{F}/F)$ whose image is contained in $\ker\{\rho : \mathrm{Aut}_F(E_{\overline{F}}) \to \mathrm{Aut}(E_{\overline{F}}[3])\}$. Thus, it follows from the fact that $\mathrm{Aut}_{\overline{F}}(E_{\overline{F}}) \cap \ker(\rho) = \{1\}$ that such a section is uniquely determined. This completes the proof of (3).

(4): Consider the twist $(E_F)^g := E_F \times_{F,g} F$ of E_F by any $g \in \mathrm{Gal}(\overline{F}/F_{\mathrm{mod}})$. (Note that F is Galois over F_{mod} by condition (2).) Then, by transport of structure, the 3-torsion points of $(E_F)^g$ are defined over F, and $F((E_F)^g[l]) = K^g := K \otimes_{F,g} F$. Thus, it follows from (3) that $(E_F)^g$ is isomorphic to E_F over F. Hence K^g is isomorphic to K over F, which implies that we have an isomorphism $K \xrightarrow{\sim} K$ over $F \overset{g}{\xrightarrow{\sim}} F$. Thus, K is Galois over F_{mod}. This completes the proof of (4). $\square$

We further assume that

(6) E_F has good reduction at all $v \in \mathbb{V}(F)^{\mathrm{good}} \cap \mathbb{V}(F)^{\mathrm{non}}$ such that $v \nmid 2l$, and

(7) $F = F_{\mathrm{tpd}}(\sqrt{-1}, E_{F_{\mathrm{tpd}}}[3{\cdot}5])$. (Here, $E_{F_{\mathrm{tpd}}}$ is a model of $E_{\overline{F}}$ over F_{tpd} which is defined by an equation in Legendre form, i.e., an equation of the form $y^2 = x(x-1)(x-\lambda)$

with $\lambda \in F_{\mathrm{tpd}}$.)

For an intermediate extension $F_{\mathrm{mod}} \subset L \subset K$ which is Galois over F_{mod}, we write $\mathfrak{d}^L \in \mathrm{ADiv}(L)$ for the effective arithmetic divisor supported in $\mathbb{V}(L)^{\mathrm{non}}$ determined by the different ideal of L over $\mathbb{Q}$. We define $\log(\mathfrak{d}^L) := \frac{1}{[L:\mathbb{Q}]}\deg_L(\mathfrak{d}^L) \in \mathbb{R}_{\geq 0}$. It makes sense to consider the q-parameters of E_F at bad places since E_F has semistable reduction at all $w \in \mathbb{V}(F)^{\mathrm{non}}$ by Lemma 1.7 (2). For an intermediate extension $F \subset L \subset K$ which is Galois over F_{mod}, we write $\mathfrak{q}^L \in \mathrm{ADiv}_{\mathbb{R}}(L)$ for the effective $\mathbb{R}$-arithmetic divisor supported in $\mathbb{V}(L)^{\mathrm{non}}$ determined by the q-parameters of $E_L := E_F \times_F L$ at primes in $\mathbb{V}(L)^{\mathrm{bad}} := \mathbb{V}(L) \times_{\mathbb{V}_{\mathrm{mod}}} \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$, and $q_{\underline{v}} \in K_{\underline{v}}$ for the q-parameter at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. (Note that $2l$ is prime to the elements in $\mathrm{Supp}(\mathfrak{q}^L)$ even if E_F has bad reduction over a place dividing $2l$.) We define $\log(\mathfrak{q}^L) := \frac{1}{[L:\mathbb{Q}]}\deg_L(\mathfrak{q}^L) \in \mathbb{R}_{\geq 0}$. Since $\log(\mathfrak{q}^L)$ does not depend on L, we shall write $\log(\mathfrak{q})$ for $\log(\mathfrak{q}^L)$. We write $\mathfrak{f}^L \in \mathrm{ADiv}(L)$ for the effective arithmetic divisor whose support coincides with $\mathrm{Supp}(\mathfrak{q}^L)$, and whose coefficients are all equal to 1. (Note that $2l$ is prime to the elements in $\mathrm{Supp}(\mathfrak{f}^L)$.) We define $\log(\mathfrak{f}^L) := \frac{1}{[L:\mathbb{Q}]}\deg_L(\mathfrak{f}^L) \in \mathbb{R}_{\geq 0}$.

For an intermediate extension $F_{\mathrm{tpd}} \subset L \subset K$ which is Galois over F_{mod}, we write

$$\mathbb{V}(L)^{\mathrm{dist}} := \{w \in \mathbb{V}(L)^{\mathrm{non}} \mid \exists v \in \mathbb{V}(K)^{\mathrm{non}}_w \text{ which is ramified over } \mathbb{Q}\}.$$

We write $\mathbb{V}^{\mathrm{dist}}_{\mathbb{Q}}$ and $\mathbb{V}^{\mathrm{dist}}_{\mathrm{mod}}$ for the images of $\mathbb{V}(F_{\mathrm{tpd}})^{\mathrm{dist}}$ in $\mathbb{V}_{\mathbb{Q}}$ and in $\mathbb{V}_{\mathrm{mod}}$, respectively, via the natural surjections $\mathbb{V}(F_{\mathrm{tpd}}) \twoheadrightarrow \mathbb{V}_{\mathrm{mod}} \twoheadrightarrow \mathbb{V}_{\mathbb{Q}}$. For $L = \mathbb{Q}, F_{\mathrm{mod}}$, we write

$$\mathfrak{s}^L := \sum_{w \in \mathbb{V}(L)^{\mathrm{dist}}} e_w w \in \mathrm{ADiv}(L),$$

where e_w denotes the ramification index of $L_w/\mathbb{Q}_{p_w}$. We define $\log(\mathfrak{s}^L) := \frac{1}{[L:\mathbb{Q}]}\deg_L(\mathfrak{s}^L) \in \mathbb{R}_{\geq 0}$. We write e_{mod} for the maximal ramification index of F_{mod} over $\mathbb{Q}$. We write

$$d^*_{\mathrm{mod}} := 2 \cdot \#(\mathbb{Z}/4\mathbb{Z})^{\times} \cdot \#\mathrm{GL}_2(\mathbb{F}_2) \cdot \#\mathrm{GL}_2(\mathbb{F}_3) \cdot \#\mathrm{GL}_2(\mathbb{F}_5) \cdot d_{\mathrm{mod}} = 2^{12} \cdot 3^3 \cdot 5 \cdot d_{\mathrm{mod}},$$

$$e^*_{\mathrm{mod}} := 2 \cdot \#(\mathbb{Z}/4\mathbb{Z})^{\times} \cdot \#\mathrm{GL}_2(\mathbb{F}_2) \cdot \#\mathrm{GL}_2(\mathbb{F}_3) \cdot \#\mathrm{GL}_2(\mathbb{F}_5) \cdot e_{\mathrm{mod}} = 2^{12} \cdot 3^3 \cdot 5 \cdot e_{\mathrm{mod}}.$$

(Note that $\#\mathrm{GL}_2(\mathbb{F}_2) = 2 \cdot 3$, $\#\mathrm{GL}_2(\mathbb{F}_3) = 2^4 \cdot 3$, and $\#\mathrm{GL}_2(\mathbb{F}_5) = 2^5 \cdot 3 \cdot 5$.) We write

$$\mathfrak{s}^{\leq} := \sum_{v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{dist}}_{\mathbb{Q}}} \frac{\iota_{v_{\mathbb{Q}}}}{\log(p_{v_{\mathbb{Q}}})} v_{\mathbb{Q}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathbb{Q}),$$

where $\iota_{v_{\mathbb{Q}}} := 1$ if $p_{v_{\mathbb{Q}}} \leq e^*_{\mathrm{mod}} l$ and $\iota_{v_{\mathbb{Q}}} := 0$ if $p_{v_{\mathbb{Q}}} > e^*_{\mathrm{mod}} l$. We define $\log(\mathfrak{s}^{\leq}) := \deg_{\mathbb{Q}}(\mathfrak{s}^{\leq}) \in \mathbb{R}_{\geq 0}$.

For L a finite extension of F, $\mathfrak{a} = \sum_{w \in \mathbb{V}(L)} c_w w \in \mathrm{ADiv}_{\mathbb{R}}(L)$, and $v \in \mathbb{V}(F)$, we define $\mathfrak{a}_v := \sum_{w \in \mathbb{V}(L)_v} c_w w$.

Lemma 1.8. ([IUTchIV, Proposition 1.8 (vi), (vii)]) *The extension F/F_{tpd} is tamely ramified outside $2 \cdot 3 \cdot 5$ and unramified outside $2 \cdot 3 \cdot 5 \cdot l$ and $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}})$. The* *extension K/F is tamely ramified outside l and unramified outside $2 \cdot l$ and* $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}})$*. In particular, the extension K/F_{tpd} is unramified outside $2 \cdot 3 \cdot 5 \cdot l$ and* $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}})$*.*

Proof. First, we show that $E_{F_{\mathrm{tpd}}} \times_{F_{\mathrm{tpd}}} F'$ has semistable reduction at $w \nmid 2$ for some extension $F' \supset F_{\mathrm{tpd},w}$ with $[F' : F_{\mathrm{tpd},w}] \leq 2$ as follows: $E_{F_{\mathrm{tpd}}}$ is defined by an equation in Legendre form $y^2 = x(x-1)(x-\lambda)$ for $\lambda \neq 0, 1 \in F_{\mathrm{tpd}}$. If $\lambda \in O_{F_{\mathrm{tpd},w}}$, then it has semistable reduction since $0 \not\equiv 1$ in any characteristic. If $\lambda \notin O_{F_{\mathrm{tpd},w}}$, then $\alpha\lambda \in O^{\times}_{F_{\mathrm{tpd},w}}$ for some $0 \neq \alpha \in O_{F_{\mathrm{tpd},w}}$, so by putting $x' := \alpha x$ and $y' := \alpha^{3/2} y$, we obtain a defining equation $(y')^2 = x'(x'-\alpha)(x'-\alpha\lambda)$ over $F_{\mathrm{tpd},w}(\sqrt{\alpha})$, which has semistable reduction. Moreover, if the model of $E_{F_{\mathrm{tpd},w}}$ defined by this equation has good reduction, then $\alpha \in O^{\times}_{F_{\mathrm{tpd},w}}$, so $\lambda \in O^{\times}_{F_{\mathrm{tpd},w}}$, a contradiction.

Take an algebraic closure $\overline{F_{\mathrm{tpd},w}}$ of $F_{\mathrm{tpd},w}$. Then the action of the inertia subgroup of $\mathrm{Gal}(\overline{F_{\mathrm{tpd},w}}/F')$ on $E[3 \cdot 5]$ is unipotent (cf. [SGA7t1, Exposé IX, Proposition 3.5]) for $w \nmid 2 \cdot 3 \cdot 5$. Hence $F = F_{\mathrm{tpd}}(\sqrt{-1}, E[3 \cdot 5])$ is tamely ramified over F_{tpd} outside $2 \cdot 3 \cdot 5$ and unramified outside $2 \cdot 3 \cdot 5 \cdot l$ and $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}})$ (cf. the assumption (6) and the definition of $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}})$). By a similar argument, since E_F has semistable reduction at all $w \in \mathbb{V}(F)^{\mathrm{non}}$ by Lemma 1.7 (2), the action of the inertia subgroup of $\mathrm{Gal}(\overline{F}/F)$ at $w \nmid l$ on $E[l]$ is unipotent. Thus, $K = F(E[l])$ is tamely ramified over F outside l and unramified outside $2 \cdot l$ and $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}})$ (cf. the assumption (6) and the definition of $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}})$). □

In inter-universal Teichmüller theory, we will think of the bijection $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$ as a kind of "analytic section" of the natural morphism $\mathrm{Spec}\, O_K \twoheadrightarrow \mathrm{Spec}\, O_{F_{\mathrm{mod}}}$. From this point of view, it is natural to think of $\frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\mu^{\log}_{K_{\underline{v}}}$ (resp. $\mu^{\log}_{\otimes_{\underline{v}\in\underline{\mathbb{V}}} K_{\underline{v}}}$) as a sort of stack-theoretic version of $\mu^{\log}_{(F_{\mathrm{mod}})_v}$ (resp. $\mu^{\log}_{\otimes_{v\in\mathbb{V}_{\mathrm{mod}}}(F_{\mathrm{mod}})_v}$). (Note that here the tensor product is taken with respect to $\underline{\mathbb{V}}$, *not* $\mathbb{V}(K)$. See also Definition 10.4 and Definition 10.5 (4).) In particular, when considering various weighted sums, we shall regard $\frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\mu^{\log}_{K_{\underline{v}}}$ or its normalised version $\frac{1}{[(F_{\mathrm{mod}})_v:\mathbb{Q}_{v_{\mathbb{Q}}}]}\frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\mu^{\log}_{K_{\underline{v}}} = \frac{1}{[K_{\underline{v}}:\mathbb{Q}_{v_{\mathbb{Q}}}]}\mu^{\log}_{K_{\underline{v}}}$ for $\underline{v} \in \underline{\mathbb{V}}$ (not for $\mathbb{V}(K)$) as having weight $[(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}]$ (*not* $[K_{\underline{v}} : \mathbb{Q}_{v_{\mathbb{Q}}}]$).

Lemma 1.9. ([IUTchIV, some portions of Steps (v), (vi), (vii) of the proof of Theorem 1.10, and Proposition 1.5]) *For $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, $1 \leq j \leq l^{*} (= \frac{l-1}{2})$, and $v_0, \ldots, v_j \in (\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}$ (where $v_0, \ldots, v_j$ are not necessarily distinct), we write $-|\log(\underline{\underline{\Theta}})|_{\{v_0,\ldots,v_j\}}$ for the log-volume (i.e., $\mu^{\log}_{\otimes_{0\leq i\leq j} K_{\underline{v}_i}}$) of the following:*

- *(Non-Archimedean* **vertical** *(Indet $\uparrow$),* **horizontal** *(Indet $\rightarrow$), and* **permutation** *(Indet $\curvearrowleft\!\!\!\curvearrowright$)* **indeterminacies***) For $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$, the holomorphic hull of the union of*

$$\phi(q^{j^2/2l}_{\underline{v}_j} \mathcal{I}_{v_0,\cdots,v_j}) \quad (\textit{resp. } \phi(\mathcal{I}_{v_0,\cdots,v_j}))$$

for $\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *(resp. for* $\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{good}}$*), and*

$$\phi\left(q_{\underline{v}_j}^{j^2/2l} O_{K_{\underline{v}_j}} \otimes_{O_{K_{\underline{v}_j}}} (\otimes_{0\le i\le j} O_{K_{\underline{v}_i}})^{\sim}\right) \quad (\textit{resp. } \phi\left((\otimes_{0\le i\le j} O_{K_{\underline{v}_i}})^{\sim}\right))$$

for $\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *(resp. for* $\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{good}}$*), where* $\phi : \mathbb{Q}_{v_\mathbb{Q}} \otimes_{\mathbb{Z}_{v_\mathbb{Q}}} \mathcal{I}_{v_0,\dots,v_j} \xrightarrow{\sim} \mathbb{Q}_{v_\mathbb{Q}} \otimes_{\mathbb{Z}_{v_\mathbb{Q}}} \mathcal{I}_{v_0,\dots,v_j}$ *varies over the automorphisms of finite-dimensional* $\mathbb{Q}_{v_\mathbb{Q}}$*-vector spaces that stabilize the submodule* $\mathcal{I}_{v_0,\dots,v_j}$*, and the tensor products* $\otimes_{0\le i\le j}$ *are taken over* $\mathbb{Z}_{v_\mathbb{Q}}$ *(cf. also the discussion of the "***Teichmüller dilation***" in Remark 1.6.1 and Section 3.5).*

- *(Archimedean* **vertical** *(Indet* $\uparrow$*),* **horizontal** *(Indet* $\rightarrow$*), and* **permutation** *(Indet* $\curvearrowleft$*)* **indeterminacies***) For* $v_\mathbb{Q} \in \mathbb{V}_\mathbb{Q}^{\mathrm{arc}}$*, the holomorphic hull of the union of*

$$\phi(\mathcal{I}_{v_0,\dots,v_j}),$$

and

$$(\otimes_{0\le i\le j}\phi_i)(B_I),$$

where $(\phi_i)_{0\le i\le j}$ *varies over the elements of* $\prod_{0\le i\le j} \mathrm{Aut}(K_{\underline{v}_i})^{\boxplus\mathrm{prim}}$*, and we recall that* $B_I = (\textit{unit ball})^{\oplus 2^j}$ *in the natural direct sum decomposition* $\otimes_{0\le i\le j} K_{\underline{v}_i} \cong \mathbb{C}^{\oplus 2^j}$ *(where the tensor products are taken as topological modules).*

Write $\mathfrak{d}_i := \mathfrak{d}_{K_{\underline{v}_i}/\mathbb{Q}_{v_\mathbb{Q}}}$ *and* $\mathfrak{d}_I := \sum_{0\le i\le j} \mathfrak{d}_i$ *for* $v_\mathbb{Q} \in \mathbb{V}_\mathbb{Q}^{\mathrm{non}}$*. Then we have the following upper bounds of* $-|\log(\underline{\underline{\Theta}})|_{\{v_0,\dots,v_j\}}$*:*

(1) For $v_\mathbb{Q} \in \mathbb{V}_\mathbb{Q}^{\mathrm{dist}}$*, we have*

$$-|\log(\underline{\underline{\Theta}})|_{\{v_0,\dots,v_j\}} \le \begin{cases} \left(-\frac{j^2}{2l}\mathrm{ord}(q_{\underline{v}_j}) + \mathfrak{d}_I + 1\right)\log p_{v_\mathbb{Q}} + 4(j+1)\iota_{v_\mathbb{Q}}\log(e^*_{\mathrm{mod}} l), & \textit{if } \underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{bad}}, \\ (\mathfrak{d}_I + 1)\log p_{v_\mathbb{Q}} + 4(j+1)\iota_{v_\mathbb{Q}}\log(e^*_{\mathrm{mod}} l), & \textit{if } \underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{good}}. \end{cases}$$

(2) For $v_\mathbb{Q} \in \mathbb{V}_\mathbb{Q}^{\mathrm{non}} \setminus \mathbb{V}_\mathbb{Q}^{\mathrm{dist}}$*, we have* $-|\log(\underline{\underline{\Theta}})|_{\{v_0,\dots,v_j\}} \le 0$.

(3) For $v_\mathbb{Q} \in \mathbb{V}_\mathbb{Q}^{\mathrm{arc}}$*, we have* $-|\log(\underline{\underline{\Theta}})|_{\{v_0,\dots,v_j\}} \le (j+1)\log(\pi)$.

Remark 1.9.1. In Section 13, we shall see that the vertical (resp. horizontal) indeterminacy arises from the vertical (resp. horizontal) arrows of the log-theta-lattice, i.e., the $\mathfrak{log}$-links (resp. the theta-links), while the permutation indeterminacy arises from the permutation symmetry of the étale picture.

Proof. (1): We apply Lemma 1.6 (1) to $\lambda := \frac{j^2}{2l}\mathrm{ord}(q_{\underline{v}_j})$ (resp. 0) for $\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{bad}}$ (resp. for $\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{good}}$), $I := \{0, 1, \dots, j\}$, $i_0 := j$, and $k_i := K_{\underline{v}_i}$. (Note that $\lambda \in \frac{1}{e_{\underline{v}_j}}\mathbb{Z}$ since $q_{\underline{v}_j}^{1/2l} \in K_{\underline{v}_j}$ by the assumptions that $K = F(E_F[l])$ and that $E_F[2]$ is rational over F, i.e., $F = F(E_F[2])$.) Then by the first inclusion of Lemma 1.6 (1), both

$\phi\left(q_{\underline{v}_j}^{j^2/2l} O_{K_{\underline{v}_j}} \otimes_{O_{K_{\underline{v}_j}}} (\otimes_{0\leq i\leq j} O_{K_{\underline{v}_i}})^{\sim}\right)$ (resp. $\phi\left((\otimes_{0\leq i\leq j} O_{K_{\underline{v}_i}})^{\sim}\right)$) ((Indet $\to$), (Indet $\curvearrowleft$)) and $q_{\underline{v}_j}^{j^2/2l}\mathcal{I}_{v_0,\dots,v_j}$ (resp. $\mathcal{I}_{v_0,\dots,v_j}$) ((Indet $\uparrow$)) are contained in $p_{v_{\mathbb{Q}}}^{\lfloor\lambda\rfloor-\lceil\mathfrak{d}_I\rceil-\lceil a_I\rceil} \otimes_{i\in I} \log_{p_{v_{\mathbb{Q}}}}(O_{K_{\underline{v}_i}}^{\times})$. By the second inclusion of Lemma 1.6 (1), the holomorphic hull of $p_{v_{\mathbb{Q}}}^{\lfloor\lambda\rfloor-\lceil\mathfrak{d}_I\rceil-\lceil a_I\rceil} \otimes_{i\in I} \log_{p_{v_{\mathbb{Q}}}}(O_{K_{\underline{v}_i}}^{\times})$ is contained in $p_{v_{\mathbb{Q}}}^{\lfloor\lambda\rfloor-\lceil\mathfrak{d}_I\rceil-\lceil a_I\rceil-\lceil b_I\rceil}(\otimes_{i\in I} O_{K_{\underline{v}_i}}^{\times})^{\sim}$, and its log-volume is $\leq (-\lambda+\mathfrak{d}_I+1)\log(p_{v_{\mathbb{Q}}}) + \sum_{i\in I^*}(3+\log(e_i))$ by Lemma 1.6 (1), where e_i denotes the ramification index of $K_{\underline{v}_i}$ over $\mathbb{Q}_{p_{\underline{v}_i}}$. If $e_i > p_{v_{\mathbb{Q}}}-2$, then $p_{v_{\mathbb{Q}}} \leq e^*_{\mathrm{mod}} l$ since, for $\underline{v}_i \nmid l$ (resp. $\underline{v}_i \mid l$), we have $p_{v_{\mathbb{Q}}} \leq 1+e_i \leq 1+e^*_{\mathrm{mod}}l/2 \leq e^*_{\mathrm{mod}}l$ (resp. $p_{v_{\mathbb{Q}}} = l \leq e^*_{\mathrm{mod}}l$). For $e_i > p_{v_{\mathbb{Q}}}-2$, we also have $\log(e_i) \leq -3+4\log(e^*_{\mathrm{mod}}l)$ since $e_i \leq e_{\mathrm{mod}}[F(=F_{\mathrm{tpd}}(\sqrt{-1}, E_{F_{\mathrm{tpd}}}[3\cdot 5])) : F_{\mathrm{tpd}}][K:F] \leq e_{\mathrm{mod}} \cdot \#(\mathbb{Z}/4\mathbb{Z})^{\times} \cdot \#GL_2(\mathbb{F}_3) \cdot \#GL_2(\mathbb{F}_5) \cdot \#GL_2(\mathbb{F}_l) < e^*_{\mathrm{mod}}(l^2-1)(l^2-l) < e^*_{\mathrm{mod}} l^4$ and $e^3 \leq (e^*_{\mathrm{mod}})^3$. Thus, we have $(-\lambda+\mathfrak{d}_I+1)\log(p_{v_{\mathbb{Q}}}) + \sum_{i\in I^*}(3+\log(e_i)) \leq (-\lambda+\mathfrak{d}_I+1)\log(p_{v_{\mathbb{Q}}}) + 4(j+1)\iota_{v_{\mathbb{Q}}}\log(e^*_{\mathrm{mod}}l)$, since if $\iota_{v_{\mathbb{Q}}}=0$, (i.e., $p_{v_{\mathbb{Q}}} > e^*_{\mathrm{mod}}l$), then $e_i \leq p_{v_{\mathbb{Q}}}-2$ for all i, hence $I^*=\emptyset$. The last equality of the claim follows from the definitions.

(2): For $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}} \setminus \mathbb{V}_{\mathbb{Q}}^{\mathrm{dist}}$, the place $v_{\mathbb{Q}}$ is unramified in K and $v_{\mathbb{Q}} \neq 2$, since 2 ramifies in K by $K \ni \sqrt{-1}$. Thus, the ramification index e_i of $K_{\underline{v}_i}$ over $\mathbb{Q}_{v_{\mathbb{Q}}}$ is 1 for each $0\leq i\leq j$, and $p_{v_{\mathbb{Q}}} > 2$. We apply Lemma 1.6 (2) to $\lambda := 0$, $I := \{0,1,\dots,j\}$, and $k_i := K_{\underline{v}_i}$. Both of $\phi\left((\otimes_{0\leq i\leq j} O_{K_{\underline{v}_i}})^{\sim}\right)$ ((Indet $\to$), (Indet $\curvearrowleft$)) and the log-shell $\mathcal{I}_{v_0,\dots,v_j}$ (Indet $\uparrow$) are contained in $\otimes_{i\in I}\log_{p_{v_{\mathbb{Q}}}}(O_{K_{\underline{v}_i}}^{\times})$. By the second inclusion of Lemma 1.6 (2), the holomorphic hull of $\otimes_{i\in I}\log_{p_{v_{\mathbb{Q}}}}(O_{K_{\underline{v}_i}}^{\times})$ is contained in $(\otimes_{i\in I} O_{K_{\underline{v}_i}}^{\times})^{\sim}$, and its log-volume is $=0$.

(3): The natural direct sum decomposition $\otimes_{0\leq i\leq j} K_{\underline{v}_i} \cong \mathbb{C}^{\oplus 2^j}$ (where the tensor products are taken as topological modules), where $K_{\underline{v}_i} \cong \mathbb{C}$, the hermitian metric on $\mathbb{C}^{\oplus 2^j}$, and the integral structure $B_I = (\text{unit ball})^{\oplus 2^j} \subset \mathbb{C}^{\oplus 2^j}$ are preserved by the automorphisms of $\otimes_{0\leq i\leq j} K_{\underline{v}_i}$ induced by any $(\phi_i)_{0\leq i\leq j} \in \prod_{0\leq i\leq j} \mathrm{Aut}(K_{\underline{v}_i})^{\boxplus\mathrm{prim}}$ ((Indet $\to$), (Indet $\curvearrowleft$)). Note that, via the natural direct sum decomposition $\otimes_{0\leq i\leq j} K_{\underline{v}_i} \cong \mathbb{C}^{\oplus(j+1)}$, the direct sum metric on $\mathbb{C}^{\oplus(j+1)}$ induced by the standard metric on $\mathbb{C}$ is 2^j times the tensor product metric on $\otimes_{0\leq i\leq j} K_{\underline{v}_i}$ induced by the standard metric on $K_{\underline{v}_i} \cong \mathbb{C}$ (Note that $|1\otimes\sqrt{-1}|^2_{\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}} = 1$ and $|(\sqrt{-1}, -\sqrt{-1})|^2_{\mathbb{C}\oplus\mathbb{C}} = 2$) (cf. also [IUTchIV, Proposition 1.5 (iii), (iv)]). The log-shell $\mathcal{I}_{v_0,\dots,v_j}$ is contained in $\pi^{j+1}B_I$ (Indet $\uparrow$). Thus, an upper bound of the log-volume is given by $(j+1)\log(\pi)$. □

Lemma 1.10. ([IUTchIV, Proposition 1.7, and some portions of Steps (v), (vi), (vii) in the proof of Theorem 1.10]) *Fix $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. For $1 \leq j \leq l^{*}(=\frac{l-1}{2})$, we take the weighted average $-|\log(\underline{\underline{\Theta}})|_{v_{\mathbb{Q}},j}$ of $-|\log(\underline{\underline{\Theta}})|_{\{v_0,\dots,v_j\}}$ with respect to all $(j+1)$-tuples of elements $\{v_i\}_{0\leq i\leq j}$ in $(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}$ with weight $w_{v_0,\dots,v_j} := \prod_{0\leq i\leq j} w_{v_i}$, where $w_v := [(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}]$ (not $[K_{\underline{v}} : \mathbb{Q}_{v_{\mathbb{Q}}}]$), i.e.,*

$$-|\log(\underline{\underline{\Theta}})|_{v_{\mathbb{Q}},j} := \frac{1}{W} \sum_{v_0,\dots,v_j \in (\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}} w_{v_0,\dots,v_j}\left(-|\log(\underline{\underline{\Theta}})|_{\{v_0,\dots,v_j\}}\right),$$

where $W := \sum_{v_0,\dots,v_j\in(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}} w_{v_0,\dots,v_j} = (\sum_{v\in(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}} w_v)^{j+1} = [F_{\mathrm{mod}} : \mathbb{Q}]^{j+1}$*, and* $\sum_{v_0,\dots,v_j\in(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}}$ *is the summation of all* $(j+1)$*-tuples of (not necessarily distinct) elements* $v_0,\dots,v_j \in (\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}$*. (We write* $\sum\limits_{v_0,\dots,v_j}$ *for it from now on to lighten the notation.) We write* $-|\log(\underline{\underline{\Theta}})|_{v_{\mathbb{Q}}}$ *for the average of* $-|\log(\underline{\underline{\Theta}})|_{v_{\mathbb{Q}},j}$ *with respect to* $1 \leq j \leq l^*$*, (which is called* **procession normalised average***), i.e.,* $-|\log(\underline{\underline{\Theta}})|_{v_{\mathbb{Q}}} := \frac{1}{l^*}\sum_{1\leq j\leq l^*}(-|\log(\underline{\underline{\Theta}})|_{v_{\mathbb{Q}},j})$*.*

(1) For $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{dist}}$*, we have*

$$-|\log(\underline{\underline{\Theta}})|_{v_{\mathbb{Q}}} \leq -\tfrac{l+1}{24}\log(\mathfrak{q}_{v_{\mathbb{Q}}}) + \tfrac{l+5}{4}\log(\mathfrak{d}^K_{v_{\mathbb{Q}}}) + \log(\mathfrak{s}^{\mathbb{Q}}_{v_{\mathbb{Q}}}) + (l+5)\log(\mathfrak{s}^{\leq}_{v_{\mathbb{Q}}})\log(e^*_{\mathrm{mod}}l).$$

(2) For $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}} \setminus \mathbb{V}_{\mathbb{Q}}^{\mathrm{dist}}$*, we have* $-|\log(\underline{\underline{\Theta}})|_{v_{\mathbb{Q}}} \leq 0$*.*

(3) For $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$*, we have* $-|\log(\underline{\underline{\Theta}})|_{v_{\mathbb{Q}}} \leq l+1$*.*

Remark 1.10.1. When we think of $\frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\mu^{\log}_{K_{\underline{v}}}$ as a stack theoretic version of $\mu^{\log}_{(F_{\mathrm{mod}})_v}$ and identify $\underline{\mathbb{V}}$ with $\mathbb{V}_{\mathrm{mod}}$ as explained before, the weighted average

$$\frac{1}{W}\sum_{v_0,\dots,v_j} w_{v_0,\dots,v_j}\sum_{0\leq i\leq j}\frac{\mu^{\log}_{K_{\underline{v}_i}}}{[K_{\underline{v}_i}:\mathbb{Q}_{v_{\mathbb{Q}}}]}$$

corresponds to

$$\begin{aligned}\frac{1}{W}\sum_{0\leq i\leq j}\sum_{v_0,\dots,v_j} w_{v_0,\dots,v_j}\frac{\mu^{\log}_{(F_{\mathrm{mod}})_{v_i}}}{[(F_{\mathrm{mod}})_{v_i}:\mathbb{Q}_{v_{\mathbb{Q}}}]} &= \frac{1}{W}\sum_{0\leq i\leq j}\left(\sum_{v\in(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}} w_v\right)^j\sum_{v\in(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}} w_v\frac{\mu^{\log}_{(F_{\mathrm{mod}})_v}}{[(F_{\mathrm{mod}})_v:\mathbb{Q}_{v_{\mathbb{Q}}}]}\\ &= \frac{j+1}{[F_{\mathrm{mod}}:\mathbb{Q}]}\sum_{v\in(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}}\mu^{\log}_{(F_{\mathrm{mod}})_v},\end{aligned}$$

which calculates $(j+1)$ times the $v_{\mathbb{Q}}$-part of the normalised degree map $\frac{1}{[F_{\mathrm{mod}}:\mathbb{Q}]}\deg_{F_{\mathrm{mod}}}$.

Proof. (1): The weighted average of the upper bound of Lemma 1.9 (1) gives us

$$\begin{aligned}-|\log(\underline{\underline{\Theta}})|_{v_{\mathbb{Q}},j} \leq &-\frac{1}{W}\frac{j^2}{2l}\sum_{v_0,\dots,v_j} w_{v_0,\dots,v_j}\frac{\mu^{\log}_{K_{\underline{v}_j}}(\mathfrak{q}_{\underline{v}_j})}{[K_{\underline{v}_j}:\mathbb{Q}_{v_{\mathbb{Q}}}]}\\ &+\frac{1}{W}\sum_{v_0,\dots,v_j} w_{v_0,\dots,v_j}\sum_{0\leq i\leq j}\left(\frac{\mu^{\log}_{K_{\underline{v}_i}}(\mathfrak{d}^K_{\underline{v}_i})}{[K_{\underline{v}_i}:\mathbb{Q}_{v_{\mathbb{Q}}}]} + \frac{\mu^{\log}_{\mathbb{Q}_{v_{\mathbb{Q}}}}(\mathfrak{s}^{\mathbb{Q}}_{v_{\mathbb{Q}}})}{j+1} + 4\mu^{\log}_{\mathbb{Q}_{v_{\mathbb{Q}}}}(\mathfrak{s}^{\leq}_{v_{\mathbb{Q}}})\log(e^*_{\mathrm{mod}}l)\right).\end{aligned}$$

Now, $-\frac{1}{W}\frac{j^2}{2l}\sum_{v_0,\dots,v_j} w_{v_0,\dots,v_j}\frac{\mu^{\log}_{K_{\underline{v}_j}}(\mathfrak{q}_{\underline{v}_j})}{[K_{\underline{v}_j}:\mathbb{Q}_{v_\mathbb{Q}}]}$ is equal to

$$
\begin{aligned}
&-\frac{1}{W}\frac{j^2}{2l}\left(\sum_{v\in(\mathbb{V}_{\mathrm{mod}})_{v_\mathbb{Q}}} w_v\right)^j \sum_{v\in(\mathbb{V}_{\mathrm{mod}})_{v_\mathbb{Q}}} w_v \frac{\mu^{\log}_{K_{\underline{v}}}(\mathfrak{q}_{\underline{v}})}{[K_{\underline{v}}:\mathbb{Q}_{v_\mathbb{Q}}]}\\
&= -\frac{1}{[F_{\mathrm{mod}}:\mathbb{Q}]}\frac{j^2}{2l}\sum_{v\in(\mathbb{V}_{\mathrm{mod}})_{v_\mathbb{Q}}}\frac{\mu^{\log}_{K_{\underline{v}}}(\mathfrak{q}_{\underline{v}})}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]} = -\frac{1}{[F_{\mathrm{mod}}:\mathbb{Q}]}\frac{j^2}{2l}\sum_{w\in\mathbb{V}(K)_{v_\mathbb{Q}}}\frac{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}{[K:F_{\mathrm{mod}}]}\frac{\mu^{\log}_{K_w}(\mathfrak{q}_w)}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\\
&= -\frac{1}{[K:\mathbb{Q}]}\frac{j^2}{2l}\sum_{w\in\mathbb{V}(K)_{v_\mathbb{Q}}}\mu^{\log}_{K_w}(\mathfrak{q}_w) = -\frac{j^2}{2l}\log(\mathfrak{q}_{v_\mathbb{Q}}),
\end{aligned}
$$

where the second equality follows from that $\mu^{\log}_{K_w}(\mathfrak{q}_w)=\mu^{\log}_{K_{\underline{v}}}(\mathfrak{q}_{\underline{v}})$, $[K_w:(F_{\mathrm{mod}})_v]=[K_{\underline{v}}:(F_{\mathrm{mod}})_v]$, and $\#\mathbb{V}(K)_v=\frac{[K:F_{\mathrm{mod}}]}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}$ for any $w\in\mathbb{V}(K)_v$ with a fixed $v\in\mathbb{V}_{\mathrm{mod}}$, since K is Galois over F_{mod} (Lemma 1.7 (4)). On the other hand, $\frac{1}{W}\sum_{v_0,\dots,v_j} w_{v_0,\dots,v_j}\sum_{0\le i\le j}\Big(\frac{\mu^{\log}_{K_{\underline{v}_i}}(\mathfrak{d}^K_{\underline{v}_i})}{[K_{\underline{v}_i}:\mathbb{Q}_{v_\mathbb{Q}}]}+\frac{\mu^{\log}_{\mathbb{Q}_{v_\mathbb{Q}}}(\mathfrak{s}^{\mathbb{Q}}_{v_\mathbb{Q}})}{j+1}+4\mu^{\log}_{\mathbb{Q}_{v_\mathbb{Q}}}(\mathfrak{s}^{\le}_{v_\mathbb{Q}})\log(e^*_{\mathrm{mod}}l)\Big)$ is equal to

$$
\begin{aligned}
&\frac{1}{W}\sum_{0\le i\le j}\left(\sum_{v\in(\mathbb{V}_{\mathrm{mod}})_{v_\mathbb{Q}}} w_v\right)^j\sum_{v\in(\mathbb{V}_{\mathrm{mod}})_{v_\mathbb{Q}}} w_v\left(\frac{\mu^{\log}_{K_{\underline{v}_i}}(\mathfrak{d}^K_{\underline{v}_i})}{[K_{\underline{v}_i}:\mathbb{Q}_{v_\mathbb{Q}}]}+\frac{\mu^{\log}_{\mathbb{Q}_{v_\mathbb{Q}}}(\mathfrak{s}^{\mathbb{Q}}_{v_\mathbb{Q}})}{j+1}+4\mu^{\log}_{\mathbb{Q}_{v_\mathbb{Q}}}(\mathfrak{s}^{\le}_{v_\mathbb{Q}})\log(e^*_{\mathrm{mod}}l)\right)\\
&=\frac{j+1}{[F_{\mathrm{mod}}:\mathbb{Q}]}\sum_{v\in(\mathbb{V}_{\mathrm{mod}})_{v_\mathbb{Q}}} w_v\left(\frac{\mu^{\log}_{K_{\underline{v}}}(\mathfrak{d}^K_{\underline{v}})}{[K_{\underline{v}}:\mathbb{Q}_{v_\mathbb{Q}}]}+\frac{\mu^{\log}_{\mathbb{Q}_{v_\mathbb{Q}}}(\mathfrak{s}^{\mathbb{Q}}_{v_\mathbb{Q}})}{j+1}+4\mu^{\log}_{\mathbb{Q}_{v_\mathbb{Q}}}(\mathfrak{s}^{\le}_{v_\mathbb{Q}})\log(e^*_{\mathrm{mod}}l)\right)\\
&=\frac{j+1}{[F_{\mathrm{mod}}:\mathbb{Q}]}\sum_{v\in(\mathbb{V}_{\mathrm{mod}})_{v_\mathbb{Q}}}\frac{\mu^{\log}_{K_{\underline{v}}}(\mathfrak{d}^K_{\underline{v}})}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}+\mu^{\log}_{\mathbb{Q}_{v_\mathbb{Q}}}(\mathfrak{s}^{\mathbb{Q}}_{v_\mathbb{Q}})+4(j+1)\mu^{\log}_{\mathbb{Q}_{v_\mathbb{Q}}}(\mathfrak{s}^{\le}_{v_\mathbb{Q}})\log(e^*_{\mathrm{mod}}l)\\
&=\frac{j+1}{[F_{\mathrm{mod}}:\mathbb{Q}]}\sum_{w\in\mathbb{V}(K)_{v_\mathbb{Q}}}\frac{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}{[K:F_{\mathrm{mod}}]}\frac{\mu^{\log}_{K_w}(\mathfrak{d}^K_w)}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}+\mu^{\log}_{\mathbb{Q}_{v_\mathbb{Q}}}(\mathfrak{s}^{\mathbb{Q}}_{v_\mathbb{Q}})+4(j+1)\mu^{\log}_{\mathbb{Q}_{v_\mathbb{Q}}}(\mathfrak{s}^{\le}_{v_\mathbb{Q}})\log(e^*_{\mathrm{mod}}l)\\
&=(j+1)\log(\mathfrak{d}^K_{v_\mathbb{Q}})+\log(\mathfrak{s}^{\mathbb{Q}}_{v_\mathbb{Q}})+4(j+1)\log(\mathfrak{s}^{\le}_{v_\mathbb{Q}})\log(e^*_{\mathrm{mod}}l),
\end{aligned}
$$

where the second equality follows from $\sum_{v\in(\mathbb{V}_{\mathrm{mod}})_{v_\mathbb{Q}}} w_v=[F_{\mathrm{mod}}:\mathbb{Q}]$ and the third equality follows from that $\mu^{\log}_{K_w}(\mathfrak{d}_w)=\mu^{\log}_{K_{\underline{v}}}(\mathfrak{d}_{\underline{v}})$, $[K_w:(F_{\mathrm{mod}})_v]=[K_{\underline{v}}:(F_{\mathrm{mod}})_v]$, and $\#\mathbb{V}(K)_v=\frac{[K:F_{\mathrm{mod}}]}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}$ for any $w\in\mathbb{V}(K)_v$ with a fixed $v\in\mathbb{V}_{\mathrm{mod}}$ as before. Thus, by combining these, we have

$$
-|\log(\underline{\underline{\Theta}})|_{v_\mathbb{Q},j}\le-\frac{j^2}{2l}\log(\mathfrak{q}_{v_\mathbb{Q}})+(j+1)\log(\mathfrak{d}^K_{v_\mathbb{Q}})+\log\mathfrak{s}^{\mathbb{Q}}_{v_\mathbb{Q}}+4(j+1)\log(\mathfrak{s}^{\le}_{v_\mathbb{Q}})\log(e^*_{\mathrm{mod}}l).
$$

Then (1) holds since we have $\frac{1}{l^*}\sum_{1\le j\le l^*}(j+1)=\frac{l^*+1}{2}+1=\frac{l+5}{4}$, and $\frac{1}{l^*}\sum_{1\le j\le l^*}j^2=\frac{(l^*+1)(2l^*+1)}{6}=\frac{(l+1)l}{12}$. Next, (2) trivally holds by Lemma 1.9 (2). Finally, (3) holds by Lemma 1.9 (3) with $\frac{l+5}{4}\log(\pi)<\frac{l+5}{4}2\le l+1$ since $l\ge 3$. □

Lemma 1.11. ([IUTchIV, Steps (ii), (iii), (viii) in the proof of Theorem 1.10, and Proposition 1.6])

(1) We have the following bound of $\log(\mathfrak{d}^K)$ *in terms of* $\log(\mathfrak{d}^{F_{\mathrm{tpd}}})$ *and* $\log(\mathfrak{f}^{F_{\mathrm{tpd}}})$*:*

$$\log(\mathfrak{d}^K) \leq \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + 2\log l + 21.$$

(2) We have the following bound of $\log(\mathfrak{s}^{\mathbb{Q}})$ *in terms of* $\log(\mathfrak{d}^{F_{\mathrm{tpd}}})$ *and* $\log(\mathfrak{f}^{F_{\mathrm{tpd}}})$*:*

$$\log(\mathfrak{s}^{\mathbb{Q}}) \leq 2d_{\mathrm{mod}}(\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + \log l + 5.$$

(3) We have the following bound of $\log(\mathfrak{s}^{\leq})\log(e^*_{\mathrm{mod}}l)$*: there is* $\eta_{\mathrm{prm}} \in \mathbb{R}_{>0}$ *(which is a constant determined by using the prime number theorem) such that*

$$\log(\mathfrak{s}^{\leq})\log(e^*_{\mathrm{mod}}l) \leq \tfrac{4}{3}(e^*_{\mathrm{mod}}l + \eta_{\mathrm{prm}}).$$

Proof. Note that $\log(\mathfrak{d}^L)+\log(\mathfrak{f}^L) = \frac{1}{[L:\mathbb{Q}]}\sum_{w\in\mathbb{V}(L)^{\mathrm{non}}} e_w\mathfrak{d}_w\log(q_w)+\frac{1}{[L:\mathbb{Q}]}\sum_{w\in\mathrm{Supp}(\mathfrak{f}^L)}\log(q_w) = \frac{1}{[L:\mathbb{Q}]}\sum_{w\in\mathbb{V}(L)^{\mathrm{non}}}(\mathfrak{d}_w + \iota_{\mathfrak{f}^L,w}/e_w)e_w\log(q_w)$ for $L = K, F, F_{\mathrm{tpd}}, F_{\mathrm{mod}}$, where q_w is the cardinality of the residue field of L_w, e_w is the ramification index of L_w over $\mathbb{Q}_{p_w}$ and $\iota_{\mathfrak{f}^L,w} := 1$ if $w \in \mathrm{Supp}(\mathfrak{f}^L)$, and $\iota_{\mathfrak{f}^L,w} := 0$ if $w \notin \mathrm{Supp}(\mathfrak{f}^L)$.

(1): The extension F/F_{tpd} is tamely ramified outside $2\cdot 3\cdot 5$ (Lemma 1.8). Then by using Lemma 1.5 (1) ($\mathfrak{d}_{L_0} + 1/e_0 = \mathfrak{d}_L + 1/e$) for the primes outside $2\cdot 3\cdot 5$ and Lemma 1.5 (2) ($\mathfrak{d}_L + 1/e \leq \mathfrak{d}_{L_0} + 1/e_0 + m + 1/e \leq \mathfrak{d}_{L_0} + 1/e_0 + (m+1)$) for the primes dividing $2\cdot 3\cdot 5$, we have $\log(\mathfrak{d}^F)+\log(\mathfrak{f}^F) \leq \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + \log(2^{11}\cdot 3^3\cdot 5^2) \leq \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + 21$ since $[F : F_{\mathrm{tpd}}] = [F_{\mathrm{tpd}}(\sqrt{-1}) : F_{\mathrm{tpd}}][F : F_{\mathrm{tpd}}(\sqrt{-1})] \leq \#(\mathbb{Z}/4\mathbb{Z})^\times\cdot\#\mathrm{GL}_2(\mathbb{F}_3)\cdot\#\mathrm{GL}_2(\mathbb{F}_5) = 2\cdot(2^4\cdot 3)\cdot(2^5\cdot 3\cdot 5) = 2^{10}\cdot 3^2\cdot 5$, and $\log 2 < 1$, $\log 3 < 2$, $\log 5 < 2$. In a similar way, we have $\log(\mathfrak{d}^K)+\log(\mathfrak{f}^K) \leq \log(\mathfrak{d}^F)+\log(\mathfrak{f}^F)+2\log l$, since K/F is tamely ramified outside l (Lemma 1.8). Then we have $\log(\mathfrak{d}^K) \leq \log(\mathfrak{d}^K) + \log(\mathfrak{f}^K) \leq \log(\mathfrak{d}^F) + \log(\mathfrak{f}^F) + 2\log l \leq \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + 2\log l + 21$.

(2): We have $\log(s^{\mathbb{Q}}_{v_{\mathbb{Q}}}) \leq d_{\mathrm{mod}}\log(s^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}})$ for $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$. By using Lemma 1.5 (1), we have $\log(s^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \leq 2(\log(\mathfrak{d}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}))$ for $\mathbb{V}^{\mathrm{non}}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \nmid 2\cdot 3\cdot 5\cdot l$, since $1 = \mathfrak{d}_{\mathbb{Q}_{v_{\mathbb{Q}}}} + 1/e_{\mathbb{Q}_{v_{\mathbb{Q}}}} \leq \mathfrak{d}_{F_{\mathrm{mod},v}} + 1/e_{F_{\mathrm{mod},v}} \leq 2(\mathfrak{d}_{F_{\mathrm{mod},v}} + \iota_{\mathfrak{f}^{F_{\mathrm{mod}}},v}/e_{F_{\mathrm{mod},v}})$, where $\iota_{\mathfrak{f}^{F_{\mathrm{mod}}},v} := 1$ for $v \in \mathrm{Supp}(\mathfrak{f}^{F_{\mathrm{mod}}})$ and $\iota_{\mathfrak{f}^{F_{\mathrm{mod}}},v} := 0$ for $v \notin \mathrm{Supp}(\mathfrak{f}^{F_{\mathrm{mod}}})$. Thus, we have $\log(s^{\mathbb{Q}}) \leq 2d_{\mathrm{mod}}(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}}))+\log(2\cdot 3\cdot 5\cdot l) \leq 2d_{\mathrm{mod}}(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}}))+\log l+5$, since $\log 2 < 1$, $\log 3 < 2$, and $\log 5 < 2$.

(3): We have $\log(\mathfrak{s}^{\leq})\log(e^*_{\mathrm{mod}}l) \leq \log(e^*_{\mathrm{mod}}l)\sum_{p\leq e^*_{\mathrm{mod}}l} 1$. By the *prime number theorem* $\lim_{n\to\infty} n\log(p_n)/p_n = 1$ (where p_n is the n-th prime number), there exists $\eta_{\mathrm{prm}} \in \mathbb{R}_{>0}$ such that $\sum_{\mathrm{prime}\ p\leq\eta} 1 \leq \frac{4\eta}{3\log(\eta)}$ for $\eta \geq \eta_{\mathrm{prm}}$. Then $\log(e^*_{\mathrm{mod}}l)\sum_{p\leq e^*_{\mathrm{mod}}l} 1 \leq \frac{4}{3}\log(e^*_{\mathrm{mod}}l)\frac{e^*_{\mathrm{mod}}l}{\log(e^*_{\mathrm{mod}}l)} = \frac{4}{3}e^*_{\mathrm{mod}}l$ if $e^*_{\mathrm{mod}}l \geq \eta_{\mathrm{prm}}$, and $\log(e^*_{\mathrm{mod}}l)\sum_{p\leq e^*_{\mathrm{mod}}l} 1 \leq \log(\eta_{\mathrm{prm}})\frac{4}{3}\frac{\eta_{\mathrm{prm}}}{\log(\eta_{\mathrm{prm}})} = \frac{4}{3}\eta_{\mathrm{prm}}$ if $e^*_{\mathrm{mod}}l < \eta_{\mathrm{prm}}$. Thus, we have $\log(\mathfrak{s}^{\leq})\log(e^*_{\mathrm{mod}}l) \leq \frac{4}{3}(e^*_{\mathrm{mod}}l + \eta_{\mathrm{prm}})$. □

Proposition 1.12. ([IUTchIV, Theorem 1.10]) *We set* $-|\log(\underline{\underline{q}})| := -\frac{1}{2l}\log(\mathfrak{q})$. *We have the following an upper bound of* $-|\log(\underline{\underline{\Theta}})| := -\sum_{v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}} |\log(\underline{\underline{\Theta}})|_{v_{\mathbb{Q}}}$*:*

$$\begin{aligned}&-|\log(\underline{\underline{\Theta}})| \leq -\tfrac{1}{2l}\log(\mathfrak{q})+\\&\tfrac{l+1}{4}\left(-\tfrac{1}{6}\left(1-\tfrac{12}{l^2}\right)\log(\mathfrak{q})+\left(1+\tfrac{12d_{\mathrm{mod}}}{l}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right)+10(e^*_{\mathrm{mod}}l+\eta_{\mathrm{prm}})\right).\end{aligned}$$

In particular, we have $-|\log(\underline{\underline{\Theta}})| < \infty$. *If* $\boxed{-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})|}$, *then we have*

$$\boxed{\tfrac{1}{6}\log(\mathfrak{q}) \leq \left(1+\tfrac{20d_{\mathrm{mod}}}{l}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right)+20(e^*_{\mathrm{mod}}l+\eta_{\mathrm{prm}})},$$

where η_{prm} *is the constant in Lemma 1.11.*

Proof. By Lemma 1.10 (1), (2), (3) and Lemma 1.11 (1), (2), (3), we have

$$\begin{aligned}-|\log(\underline{\underline{\Theta}})| \leq& -\tfrac{l+1}{24}\log(\mathfrak{q})+\tfrac{l+5}{4}\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}})+2\log l+21\right)\\&+\left(2d_{\mathrm{mod}}(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}}))+\log l+5\right)+(l+5)\tfrac{4}{3}(e^*_{\mathrm{mod}}l+\eta_{\mathrm{prm}})+l+1.\end{aligned}$$

Since $\frac{l+5}{4}=\frac{l^2+5l}{4l}<\frac{l^2+5l+4}{4l}=\frac{l+1}{4}(1+\frac{4}{l})$, $4<4\frac{l+1}{l}=\frac{l+1}{4}\frac{16}{l}$, and $l+5\leq\frac{20}{3}\frac{l+1}{4}$ (for $l\geq 5$), this is bounded above by

$$\begin{aligned}<&\tfrac{l+1}{4}\left(-\tfrac{1}{6}\log(\mathfrak{q})+\left(1+\tfrac{4}{l}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}})+2\log l+21\right)\right.\\&\left.+\tfrac{4}{l}\left(2d_{\mathrm{mod}}(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}}))+\log l+5\right)+\tfrac{20}{3}\tfrac{4}{3}(e^*_{\mathrm{mod}}l+\eta_{\mathrm{prm}})+4\right)\\=&\tfrac{l+1}{4}\left(-\tfrac{1}{6}\log(\mathfrak{q})+\left(1+\tfrac{4}{l}+\tfrac{8d_{\mathrm{mod}}}{l}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right)\right.\\&\left.+\left(1+\tfrac{4}{l}\right)(2\log l+21)+\tfrac{16}{l}(\log l+5)+\tfrac{80}{9}(e^*_{\mathrm{mod}}l+\eta_{\mathrm{prm}})+4\right).\end{aligned}$$

Since $4+8d_{\mathrm{mod}}\leq 12d_{\mathrm{mod}}$, $(1+\frac{4}{l})(2\log l+21)=2\log l+8\frac{\log l}{l}+(1+\frac{4}{l})21<2\log l+8\frac{1}{2}+(1+1)21=2\log l+46$ (for $l\geq 5$), $16\frac{\log l}{l}<16\frac{1}{2}=8$, and $\frac{16}{l}5\leq 16$ (for $l\geq 5$), this is bounded above by

$$<\tfrac{l+1}{4}\left(-\tfrac{1}{6}\log(\mathfrak{q})+\left(1+\tfrac{12d_{\mathrm{mod}}}{l}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right)+2\log l+\tfrac{80}{9}(e^*_{\mathrm{mod}}l+\eta_{\mathrm{prm}})+74\right).$$

Since $2\log l+74<2l+74<2\cdot 74l+2\cdot 74l=2^2\cdot 74l<2^2\cdot 2^{12}\cdot 3\cdot 5l<\frac{4}{9}e^*_{\mathrm{mod}}l<\frac{4}{9}(e^*_{\mathrm{mod}}l+\eta_{\mathrm{prm}})$, and $\frac{80}{9}+\frac{4}{9}<10$, this is bounded above by

$$<\tfrac{l+1}{4}\left(-\tfrac{1}{6}\log(\mathfrak{q})+\left(1+\tfrac{12d_{\mathrm{mod}}}{l}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right)+10(e^*_{\mathrm{mod}}l+\eta_{\mathrm{prm}})\right).$$

Since $\frac{l+1}{4}\frac{1}{6}\frac{12}{l^2}=\frac{1}{2}(1+\frac{1}{l})>\frac{1}{2l}$, this is bounded above by

$$\begin{aligned}<&\tfrac{l+1}{4}\left(-\tfrac{1}{6}\left(1-\tfrac{12}{l^2}\right)\log(\mathfrak{q})+\left(1+\tfrac{12d_{\mathrm{mod}}}{l}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right)+10(e^*_{\mathrm{mod}}l+\eta_{\mathrm{prm}})\right)\\&-\tfrac{1}{2l}\log(\mathfrak{q}).\end{aligned}$$

If $-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})|$, then for any $-|\log(\underline{\underline{\Theta}})| \leq C_\Theta \log(\underline{\underline{q}})$, we have $-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})| \leq C_\Theta \log(\underline{\underline{q}})$, hence $\boxed{C_\Theta \geq -1}$ since $|\log(\underline{\underline{q}})| = \frac{1}{2l}\log(\mathfrak{q}) > 0$. By taking C_Θ to be

$$\frac{2l(l+1)}{4\log(\mathfrak{q})}\left(-\frac{1}{6}\left(1-\frac{12}{l^2}\right)\log(\mathfrak{q}) + \left(1+\frac{12d_{\mathrm{mod}}}{l}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right) + 10(e^*_{\mathrm{mod}}l + \eta_{\mathrm{prm}})\right) - 1,$$

we have

$$\frac{1}{6}\log(\mathfrak{q}) \leq \left(1-\frac{12}{l^2}\right)^{-1}\left(\left(1+\frac{12d_{\mathrm{mod}}}{l}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right) + 10(e^*_{\mathrm{mod}}l + \eta_{\mathrm{prm}})\right).$$

Since $(1-\frac{12}{l^2})^{-1} \leq 2$ and $(1-\frac{12}{l^2})(1+\frac{20d_{\mathrm{mod}}}{l}) \geq 1+\frac{12d_{\mathrm{mod}}}{l} \Leftrightarrow 12 \leq d_{\mathrm{mod}}(8l-\frac{240}{l})$ which holds for $l \geq 7$ (by $d_{\mathrm{mod}}(8l-\frac{240}{l}) \geq 8l-\frac{240}{l} \geq 56-\frac{240}{7} > 12$), we have

$$\frac{1}{6}\log(\mathfrak{q}) \leq \left(1+\frac{20d_{\mathrm{mod}}}{l}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right) + 20(e^*_{\mathrm{mod}}l + \eta_{\mathrm{prm}}).$$

□

§ 1.4. Third Reduction — Choice of Initial Θ-Data.

In this subsection, we regard $U_{\mathbb{P}^1}$ as the λ-line, i.e., the fine moduli *scheme* whose S-valued points (where S is an arbitrary scheme) are the isomorphism classes of the triples $[E, \phi_2, \omega]$, where E is an elliptic curve $f: E \to S$ equipped with an isomorphism $\phi_2 : (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \xrightarrow{\sim} E[2]$ of S-group schemes, and an S-basis ω of $f_*\Omega^1_{E/S}$ to which an adapted $x \in f_*\mathcal{O}_E(-2(\text{origin}))$ satisfies $x(\phi_2(1,0)) = 0$, $x(\phi_2(0,1)) = 1$. Here, a section $x \in f_*\mathcal{O}_E(-2(\text{origin}))$, for which $\{1, x\}$ forms Zariski locally a basis of $f_*\mathcal{O}_E(-2(\text{origin}))$, is called adapted to an S-basis ω of $f_*\Omega^1_{E/S}$, if Zariski locally, there is a formal parameter T at the origin such that $\omega = (1 + \text{higher terms})dT$ and $x = \frac{1}{T^2}(1 + \text{higher terms})$ (cf. [KM, (2.2), (4.6.2)]). Then $\lambda \in U_{\mathbb{P}^1}(S)$ corresponds to $E: y^2 = x(x-1)(x-\lambda)$, $\phi_2((1,0)) = (x=0, y=0)$, $\phi_2((0,1)) = (x=1, y=0)$, and $\omega = -\frac{dx}{2y}$. For a cyclic subgroup scheme $H \subset E[l]$ of order $l > 2$, a level 2 structure ϕ_2 gives us a level 2 structure $\mathrm{Im}(\phi_2)$ of E/H. An S-basis ω also gives us an S-basis $\mathrm{Im}(\omega)$ of $f_*\Omega^1_{(E/H)/S}$. For $\alpha = (\phi_2, \omega)$, write $\mathrm{Im}(\alpha) := (\mathrm{Im}(\phi_2), \mathrm{Im}(\omega))$.

Let F be a number field. For a semi-abelian variety E of relative dimension 1 over a number $\operatorname{Spec} O_F$ whose generic fiber E_F is an elliptic curve, we define Faltings height of E as follows: Let ω_E be the module of invariant differentials on E (i.e., the pull-back of Ω^1_{E/O_F} via the zero section), which is finite flat of rank 1 over O_F. We equip an hermitian metric $||-||^{\mathrm{Falt}}_{E_v}$ on $\omega_{E_v} := \omega_E \otimes_{O_F} \overline{F_v}$ for $v \in \mathbb{V}(F)^{\mathrm{arc}}$ by $(||a||^{\mathrm{Falt}}_{E_v})^2 := \frac{\sqrt{-1}}{2}\int_{E_v} a \wedge \overline{a}$, where $E_v := E \times_F \overline{F_v}$ and $\overline{a}$ is the complex conjugate of a. We also equip an hermitian metric $||-||^{\mathrm{Falt}}_E$ on $\omega_E \otimes_{\mathbb{Z}} \mathbb{C} \cong \oplus_{\mathrm{real}: v \in \mathbb{V}(F)^{\mathrm{arc}}} \omega_{E_v} \oplus \oplus_{\mathrm{complex}: v \in \mathbb{V}(F)^{\mathrm{arc}}} (\omega_{E_v} \oplus \overline{\omega_{E_v}})$, by $||-||^{\mathrm{Falt}}_{E_v}$ (resp. $||-||^{\mathrm{Falt}}_{E_v}$ and its complex conjugate) for real $v \in \mathbb{V}(F)^{\mathrm{arc}}$ (resp. for complex $v \in \mathbb{V}(F)^{\mathrm{arc}}$), where $\overline{\omega_{E_v}}$ is the complex conjugate of ω_{E_v}. Then we obtain an arithmetic line bundle $\overline{\omega}_E := (\omega_E, ||-||_E^{\mathrm{Falt}})$. We define **Faltings height** of E by $\mathrm{ht}^{\mathrm{Falt}}(E) := \frac{1}{[F:\mathbb{Q}]}\deg_F(\overline{\omega}_E) \in \mathbb{R}$. Note that for any $0 \neq a \in \omega_E$, the non-Archimedean (resp. Archimedean) portion $\mathrm{ht}^{\mathrm{Falt}}(E,a)^{\mathrm{non}}$ (resp. $\mathrm{ht}^{\mathrm{Falt}}(E,a)^{\mathrm{arc}}$) of $\mathrm{ht}^{\mathrm{Falt}}(E)$ is given by $\frac{1}{[F:\mathbb{Q}]}\sum_{v\in\mathbb{V}(F)^{\mathrm{non}}} v(a)\log p_v^{f_v} = \frac{1}{[F:\mathbb{Q}]}\log\#(\omega_E/a\omega_E)$ (resp. $-\frac{1}{[F:\mathbb{Q}]}\sum_{v\in\mathbb{V}(F)^{\mathrm{arc}}}[F_v : \mathbb{R}]\log\left(\frac{\sqrt{-1}}{2}\int_{E_v} a\wedge\overline{a}\right)^{1/2} = -\frac{1}{2[F:\mathbb{Q}]}\sum_{v\in\mathbb{V}(F)^{\mathrm{arc}}}[F_v:\mathbb{R}]\log\left(\frac{\sqrt{-1}}{2}\int_{E_v} a\wedge\overline{a}\right)$), where $\mathrm{ht}^{\mathrm{Falt}}(E) = \mathrm{ht}^{\mathrm{Falt}}(E,a)^{\mathrm{non}} + \mathrm{ht}^{\mathrm{Falt}}(E,a)^{\mathrm{arc}}$ is independent of the choice of $0 \neq a \in \omega_E$ (cf. Section 1.1).

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. For any point $[E,\alpha] \in U_{\mathbb{P}^1}(\overline{\mathbb{Q}})$ of the λ-line, we define $\mathrm{ht}^{\mathrm{Falt}}([E,\alpha]) := \mathrm{ht}^{\mathrm{Falt}}(E)$. When $[E,\alpha] \in U_{\mathbb{P}^1}(\mathbb{C})$ varies, the hermitian metric $||-||_E^{\mathrm{Falt}}$ on ω_E continuously varies, and gives a hermitian metric on the line bundle $\omega_{\mathcal{E}}$ on $U_{\mathbb{P}^1}(\mathbb{C})$, where $\mathcal{E}$ is the universal elliptic curve of the λ-line. Note that this metric cannot be extended to the compactification $\mathbb{P}^1$ of the λ-line, and the Faltings height has logarithmic singularity at $\{0,1,\infty\}$. (See also Lemma 1.13 (1) and its proof below.)

We also introduce some notation. We write $\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$ for the non-Archimedean portion of $\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E,\alpha])$, i.e., $\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E,\alpha]) := \frac{1}{[F:\mathbb{Q}]}\deg_F(x_F^{-1}(\{0,1,\infty\}))$ for $x_F : \operatorname{Spec} O_F \to \mathbb{P}^1$ representing $[E,\alpha] \in \mathbb{P}^1(F) \cong \mathbb{P}^1(O_F)$ (Note that $x_F^{-1}(\{0,1,\infty\})$ is supported in $\mathbb{V}(F)^{\mathrm{non}}$ and $\deg_F$ is the degree map on $\mathrm{ADiv}(F)$, not on $\mathrm{APic}(\operatorname{Spec} O_F)$). Note that we have

$$\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})} \lesssim \mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$$

on $\mathbb{P}^1(\overline{\mathbb{Q}})$, since the Archimedean portion is bounded below in light of the compactness of $(\mathbb{P}^1)^{\mathrm{arc}}$ (cf. the proof of Proposition 1.2).

We also note that ht_∞ in [GenEll, Section 3] is a function on $\overline{\mathcal{M}_{\mathrm{ell}}}(\overline{\mathbb{Q}})$, on the other hand, our $\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$ is a function on λ-line $\mathbb{P}^1(\overline{\mathbb{Q}})$, and that the pull-back of ht_∞ to the λ-line is equal to 6 times our $\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$ ([IUTchIV, Corollary 2.2 (i)], cf. also the proof of Lemma 1.13 (1) below).

Lemma 1.13. ([GenEll, Proposition 3.4, Lemma 3.5], [Silv2, Proposition 2.1, Corollary 2.3]) *Let $l > 2$ be a prime, E an elliptic curve over a number field F such that E has everywhere semistable reduction, and $H \subset E[l]$ a cyclic subgroup scheme of order l. Then we have*

(1) (relation between $\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$ and $\mathrm{ht}^{\mathrm{Falt}}$)

$$2\mathrm{ht}^{\mathrm{Falt}} \lesssim \mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})} \lesssim 2\mathrm{ht}^{\mathrm{Falt}} + \log(\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}) \lesssim 2\mathrm{ht}^{\mathrm{Falt}} + \epsilon\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$$

for any $\epsilon \in \mathbb{R}_{>0}$ on $U_{\mathbb{P}^1}(\overline{\mathbb{Q}})$,

(2) (relation betwen $\mathrm{ht}^{\mathrm{Falt}}([E,\alpha])$ and $\mathrm{ht}^{\mathrm{Falt}}([E/H,\mathrm{Im}(\alpha)])$)

$$\mathrm{ht}^{\mathrm{Falt}}([E,\alpha]) - \frac{1}{2}\log l \leq \mathrm{ht}^{\mathrm{Falt}}([E/H,\mathrm{Im}(\alpha)]) \leq \mathrm{ht}^{\mathrm{Falt}}([E,\alpha]) + \frac{1}{2}\log l.$$

(3) (relation between $\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E,\alpha])$ *and* $\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E/H,\mathrm{Im}(\alpha)])$*)*
Furthermore, we assume that l *is prime to* $v(q_{E,v}) \in \mathbb{Z}_{>0}$ *for any* $v \in \mathbb{V}(F)$*, where* E *has bad reduction with q-parameter* $q_{E,v}$ *(e.g.,* $l > v(q_{E,v})$ *for any such* v*'s). Then we have*

$$l \cdot \mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E,\alpha]) = \mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E/H,\mathrm{Im}(\alpha)]).$$

Proof. (1): We have the Kodaira-Spencer isomorphism $\omega_{\overline{\mathcal{E}}}^{\otimes 2} \cong \omega_{\mathbb{P}^1}(\{0,1,\infty\})$, where $\overline{\mathcal{E}}$ is the universal generalised elliptic curve over the compactification $\mathbb{P}^1$ of the λ-line, which extends $\mathcal{E}$ over the λ-line $U_{\mathbb{P}^1}$. Thus we have $\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})} \approx 2\mathrm{ht}_{\omega_{\overline{\mathcal{E}}}}$ on $\mathbb{P}^1(\overline{\mathbb{Q}})$. Thus, it is reduced to compare $\mathrm{ht}_{\omega_{\overline{\mathcal{E}}}}$ and $\mathrm{ht}^{\mathrm{Falt}}$. Here, $\mathrm{ht}_{\omega_{\overline{\mathcal{E}}}}$ is defined by equipping a hermitian metric on the line bundle $\omega_{\overline{\mathcal{E}}}$. On the other hand, $\mathrm{ht}^{\mathrm{Falt}}$ is defined by equipping a hermitian metric on the line bundle $\omega_{\mathcal{E}}$, which is the restriction of $\omega_{\overline{\mathcal{E}}}$ ot $U_{\mathbb{P}^1}$. Thus, it is reduced to compare the Archimedean contributions of $\mathrm{ht}_{\omega_{\overline{\mathcal{E}}}}$ and $\mathrm{ht}^{\mathrm{Falt}}$. The former metric is bounded on the compact space $(\mathbb{P}^1)^{\mathrm{arc}}$. On the other hand, we show the latter metric defined on the non-compact space $(U_{\mathbb{P}^1})^{\mathrm{arc}}$ has logarithmic singularity along $\{0,1,\infty\}$. Let $0 \neq dz \in \omega_E$ be an invariant differential over O_F. Then dz decomposes as $((dz_v)_{\mathrm{real}:v\in\mathbb{V}(F)^{\mathrm{arc}}}, (dz_v, \overline{dz_v})_{\mathrm{complex}:v\in\mathbb{V}(F)^{\mathrm{arc}}})$ on $E^{\mathrm{arc}} \cong \coprod_{\mathrm{real}:v\in\mathbb{V}(F)^{\mathrm{arc}}} E_v \coprod \coprod_{\mathrm{complex}:v\in\mathbb{V}(F)^{\mathrm{arc}}}(E_v \coprod \overline{E_v})$, where $\overline{dz_v}$, $\overline{E_v}$ are the complex conjugates of dz_v, E_v respectively. For $v \in \mathbb{V}(F)^{\mathrm{arc}}$, we have $E_v \cong \overline{F_v}^{\times}/q_{E,v}^{\mathbb{Z}} \cong \overline{F_v}/(\mathbb{Z}\oplus\tau_v\mathbb{Z})$ and dz_v is the descent of the usual Haar measure on $\overline{F_v}$, where $q_{E,v} = e^{2\pi i\tau_v}$ and τ_v is in the upper half plane. Then $||dz_v||^{\mathrm{Falt}}_{E_v} = (\frac{\sqrt{-1}}{2}\int_{E_v} dz_v \wedge \overline{dz_v})^{1/2} = (\mathrm{Im}(\tau_v))^{1/2} = (-\frac{1}{4\pi}\log(|q_{E,v}|_v^2))^{1/2}$ and $\mathrm{ht}^{\mathrm{Falt}}(E,dz)^{\mathrm{arc}} \approx -\frac{1}{2[F:\mathbb{Q}]}\sum_{v\in\mathbb{V}(F)^{\mathrm{arc}}}[F_v:\mathbb{R}]\log(-\log|q_{E,v}|_v)$ has a logarithmic singularity at $|q_{E,v}|_v = 0$. Thus, it is reduced to calculate the logarithmic singularity of $\mathrm{ht}^{\mathrm{Falt}}(E,dz)^{\mathrm{arc}}$ in terms of $\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$. We have $|j_E|_v = |j_{E_v}|_v \approx |q_{E,v}|_v^{-1}$ near $|q_{E,v}|_v = 0$, where j_E is the j-invariant of E. Then by the arithmetic-geometric inequality, we have $\mathrm{ht}^{\mathrm{Falt}}(E,dz)^{\mathrm{arc}} \approx -\frac{1}{2[F:\mathbb{Q}]}\log\prod_{v\in\mathbb{V}(F)^{\mathrm{arc}}}(\log|j_E|_v)^{[F_v:\mathbb{R}]} \geq -\frac{1}{2}\log\left(\frac{1}{[F:\mathbb{Q}]}\sum_{v\in\mathbb{V}(F)^{\mathrm{arc}}}\log|j_E|_v\right)$ near $\prod_{v\in\mathbb{V}(F)^{\mathrm{arc}}}|j_E|_v = \infty$. On the other hand, we have $|j|_v^{-1} \approx |\lambda|_v^2, |\lambda-1|_v^2, 1/|\lambda|_v^2$ near $|\lambda|_v = 0,1,\infty$ respectively for $v \in \mathbb{V}(F)^{\mathrm{arc}}$, since $j = 2^8(\lambda^2-\lambda+1)^3/\lambda^2(\lambda-1)^2$. Thus, we have $\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E,\alpha]) = \frac{1}{[F:\mathbb{Q}]}\sum_{v\in\mathbb{V}(F)^{\mathrm{non}}}(v(\lambda_E)+v(\lambda_E-1)+v(1/\lambda_E))\log q_v = \frac{1}{2[F:\mathbb{Q}]}\sum_{v\in\mathbb{V}(F)^{\mathrm{non}}}v(j_E^{-1})\log q_v = \frac{1}{2[F:\mathbb{Q}]}\sum_{v\in\mathbb{V}(F)^{\mathrm{non}}}\log|j_E^{-1}|_v$. By the product formula, this is equal to $\frac{1}{2[F:\mathbb{Q}]}\sum_{v\in\mathbb{V}(F)^{\mathrm{arc}}}\log|j_E|_v$. By combining these, we obtain $\mathrm{ht}^{\mathrm{Falt}}(E,dz)^{\mathrm{arc}} \gtrsim -\frac{1}{2}\log(2\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E,\alpha])) \approx -\frac{1}{2}\log(\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E,\alpha]))$ near $\prod_{v\in\mathbb{V}(F)^{\mathrm{arc}}}|j_E|_v = \infty$, or equivalently, near $\prod_{v\in\mathbb{V}(F)^{\mathrm{non}}}|j_E|_v = 0$. We also have $\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})} \lesssim \mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$ on $\mathbb{P}^1(\overline{\mathbb{Q}})$ since the Archimedean contribution is bounded below in light of the compactness of $(\mathbb{P}^1)^{\mathrm{arc}}$. Therefore, we have $\mathrm{ht}^{\mathrm{Falt}} \lesssim \mathrm{ht}_{\omega_{\overline{\mathcal{E}}}} \lesssim \mathrm{ht}^{\mathrm{Falt}} + \frac{1}{2}\log(\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})})$. This implies $2\mathrm{ht}^{\mathrm{Falt}} \lesssim \mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})} \lesssim 2\mathrm{ht}^{\mathrm{Falt}} + \log(\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})})$. The remaining portion comes from $\log(1+x) \lesssim \epsilon x$ for any $\epsilon \in \mathbb{R}_{>0}$.

(2): We have $\mathrm{ht}^{\mathrm{Falt}}([E,\alpha])^{\mathrm{non}} - \log l \leq \mathrm{ht}^{\mathrm{Falt}}([E/H, \mathrm{Im}(\alpha)])^{\mathrm{non}} \leq \mathrm{ht}^{\mathrm{Falt}}([E,\alpha])^{\mathrm{non}}$ since $\#\mathrm{coker}\{\omega_{E/H} \hookrightarrow \omega_E\}$ is killed by l. We also have $\mathrm{ht}^{\mathrm{Falt}}([E/H, \mathrm{Im}(\alpha)])^{\mathrm{arc}} = \mathrm{ht}^{\mathrm{Falt}}([E,\alpha])^{\mathrm{arc}} + \frac{1}{2}\log l$, since $(||-||^{\mathrm{Falt}}_{E/H})^2 = l(||-||^{\mathrm{Falt}}_{E})^2$ by the definition of $||-||^{\mathrm{Falt}}$ by the integrations on $E(\mathbb{C})$ and $(E/H)(\mathbb{C})$. By combining the non-Archimedean portion and the Archimedean portion, we have the second claim.

(3): Let $v \in \mathbb{V}(F)^{\mathrm{non}}$ where E has bad reduction. Then the l-cyclic subgroup $H \times_F F_v$ is the canonical multiplicative subgroup $\mathbb{F}_l(1)$ in the Tate curve $E \times_F F_v$, by the assumption $l \nmid v(q_{E,v})$. Then the claim follows from that the Tate parameter of E/H is equal to l-th power of the one of E. $\square$

Corollary 1.14. ([GenEll, Lemma 3.5]) *In the situation of Lemma 1.13 (3), we have*

$$\frac{l}{1+\epsilon}\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E,\alpha]) \leq \mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E,\alpha]) + \log l + C_\epsilon$$

for some constant $C_\epsilon \in \mathbb{R}$ which (may depend on ϵ, however) is independent of E, F, H and l.

Remark 1.14.1. The above corollary says that if $E[l]$ has a global multiplicative subgroup, then the height of E is bounded. Therefore, a global multiplicative subspace $M \subset E[l]$ does not exist for general E in the moduli of elliptic curves. A "global multiplicative subgroup" is one of the main themes of inter-universal Teichmüller theory. In inter-universal Teichmüller theory, we construct a kind of "global multiplicative subgroup" for sufficiently general E in the moduli of elliptic curves, by going out the scheme theory. cf. also Appendix A.

Proof. For $\epsilon > 0$, take $\epsilon' > 0$ such that $\frac{1}{1-\epsilon'} < 1+\epsilon$. There is a constant $A'_\epsilon \in \mathbb{R}$ such that $\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})} \leq 2\mathrm{ht}^{\mathrm{Falt}} + \epsilon'\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})} + A'_\epsilon$ on $U_{\mathbb{P}^1}(\overline{\mathbb{Q}})$ by the second and the third inequalities of Lemma 1.13 (1). We have $\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})} \leq 2(1+\epsilon)\mathrm{ht}^{\mathrm{Falt}} + A_\epsilon$ on $U_{\mathbb{P}^1}(\overline{\mathbb{Q}})$ by the choice of $\epsilon' > 0$, where $A_\epsilon := \frac{1}{1-\epsilon'}A'_\epsilon$. By the first inequality of Lemma 1.13 (1), we have $2\mathrm{ht}^{\mathrm{Falt}} \leq \mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})} + B$ for some constant $B \in \mathbb{R}$. Write $C_\epsilon := A_\epsilon + B$. Then we have $\frac{l}{1+\epsilon}\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E,\alpha]) = \frac{1}{1+\epsilon}\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E/H, \mathrm{Im}(\alpha)]) \leq 2\mathrm{ht}^{\mathrm{Falt}}([E/H, \mathrm{Im}(\alpha)]) + A_\epsilon \leq 2\mathrm{ht}^{\mathrm{Falt}}([E,\alpha]) + \log l + A_\epsilon \leq \mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}([E,\alpha]) + \log l + C_\epsilon$, where the equality follows from Lemma 1.13 (3), and the first inequality follows from Lemma 1.13 (2). $\square$

From now on, we use the assumptions and the notation in the previous subsection. We also write $\log(\mathfrak{q}^{\forall})$ (resp. $\log(\mathfrak{q}^{\dagger 2})$) for the $\mathbb{R}$-valued function on the λ-line $U_{\mathbb{P}^1}$ obtained by the normaised degree $\frac{1}{[L:\mathbb{Q}]}\deg_L$ of the effective ($\mathbb{Q}$-)arithmetic divisor determined by the q-parameters of an elliptic curve over a number field L at arbitrary non-Archimedean primes. (resp. non-arcihmedean primes which do not divide 2). Note that $\log(\mathfrak{q})$ in the previous subsection avoids the primes dividing $2l$, and that for a compactly bounded subset $\mathcal{K} \subset U_{\mathbb{P}^1}(\overline{\mathbb{Q}})$ whose support contains the prime 2, we have $\log(\mathfrak{q}^{\forall}) \approx \log(\mathfrak{q}^{\dagger 2})$ on $\mathcal{K}$ (cf. [IUTchIV, Corolarry 2.2 (i)]). We also note that we have

$$\frac{1}{6}\log(\mathfrak{q}^{\forall}) \approx \mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})} \approx \mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$$

on $\mathcal{K}\,(\subset U_{\mathbb{P}^1}(\overline{\mathbb{Q}}))$. (For the first equivalence, see the argument just before Lemma 1.13, and the proof of Lemma 1.13 (1); For the second equivalence, note that $\mathcal{K}$ includes the unique Archimedean place of $\mathbb{Q}$.)

Proposition 1.15. ([IUTchIV, Corollary 2.2]) *Let $\mathcal{K} \subset U_{\mathbb{P}^1}(\overline{\mathbb{Q}})$ be a compactly bounded subset with support containing $\mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$ and $2 \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$, and $A \subset U_{\mathbb{P}^1}(\overline{\mathbb{Q}})$ a finite set containing $\Big\{[(E,\alpha)] \mid \#\mathrm{Aut}_{\overline{\mathbb{Q}}}(E) \neq \{\pm 1\}\Big\}$. Then there exists $C_{\mathcal{K}} \in \mathbb{R}_{>0}$, which depends only on $\mathcal{K}$, satisfying the following property: Let $d \in \mathbb{Z}_{>0}$, $\epsilon \in \mathbb{R}_{>0}$, and set $d^* := 2^{12} \cdot 3^3 \cdot 5 \cdot d$. Then there exists a finite subset $\mathfrak{Exc}_{\mathcal{K},d,\epsilon} \subset U_{\mathbb{P}^1}(\overline{\mathbb{Q}})^{\leq d}$ such that $\mathfrak{Exc}_{\mathcal{K},d,\epsilon} \supset A$ and satisfies the following property: Let $x = [(E_F,\alpha)] \in (U_{\mathbb{P}^1}(F) \cap \mathcal{K}) \setminus \mathfrak{Exc}_{\mathcal{K},d,\epsilon}$ with $[F:\mathbb{Q}] \leq d$. Write F_{mod} for the field of moduli of $E_{\overline{F}} := E_F \times_F \overline{F}$, and $F_{\mathrm{tpd}} := F_{\mathrm{mod}}(E_{F_{\mathrm{mod}}}[2]) \subset \overline{F}$ where $E_{F_{\mathrm{mod}}}$ is a model of $E_{\overline{F}}$ over F_{mod} (Note that $F_{\mathrm{mod}}(E_{F_{\mathrm{mod}}}[2])$ is independent of the choice of the model $E_{F_{\mathrm{mod}}}$ by the assumption of $\mathrm{Aut}_{\overline{F}}(E_{\overline{F}}) \neq \{\pm 1\}$, and that $F_{\mathrm{mod}}(E_{F_{\mathrm{mod}}}[2]) \subset F$ since $[(E_F,\alpha)] \in U_{\mathbb{P}^1}(F)$. cf. Lemma 1.7 (1)). We assume that all the points of $E_F[3\cdot 5]$ are rational over F and that $F = F_{\mathrm{tpd}}(\sqrt{-1}, E_{F_{\mathrm{tpd}}}[3\cdot 5])$, where $E_{F_{\mathrm{tpd}}}$ is a model of $E_{\overline{F}}$ over F_{tpd} which is defined by the Legendre form (Note that $E_F \cong E_{F_{\mathrm{tpd}}} \times_{F_{\mathrm{tpd}}} F$ and E_F has semistable reduction for all $w \in \mathbb{V}(F)^{\mathrm{non}}$ by Lemma 1.7 (2), (3)). Then E_F and F_{mod} arise from an* ***initial* $\boldsymbol{\Theta}$*-data*** *(cf. Definition 10.1)*

$$(\overline{F}/F, X_F, l, \underline{C}_K, \underline{\mathbb{V}}, \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}, \underline{\epsilon}).$$

(Note that it is included in the definition of initial Θ-data that the image of the outer homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\mathbb{F}_l)$ determined by $E_F[l]$ contains $\mathrm{SL}_2(\mathbb{F}_l)$.) Furthermore, we assume that $\boxed{-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})|}$ for E_F and F_{mod}, which arise from an initial Θ-data. Then we have

$$\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}(x) \leq (1+\epsilon)(\text{log-diff}_{\mathbb{P}^1}(x) + \text{log-cond}_{\{0,1,\infty\}}(x)) + C_{\mathcal{K}}.$$

Remark 1.15.1. We take $A = \{[(E,\alpha)] \in U_{\mathbb{P}^1}(\overline{\mathbb{Q}}) \mid E$ does not admit $\overline{\mathbb{Q}}$-core$\}$. cf. Definition 3.3 and Lemma C.3 for the definition of k-core, the finiteness of A, and that $A \supset \Big\{[(E,\alpha)] \mid \#\mathrm{Aut}_{\overline{\mathbb{Q}}}(E) \neq \{\pm 1\}\Big\}$.

Remark 1.15.2. By Proposition 1.15, Theorem 0.1 is reduced to show $-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})|$ for E_F and F_{mod}, which arise from an initial Θ-data. The inequality $-|\log(\underline{\underline{q}})| \leq$

$-|\log(\underline{\underline{\Theta}})|$ is almost a tautological translation of the inequality which we want to show (cf. also Appendix A). In this sense, these reduction steps are just calculations to reduce the main theorem to the situation where we can take an initial Θ-data, i.e., the situation where the inter-universal Teichmüller theory works, and no deep things happen in these reduction steps.

Proof. First we write $\mathfrak{Exc}_{\mathcal{K},d} := A$, and we enlarge the finite set $\mathfrak{Exc}_{\mathcal{K},d}$ several times in the rest of the proof in the manner that depends only on $\mathcal{K}$ and d, but not on x. When it will depend on $\epsilon > 0$, then we will change the notation $\mathfrak{Exc}_{\mathcal{K},d}$ by $\mathfrak{Exc}_{\mathcal{K},d,\epsilon}$. Let $x = [(E_F, \alpha)] \in (U_{\mathbb{P}^1}(F) \cap \mathcal{K}) \setminus \mathfrak{Exc}_{\mathcal{K},d}$.

Let $\eta_{\text{prm}} \in \mathbb{R}_{>0}$ be the constant in Lemma 1.11. We take another constant $\xi_{\text{prm}} \in \mathbb{R}_{>0}$ determined by using the prime number theorem as follows (cf. [GenEll, Lemma 4.1]): We define $\vartheta(x) := \sum_{\text{prime}:\, p \leq x} \log p$ (Chebychev's ϑ-function). By the *prime number theorem* (and Lemma C.4), we have $\vartheta(x) \sim x$ $(x \to \infty)$, where $\sim$ means that the ration of the both side goes to 1. Hence there exists a constant $\mathbb{R} \ni \xi_{\text{prim}} \geq 5$ such that

(s0) $$\frac{2}{3}x < \vartheta(x) \leq \frac{4}{3}x$$

for any $x \geq \xi_{\text{prm}}$.

Let $h := h(E_F) = \log(\mathfrak{q}^{\forall}) = \frac{1}{[F:\mathbb{Q}]} \sum_{v \in \mathbb{V}(F)^{\text{non}}} h_v f_v \log(p_v)$ be the summation of the contributions from $\mathfrak{q}_v$ for $v \in \mathbb{V}(F)^{\text{non}}$. and the degree of extension of the residue field over $\mathbb{F}_{p_v}$ respectively. Note also that $h_v \in \mathbb{Z}_{\geq 0}$ and that $h_v = 0$ if and only if E_F has good reduction at v. By $\frac{1}{6}\log(\mathfrak{q}^{\forall}) \approx \text{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$ and Proposition C.1, we there are only finiely many isomorphism classes of E_F (hence finiely many $x = [E_F, \alpha]$) satisfying $h^{\frac{1}{2}} < \xi_{\text{prm}} + \eta_{\text{prm}}$. Therefore, by enlarging the finite set $\mathfrak{Exc}_{\mathcal{K},d}$, we may assume that

(s1) $$h^{\frac{1}{2}} \geq \xi_{\text{prm}} + \eta_{\text{prm}}.$$

Note that $h^{\frac{1}{2}} \geq 5$ since $\xi_{\text{prm}} \geq 5$ and $\eta_{\text{prm}} > 0$. We have

(s2)
$$2d^* h^{\frac{1}{2}} \log(2d^* h) \geq 2[F:\mathbb{Q}] h^{\frac{1}{2}} \log(2[F:\mathbb{Q}]h) \geq \sum_{h_v \neq 0} 2h^{-\frac{1}{2}} \log(2h_v f_v \log(p_v)) h_v f_v \log(p_v)$$
$$\geq \sum_{h_v \neq 0} h^{-\frac{1}{2}} \log(h_v) h_v \geq \sum_{h_v \geq h^{1/2}} h^{-\frac{1}{2}} \log(h_v) h_v \geq \sum_{h_v \geq h^{1/2}} \log(h_v),$$

where the third inequality follows from $2\log(p_v) \geq 2\log 2 = \log 4 > 1$. By $[F:\mathbb{Q}] \leq d^*$, we also have

(s3) $$d^*h^{\frac{1}{2}} \geq [F:\mathbb{Q}]h^{\frac{1}{2}} = \sum_{v \in \mathbb{V}(F)^{\mathrm{non}}} h^{-\frac{1}{2}} h_v f_v \log(p_v) \geq \sum_{v \in \mathbb{V}(F)^{\mathrm{non}}} h^{-\frac{1}{2}} h_v \log(p_v)$$
$$\geq \sum_{h_v \geq h^{1/2}} h^{-\frac{1}{2}} h_v \log(p_v) \geq \sum_{h_v \geq h^{1/2}} \log(p_v).$$

Let $\mathcal{A}$ be the set of prime numbers satisfying either

(S1) $p \leq h^{\frac{1}{2}}$,

(S2) $p \mid h_v \neq 0$ for some $v \in \mathbb{V}(F)^{\mathrm{non}}$, or

(S3) $p = p_v$ for some $v \in \mathbb{V}(F)^{\mathrm{non}}$ and $h_v \geq h^{\frac{1}{2}}$.

Then we have

(S'1) $\sum_{p:(\mathrm{S1})} \log p = \vartheta(h^{\frac{1}{2}}) \leq \frac{4}{3} h^{\frac{1}{2}}$ by the second inequality of (s0), and $h^{\frac{1}{2}} \geq \xi_{\mathrm{prm}}$, which follows from (s1),

(S'2) $\sum_{p:(\mathrm{S2}),\,\mathrm{not}\,(\mathrm{S3})} \log p \leq \sum_{h_v > h^{1/2}} \log(h_v) \leq 2d^* h^{\frac{1}{2}} \log(2d^*h)$ by (s2), and

(S'3) $\sum_{p:(\mathrm{S3})} \log p \leq d^* h^{\frac{1}{2}}$ by (s3).

Then we obtain

(S'123) $$\vartheta_{\mathcal{A}} := \sum_{p \in \mathcal{A}} \log(p) \leq 2h^{\frac{1}{2}} + d^*h^{\frac{1}{2}} + 2d^*h^{\frac{1}{2}} \log(2d^*h)$$
$$\leq 4d^*h^{\frac{1}{2}} \log(2d^*h) \leq -\xi_{\mathrm{prm}} + 5d^*h^{\frac{1}{2}} \log(2d^*h),$$

where the first inequality follows from (S'1), (S'2), and (S'3), the second inequality follows from $2h^{\frac{1}{2}} \leq d^*h^{\frac{1}{2}}$ and $\log(2d^*h^{\frac{1}{2}}) \geq \log 4 > 1$, and the last inequality follows from (s1). Then there exists a prime number $l \notin \mathcal{A}$ such that $l \leq 2(\vartheta_{\mathcal{A}} + \xi_{\mathrm{prm}})$, because otherwise we have $\vartheta_{\mathcal{A}} \geq \vartheta(2(\vartheta_{\mathcal{A}} + \xi_{\mathrm{prm}})) \geq \frac{2}{3}(2(\vartheta_{\mathcal{A}} + \xi_{\mathrm{prm}})) \geq \frac{4}{3}\vartheta_{\mathcal{A}}$, by the second inequality of (s0), which is a contradiction. Since $l \notin \mathcal{A}$, we have

(P1) (upper bound of l)
$(5 \leq)\, h^{\frac{1}{2}} < l \leq 10d^*h^{\frac{1}{2}} \log(2d^*h)\, (\leq 20(d^*)^2h^2)$,
where the second inequality follows from that l does not satisfy (S1), the third inequality follows from $l \leq 2(\vartheta_{\mathcal{A}} + \xi_{\mathrm{prm}})$ and (S'123), and the last inequality follows from $\log(2d^*h) \leq 2d^*h \leq 2d^*h^{\frac{3}{2}}$ (since $\log x \leq x$ for $x \geq 1$),

(P2) (monodromy non-vanishing modulo l)
$l \nmid h_v$ for any $v \in \mathbb{V}(F)^{\mathrm{non}}$ such that $h_v \neq 0$, since l does not satisfy (S2), and

(P3) (upper bound of monodromy at l)
if $l = p_v$ for some $v \in \mathbb{V}(F)^{\mathrm{non}}$, then $h_v < h^{\frac{1}{2}}$, since l does not satisfy (S3).

Claim 1: We claim that, by enlarging the finite set $\mathfrak{Erc}_{\mathcal{K},d}$, we may assume that

(P4) there does not exist l-cyclic subgroup scheme in $E_F[l]$.

Proof of Claim 1: If there exists an l-cyclic subgroup scheme in $E_F[l]$, then by applying Corollary 1.14 for $\epsilon = 1$, we have $\frac{l-2}{2}\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}(x) \leq \log l + T_{\mathcal{K}} \leq l + T_{\mathcal{K}}$ (since $\log x \leq x$ for $x \geq 1$) for some $T_{\mathcal{K}} \in \mathbb{R}_{>0}$, where $T_{\mathcal{K}}$ depends only on $\mathcal{K}$. Thus, $\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}(x)$ is bounded because we have $\mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}(x) \leq \frac{2l}{l-2} + \frac{2}{l-2}T_{\mathcal{K}} < \frac{14}{7-2} + \frac{2}{7-2}T_{\mathcal{K}}$. Note also that $\mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})} \approx \mathrm{ht}^{\mathrm{non}}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$ on $\mathcal{K}$, since $\mathcal{K}$ includes the unique Archimedean place of $\mathbb{Q}$. Therefore, there exist only finitely many such $x = [E_F, \alpha]$'s by Proposition C.1. The claim is proved.

Claim 2: Next, we claim that, by enlarging the finite set $\mathfrak{Erc}_{\mathcal{K},d}$, we may assume that

(P5) $\emptyset \neq \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} := \{v \in \mathbb{V}^{\mathrm{non}}_{\mathrm{mod}} \mid v \nmid 2l, \text{ and } E_F \text{ has bad multiplicative reduction at } v\}$

Proof of Claim 2: First, we note that we have

$$h^{\frac{1}{2}} \log l \leq h^{\frac{1}{2}} \log(20(d^*)^2 h^2) \leq 2h^{\frac{1}{2}} \log(5d^* h) \tag{p5a}$$
$$\leq 8h^{\frac{1}{2}} \log(2(d^*)^{\frac{1}{4}} h^{\frac{1}{4}}) \leq 8h^{\frac{1}{2}} 2(d^*)^{\frac{1}{4}} h^{\frac{1}{4}} = 16(d^*)^{\frac{1}{4}} h^{\frac{3}{4}}. \tag{p5b}$$

where the first inequality follows from (P1). If $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} = \emptyset$, then we have $h \approx \log(\mathfrak{q}^{\dagger 2}) \leq h^{\frac{1}{2}} \log l \leq 16(d^*)^{\frac{1}{4}} h^{\frac{3}{4}}$ on $\mathcal{K}$, where the first inequality follows from (P3), and the last inequality is (p5b). Thus, $h^{\frac{1}{4}}$, hence h as well, is bounded. Therefore, there exist only finitely many such $x = [E_F, \alpha]$'s by Proposition C.1. The claim is proved.

Claim 3: We also claim that, by enlarging the finite set $\mathfrak{Erc}_{\mathcal{K},d}$, we may assume that

(P6) The image of the outer homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\mathbb{F}_l)$ determined by $E_F[l]$ contains $\mathrm{SL}_2(\mathbb{F}_l)$.

Proof of Claim 3 (cf. [GenEll, Lemma 3.1 (i), (iii)]): By (P2) $l \nmid h_v \neq 0$ and (P5) $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \neq \emptyset$, the image H of the outer homomorphism contains the matrix $N_+ := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Here, N_+ generates an l-Sylow subgroup S of $\mathrm{GL}_2(\mathbb{F}_l)$, and the number of l-Sylow subgroups of $\mathrm{GL}_2(\mathbb{F}_l)$ is precisely $l+1$. Note that the normaliser of S in $\mathrm{GL}_2(\mathbb{F}_l)$ is the subgroup of the upper triangular matrices. By (P4) $E[l] \not\supset$ (l-cyclic subgroup), the image contains a matrix which is not upper triangluar. Thus, the number n_H of

l-Sylow subgroups of H is greater than 1. On the other hand, $n_H \equiv 1 \,(\mathrm{mod}\, l)$ by the general theory of Sylow subgroups. Then we have $n_H = l+1$ since $1 < n_H \leq l+1$. In particular, we have $N_+ = \begin{pmatrix} 1\,1 \\ 0\,1 \end{pmatrix}, N_- := \begin{pmatrix} 1\,0 \\ 1\,1 \end{pmatrix} \in H$. Let $G \subset \mathrm{SL}_2(\mathbb{F}_l)$ be the subgroup generated by N_+ and N_-. Then it suffices to show that $G = \mathrm{SL}_2(\mathbb{F}_l)$. We note that for $a, b \in \mathbb{F}_l$, the matrix $N_-^b N_+^a$ (this makes sense since $N_+^l = N_-^l = 1$) takes the vector $v := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} a \\ ab+1 \end{pmatrix}$. This implies that we have $\left(\mathbb{F}_l^\times \times \mathbb{F}_l\right) \subset G$. This also implies that for $c \in \mathbb{F}_l^\times$, there exists $A_c \in G$ such that $A_c v = \begin{pmatrix} c \\ 0 \end{pmatrix} (= cA_1 v)$. Then we have $cv = A_1^{-1} A_c v \in Gv$. Thus, we proved that $(\mathbb{F}_l \times \mathbb{F}_l) \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \subset Gv$. Let $M \in \mathrm{SL}_2(\mathbb{F}_l)$ be any matrix. By multiplying M by an element in G, we may assume that $Mv = v$, since $(\mathbb{F}_l \times \mathbb{F}_l) \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \subset Gv$. This means that $M \subset \left\{ \begin{pmatrix} 1\,0 \\ *\,1 \end{pmatrix} \right\}$. Thus, M is a power of N_-. The claim is proved.

Then we take, as parts of initial Θ-data, $\overline{F}$ to be $\overline{\mathbb{Q}}$ so far, F, X_F, l to be the number field F, once-punctured elliptic curve associated to E_F, and the prime number, respectively, in the above discussion, and $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$ to be the set $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$ of $(P5)$. By using $(P1)$, $(P2)$, $(P5)$, and $(P6)$, there exist data $\underline{C}_K$, $\underline{\mathbb{V}}$, and $\underline{\epsilon}$, which satisfy the conditions of initial Θ-data (cf. Definition 10.1. The existence of $\underline{\mathbb{V}}$ and $\underline{\epsilon}$ is a consequence of $(P6)$), and moreover,

(P7) the resulting initial Θ-data $(\overline{F}/F, X_F, l, \underline{C}_K, \underline{\mathbb{V}}, \mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}, \underline{\epsilon})$ satisfies the conditions in Section 1.3.

Now, we have $\boxed{-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})|}$ by assumption, and apply Proposition 1.12. (Note that we are in the situation where we can apply it.)
Then we obtain

$$\begin{aligned} &\frac{1}{6}\log(\mathfrak{q}) \leq \left(1 + \frac{20 d_{\mathrm{mod}}}{l}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right) + 20(e^*_{\mathrm{mod}} l + \eta_{\mathrm{prm}}) \\ \text{(A)} \qquad &\leq \left(1 + d^* h^{-\frac{1}{2}}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right) + 200(d^*)^2 h^{\frac{1}{2}} \log(2d^* h) + 20\eta_{\mathrm{prm}}, \end{aligned}$$

where the second inequality follows from the second and third inequalities in (P1) and $20 d_{\mathrm{mod}} < d^*_{\mathrm{mod}} (:= 2^{12} \cdot 3^3 \cdot 5 \cdot d_{\mathrm{mod}}) \leq d^* (:= 2^{12} \cdot 3^3 \cdot 5 \cdot d_{\mathrm{mod}})$. We also have

$$\text{(B)} \qquad \frac{1}{6}\log(\mathfrak{q}^{\dagger 2}) - \frac{1}{6}\log(\mathfrak{q}) \leq \frac{1}{6} h^{\frac{1}{2}} \log l \leq \frac{1}{3} h^{\frac{1}{2}} \log(5d^* h) \leq h^{\frac{1}{2}} \log(2d^* h),$$

where the first inequality follows from (P3) and (P5), the second inequality follows from (p5a), and the last inequality follows from $5 < 2^3$. We also note that

$$\text{(C)} \qquad \frac{1}{6}\log(\mathfrak{q}^{\forall}) - \frac{1}{6}\log(\mathfrak{q}^{\dagger 2}) \leq B_{\mathcal{K}}$$

for some constant $B_{\mathcal{K}} \in \mathbb{R}_{>0}$, which depends only on $\mathcal{K}$, since $\log(\mathfrak{q}^{\forall}) \approx \log(\mathfrak{q}^{\dagger 2})$ on $\mathcal{K}$ as remarked when we introduced $\log(\mathfrak{q}^{\forall})$ and $\log(\mathfrak{q}^{\dagger 2})$ just before this proposition. By combining (A), (B), and (C), we obtain

$$\begin{aligned}
\frac{1}{6}h = \frac{1}{6}\log(\mathfrak{q}^{\forall}) &\leq \left(1 + d^* h^{-\frac{1}{2}}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right) + (15d^*)^2 h^{\frac{1}{2}} \log(2d^*h) + \frac{1}{2}C_{\mathcal{K}} \\
\text{(ABC)} \qquad &\leq \left(1 + d^* h^{-\frac{1}{2}}\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right) + \frac{1}{6}h\frac{2}{5}(60d^*)^2 h^{-\frac{1}{2}} \log(2d^*h) + \frac{1}{2}C_{\mathcal{K}},
\end{aligned}$$

where we write $C_{\mathcal{K}} := 40\eta_{\mathrm{prm}} + 2B_{\mathcal{K}}$, the first inequality follows from $200 < 15^2$, the second inequality follows from $1 < \frac{32}{30} = \frac{1}{6}\frac{2}{5}4^2$. Here, we put $\epsilon_E := (60d^*)^2 h^{-\frac{1}{2}} \log(2d^*h)$ $(\geq 5d^* h^{-\frac{1}{2}})$. We have

$$\text{(Epsilon)} \qquad \epsilon_E \leq 4(60d^*)^2 h^{-\frac{1}{2}} \log(2(d^*)^{\frac{1}{4}} h^{\frac{1}{4}}) \leq 4(60d^*)^3 h^{-\frac{1}{2}} h^{\frac{1}{4}} = 4(60d^*)^3 h^{-\frac{1}{4}}.$$

Let $\epsilon > 0$. If $\epsilon_E > \min\{1, \epsilon\}$, then $h^{\frac{1}{4}}$, hence h as well, is bounded by (Epsilon). Therefore, by Proposition C.1, by replacing the finite set $\mathfrak{Exc}_{\mathcal{K},d}$ by a finite set $\mathfrak{Exc}_{\mathcal{K},d,\epsilon}$, we may assume that $\epsilon_E \leq \min\{1, \epsilon\}$. Then finally we obtain

$$\begin{aligned}
\frac{1}{6}h &\leq \left(1 - \frac{2}{5}\epsilon_E\right)^{-1}\left(1 + \frac{1}{5}\epsilon_E\right)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right) + \left(1 - \frac{2}{5}\epsilon_E\right)^{-1}\frac{1}{2}C_{\mathcal{K}} \\
&\leq (1 + \epsilon_E)\left(\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})\right) + C_{\mathcal{K}} \\
&\leq (1 + \epsilon)\left(\text{log-diff}_{\mathbb{P}^1}(x_E) + \text{log-cond}_{\{0,1,\infty\}}(x_E)\right) + C_{\mathcal{K}},
\end{aligned}$$

where the first inequality follows from the definition of ϵ_E and $\epsilon \geq 5d^* h^{-\frac{1}{2}}$, the second inequality follows from $\frac{1+\frac{1}{5}\epsilon_E}{1-\frac{2}{5}\epsilon_E} \leq 1 + \epsilon_E$ (i.e., $\epsilon_E(1 - \epsilon_E) \geq 0$, which holds since $\epsilon_E \leq 1$), and $1 - \frac{2}{5}\epsilon_E \geq \frac{1}{2}$ (i.e., $\epsilon_E \leq \frac{5}{4}$, which holds since $\epsilon_E \leq 1$), and the third inequality follows from $\epsilon_E \leq \epsilon$, $\text{log-diff}_{\mathbb{P}^1}(x_E) = \log(\mathfrak{d}^{F_{\mathrm{tpd}}})$ by definition, and $\log(\mathfrak{f}^{F_{\mathrm{tpd}}}) \leq \text{log-cond}_{\{0,1,\infty\}}(x_E)$. (Note that $\mathrm{Supp}(\mathfrak{f})$ excludes the places dividing $2l$ in the definition.) Now the proposition follows from $\frac{1}{6}\log(\mathfrak{q}^{\forall}) \approx \mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$ on $\mathbb{P}^1(\overline{\mathbb{Q}})$ as remarked just before this proposition (by the effect of this $\approx$, the $C_{\mathcal{K}}$ in the statement of the proposition may differ from the $C_{\mathcal{K}}$ in the proof). $\square$

Remark 1.15.3. (**Miracle Identity**) As shown in the proof, the reason that the main term of the inequality is 1 (i.e., ht $\leq$ ($\boxed{1}+\epsilon$)(log-diff + log-cond) + bounded term) is as follows (cf. the calculations in the proof of Lemma 1.10): On one hand (ht-side), we have an average $6\frac{1}{2l}\frac{1}{l/2}\sum_{j=1}^{l/2} j^2 \approx 6\frac{1}{2l}\frac{1}{l/2}\frac{1}{3}\left(\frac{l}{2}\right)^3 = \frac{l}{4}$. Note that we multiply $\frac{1}{2l}$ since the theta function under consideration lives in a covering of degree $2l$, and that we multiply 6 since the degree of λ-line over j-line is 6. On the other hand ((log-diff + log-cond)-side), we have an average $\frac{1}{l/2}\sum_{j=1}^{l/2} j \approx \frac{1}{l/2}\frac{1}{2}\left(\frac{l}{2}\right)^2 = \frac{l}{4}$. *These two values miraculously coincide!* In other words, the reason that the main term of the inequality is 1 comes from the equality

$$6 \text{ (the degree of } \lambda\text{-line over } j\text{-line)} \times \frac{1}{2} \text{ (theta function involves a double covering)}$$
$$\times \frac{1}{2^2} \text{ (the exponent of theta series is quadratic)} \times \frac{1}{3} \text{ (the main term of } \sum_{j=1}^{n} j^2 \approx n^3/3)$$
$$= \frac{1}{2^1} \text{ (the terms of differents are linear)} \times \frac{1}{2} \text{ (the main term of } \sum_{j=1}^{n} j \approx n^2/2).$$

This equality was already observed in Hodge-Arakelove theory, and motivates the definition of the Θ-link (cf. also Appendix A). *Mochizuki firstly observed this equality, and next he established the framework (i.e. going out of the scheme theory and studying inter-universal geometry) in which these calculations work* (cf. also [IUTchIV, Remark 1.10.1]).

Note also that it is already known that this main term 1 cannot be improved by Masser's calculations in analytic number theory (cf. [Mass2]).

Remark 1.15.4. (ϵ-term) In the proof of Proposition 1.15, we also obtained an upper bound of the second main term (i.e., the main behaviour of the term involved to ϵ) of the Diophantine inequality (when restricted to $\mathcal{K}$):

$$ht \leq \delta + *\delta^{\frac{1}{2}} \log(\delta)$$

on $\mathcal{K}$, where $*$ is a positive real constant, $ht := \mathrm{ht}_{\omega_{\mathbb{P}^1}(\{0,1,\infty\})}$ and $\delta := \text{log-diff}_{\mathbb{P}^1} + \text{log-cond}_{\{0,1,\infty\}}$ (cf. (ABC) in the proof of Proposition 1.15) It seems that the exponent $\frac{1}{2}$ suggests a possible relation to **Riemann hypothesis**. For more informations, see [IUTchIV, Remark 2.2.1] for remarks on a possible relation to **inter-universal Melline transformation**, and [vFr], [Mass2] for lower bounds of the ϵ-term from analytic number theory.

Remark 1.15.5. (Uniform ABC) So-called the **uniform** *abc* **Conjecture** (uniformity with respect to d of the bounded discrepancy in the Diophantine inequality) is not proved yet; however, we have an estimate of the dependence on d of our upper bound as follows (cf. [IUTchIV, Corollary 2.2 (ii), (iii)]): For any $0 < \epsilon_d \leq 1$, write

$\epsilon_d^* := \frac{1}{16}\epsilon_d (< \frac{1}{2})$. Then we have

$$\begin{aligned}
&\min\{1,\epsilon\}^{-1}\epsilon_E = \min\{1,\epsilon\}^{-1}(60d^*)^2 h^{-\frac{1}{2}}\log(2d^*h)\\
&= (\min\{1,\epsilon\}\epsilon_d^*)^{-1}(60d^*)^2 h^{-\frac{1}{2}}\log(2^{\epsilon_d^*}(d^*)^{\epsilon_d^*}h^{\epsilon_d^*})\\
&\leq (\min\{1,\epsilon\}\epsilon_d^*)^{-1}(60d^*)^{2+\epsilon_d^*}h^{-(\frac{1}{2}-\epsilon_d^*)} \leq \left((\min\{1,\epsilon\}\epsilon_d^*)^{-3}(60d^*)^{4+\epsilon_d}h^{-1}\right)^{\frac{1}{2}-\epsilon_d^*},
\end{aligned}$$

where the first inequality follows from $h^{\frac{1}{2}} \geq 5$, and $x \leq \log x$ for $x \geq 1$, and the second inequality follows from $-3(\frac{1}{2}-\epsilon_d^*) = -\frac{3}{2}+\frac{3}{16}\epsilon_d \leq -\frac{21}{16} < -1$ and $(\frac{1}{2}-\epsilon_d^*)(4+\epsilon_d) = -\frac{1}{16}\epsilon_d^2 + \frac{1}{4}\epsilon_d + 2 \geq \frac{1}{4}\epsilon_d + 2 \geq \epsilon_d^* + 2$. We recall that, at the final stage of the proof of Proposition 1.15, we enlarged $\mathfrak{Exc}_{\mathcal{K},d}$ to $\mathfrak{Exc}_{\mathcal{K},d,\epsilon}$ so that it includes the points satisfying $\epsilon_E > \min\{1,\epsilon\}$. Now, we enlarge $\mathfrak{Exc}_{\mathcal{K},d}$ to $\mathfrak{Exc}_{\mathcal{K},d,\epsilon,\epsilon_d}$, which depends only on $\mathcal{K}$, d, ϵ, and ϵ_d, so that it includes the points satisfying $\epsilon_E > \min\{1,\epsilon\}$. Therefore, we obtain an inequality

$$ht := \frac{1}{6}h \leq H_{\mathrm{unif}}\min\{1,\epsilon\}^{-3}\epsilon_d^{-3}d^{4+\epsilon_d} + H_{\mathcal{K}}$$

on $\mathfrak{Exc}_{\mathcal{K},d,\epsilon,\epsilon_d}$, where $H_{\mathrm{unif}} \in \mathbb{R}_{>0}$ is independent of $\mathcal{K}$, d, ϵ, and ϵ_d, and $H_{\mathcal{K}} \in \mathbb{R}_{>0}$ depends only on $\mathcal{K}$. The above inequality shows an explicit dependence on d of our upper bound.

§ 2. Preliminaries on Anabelian Geometry.

In this section, we give some reviews on the preliminaries on anabelian geometry which will be used in the subsequent sections.

§ 2.1. Some Basics on Galois Groups of Local Fields.

Proposition 2.1. ([AbsAnab, Proposition 1.2.1]) *For $i = 1,2$, let K_i be a finite extension of $\mathbb{Q}_{p_i}$ with residue field k_i, and $\overline{K_i}$ be an algebraic closure of K_i with residue field $\overline{k_i}$ (which is an algebraic closure of k_i). We write $e(K_i)$ for the ramification index of K_i over $\mathbb{Q}_{p_i}$ and write $f(K_i) := [k_i : \mathbb{F}_{p_i}]$. Write $G_{K_i} := \mathrm{Gal}(\overline{K_i}/K_i)$, and we write $P_{K_i} \subset I_{K_i}(\subset G_{K_i})$ for the wild inertia subgroup and the inertia subgroup of G_{K_i} respectively. Let $\alpha : G_{K_1} \xrightarrow{\sim} G_{K_2}$ be an isomorphism of profinite groups. Then we have the following:*

(1) $p_1 = p_2\ (=: p)$.

(2) The abelianisation $\alpha^{\mathrm{ab}} : G_{K_1}^{\mathrm{ab}} \xrightarrow{\sim} G_{K_2}^{\mathrm{ab}}$, and the inclusions $k_i^\times \subset O_{K_i}^\times \subset K_i^\times \subset G_{K_i}^{\mathrm{ab}}$, where the last inclusion is defined by the local class field theory, induce isomorphisms

(a) $\alpha^{\mathrm{ab}} : k_1^\times \xrightarrow{\sim} k_2^\times$,

(b) $\alpha^{\mathrm{ab}} : O_{K_1}^\times \xrightarrow{\sim} O_{K_2}^\times$,

(c) $\alpha^{\mathrm{ab}} : O^{\triangleright}_{K_1} \xrightarrow{\sim} O^{\triangleright}_{K_2}$ *(cf. Section 0.2 for the notation* $O^{\triangleright}_{K_i}$*), and*

(d) $\alpha^{\mathrm{ab}} : K_1^{\times} \xrightarrow{\sim} K_2^{\times}$*.*

(3) *(a)* $[K_1 : \mathbb{Q}_p] = [K_2 : \mathbb{Q}_p]$*,*

(b) $f(K_1) = f(K_2)$*, and*

(c) $e(K_1) = e(K_2)$*.*

(4) The restrictions of α *induce*

(a) $\alpha|_{I_{K_1}} : I_{K_1} \xrightarrow{\sim} I_{K_2}$*, and*

(b) $\alpha|_{P_{K_1}} : P_{K_1} \xrightarrow{\sim} P_{K_2}$*.*

(5) The induced map $G^{\mathrm{ab}}_{K_1}/I_{K_1} \xrightarrow{\sim} G^{\mathrm{ab}}_{K_2}/I_{K_2}$ *preserves the Frobenius element* Frob_{K_i} *(i.e., the automorphism given by* $\overline{k_i} \ni x \mapsto x^{\#k_i}$*).*

(6) The collection of the isomorphisms $\left\{(\alpha|_{U_1})^{\mathrm{ab}} : U_1^{\mathrm{ab}} \xrightarrow{\sim} U_2^{\mathrm{ab}}\right\}_{G_{K_1} \overset{open}{\supset} U_1 \overset{\alpha}{\xrightarrow{\sim}} U_2 \subset G_{K_2}}$ *induces an isomorphism* $\mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K_1}) \xrightarrow{\sim} \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K_2})$*, which is compatible with the actions of* G_{K_i} *for* $i = 1, 2$*, via* $\alpha : G_{K_1} \xrightarrow{\sim} G_{K_2}$*. In particular,* α *preserves the cyclotomic characters* $\chi_{\mathrm{cyc},i}$ *for* $i = 1, 2$*.*

(7) The isomorphism $\alpha^* : H^2(\mathrm{Gal}(\overline{K_2}/K_2), \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K_2})) \xrightarrow{\sim} H^2(\mathrm{Gal}(\overline{K_1}/K_1), \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K_1}))$ *induced by* α *is compatible with the isomorphisms* $H^2(\mathrm{Gal}(\overline{K_i}/K_i), \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K_i})) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ *in the local class field theory for* $i = 1, 2$*.*

Remark 2.1.1. In the proof, we can see that the objects in the above (1)–(7) are functorially reconstructed by using only K_1 (or K_2), and we have no need of both K_1 and K_2, nor the isomorphism α (i.e., no need of referred models). In this sense, the reconstruction algorithms in the proof are in the "**mono-anabelian philosophy**" of Mochizuki (cf. also Remark 3.4.4 (2), (3)).

Proof. We can *group-theoretically* reconstruct the objects in (1)-(7) from G_{K_i} as follows:

(1): p_i is the unique prime number which attains the maximum of $\left\{\mathrm{rank}_{\mathbb{Z}_l} G^{\mathrm{ab}}_{K_i}\right\}_{l:\,\mathrm{prime}}$, by the local class field theory $G^{\mathrm{ab}}_{K_i} \cong (K_i^{\times})^{\wedge}$.

(2a): $k_i^{\times} \cong (G^{\mathrm{ab}}_{K_i})^{\text{prime-to-}p}_{\mathrm{tors}}$ the prime-to-p part of the torsion subgroup of $G^{\mathrm{ab}}_{K_i}$, where p is group-theoretically reconstructed in (1).

(3a): $[K_i : \mathbb{Q}_p] = \mathrm{rank}_{\mathbb{Z}_p} G^{\mathrm{ab}}_{K_i} - 1$, where p is group-theoretically reconstructed in (1).

(3b): $p^{f(K_i)} = \#(k_i^{\times}) + 1$, where k_i and p are group-theoretically reconstructed in (2a) and (1) respectively.

(3c): $e(K_i) = [K_i : \mathbb{Q}_p]/f(K_i)$, where the numerator and the denominator are group-theoretically reconstructed in (3a) and (3b) respectively.

(4a): $I_{K_i} = \bigcap_{G_{K_i} \supset U:\, \mathrm{open},\, e(U)=e(G_{K_i})} U$, where we write $e(U)$ for the number group-theoretically constructed from U in (3c) (i.e., $e(U) := (\mathrm{rank}_{\mathbb{Z}_p} U^{\mathrm{ab}} - 1)/\log_p(\#(U^{\mathrm{ab}})^{\text{prime-to-}p}_{\mathrm{tors}} + 1)$, where $\{p\} := \left\{p \mid \mathrm{rank}_{\mathbb{Z}_p} G^{\mathrm{ab}}_{K_i} = \max_l \mathrm{rank}_{\mathbb{Z}_l} G^{\mathrm{ab}}_{K_i}\right\}$ and $\log_p$ is the (real) logarithm with base p).

(4b): $P_{K_i} = (I_{K_i})^{\text{pro-}p}$ the pro-p part of I_{K_i}, where I_{K_i} is group-theoretically reconstructed in (4a).

(2b): $O^{\times}_{K_i} \cong \mathrm{Im}\,(I_{K_i}) := \mathrm{Im}\left\{I_{K_i} \hookrightarrow G_{K_i} \twoheadrightarrow G^{\mathrm{ab}}_{K_i}\right\}$ by the local class field theory, where I_{K_i} is group-theoretically reconstructed in (4a).

(5): The Fronbenius element Frob_{K_i} is characterised by the element in $G_{K_i}/I_{K_i} (\cong G^{\mathrm{ab}}_{K_i}/\mathrm{Im}\,(I_{K_i}))$ such that the conjugate action on I_{K_i}/P_{K_i} is a multiplication by $p^{f(K_i)}$ (Here we regard the topological group I_{K_i}/P_{K_i} additively), where I_{K_i} and P_{K_i} are group-theoretically reconstructed in (4a) and (4b) respectively.

(2c): We reconstruc $O^{\triangleright}_{K_i}$ by the following pull-back diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}\,(I_{K_i}) \arrow[r] & G^{\mathrm{ab}}_{K_i} \arrow[r] & G^{\mathrm{ab}}_{K_i}/\mathrm{Im}\,(I_{K_i}) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Im}\,(I_{K_i}) \arrow[r] \arrow[u, "="'] & O^{\triangleright}_{K_i} \arrow[r] \arrow[u, hook] & \mathbb{Z}_{\geq 0}\mathrm{Frob}_{K_i} \arrow[r] \arrow[u, hook] & 0,
\end{tikzcd}$$

where I_{K_i} and Frob_{K_i} are group-theoretically reconstructed in (4a) and (5) respectively.

(2d): In the same way as in (2c), we reconstruc $K_i^{\times}$ by the following pull-back diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}\,(I_{K_i}) \arrow[r] & G^{\mathrm{ab}}_{K_i} \arrow[r] & G^{\mathrm{ab}}_{K_i}/\mathrm{Im}\,(I_{K_i}) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Im}\,(I_{K_i}) \arrow[r] \arrow[u, "="'] & K_i^{\times} \arrow[r] \arrow[u, hook] & \mathbb{Z}\mathrm{Frob}_{K_i} \arrow[r] \arrow[u, hook] & 0,
\end{tikzcd}$$

where I_{K_i} and Frob_{K_i} are group-theoretically reconstructed in (4a) and (5) respectively.

(6): Let L be a finite extension of K_i. Then we have the Verlangerung (or transfer) $G^{\mathrm{ab}}_{K_i} \to G^{\mathrm{ab}}_L$ of $G_L \subset G_{K_i}$ by the norm map $G^{\mathrm{ab}}_{K_i} \cong H_1(G_{K_i}, \mathbb{Z}) \to H_1(G_L, \mathbb{Z}) \cong G^{\mathrm{ab}}_L$ in group homology, which is a group-theoretic construction (Or, we can explicitly construct the Verlangerung $G^{\mathrm{ab}}_{K_i} \hookrightarrow G^{\mathrm{ab}}_L$ without group homology as follows: For $x \in G_{K_i}$, take a lift $\widetilde{x} \in G_{K_i}$ of x. We write $G_{K_i} = \coprod_i g_i G_L$ for the coset decomposition, and we write $\widetilde{x} g_i = g_{j(i)} x_i$ for each i, where $x_i \in G_L$. Then the Verlangerung is given by $G^{\mathrm{ab}}_{K_i} \ni x \mapsto (\prod_i x_i \bmod \overline{[G_L, G_L]}) \in G^{\mathrm{ab}}_L$, where we write $\overline{[G_L, G_L]}$ for the topological closure of the commutator subgroup $[G_L, G_L]$ of G_L). Then this reconstructs the inclusion $K_i^{\times} \hookrightarrow L^{\times}$, by the local class field theory and the reconstruction in (2d). The conjugate action of G_{K_i} on $G_L \twoheadrightarrow G^{\mathrm{ab}}_L$ preserves $L^{\times} \subset G^{\mathrm{ab}}_L$ by the reconstruction of (2d). This reconstructs the action of G_{K_i} on $L^\times$. By taking the limit, we reconstruct $\overline{K_i}^\times$, hence $\mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K_i}) = \mathbb{Q}/\mathbb{Z} \otimes_{\widehat{\mathbb{Z}}} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \overline{K_i}^\times)$ equipped with the action of G_{K_i}.

(7): The isomorphism $H^2(\mathrm{Gal}(\overline{K_i}/K_i), \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K_i})) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ is defined by the composition

$$\begin{aligned} &H^2(\mathrm{Gal}(\overline{K_i}/K_i), \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K_i})) \xrightarrow{\sim} H^2(\mathrm{Gal}(\overline{K_i}/K_i), \overline{K_i}^\times) \xleftarrow{\sim} H^2(\mathrm{Gal}(K_i^{\mathrm{ur}}/K_i), (K_i^{\mathrm{ur}})^\times) \\ &\xrightarrow{\sim} H^2(\mathrm{Gal}(K_i^{\mathrm{ur}}/K_i), \mathbb{Z}) \xleftarrow{\sim} H^1(\mathrm{Gal}(K_i^{\mathrm{ur}}/K_i), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\mathrm{Gal}(K_i^{\mathrm{ur}}/K_i), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}, \end{aligned}$$

where the first isomorphism is induced by the canonical inclusion $\mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K_i}) \hookrightarrow \overline{K_i}^\times$, the multiplicative group $(K_i^{\mathrm{ur}})^\times$ (*not* the field K_i^{ur}) of the maximal unramified extension K_i^{ur} of K_i and the Galois group $\mathrm{Gal}(K_i^{\mathrm{ur}}/K)$ are group-theoretically reconstructed in (2d) and (4a) respectively, the third isomorphism is induced by the valuation $(K_i^{\mathrm{ur}})^\times \twoheadrightarrow \mathbb{Z}$, which is group-theoretically reconstructed in (2b) and (2d), the fourth isomorphism is induced by the long exact sequence associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, and the last isomorphism is induced by the evaluation at Frob_{K_i}, which is group-theoretically reconstructed in (5). Thus, the above composition is group-theoretically reconstructed. $\square$

§ 2.2. Arithmetic Quotients.

Proposition 2.2. ([AbsAnab, Lemma 1.1.4]) *Let F be a field, and write $G := \mathrm{Gal}(\overline{F}/F)$ for a separable closure $\overline{F}$ of F. Let*

$$1 \to \Delta \to \Pi \to G \to 1$$

be an exact sequence of profinite groups. We assume that Δ is topologically finitely generated.

(1) Assume that F is a number field. Then Δ is group-theoretically characterised in Π by the maximal closed normal subgroup of Π which is topologically finitely generated.

(2) (Tamagawa) Assume that F is a finite extension of $\mathbb{Q}_p$. For an open subgroup $\Pi' \subset \Pi$, we write $\Delta' := \Pi' \cap \Delta$ and $G' := \Pi'/\Delta'$, and let G' act on $(\Delta')^{\mathrm{ab}}$ by the conjugate. We also assume that

(Tam1)

$$\forall \Pi' \subset \Pi: \text{ open}, \ Q := \left((\Delta')^{\mathrm{ab}}\right)_{G'} / (\mathrm{tors}) \text{ is a finitely generated free } \widehat{\mathbb{Z}}\text{-module},$$

where we write $(-)_{G'}$ for the G'-coinvariant quotient, and (tors) *for the torsion part of the numerator. Then Δ is group-theoretically characterised in Π as the intersection of those open subgroups $\Pi' \subset \Pi$ such that, for any prime number $l \neq p$,* *we have*

$$\text{(Tam2)}\qquad \begin{aligned}&\dim_{\mathbb{Q}_p}(\Pi')^{\mathrm{ab}}\otimes_{\widehat{\mathbb{Z}}}\mathbb{Q}_p-\dim_{\mathbb{Q}_l}(\Pi')^{\mathrm{ab}}\otimes_{\widehat{\mathbb{Z}}}\mathbb{Q}_l\\&=[\Pi:\Pi']\left(\dim_{\mathbb{Q}_p}(\Pi)^{\mathrm{ab}}\otimes_{\widehat{\mathbb{Z}}}\mathbb{Q}_p-\dim_{\mathbb{Q}_l}(\Pi)^{\mathrm{ab}}\otimes_{\widehat{\mathbb{Z}}}\mathbb{Q}_l\right),\end{aligned}$$

where p is also group-theoretically characterised as the unique prime number such that $\dim_{\mathbb{Q}_p}(\Pi)^{\mathrm{ab}}\otimes_{\widehat{\mathbb{Z}}}\mathbb{Q}_p-\dim_{\mathbb{Q}_l}(\Pi)^{\mathrm{ab}}\otimes_{\widehat{\mathbb{Z}}}\mathbb{Q}_l\neq 0$ *for infinitely many prime numbers l.*

Proof. (1): This follows from the fact that every topologically finitely generated closed normal subgroup of $\mathrm{Gal}(\overline{F}/F)$ is trivial (cf. [FJ, Theorem 15.10]).

(2): We have the inflation-restriction sequence associated to $1\to\Delta\to\Pi\to G\to 1$:

$$1\to H^1(G,\mathbb{Q}/\mathbb{Z})\to H^1(\Pi,\mathbb{Q}/\mathbb{Z})\to H^1(\Delta,\mathbb{Q}/\mathbb{Z})^G\to H^2(G,\mathbb{Q}/\mathbb{Z}),$$

where we write $(-)^G$ for the G-invariant submodule. For the last term $H^2(G,\mathbb{Q}/\mathbb{Z})$, we also have $H^2(G,\mathbb{Q}/\mathbb{Z})=\varinjlim_n H^2(G,\frac{1}{n}\mathbb{Z}/\mathbb{Z})\cong\varinjlim_n\mathrm{Hom}(H^0(G,\mu_n),\mathbb{Q}/\mathbb{Z})\cong\mathrm{Hom}(\varprojlim_n H^0(G,\mu_n),\mathbb{Q}/\mathbb{Z})=0$ by the local class field theory. Thus, by taking $\mathrm{Hom}(-,\mathbb{Q}/\mathbb{Z})$ of the above exact sequence, we obtain an exact sequence

$$0\to\left(\Delta^{\mathrm{ab}}\right)_G\to\Pi^{\mathrm{ab}}\to G^{\mathrm{ab}}\to 0.$$

Let F' denote the finite extension corresponding to an open subgroup $G'\subset G$. Then by the assumption of (Tam1), we obtain

$$\begin{aligned}&\dim_{\mathbb{Q}_p}(\Pi')^{\mathrm{ab}}\otimes_{\widehat{\mathbb{Z}}}\mathbb{Q}_p-\dim_{\mathbb{Q}_l}(\Pi')^{\mathrm{ab}}\otimes_{\widehat{\mathbb{Z}}}\mathbb{Q}_l\\&=\dim_{\mathbb{Q}_p}(G')^{\mathrm{ab}}\otimes_{\widehat{\mathbb{Z}}}\mathbb{Q}_p-\dim_{\mathbb{Q}_l}(G')^{\mathrm{ab}}\otimes_{\widehat{\mathbb{Z}}}\mathbb{Q}_l=[F':\mathbb{Q}_p],\end{aligned}$$

where the last equality follows from the local class field theory. The group-theoretic characterisation of p follows from the above equalies. The above equalites also imply that (Tam2) is equivalent to $[F':\mathbb{Q}_p]=[\Pi:\Pi'][F:\mathbb{Q}_p]$, which is equivalent to $[\Pi:\Pi']=[G:G']$, i.e., $\Delta=\Delta'$. This proves the second claim of the proposition. □

Lemma 2.3. ([AbsAnab, Lemma 1.1.5]) *Let F be a non-Archimedean local field, and A a semi-abelian variety over F. Let $\overline{F}$ be an algebraic closure of F, and write $G:=\mathrm{Gal}(\overline{F}/F)$. We write $T(A):=\mathrm{Hom}(\mathbb{Q}/\mathbb{Z},A(\overline{F}))$ for the Tate module of A. Then $Q:=T(A)_G/(\mathrm{tors})$ is a finitely generated free $\widehat{\mathbb{Z}}$-module.*

Proof. We have an extension $0\to S\to A\to A'\to 0$ of group schemes over F, where S is a torus and A' is an abelian variety over F. Then $T(S)\cong\widehat{\mathbb{Z}}(1)^{\oplus n}$ for some n after restristing on an open subgroup of G, where $T(S)$ is the Tate module of T.

Thus, the image of $T(S)$ in Q is trivial. Therefore, we may assume that A is an abelian variety. By [SGA7t1, Exposé IX §2], we have extensions

$$0 \to T(A)^{\leq -1} \to T(A) \to T(A)^0 \to 0,$$
$$0 \to T(A)^{-2} \to T(A)^{\leq -1} \to T(A)^{-1} \to 0$$

of G-modules, where $T(A)^{\leq -1}$ and $T(A)^{\leq -2}$ are the "fixed part" and the "toric part" of $T(A)$ respectively in the terminology of [SGA7t1, Exposé IX §2], and we have isomorphisms $T(A)^{-1} \cong T(B)$ for an abelian variety B over F which has potentially good reduction, and $T(A)^0 \cong M^0 \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, $T(A)^{-2} \cong M^{-2} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}(1)$, where M^0 and M^{-2} are finitely generated free $\mathbb{Z}$-modules and G acts both on M^0 and M^{-2} via finite quotients. Thus, the images of $T(A)^{-2}$ and $T(A)^{-1}$ in Q are trivial (by the Weil conjecture proved by Weil for abelian varieties in the latter case). Therefore, we obtain $Q \cong (T(A)^0)_G/(\text{tors})$, which is isomorphic to $(M^0)_G/(\text{tors}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, since $\widehat{\mathbb{Z}}$ is flat over $\mathbb{Z}$. Now the lemma follows since $(M^0)_G/(\text{tors})$ is free over $\mathbb{Z}$. □

Corollary 2.4. *We have a group-theoretic characterisation of $\Delta = \pi_1(X_{\overline{F}}, \overline{x})$ in $\Pi = \pi_1(X, \overline{x})$ as Proposition 2.2 (2) (Tam2), where X is a geometrically connected smooth hyperbolic curve over a finite extension F of $\mathbb{Q}_p$, and $\overline{s} : \operatorname{Spec} \overline{F} \to X$ a geometric point lying over $\operatorname{Spec} \overline{F}$ (which gives a geometric point $\overline{s}$ on $X_{\overline{F}} := X \times_F \overline{F}$ via $X_{\overline{F}} \to X$).*

Remark 2.4.1. Let Σ be a set of prime numbers such that $p \in \Sigma$ and $\#\Sigma \geq 2$. In the situation of Corollary 2.4, let Δ^{Σ} be the maximal pro-Σ quotient, and write $\Pi^{\Sigma} := \Pi/\ker(\Delta \twoheadrightarrow \Delta^{\Sigma})$. Then the algorithm of Proposition 2.2 (2) works for Π^{Σ} as well, hence Corollary 2.4.1 holds for Π^{Σ} as well.

Proof. The corollary immediately follows from Proposition 2.2 (2) and Lemma 2.3. □

§ 2.3. Slimness and Commensurable Terminality.

Definition 2.5.

(1) Let G be a profinite group. We say that G is **slim** if we have $Z_G(H) = \{1\}$ for any open subgroup $H \subset G$.

(2) Let $f : G_1 \to G_2$ be a continuous homomorhism of profinite groups. We say that G_1 **relatively slim** over G_2 (via f) if we have $Z_{G_2}(\operatorname{Im}\{H \to G_2\}) = \{1\}$ for any open subgroup $H \subset G_1$.

Lemma 2.6. ([AbsAnab, Remark 0.1.1, Remark 0.1.2]) *Let G be a profinite gruop, and $H \subset G$ a closed subgroup of G.*

(1) If $H \subset G$ is relatively slim, then both H and G are slim.

(2) If $H \subset G$ is commensurably terminal and H is slim, then $H \subset G$ is relatively slim.

Proof. (1): For any open subgroup $H' \subset H$, we have $Z_H(H') \subset Z_G(H') = \{1\}$. For any open subgroup $G' \subset G$, we have $Z_G(G') \subset Z_G(H \cap G') = \{1\}$ since $H \cap G'$ is open in H.

(2): Let $H' \subset H$ be an open subgroup. The natural inclusion $C_G(H) \subset C_G(H')$ is an equality since H' is open in H. Then we have $Z_G(H') \subset C_G(H') = C_G(H) = H$. This combined with $Z_H(H') = \{1\}$ implies $Z_G(H') = \{1\}$. $\square$

Proposition 2.7. ([AbsAnab, Theorem 1.1.1, Corollary 1.3.3, Lemma 1.3.1, Lemma 1.3.7]) *Let F be a number field, and v a non-Archimedean place. Let $\overline{F_v}$ be an algebraic closure of F_v, $\overline{F}$ the algebraic closure of F in $\overline{F_v}$.*

(1) Write $G := \mathrm{Gal}(\overline{F}/F) \supset G_v := \mathrm{Gal}(\overline{F_v}/F_v)$.

(a) $G_v \subset G$ is commensurably terminal,

(b) $G_v \subset G$ is relatively slim,

(c) G_v is slim, and

(d) G is slim.

(2) Let X be a hyperbolic curve over F. Let $\overline{s} : \mathrm{Spec}\,\overline{F_v} \to X_{\overline{F_v}} := X \times_F \overline{F_v}$ be a geometric point lying over $\mathrm{Spec}\,\overline{F_v}$ (which gives geometric points $\overline{s}$ on $X_{\overline{F}} := X \times_F \overline{F}$, $X_{F_v} := X \times_F F_v$, and X via $X_{\overline{F_v}} \to X_{\overline{F}} \to X$, and $X_{\overline{F_v}} \to X_{F_v} \to X$). Write $\Delta := \pi_1(X_{\overline{F}}, \overline{s}) \cong \pi_1(X_{\overline{F_v}}, \overline{s})$, $\Pi := \pi_1(X, \overline{s})$, and $\Pi_v := \pi_1(X_{F_v}, \overline{s})$. Let x be any cusp of $X_{\overline{F}}$ (i.e., a point of the unique smooth compactification of $X_{\overline{F}}$ over $\overline{F}$ which does not lie in $X_{\overline{F}}$), and we write $I_x \subset \Delta$ (well-defined up to conjugates) for the inertia subgroup at x (Note that I_x is isomorphic to $\widehat{\mathbb{Z}}(1)$). For any prime number l, we write $I_x^{(l)} \to \Delta^{(l)}$ for the maximal pro-l quotient of $I_x \subset \Delta$ (Note that $I_x^{(l)}$ is isomorphic to $\mathbb{Z}_l(1)$ and that it is easy to see that $I_x^{(l)} \to \Delta^{(l)}$ is injective).

(a) Δ is slim,

(b) Π and Π_v are slim, and

(c) $I_x^{(l)} \subset \Delta^{(l)}$ and $I_x \subset \Delta$ are commensurably terminal.

Remark 2.7.1. Furthermore, we can show that $\mathrm{Gal}(\overline{F}/F)$ is slim for any Kummer-faithful field F (Remark 3.17.3).

Proof. (1a)(cf. also [NSW, Corollary 12.1.3, Corollary 12.1.4]): First, we claim that any subfield $K \subset \overline{F}$ with $K \neq \overline{F}$ has at most one prime ideal which is indecomposable in $\overline{F}$. Proof of the claim: Let $\mathfrak{p}_1 \neq \mathfrak{p}_2$ be prime ideals in K which do not split in $\overline{F}$. Let $f_1 \in K[X]$ be any irreducible polynomial of degree $d > 0$, and $f_2 \in K[X]$ a completely split separable polynomial of the same degree d. By the approximation theorem, for any $\epsilon > 0$ there exists $f \in K[X]$ a polynomial of degree d, such that $|f - f_1|_{\mathfrak{p}_1} < \epsilon$ and $|f - f_2|_{\mathfrak{p}_2} < \epsilon$. Then for sufficiently small $\epsilon > 0$ the splitting fields of f and f_i over $K_{\mathfrak{p}_i}$ coincide for $i = 1, 2$ by Krasner's lemma. By assumption that $\mathfrak{p}_1 \neq \mathfrak{p}_2$ do not split in $\overline{F}$, the splitting fields of f_1 and f_2 over K coincide. Then we have $K = \overline{F}$ since splitting field of f_2 is K, and f_1 is any irreducible polynomial. The claim is proved. We show (1a). We specify a base point of G_v to kill the conjugacy indeterminacy, that is, we take a place $\widetilde{v}$ in $\overline{K_v}$ over v, and we use $G_{\widetilde{v}}$ instead of G_v. Let $g \in C_G(G_{\widetilde{v}})$. Then $G_{\widetilde{v}} \cap G_{g\widetilde{v}} \neq \{1\}$, since $G_{\widetilde{v}} \cap gG_{\widetilde{v}}g^{-1} = G_{\widetilde{v}} \cap G_{g\widetilde{v}}$ has finite index in $G_{\widetilde{v}}$. Then the above claim implies that $g\widetilde{v} = \widetilde{v}$. Thus, we have $g \in G_{\widetilde{v}}$.

(c): Let $G_K \subset G_v$ be an open subgroup, and $g \in Z_{G_v}(G_K)$. Then for any finite Galois extension L over K, the action of g on G_L, hence on G_L^{ab}, is trivial. By the local class field theory, the action of g on $L^\times$ is also trivial. Thus, we have $g = 1$ since L is any extension over K.

(b) follows from (a), (c), and Lemma 2.6 (2).

(d) follows from (b) and Lemma 2.6 (1).

(2a): This is similar to the proof of (1c). Let $H \subset \Delta$ be an open subgroup. We write $X_H \to X_{\overline{F}}$ for the finite étale covering corresponding to H. We take any sufficiently small open normal subgroup $H' \subset \Delta$ such that $H' \subset H$ and the corresponding finite étale covering $X_{H'} \to X_H$ has the canonical compactification $\overline{X_{H'}}$ of genus > 1. We have an identification $H' = \pi_1(X_{H'}, y)$ for a basepoint y. We write $J_{H'} := \mathrm{Jac}(\overline{X_{H'}})$ with the origin O for the Jacobian variety of $\overline{X_{H'}}$. Let $g \in \Delta$. Then we have the following commutative diagram of pointed schemes:

$$\begin{tikzcd}
(X_{H'}, y) \arrow[r, hook] \arrow[d, "g^X"'] & (\overline{X_{H'}}, y) \arrow[r, "f_y"] \arrow[d, "g^{\overline{X}}"'] & (J_{H'}, O) \arrow[d, "g^J"'] \\
(X_{H'}, g(y)) \arrow[r, hook] & (\overline{X_{H'}}, g(y)) \arrow[r, "f_{g(y)}"] & (J_{H'}, g(O)),
\end{tikzcd}$$

which induces

$$\begin{tikzcd}
\pi_1(X_{H'}, y) \arrow[r, two heads] \arrow[d, "g_*^X"'] & \pi_1(J_{H'}, O) \arrow[r, "\sim"] \arrow[d, "g_*^J"'] & T(J_{H'}, O) \arrow[d, "g_*^J"'] \\
\pi_1(X_{H'}, g(y)) \arrow[r, two heads] & \pi_1(J_{H'}, g(O)) \arrow[r, "\sim"] & T(J_{H'}, g(O)),
\end{tikzcd}$$

where we write $T(J_{H'}, O)$ and $T(J_{H'}, g(O))$ for the Tate modules of $J_{H'}$ with origin O and $g(O)$ respectively (Note that we have the isomorphisms from π_1 to the Tate modules, since $\overline{F}$ is of characteristic 0). Here, the morphism $g^J : (J_{H'}, O) \to (J_{H'}, g(O))$ is the composite of an automorphism $(g^J)' : (J_{H'}, O) \to (J_{H'}, O)$ of abelian varieties and an addition by $g(O)$. We also have a conjugate action $\mathrm{conj}(g) : H' = \pi_1(X_{H'}, y) \to \pi_1(X_{H'}, g^*(y)) = gH'g^{-1} = H'$, which induces an action $\mathrm{conj}(g)^{\mathrm{ab}} : (H')^{\mathrm{ab}} \to (H')^{\mathrm{ab}}$. This is also compatible with the homomorphism induced by $(g^J)'$:

$$\begin{tikzcd}
(H')^{\mathrm{ab}} \arrow[r, two heads] \arrow[d, "\mathrm{conj}(g)^{\mathrm{ab}}"'] & T(J_{H'}, O) \arrow[d, "(g^J)'_*"'] \\
(H')^{\mathrm{ab}} \arrow[r, two heads] & T(J_{H'}, O).
\end{tikzcd}$$

Assume that $g \in Z_\Delta(H)$. Then the conjugate action of g on H', hence on $(H')^{\mathrm{ab}}$, is trivial. By the surjection $(H')^{\mathrm{ab}} \twoheadrightarrow T(J_{H'}, O)$, the action $(g^J)'_* : T(J_{H'}, O) \to T(J_{H'}, O)$ is trivial. Thus, the action $(g^J)' : (J_{H'}, O) \to (J_{H'}, O)$ is also trivial, since the torsion points of $J_{H'}$ are dense in $J_{H'}$. Therefore, the morphism $g^J : (J_{H'}, O) \to (J_{H'}, g^*(O))$ of pointed schemes is the addition by $g(O)$. Then the compatibility of $g^{\overline{X}} : (\overline{X_{H'}}, y) \to (\overline{X_{H'}}, g(y))$ and $g^J : (J_{H'}, O) \to (J_{H'}, g(O))$ with respect to f_y and $f_{g(y)}$ (i.e., the first commutative diagram) implies that $g^{\overline{X}} : (\overline{X_{H'}}, y) \to (\overline{X_{H'}}, g(y))$, hence $g^X : (X_{H'}, y) \to (X_{H'}, g(y))$, is an identity morphism by (the uniqueness assertion of) Torelli's theorem (cf. [Mil, Theorem 12.1 (b)]). Then we have $g = 1$ since H' is any sufficiently small open subgroup in H.

(b) follows from (a), (1c), and (1d).

(c): This is similar to the proof of (1a). We assume that $C_\Delta(I_x) \neq I_x$ (resp. $C_{\Delta^{(l)}}(I_x^{(l)}) \neq I_x^{(l)}$). Let $g \in C_\Delta(I_x)$ (resp. $C_{\Delta^{(l)}}(I_x^{(l)})$) be such that g is not in I_x (resp. $I_x^{(l)}$). Since $g \notin I_x$ (resp. $g \notin I_x^{(l)}$), we have a finite Galois covering (resp. a finite Galois covering of degree a power of l) $Y \to X_{\overline{F}}$ (which is unramified over x) and a cusp y of Y over x such that $y \neq g(y)$. By taking sufficiently small $\Delta_Y \subset \Delta$ (resp. $\Delta_Y \subset \Delta^{(l)}$), we may assume that Y has a cusp $y' \neq y, g(y)$. We have $I_{g(y)} = gI_yg^{-1}$ (resp. $I_{g(y)}^{(l)} = gI_y^{(l)}g^{-1}$). Since $I_y \cap I_{g(y)}$ (resp. $I_y^{(l)} \cap I_{g(y)}^{(l)}$) has a finite index in I_y (resp. $I_y^{(l)}$), we have a finite Galois covering (resp. a finite Galois covering of degree a power of l) $Z \to Y$ such that Z has cusps z, $g(z)$, and z' lying over y, $g(y)$, and y' respectively, and $I_z = I_{g(z)}$ (resp. $I_z^{(l)} = I_{g(z)}^{(l)}$), i.e., z and $g(z)$ have conjugate inertia subgroups in Δ_Z (resp. $\Delta_Z^{(l)}$) (Note that inertia subgroups are well-defined up to inner conjugate). On the other hand, we have abelian coverings of Z which are totally ramified over z and not ramified over $g(z)$, since we have a cusp z' other than z and $g(z)$ (Note that the abelianisation of a surface relation $\gamma_1 \cdots \gamma_n \prod_{i=1}^g [\alpha_i, \beta_i] = 1$ is $\gamma_1 \cdots \gamma_n = 1$, and that if $n \geq 3$, then we can choose the ramifications at γ_1 and γ_2 independently). This contradicts that z and $g(z)$ have conjugate inertia subgroups in Δ_Z (resp. $\Delta_Z^{(l)}$). $\square$

§ 2.4. Characterisation of Cuspidal Decomposition Groups.

Let k a finite extension of $\mathbb{Q}_p$. For a hyperbolic curve X of type (g,r) over k, we write Δ_X and Π_X for the geometric fundamental group (i.e., π_1 of $X_{\overline{k}} := X \times_k \overline{k}$) and the arithmetic fundamental group (i.e., π_1 of X) of X for some basepoint, respectively. Note that we have a group-theoretic characterisation of the subgroup $\Delta_X \subset \Pi_X$ (hence the quotient $\Pi_X \twoheadrightarrow G_k$) by Corollary 2.4. For a cusp x, we write I_x and D_x for the inertia subgroup and the decomposition subgroup at x in Δ_X and in Π_X respectively (they are well-defined up to inner automorphism). For a prime number l, we also write $I_x^{(l)}$ and $\Delta_X^{(l)}$ for the maximal pro-l quotient of I_x and Δ_X, respectively. Write also $\Pi_X^{(l)} := \Pi_X/\ker(\Delta_X \twoheadrightarrow \Delta_X^{(l)})$. Then we have a short exact sequence $1 \to \Delta_X^{(l)} \to \Pi_X^{(l)} \to G_k \to 1$.

Lemma 2.8. ([AbsAnab, Lemma 1.3.9], [AbsTopI, Lemma 4.5]) *Let X be a hyperbolic curve of type (g,r) over k.*

(1) X is not proper (i.e., $r > 0$) if and only if Δ_X is a free profinite group (Note that this criterion is group-theoretic).

(2) We can group-theoretically reconstruct (g,r) from Π_X as follows:

$$r = \dim_{\mathbb{Q}_l} \left(\Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l\right)^{\mathrm{wt}=2} - \dim_{\mathbb{Q}_l} \left(\Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l\right)^{\mathrm{wt}=0} + 1 \quad \textit{if } r > 0, \quad \textit{for } l \neq p,$$
$$g = \begin{cases} \frac{1}{2}\left(\dim_{\mathbb{Q}_l} \Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l - r + 1\right) & \textit{if } r > 0, \\ \frac{1}{2} \dim_{\mathbb{Q}_l} \Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l & \textit{if } r = 0 \quad \textit{for any } l, \end{cases}$$

where $(-)^{\mathrm{wt}=w}$ with $w \in \mathbb{Z}$ is the subspace on which the Frobenius at p acts with eigenvalues of weight w, i.e., algebraic numbers with absolute values $q^{\frac{w}{2}}$ (Note that the weight is independent of the choice of a lifting of the Frobenius element Frob_k to G_k in the extension $1 \to I_k \to G_k \to \widehat{\mathbb{Z}}\mathrm{Frob}_k \to 1$, since the action of the inertia subgroup on Δ_X^{ab} is quasi-unipotent). Here, note also that G_k and Δ_X are group-theoretically reconstructed from Π_X by Corollary 2.4, the prime number p, the cardinality q of the residue field, and the Frobenius element Frob_k are group-theoretically reconstructed from G_k by Proposition 2.1 (1), (1) and (3b), and (5) respectively (cf. also Remark 2.1.1).

Remark 2.8.1. By the same group-theoretic algorithm as in Lemma 2.8, we can also group-theoretically reconstruct (g,r) from the extension datum $1 \to \Delta_X^{(l)} \to \Pi_X^{(l)} \to G_k \to 1$ for any $l \neq p$ (i.e., in the case where the quotient $\Pi_X^{(l)} \twoheadrightarrow G_k$ is given).

Proof. (1): Trivial (Note that, in the proper case, the non-vanishing of H^2 implies the non-freeness of Δ_X). (2): Let $X \hookrightarrow \overline{X}$ be the canonical smooth compactification.

Then we have

$$
\begin{aligned}
r-1 &= \dim_{\mathbb{Q}_l} \ker\left\{\Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l \twoheadrightarrow \Delta_{\overline{X}}^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l\right\} = \dim_{\mathbb{Q}_l} \ker\left\{\Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l \twoheadrightarrow \Delta_{\overline{X}}^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l\right\}^{\mathrm{wt}=2} \\
&= \dim_{\mathbb{Q}_l}(\Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l)^{\mathrm{wt}=2} - \dim_{\mathbb{Q}_l}(\Delta_{\overline{X}}^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l)^{\mathrm{wt}=2} \\
&= \dim_{\mathbb{Q}_l}(\Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l)^{\mathrm{wt}=2} - \dim_{\mathbb{Q}_l}(\Delta_{\overline{X}}^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l)^{\mathrm{wt}=0} \\
&= \dim_{\mathbb{Q}_l}(\Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l)^{\mathrm{wt}=2} - \dim_{\mathbb{Q}_l}(\Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l)^{\mathrm{wt}=0},
\end{aligned}
$$

where the forth equality follows from the self-duality of $\Delta_{\overline{X}}$. The rest of the lemma (the formula for g) is trivial. $\square$

Corollary 2.9. ([NodNon, Lemma 1.6 (ii)⇒(i)]) *Let X be an affine hyperbolic curves over k, and $\overline{X}$ the canonical smooth compactification. We have the following group-theoretic characterisations or reconstruction algorithms from Π_X:*

(1) The natural surjection $\Delta_X \twoheadrightarrow \Delta_{\overline{X}}$ (resp. $\Delta_X^{(l)} \twoheadrightarrow \Delta_{\overline{X}}^{(l)}$ for any $l \neq p$) is group-theoretically characterised as follows: An open subgroup $H \subset \Delta_X$ (resp. $H \subset \Delta_X^{(l)}$) is contained in $\ker(\Delta_X \twoheadrightarrow \Delta_{\overline{X}})$ (resp. $\ker(\Delta_X^{(l)} \twoheadrightarrow \Delta_{\overline{X}}^{(l)})$) if and only if $r(X_H) = [\Delta_X : H]r(X)$ (resp. $r(X_H) = [\Delta_X^{(l)} : H]r(X)$), where X_H is the coverings corresponding to $H \subset \Delta_X$, and $r(-)$'s are their number of cusps (Note that $r(-)$'s are group-theoretically computed by Lemma 2.8 (2) and Remark 2.8.1.

(2) The inertia subgroups of cusps in $\Delta_X^{(l)}$ for any $l \neq p$ are characterised as follows: A closed subgroup $A \subset \Delta_X^{(l)}$ which is isomorphic to $\mathbb{Z}_l$ is contained in the inertia subgroup of a cusp if and only if, for any open subgroup $\Delta_Y^{(l)} \subset \Delta_X^{(l)}$, the composite

$$A \cap \Delta_Y^{(l)} \subset \Delta_Y^{(l)} \twoheadrightarrow \Delta_{\overline{Y}}^{(l)} \twoheadrightarrow (\Delta_{\overline{Y}}^{(l)})^{\mathrm{ab}}$$

vanishes. Here, we write $\overline{Y}$ for the canonical smooth compactification of Y (Note that the natural surjection $\Delta_Y^{(l)} \twoheadrightarrow \Delta_{\overline{Y}}^{(l)}$ has a group-theoretic characterisation in (1)).

(3) We can reconstruct the set of cusps of X as the set of $\Delta_X^{(l)}$-orbits of the inertia subgroups in $\Delta_X^{(l)}$ via conjugate actions by Proposition 2.7 (2c) (Note that inertia subgroups in $\Delta_X^{(l)}$ have a group-theoretic characterisation in (2)).

(4) By functorially reconstructing the cusps of any covering $Y \to X$ from $\Delta_Y \subset \Delta_X \subset \Pi_X$, we can reconstruct the set of cusps of the universal pro-covering $\widetilde{X} \to X$ (Note that the set of cusps of Y is reconstructed in (3)).

(5) We can reconstruct inertia subgroups in Δ_X as the subgroups that fix some cusp of the universal pro-covering $\widetilde{X} \to X$ of X determined by the basepoint under consideration (Note that the set of cusps of $\widetilde{X}$ is reconstructed in (4)).

(6) We have a characterisation of decomposition groups D of cusps in Π_X (resp. in $\Pi_X^{(l)}$ for any $l \neq p$) as $D = N_{\Pi_X}(I)$ (resp. $D = N_{\Pi_X^{(l)}}(I)$) for some inertia subgroup in Δ_X (resp. in $\Delta_X^{(l)}$) by Proposition 2.7 (2c) (Note that inertia subgroups in Δ_X and $\Delta_X^{(l)}$ are reconstructed in (5) and in (2) respectively).

Remark 2.9.1. (cf. also [IUTchI, Remark 1.2.2, Remark 1.2.3]) The arguments in [AbsAnab, Lemma 1.3.9], [AbsTopI, Lemma 4.5 (iv)], and [CombGC, Theorem 1.6 (i)] are wrong, because there is no covering of degree l of proper curves, which is ramified at one point and unramified elsewhere (Note that the abelianisations of the geometric fundamental group of a proper curve is equal to the one of the curve obtained by removing one point from the curve).

Proof. The claims (1) is trivial. (2): The "only if" part is trivial since an inertia subgroup is killed in $\Delta_{\overline{Y}}$. We show the "if" part. Write $\Delta_Z^{(l)} := A\Delta_Y^{(l)} \subset \Delta_X^{(l)}$. The natural surjection $\Delta_Z^{(l)} \twoheadrightarrow \Delta_Z^{(l)}/\Delta_Y^{(l)} \cong A/(A \cap \Delta_Y^{(l)})$ factors as $\Delta_Z^{(l)} \twoheadrightarrow (\Delta_Z^{(l)})^{\mathrm{ab}} \twoheadrightarrow A/(A\cap\Delta_Y^{(l)})$, since $A/(A\cap\Delta_Y^{(l)})$ is isomorphic to an abelian group $\mathbb{Z}/l^N\mathbb{Z}$ for some N. By the assumption of the vanishing of $A\cap\Delta_Y^{(l)}$ in $(\Delta_{\overline{Y}})^{\mathrm{ab}}$, the image $\mathrm{Im}\{A\cap\Delta_Y^{(l)} \to (\Delta_Y^{(l)})^{\mathrm{ab}}\}$ is contained in the subgroup generated by the image of the inertia subgroups in $\Delta_Y^{(l)}$. Hence the image $\mathrm{Im}\{A \cap \Delta_Y^{(l)} \to (\Delta_Y^{(l)})^{\mathrm{ab}} \to (\Delta_Z^{(l)})^{\mathrm{ab}} \twoheadrightarrow A/(A \cap \Delta_Y^{(l)})(\cong \mathbb{Z}/l^N\mathbb{Z})\}$ is contained in the image of the subgroup in $A/(A \cap \Delta_Y^{(l)})(\cong \mathbb{Z}/l^N\mathbb{Z})$ generated by the image of the inertia subgroups in $\Delta_Y^{(l)}$. Since the composite $A \subset \Delta_Z^{(l)} \twoheadrightarrow \Delta_Z^{(l)}/\Delta_Y^{(l)} \cong A/(A \cap \Delta_Y^{(l)})(\cong \mathbb{Z}/l^N\mathbb{Z})$ is a surjection, and since $\mathbb{Z}/l^N\mathbb{Z}$ is cyclic, there exists the image $\overline{I}_z \subset (\Delta_Z^{(l)})^{\mathrm{ab}}$ of the inertia subgroup of a cusp z in Z, such that the composite $\overline{I}_z \subset (\Delta_Z^{(l)})^{\mathrm{ab}} \twoheadrightarrow A/(A\cap\Delta_Y^{(l)})(\cong \mathbb{Z}/l^N\mathbb{Z})$ is surjective (Note that if we are working in the profinite geometric fundamental groups, instead of pro-l geometric fundamental groups, then the cyclicity does not hold, and we cannot use the same argument). This means that the corresponding subcovering $Y \to Z(\to X)$ is totally ramified at z. The claims (3), (4), (5), and (6) are trivial. $\square$

Remark 2.9.2. (Generalisation to l-cyclotomically full fields, cf. also [AbsTopI, Lemma 4.5 (iii)], [CombGC, Proposition 2.4 (iv), (vii), proof of Corollary 2.7 (i)]) We can generalise the results in this subsection for an l-cyclotomically full field k for some l (cf. Definition 3.1 (3) below), *under the assumption that the quotient* $\Pi_X \twoheadrightarrow G_k$ *is given*, as follows: For the purpose of a characterisation of inertia subgroups of cusps, it is enough to consider the case where X is affine. First, we obtain a group-theoretic reconstruction of a positive power $\chi^{+}_{\mathrm{cyc},l,\text{up to fin}}$ of the l-adic cyclotomic character up to a character of finite order by the actions of G_k on $\bigwedge^{\dim_{\mathbb{Q}_l}(H^{\mathrm{ab}}\otimes_{\widehat{\mathbb{Z}}}\mathbb{Q}_l)}(H^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l)$ for characteristic open torsion-free subgroups $H \subset \Delta_X$. Next, we group-theoretically reconstruct the l-adic cyclotomic character $\chi_{\mathrm{cyc},l,\text{up to fin}}$ up to a character of finite order as $\chi_{\mathrm{cyc},l,\text{up to fin}} =$

$\chi_{\max}$, where $\chi_{\max}$ is the maximal power of $\chi^{+}_{\text{cyc,up to fin}}$ by which G_k acts in some subquotient of $H^{\text{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l$ for sufficiently small characteristic open torsion-free subgroups $H \subset \Delta_X$. Once we reconstruct the l-adic cyclotomic character $\chi_{\text{cyc},l,\text{up to fin}}$ up to a character of finite order, then, for a finite-dimensional $\mathbb{Q}_l$-vector space V with continuous G_k-action, we take any filtration $V = V^0 \supset V^1 \supset \dots$ (resp. $V(\chi^{-1}_{\text{cyc},l,\text{up to fin}}) = V^0 \supset V^1 \supset \dots$) of $\mathbb{Q}_l[G_k]$-modules (Here we write $V(\chi^{-1})$ for the twist of V by χ^{-1}) such that each graded quotient either has the action of G_k factoring through a finite quotient or has no nontrivial subquotients, and we use, instead of $\dim_{\mathbb{Q}_l} V^{\text{wt}=0}$ (resp. $\dim_{\mathbb{Q}_l} V^{\text{wt}=2}$) in Lemma 2.8, the summation of $\dim_{\mathbb{Q}_l} V^j/V^{j+1}$, where the G_k-action on V^j/V^{j+1} factors through a finite quotient of G_k, and the rest is the same.

§ 3. Mono-anabelian Reconstruction Algorithms.

In this section, we show mono-anabelian reconstruction algorithms, which are crucial ingredients of inter-universal Teichmüller theory.

§ 3.1. Some Definitions.

Definition 3.1. ([pGC, Definition 1.5.4 (i)], [AbsTopIII, Definition 1.5], [CombGC, Definition 2.3 (ii)]) Let k be a field.

(1) We say that k is **sub-p-adic**, if there is a finitely generated field L over $\mathbb{Q}_p$ for some p such that we have an injective homomorphism $k \hookrightarrow L$ of fields.

(2) We say that k is **Kummer-faithful**, if k is of characteristic 0, and if for any finite extension k' of k and any semi-abelian variety A over k', the Kummer map $A(k') \to H^1(k', T(A))$ is injective (which is equivalent to $\bigcap_{N \geq 1} NA(k') = \{0\}$), where we write $T(A)$ for the Tate module of A.

(3) We say that k is **l-cyclotomically full**, if the l-adic cyclotomic character $\chi_{\text{cyc},l} : G_k \to \mathbb{Z}_l^{\times}$ has an open image.

Remark 3.1.1. ([pGC, remark after Definition 15.4]) For example, the following fields are sub-p-adic:

(1) finitely generated extensions of $\mathbb{Q}_p$, in particular, finite extensions of $\mathbb{Q}_p$,

(2) finite extensions of $\mathbb{Q}$, and

(3) the subfield of an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ which is the composite of all number fields of degree $\leq n$ over $\mathbb{Q}$ for some fixed integer n (Note that such a field can be embedded into a finite extension of $\mathbb{Q}_p$ by Krasner's lemma).

Lemma 3.2. ([AbsTopIII, Remark 1.5.1, Remark 1.5.4 (i), (ii)])

(1) If k is sub-p-adic, then k is Kummer-faithful.

(2) If k is Kummer-faithfull, then k is l-cyclotomically full for any l.

(3) If k is Kummer-faithfull, then any finitely generated field over k is also Kummer-faithful.

Proof. (3): Let L be a finitely generated extension of k. By Weil restriction, the injectivity of the Kummer map for a finite extension L' of L is reduced to the one for L, i.e., we may assume that $L' = L$. Let A be a semi-abelian variety over L. Let U be an integral smooth scheme over k such that A extends to a semi-abelian scheme $\mathcal{A}$ over U and the function field of U is L. By a commutative diagram

$$\begin{tikzcd} A(L) \arrow[r] \arrow[d, hook] & H^1(L, T(A)) \arrow[d] \\ \prod_{x\in|U|} \mathcal{A}_x(L_x) \arrow[r] & \prod_{x\in|U|} H^1(L_x, T(\mathcal{A}_x)), \end{tikzcd}$$

where we write $|U|$ for the set of closed points, L_x is the residue field at x, and $\mathcal{A}_x$ is the fiber at x (Note that $a \in A(L)$ is zero on any fiber of $x \in |U|$, then a is zero since $|U|$ is dense in U), we may assume that L is a finite extension of k. In this case, again by Weil restriction, the injectivity of the Kummer map for a finite extension L is reduced to the one for k, which holds by assumption.

(1): By the same way as in (3), by Weil restriction, the injectivity of the Kummer map for a finite extension k' of k is reduced to the one for k, i.e., we may assume that $k' = k$. Let k embed into a finitely generated field L over $\mathbb{Q}_p$. By the base change from k to L and the following commutative diagram

$$\begin{tikzcd} A(k) \arrow[r] \arrow[d, hook] & H^1(k, T(A)) \arrow[d] \\ A(L) \arrow[r] & H^1(L, T(A)), \end{tikzcd}$$

the injectivity of the Kummer map for k is reduced to the one for L, i.e., we may assume that k is a finitely generated extension over $\mathbb{Q}_p$. Then by (3), we may assume that $k = \mathbb{Q}_p$. If A is a torus, then $\bigcap_{N\geq 1} NA(\mathbb{Q}_p) = \{0\}$ is trivial. Hence the claim is reduced to the case where A is an abelian variety. Then $A(\mathbb{Q}_p)$ is a compact abelian p-adic Lie group, which contains $\mathbb{Z}_p^{\oplus n}$ for some n as an open subgroup. Hence we have $\bigcap_{N\geq 1} NA(\mathbb{Q}_p) = 0$. Thus, the Kummer map is injective. We are done.

(2): For any finite extensin k' over k, the Kummer map for $\mathbb{G}_m$ over k' is injective by the assumption. This implies that the image of l-adic cyclotomic character $G_k \to \mathbb{Z}_l^\times$ has an open image. $\square$

Definition 3.3. ([CanLift, Section 2]) Let k be a field. Let X be a geometrically normal, geometrically connected algebraic stack of finite type over k.

(1) We write $\overline{\mathrm{Loc}}_k(X)$ for the category whose objects are generically scheme-like algebraic stacks over k which are finite étale quotients (in the sense of stacks) of (necessarily generically scheme-like) algebraic stacks over k that admit a finite étale morphism to X over k, and whose morphisms are finite étale morphisms of stacks over k.

(2) We say X **admits k-core** if there exists a terminal object in $\overline{\mathrm{Loc}}_k(X)$. We shall refer to a terminal object in $\overline{\mathrm{Loc}}_k(X)$ as a **k-core**.

For an elliptic curve E over k with the origin O, we shall refer to the hyperbolic orbicurve (cf. Section 0.2) obtained as the quotient $(E \setminus \{O\})//\pm 1$ in the sense of stacks as a **semi-elliptic orbicurve** over k (cf. [AbsTopII, §0]. It is also called "punctured hemi-elliptic orbicurve" in [CanLift, Definition 2.6 (ii)]).

Definition 3.4. ([AbsTopII, Definition 3.5, Definition 3.1]) Let X be a hyperbolic orbicurve (cf. Section 0.2) over a field k of characteristic 0.

(1) We say that X is **of strictly Belyi type** if (a) X is defined over a number field, and if (b) there exist a hyperbolic orbicurve X' over a finite extension k' of k, a hyperbolic curve X'' of genus 0 over a finite extension k'' of k, and finite étale coverings $X \twoheadleftarrow X' \twoheadrightarrow X''$.

(2) We say that X is **elliptically admissible** if X admits k-core $X \twoheadrightarrow C$, where C is a semi-elliptic orbicurve.

Remark 3.4.1. In the moduli space $\mathcal{M}_{g,r}$ of curves of genus g with r cusps, the set of points corresponding to the curves of strictly Belyi type is *not* Zariski open for $2g-2+r \geq 3$, $g \geq 1$. cf. [Cusp, Remark 2.13.2] and [Corr, Theorem B].

Remark 3.4.2. If X is elliptically admissible and defined over a number field, then X is of strictly Belyi type (cf. also [AbsTopIII, Remark 2.8.3]), since, for an elliptic curve E, we have a diagram of finite étale coverings $E \setminus \{0\} \overset{[2]}{\twoheadleftarrow} E \setminus E[2] \underset{\text{cover}}{\overset{\text{double}}{\twoheadrightarrow}} \mathbb{P}^1 \setminus \{4\,\text{pts}\}$.

For a hyperbolic curve X over a field k of characteristic zero with the canonical smooth compactification $\overline{X}$. A closed point x in $\overline{X}$ is called **algebraic**, if there are a finite extension K of k, a hyperbolic curve Y over a number field $F \subset K$ with the canonical smooth compactification $\overline{Y}$, and an isomorphism $X \times_k K \cong Y \times_F K$ over K such that x maps to a closed point under the composition $\overline{X} \times_k K \cong \overline{Y} \times_F K \to \overline{Y}$.

§ 3.2. Belyi and Elliptic Cuspidalisations — Hidden Endomorphisms.

Let k be a field of characteristic 0, and $\overline{k}$ an algebraic closure of k. Write $G_k := \mathrm{Gal}(\overline{k}/k)$. Let X be a hyperbolic orbicurve over k (cf. Section 0.2). We write Δ_X and Π_X for the geometric fundamental group (i.e., π_1 of $X_{\overline{k}} := X \times_k \overline{k}$) and the arithmetic fundamental group (i.e., π_1 of X) of X for some basepoint, respectively. Note that we have an exact sequence $1 \to \Delta_X \to \Pi_X \to G_k \to 1$. We consider the following conditions on k and X:

(Delta)$_X$: We have a "group-theoretic characterisation" (for example, like Proposition 2.2 (1), (2)) of the subgroup $\Delta_X \subset \Pi_X$ (or equivalently, the quotient $\Pi_X \twoheadrightarrow G_k$).

(GC): Isom-version of the relative Grothendieck conjecture (cf. also Theorem B.1) for the profinite fundamental groups of any hyperbolic (orbi)curves over k holds, i.e., the natural map $\mathrm{Isom}_k(X, Y) \to \mathrm{Isom}^{\mathrm{Out}}_{G_k}(\Delta_X, \Delta_Y) := \mathrm{Isom}_{G_k}(\Delta_X, \Delta_Y)/\mathrm{Inn}(\Delta_Y)$ is bijective for any hyperbolic (orbi)curve X, Y over k.

(slim): G_k is slim (Definition 2.5 (1)).

(Cusp)$_X$: We have a "group-theoretic characterisation" (for example, like Proposition 2.9 (3)) of decomposition groups in Π_X of cusps.

We also consider the following condition (of different nature):

(Delta)'$_X$: Either

- Π_X is given and (Delta)$_X$ holds, or
- $\Delta_X \subset \Pi_X$ are given.

Note that (Delta)$_X$, (GC), and (slim) are conditions on k and X; however, as for (Delta)'$_X$, "the content of a theorem" depends on which case of (Delta)'$_X$ is satisfied, i.e., in the former case, the algorithm in a theorem requires only Π_X as (a part of) an input datum, on the other hand, in the latter case, the algorithm in a theorem requires both $\Delta_X \subset \Pi_X$ as (a part of) input data.

Remark 3.4.3.

(1) (Delta)$_X$ holds for any X in the case where k is an NF by Proposition 2.2 (1) or k is an MLF by Corollary 2.4.

(2) (GC) holds in the case where k is sub-p-adic by Theorem B.1.

(3) (slim) holds in the case where k is an NF by Proposition 2.7 (1) (d) or k is an MLF by Proposition 2.7 (1) (c). More generally, it holds for Kummmer-faithful field k by Remark 3.17.3, which is shown *without using the results in this subsection.*

(4) $(\text{Cusp})_X$ holds for any X in the case where k is an MLF by Corollary 2.9. More generally, $(\text{Cusp})_X$ holds for l-cyclotomically full field k for some l under the assumption $(\text{Delta})'_X$ by Remark 2.9.2.

In short, we have the following table (cf. also Lemma 3.2):

NF, MLF	$\Rightarrow$	sub-p-adic	$\Rightarrow$	Kummer-faithful	$\Rightarrow$	l-cyclotomically full
$(\text{Delta})_X$ holds for any X		(GC) holds		(slim) holds		$(\text{Cusp})_X$ holds under $(\text{Delta})'_X$.

Remark 3.4.4.

(1) It seems difficult to rigorously formulate the meaning of "group-theoretic characterisation". Note that the formulation for $(\text{Delta})_X$ like "any isomorphism $\Pi_{X_1} \cong \Pi_{X_2}$ of topological groups induces an isomorphism $\Delta_{X_1} \cong \Delta_{X_2}$ of topological groups" (it is called **bi-anabelian** approach) is *a priori* weaker than the notion of "group-theoretic characterisation" of Δ_X in Π_X (this is called **mono-anabelian** approach), which allows us to reconstruct the object itself (*not* the morphism between two objects).

(2) (Important Convention) In the same way, it also seems difficult to rigorously formulate "there is a group-theoretic algorithm to reconstruct" something in the sense of mono-anabelian approach (Note that it is easy to rigorously formulate it in the sense of bi-anabelian approach). To rigorously settle the meaning of it, it seems that we have to state the algorithm itself, i.e., *the algorithm itself have to be a part of the statement.* However, in this case, the statement must be often rather lengthy and complicated. In this survey, we use the phrase "group-theoretic algorithm" loosely in some sense, for the purpose of making the input data and the output data of the algorithms in the statement clear. However, the rigorous meaning will be clear in the proof, since the proof shows concrete constructions, which, properly speaking, should be included in the statement itself. We sometimes employ this convention of stating propositions and theorems in this survey (If we use the language of **species** and **mutations** (cf. [IUTchIV, §3]), then we can rigorously formulate mono-anabelian statements without mentioning the contents of algorithms).

(3) Mono-anabelian reconstruction algorithms have an advantage, as contrasted with bi-anabelian approach, of avoiding "a referred model" of a mathematical object like

"*the* $\mathbb{C}$", i.e., it is a "model-free" (or "model-implicit") approach. For more informations on Mochizuki's philosophy of mono-anabelian reconstruction algorithms versus bi-anabelian reconstruction algorithms, see [AbsTopIII, §I.3, Remark 3.7.3, Remark 3.7.5].

In this subsection, to avoid settling the meaning of "group-theoretic characterisation" in $(\text{Delta})_X$ and $(\text{Cusp})_X$ (cf. Remark 3.4.4 (1)), we assume that k is sub-p-adic, and we include the subgroup Δ_X $(\subset \Pi_X)$ as an input datum. More generally, the results in this section hold in the case where k and X satisfy $(\text{Delta})'_X$, (GC), (slim), and $(\text{Cusp})_X$. Note that if we assume that k is an NF or an MLF, then $(\text{Delta})_X$, (GC), (slim), and $(\text{Cusp})_X$ hold for any X, and we do not need include the subgroup Δ_X $(\subset \Pi_X)$ as an input datum.

Lemma 3.5. *Let $\psi : H \to \Pi$ be an open homomorphism of profinite groups, and $\phi_1, \phi_2 : \Pi \to G$ two open homomorphisms of profinite groups. We assume that G is slim. If $\phi_1 \circ \psi = \phi_2 \circ \psi$, then we have $\phi_1 = \phi_2$.*

Proof. By replacing H by the image of ψ, we may assume that H is an open subgroup of Π. By replacing H by $\cap_{g \in \Pi/H} gHg^{-1}$, we may assume that H is an open normal subgroup of Π. For any $g \in \Pi$ and $h \in H$, we have $ghg^{-1} \in H$, and $\phi_1(ghg^{-1}) = \phi_2(ghg^{-1})$ by assumption. This implies that $\phi_1(g)\phi_1(h)\phi_1(g)^{-1} = \phi_2(g)\phi_2(h)\phi_2(g)^{-1} = \phi_2(g)\phi_1(h)\phi_2(g)^{-1}$. Hence we have $\phi_1(g)\phi_2(g)^{-1} \in Z_{\text{Im}(\Pi)}(G)$. By the assumption of the slimness of G, we have $Z_{\text{Im}(\Pi)}(G) = \{1\}$, since $\text{Im}(\Pi)$ is open in G. Therefore, we obtain $\phi_1(g) = \phi_2(g)$, as desired. $\square$

Remark 3.5.1. In the algebraic geometry, a finite étale covering $Y \twoheadrightarrow X$ is an epimorphism. The above lemma says that the inclusion map $\Pi_Y \subset \Pi_X$ correspoinding to $Y \twoheadrightarrow X$ is also an epimorphism if Π_X is slim. This enables us to make a theory for profinite groups (without using 2-categories and so on.) which is parallel to geometry, when all involved profinite groups are slim. This is a philosophy behind the geometry of anabelioids ([Anbd]).

Choose a hyperbolic orbicurve X over k, and we write Π_X for the arithmetic fundamental group of X for some basepoint. We have the surjection $\Pi_X \twoheadrightarrow G_k$ determined by $(\text{Delta})'_X$. Note that now we are assuming that k is sub-p-adic, hence G_k is slim by Lemma 3.2 (1) and Remark 3.17.3. Let $G \subset G_k$ be an open subgroup, and write $\Pi := \Pi_X \times_{G_k} G$, and $\Delta := \Delta_X \cap \Pi$. In this survey, we *do not* adopt the convention that we always write $(-)'$ for the commutator subgroup for a group $(-)$.

In the elliptic and Belyi cuspidalisations, we use the following three types of operations:

Lemma 3.6. *Write $\Pi' := \Pi_{X'}$ to be the arithmetic fundamental group of a hyperbolic orbicurve X' over a finite extension k' of k. Write $\Delta' := \ker(\Pi' \twoheadrightarrow G_{k'})$.*

(1) Let $\Pi'' \hookrightarrow \Pi'$ be an open immersion of profinite groups. Then Π'' arises as a finite étale covering $X'' \twoheadrightarrow X'$ of X', and $\Delta'' := \Pi'' \cap \Delta'$ reconstructs $\Delta_{X''}$.

(2) Let $\Pi' \hookrightarrow \Pi''$ be an open immersion of profinite groups such that there exists a surjection $\Pi'' \twoheadrightarrow G''$ to an open subgroup of G, whose restriction to Π' is equal to the given homomorphism $\Pi' \twoheadrightarrow G' \subset G$. Then the surjection $\Pi'' \twoheadrightarrow G''$ is uniquely determined (hence we reconstruct the quotient $\Pi'' \twoheadrightarrow G''$ as the unique quotient of Π'' having this property), and Π'' arises as a finite étale quotient $X' \twoheadrightarrow X''$ of X'.

(3) Assume that X' is a scheme i.e., not a (non-scheme-like) stack (We can treat orbicurves as well; however, we do not use this generalisation in this survey. cf. [AbsTopI, Definition 4.2 (iii) (c)]). Let $\Pi' \twoheadrightarrow \Pi''$ be a surjection of profinite groups such that the kernel is generated by a cuspidal inertia subgroup group-theoretically characterised by Corollary 2.9 and Remark 2.9.2 (We shall refer to it as a **cuspidal quotient***). Then Π'' arises as an open immersion $X' \hookrightarrow X''$, and we reconstruct $\Delta_{X''}$ as $\Delta'/\Delta' \cap \ker(\Pi' \twoheadrightarrow \Pi'')$.*

Proof. (1) is trivial by the definition of $\Pi_{X'}$.

The first asserion of (2) comes from Lemma 3.5, since G is slim. Write $(\Pi')^{\mathrm{Gal}} := \cap_{g\in\Pi''/\Pi'} g\Pi' g^{-1} \subset \Pi'$, which is normal in Π'' by definition. Then $(\Pi')^{\mathrm{Gal}}$ arises from a finite étale covering $(X')^{\mathrm{Gal}} \twoheadrightarrow X'$ by (1). By the conjugation, we have an action of Π'' on $(\Pi')^{\mathrm{Gal}}$. By (GC), this action determines an action of $\Pi''/(\Pi')^{\mathrm{Gal}}$ on $(X')^{\mathrm{Gal}}$. We take the quotient $X'' := (X')^{\mathrm{Gal}}//(\Pi''/(\Pi')^{\mathrm{Gal}})$ in the sense of stacks. Then $\Pi_{X''}$ is isomorphic to Π'' by definition, and the quotinet $(X')^{\mathrm{Gal}} \twoheadrightarrow X''$ factors as $(X')^{\mathrm{Gal}} \twoheadrightarrow X' \twoheadrightarrow X''$ since the intermediate quotient $(X')^{\mathrm{Gal}}//(\Pi'/(\Pi')^{\mathrm{Gal}})$ is isomorphic to X'. This proves the second assertion of (2).

(3) is also trivial. □

3.2.1. Elliptic Cuspidalisation. Let X be an elliptically admissible orbicurve over k. By definition, we have a k-core $X \twoheadrightarrow C = (E \setminus \{O\})//\{\pm 1\}$ where we write E for an elliptic curve over k with the origin O. Let $N \geq 1$ be a positive integer. We write $U_{C,N} := (E \setminus E[N])//\{\pm 1\} \subset C$ for the open sub-orbicurve of C determined by the image of $E\setminus E[N]$. Write $U_{X,N} := U_{C,N} \times_C X \subset X$, which is an open suborbicurve of X. For a finite extension K of k, write $X_K := X \times_k K$, $C_K := C \times_k K$, and $E_K := E \times_k K$. For a sufficiently large finite extension K of k, all points of $E_K[N]$ are rational over K.

We have the following key diagram for elliptic cuspidalisation:

$$\text{(EllCusp)} \qquad \begin{tikzcd} X \arrow[r, two heads] & C & E \setminus \{O\} \arrow[l, two heads] & E \setminus E[N] \arrow[l, two heads, "N"'] \arrow[r, two heads] \arrow[d, hook] & U_{C,N} \arrow[d, hook] & U_{X,N} \arrow[l, two heads] \arrow[d, hook] \\ & & & E \setminus \{O\} \arrow[r, two heads] & C & X, \arrow[l, two heads] \end{tikzcd}$$

where $\twoheadrightarrow$'s are finite étale coverings, $\hookrightarrow$'s are open immersions, and two sqauares are cartesian.

We will use the technique of elliptic cuspidalisation *three times*:

(1) Firstly, in the theory of Aut-holomorphic space in Section 4, we will use it for the reconstruction of "local linear holomorphic structure" of an Aut-holomorphic space (cf. Proposition 4.5 (Step 2)).

(2) (This is the most important usage) Secondly, in the theory of the étale theta function in Section 7, we will use it for the *constant multiple rigidity* of the étale theta function (cf. Proposition 7.9).

(3) Thirdly, we will use it for the reconstruction of "pseudo-monoids" (cf. Section 9.2).

Theorem 3.7. (Elliptic Cuspidalisation, [AbsTopII, Corollary 3.3]) *Let X be an elliptically admissible orbicurve over a sub-p-adic field k. Let $N \geq 1$ be a positive integer, and we write $U_{X,N}$ for the open sub-orbicurve of X defined as above. Then from the profinite groups $\Delta_X \subset \Pi_X$, we can group-theoretically reconstruct (cf. Remark 3.4.4 (2)) the surjection*

$$\pi_X : \Pi_{U_{X,N}} \twoheadrightarrow \Pi_X$$

of profinite groups, which is induced by the open immersion $U_{X,N} \hookrightarrow X$, and the set of the decomposition groups in Π_X at the points in $X \setminus U_{X,N}$.

We shall refer to $\pi_X : \Pi_{U_{X,N}} \twoheadrightarrow \Pi_X$ as an **elliptic cuspidalisation**.

Proof. (Step 1): By (Delta)'$_X$, we have the quotient $\Pi_X \twoheadrightarrow G_k$ with kernel Δ_X. Let $G \subset G_k$ be a sufficiently small (which will depend on N later) open subgroup, and write $\Pi := \Pi_X \times_{G_k} G$, and $\Delta := \Delta_X \cap \Pi$.

(Step 2): We define a category $\overline{\mathrm{Loc}}_G(\Pi)$ as follows: The objects are profinite groups Π' such that there exist open immersions $\Pi \hookleftarrow \Pi'' \hookrightarrow \Pi'$ of profinite groups and surjections $\Pi' \twoheadrightarrow G'$, $\Pi'' \twoheadrightarrow G''$ to open subgroups of G, and that the diagram

$$\begin{tikzcd}
\Pi \arrow[d, two heads] & \Pi'' \arrow[l, hook'] \arrow[r, hook] \arrow[d, two heads] & \Pi' \arrow[d, two heads] \\
G \arrow[d, "="] & G'' \arrow[d, hook] & G' \arrow[d, hook] \\
G & G \arrow[l, "="'] \arrow[r, "="] & G.
\end{tikzcd}$$

is commutative. Note that, by this compatibility, the surjections $\Pi' \twoheadrightarrow G'$ and $\Pi'' \twoheadrightarrow G''$ are uniquely determined by Lemma 3.6 (1), (2) (or Lemma 3.5). The morphisms from Π_1 to Π_2 are open immersions $\Pi_1 \hookrightarrow \Pi_2$ of profinite groups up to inner conjugates by $\ker(\Pi_2 \twoheadrightarrow G_2)$ such that the uniquely determined homomorphisms $\Pi_1 \twoheadrightarrow G_1 \subset G$ and $\Pi_2 \twoheadrightarrow G_2 \subset G$ are compatible. The definition of the category $\overline{\mathrm{Loc}}_G(\Pi)$ depends only on the topological group structure of Π and the surjection $\Pi \twoheadrightarrow G$ of profinite groups. By (GC), the functor $X' \mapsto \Pi_{X'}$ gives us an equivalence $\overline{\mathrm{Loc}}_K(X_K) \xrightarrow{\sim} \overline{\mathrm{Loc}}_G(\Pi)$ of categories, where K is the finite extension of k corresponding to $G \subset G_k$. Then we group-theoretically reconstruct $(\Pi_{X_K} \subset)\Pi_{C_K}$ as the terminal object $(\Pi \subset)\Pi_{\mathrm{core}}$ of the category $\overline{\mathrm{Loc}}_G(\Pi)$.

(Step 3): We group-theoretically reconstruct Δ_{C_K} ($\subset \Pi_{C_K}$) as the kernel $\Delta_{\mathrm{core}} := \ker(\Pi_{\mathrm{core}} \to G)$. We group-theoretically reconstruct $\Delta_{E_K \setminus \{O\}}$ as an open subgroup Δ_{ell} of Δ_{core} of index 2 such that Δ_{ell} is torsion-free (i.e., the corresponding covering is a scheme, not a (non-scheme-like) stack), since the covering is a scheme if and only if the geometric fundamental group is torsion-free (cf. also [AbsTopI, Lemma 4.1 (iv)]). We take any (not necessarily unique) extension $1 \to \Delta_{\mathrm{ell}} \to \Pi_{\mathrm{ell}} \to G \to 1$ such that the push-out of it via $\Delta_{\mathrm{ell}} \subset \Delta_{\mathrm{core}}$ is isomorphic to the extension $1 \to \Delta_{\mathrm{core}} \to \Pi_{\mathrm{core}} \to G \to 1$ (Note that Π_{ell} is isomorphic to $\Pi_{E'_K \setminus \{O\}}$, where $E'_K \setminus \{O\}$ is a twist of order 1 or 2 of $E_K \setminus \{O\}$). We group-theoretically reconstruct $\Pi_{E'_K \setminus \{O\}}$ as Π_{ell} (Note that if we replace G by a subgroup of index 2, then we may reconstruct $\Pi_{E_K \setminus \{O\}}$; however, we do not detect group-theoretically which subgroup of index 2 is correct. However, the final output does not depend on the choice of Π_{ell}).

(Step 4): Let

(a) $\Pi_{\mathrm{ell},N} \hookrightarrow \Pi_{\mathrm{ell}}$ be an open immersion of profinite groups with $\Pi_{\mathrm{ell}}/\Pi_{\mathrm{ell},N} \cong (\mathbb{Z}/N\mathbb{Z})^{\oplus 2}$ such that the composite $\Pi_{\mathrm{ell},N} \hookrightarrow \Pi_{\mathrm{ell}} \twoheadrightarrow \Pi_{\mathrm{ell}}^{\mathrm{cpt}}$ factors through as $\Pi_{\mathrm{ell},N} \twoheadrightarrow \Pi_{\mathrm{ell},N}^{\mathrm{cpt}} \to \Pi_{\mathrm{ell}}^{\mathrm{cpt}}$, where we write $\Pi_{\mathrm{ell}} \twoheadrightarrow \Pi_{\mathrm{ell}}^{\mathrm{cpt}}$, $\Pi_{\mathrm{ell},N} \twoheadrightarrow \Pi_{\mathrm{ell},N}^{\mathrm{cpt}}$ for the quotients by all of the conjugacy classes of the cuspidal inertia subgroups in Π_{ell}, $\Pi_{\mathrm{ell},N}$ respectively, and

(b) $\Pi_{\mathrm{ell},N} \twoheadrightarrow \Pi'$ a composite of (N^2-1) cuspidal quotients of profinite groups such that there exists an isomorphism $\Pi' \cong \Pi_{\mathrm{ell}}$ of profinite groups.

Note that the factorisation $\Pi_{\mathrm{ell},N} \twoheadrightarrow \Pi_{\mathrm{ell},N}^{\mathrm{cpt}} \to \Pi_{\mathrm{ell}}^{\mathrm{cpt}}$ means that the finite étale covering corresponding to $\Pi_{\mathrm{ell},N} \hookrightarrow \Pi_{\mathrm{ell}}$ extends to a finite étale covering of their compactifications i.e., the covering corresponding to $\Pi_{\mathrm{ell},N} \hookrightarrow \Pi_{\mathrm{ell}}$ is unramified at all cusps as well. Note that there exists such a diagram

$$\Pi_{\mathrm{ell}} \hookleftarrow \Pi_{\mathrm{ell},N} \twoheadrightarrow \Pi' \cong \Pi_{\mathrm{ell}}$$

by (EllCusp). Note that for any intermediate composite $\Pi_{\mathrm{ell},N} \twoheadrightarrow \Pi^* \twoheadrightarrow \Pi'$ of cuspidal quotients in the composite $\Pi_{\mathrm{ell},N} \twoheadrightarrow \Pi'$ of cuspidal quotients, and for the uniquely determined quotient $\Pi^* \twoheadrightarrow G^*$, we have $G^* = G$ for sufficiently small open subgroup $G \subset G_k$, and we take such an open subgroup $G \subset G_k$.

We group-theoretically reconstruct the surjection $\pi_{E'} : \Pi_{E'_K \setminus E'_K[N]} \twoheadrightarrow \Pi_{E'_K \setminus \{O\}}$ induced by the open immersion $E'_K \setminus E'_K[N] \hookrightarrow E'_K \setminus \{O\}$ as the composite $\pi_{E'?} : \Pi_{\mathrm{ell},N} \twoheadrightarrow \Pi' \cong \Pi_{\mathrm{ell}}$, since we can identify $\pi_{E'?}$ with $\pi_{E'}$ by (GC).

(Step 5): We write $\Pi_{\mathrm{core},1}$ for Π_{core} for $G = G_k$. If necessary, by changing Π_{ell}, we may take Π_{ell} such that there exists a *unique* lift of $\Pi_{\mathrm{core},1}/\Pi_{\mathrm{ell}} \to \mathrm{Out}(\Pi_{\mathrm{ell}})$ to $\mathrm{Out}(\Pi_{\mathrm{ell},N})$ by (EllCusp). We form $\overset{\mathrm{out}}{\rtimes}(\Pi_{\mathrm{core},1}/\Pi_{\mathrm{ell}})$ (cf. Section 0.2) to the surjection $\Pi_{\mathrm{ell},N} \twoheadrightarrow \Pi_{\mathrm{ell}}$ i.e., $\Pi_{\mathrm{ell},N} \overset{\mathrm{out}}{\rtimes} (\Pi_{\mathrm{core},1}/\Pi_{\mathrm{ell}}) \twoheadrightarrow \Pi_{\mathrm{ell}} \overset{\mathrm{out}}{\rtimes} (\Pi_{\mathrm{core},1}/\Pi_{\mathrm{ell}}) = \Pi_{\mathrm{core},1}$, where $\Pi_{\mathrm{core},1}/\Pi_{\mathrm{ell}} \to \mathrm{Out}(\Pi_{\mathrm{ell}})$ (in the definition of $\overset{\mathrm{out}}{\rtimes}(\Pi_{\mathrm{core},1}/\Pi_{\mathrm{ell}})$) is the natural one, and $\Pi_{\mathrm{core},1}/\Pi_{\mathrm{ell}} \to \mathrm{Out}(\Pi_{\mathrm{ell},N})$ (in the definition of $\overset{\mathrm{out}}{\rtimes}(\Pi_{\mathrm{core},1}/\Pi_{\mathrm{ell}})$) is the *unique* lift of $\Pi_{\mathrm{core},1}/\Pi_{\mathrm{ell}} \to \mathrm{Out}(\Pi_{\mathrm{ell}})$ to $\mathrm{Out}(\Pi_{\mathrm{ell},N})$. Then we obtain a surjection $\pi_{C?} : \Pi_{\mathrm{core},N} := \Pi_{\mathrm{ell},N} \overset{\mathrm{out}}{\rtimes} (\Pi_{\mathrm{core},1}/\Pi_{\mathrm{ell}}) \twoheadrightarrow \Pi_{\mathrm{core},1}$. We group-theoretically reconstruct the surjection $\pi_C : \Pi_{U_{C,N}} \twoheadrightarrow \Pi_C$ induced by the open immersion $U_{C,N} \hookrightarrow C$ as the surjection $\pi_{C?} : \Pi_{\mathrm{core},N} \twoheadrightarrow \Pi_{\mathrm{core},1}$, since we can identify $\pi_{C?}$ with π_C by (GC).

(Step 6): We form a fiber product $\times_{\Pi_{\mathrm{core},1}} \Pi_X$ to the surjection $\Pi_{\mathrm{core},N} \twoheadrightarrow \Pi_{\mathrm{core},1}$ i.e., $\Pi_{X,N} := \Pi_{\mathrm{core},N} \times_{\Pi_{\mathrm{core},1}} \Pi_X \twoheadrightarrow \Pi_{\mathrm{core},1} \times_{\Pi_{\mathrm{core},1}} \Pi_X = \Pi_X$. Then we obtain a surjection $\pi_{X?} : \Pi_{X,N} \twoheadrightarrow \Pi_X$. We group-theoretically reconstruct the surjection $\pi_X : \Pi_{U_{X,N}} \twoheadrightarrow \Pi_X$ induced by the open immersion $U_{X,N} \hookrightarrow X$ as the surjection $\pi_{X?} : \Pi_{X,N} \twoheadrightarrow \Pi_X$, since the identification of $\pi_{C?}$ with π_C induces an identification of $\pi_{X?}$ with π_X.

(Step 7): We group-theretically reconstruct the decomposition groups at the points of $X \setminus U_{X,N}$ in Π_X as the image of the cuspidal decomposition groups in $\Pi_{X,N}$, which are group-theoretically characterised by Corollary 2.9, via the surjection $\Pi_{X,N} \twoheadrightarrow \Pi_X$. □

3.2.2. Belyi Cuspidalisation. Let X be a hyperbolic orbicurve of strictly Belyi type over k. We have finite étale coverings $X \twoheadleftarrow Y \twoheadrightarrow \mathbb{P}^1 \setminus (N \text{ points})$, where Y is a hyperbolic curve over a finite extension k' of k, and $N \geq 3$. We assume that $Y \twoheadrightarrow X$ is Galois. For any open sub-orbicurve $U_X \subset X$ defined over a number field, write $U_Y := Y \times_X U_X$. Then by the theorem of Belyi (cf. also Theorem C.2 for its refinement), we have a finite étale covering $U'_Y \twoheadrightarrow U_{\mathbb{P}^1}$ from an open sub-orbicurve $U'_Y \subset U_Y$ to the tripod $U_{\mathbb{P}^1}$ (cf. Section 0.2) over k'. For a sufficiently large finite extension K of k', all the points of $Y \setminus U'_Y$ are defined over K. We have the following key diagram for Belyi cuspidalisation:

(BelyiCusp)
$$\begin{tikzcd}
 & & & U'_Y \arrow[r, hook] \arrow[d, two heads] & U_Y \arrow[r, hook] \arrow[d, two heads] & Y \arrow[d, two heads] \\
X & Y \arrow[l, two heads] \arrow[r, two heads] & \mathbb{P}^1 \setminus (N\,\text{points}) \arrow[r, hook] & U_{\mathbb{P}^1} & U_X \arrow[r, hook] & X,
\end{tikzcd}$$

where $\twoheadrightarrow$'s are finite étale coverings, $\hookrightarrow$'s are open immersions, and the square is cartesian.

Theorem 3.8. (Belyi Cuspidalisation, [AbsTopII, Corollary 3.7]) *Let X be an orbicurve over a sub-p-adic field k. We assume that X is of strictly Belyi type. Then from the profinite groups $\Delta_X \subset \Pi_X$, we can group-theoretically reconstruct (cf. Remark 3.4.4 (2)) the set*
$$\{\Pi_{U_X} \twoheadrightarrow \Pi_X\}_{U_X}$$
of the surjections of profinite groups, where U_X runs through the open subschemes of X defined over a number field. We can also group-theoretically reconstruct the set of the decomposition groups in Π_X at the points in $X \setminus U_X$, where U_X runs through the open subschemes of X defined over a number field.

We shall refer to $\Pi_{U_X} \twoheadrightarrow \Pi_X$ as a **Belyi cuspidalisation**.

Proof. (Step 1): By (Delta)'$_X$, we have the quotient $\Pi_X \twoheadrightarrow G_k$ with kernel Δ_X. For sufficiently small (which will depend on U later) open subgroup $G \subset G_k$, write $\Pi := \Pi_X \times_{G_k} G$.

(Step 2): Let

(a) $\Pi \hookleftarrow \Pi^*$ be an open immersion of profinite grouops,

(b) $\Pi^* \hookrightarrow \Pi^{\mathrm{tpd},U}$ an open immersion of profinite groups, such that the group-theoretic algorithms described in Lemma 2.8 and Remark 2.9.2 tell us that the hyperbolic curve corresponding to $\Pi^{\mathrm{tpd},U}$ has genus 0,

(c) $\Pi^{\mathrm{tpd},U} \twoheadrightarrow \Pi^{\mathrm{tpd}}$ a composite of cuspidal quotients of profinite groups, such that the number of the conjugacy classes of cuspidal inertia subgroups of Π^{tpd} is three,

(d) $\Pi^{\mathrm{tpd}} \hookleftarrow \Pi^{*,U'}$ an open immersion of profinite groups,

(e) $\Pi^{*,U'} \twoheadrightarrow \Pi^{*,U}$ a composite of cuspidal quotients of profinite groups, and

(f) $\Pi^{*,U} \twoheadrightarrow \Pi^{**}$ a composite of cuspidal quotients of profinite groups such that there exists an isomorphism $\Pi^{**} \cong \Pi^*$ of profinite groups.

Note that there exists such a diagram

$$\Pi \hookleftarrow \Pi^* \hookrightarrow \Pi^{\mathrm{tpd},U} \twoheadrightarrow \Pi^{\mathrm{tpd}} \hookleftarrow \Pi^{*,U'} \twoheadrightarrow \Pi^{*,U} \twoheadrightarrow \Pi^{**} \cong \Pi^*$$

by (BelyiCusp). Note also that any algebraic curve over an algebraically closed field of characteristic 0, which is finite étale over a tripod, is defined over a number field (i.e., converse of Belyi's theorem, essentially the descent theory) and that algebraic points in a hyperbolic curve are sent to algebraic points via any isomorphism of hyperbolic curves over the base field (cf. [AbsSect, Remark 2.7.1]). Write $\pi_{Y?} : \Pi^{*,U} \twoheadrightarrow \Pi^{**} \cong \Pi^*$ to be the composite. Note that for any intermediate composite $\Pi^{*,U'} \twoheadrightarrow \Pi^{\#} \twoheadrightarrow \Pi^{**}$ in the composite $\Pi^{*,U'} \twoheadrightarrow \Pi^{**}$ of cuspidal quotients and for the uniquely determined quotient $\Pi^{\#} \twoheadrightarrow G^{\#}$, we have $G^{\#} = G$ for sufficiently small open subgroup $G \subset G_k$, and we take such an open subgroup $G \subset G_k$.

We group-theoretically reconstruct the surjection $\pi_Y : \Pi_{U_Y} \twoheadrightarrow \Pi_Y$ induced by *some* open immersion $U_Y \hookrightarrow Y$ as $\pi_{Y?} : \Pi^{*,U} \twoheadrightarrow \Pi^*$, since we can identify $\pi_{Y?}$ with π_Y by (GC) (Note that we *do not* prescribe the open immersion $U_Y \hookrightarrow Y$).

(Step 3): We choose the data (a)-(e) such that the natural homomorphism $\Pi_X/\Pi^* \to \mathrm{Out}(\Pi^*)$ has a *unique* lift $\Pi_X/\Pi^* \to \mathrm{Out}(\Pi^{*,U})$ to $\mathrm{Out}(\Pi^{*,U})$ (Note that this corresponds to that $U_Y \subset Y$ is stable under the action of $\mathrm{Gal}(Y/X)$, thus descends to $U_X \subset X$). We form $\overset{\mathrm{out}}{\rtimes}(\Pi_X/\Pi^*)$ to the surjection $\Pi^{*,U} \twoheadrightarrow \Pi^*$ i.e., $\Pi^{X,U} := \Pi^{*,U} \overset{\mathrm{out}}{\rtimes} (\Pi_X/\Pi^*) \twoheadrightarrow \Pi^* \overset{\mathrm{out}}{\rtimes} (\Pi_X/\Pi^*) = \Pi_X$. Then we obtain a surjection $\pi_{X?} : \Pi^{X,U} \twoheadrightarrow \Pi_X$. We group-theretically reconstruct the surjection $\pi_X : \Pi_{U_X} \twoheadrightarrow \Pi_X$ induced by the open immersion $U_X \hookrightarrow X$ as the surjection $\pi_{X?} : \Pi^{X,U} \twoheadrightarrow \Pi_X$, since we can identify $\pi_{X?}$ with π_X by (GC) (Note again that we *do not* prescribe the open immersion $U_X \hookrightarrow X$. We just group-theoretically reconstruct a surjection $\Pi_{U_X} \twoheadrightarrow \Pi_X$ for *some* $U_X \subset X$ such that all of the points in $X \setminus U_X$ are defined over a number field).

(Step 4): We group-theretically reconstruct the decomposition groups at the points of $X \setminus U_X$ in Π_X as the image of the cuspidal decomposition groups in $\Pi^{X,U}$, which are group-theoretically characterised by Corollary 2.9, via the surjection $\Pi_{U_X} \twoheadrightarrow \Pi_X$. □

Corollary 3.9. ([AbsTopII, 3.7.2]) *Let X be a hyperbolic orbicurve over a non-Archimedean local field k. We assume that X is of strictly Belyi type. Then from the profinite group Π_X, we can reconstruct the set of the decomposition groups at all closed points in X.*

Proof. The corollary follows from Theorem 3.8 and the approximation of a decomposition group in (the proof of) Lemma 3.10 below. □

Since the geometric fundamental group Δ_X of X (for some basepoint) is topologically finitely generated, there exist characteristic open subgroups

$$\ldots \subset \Delta_X[j+1] \subset \Delta_X[j] \subset \ldots \subset \Delta_X$$

of Δ_X for $j \geq 1$ such that $\bigcap_j \Delta_X[j] = \{1\}$. Let $\overline{k}$ be an algebraic closure of k and write $G_k := \mathrm{Gal}(\overline{k}/k)$. For any section $\sigma : G_k \to \Pi_X$, we write

$$\Pi_{X[j,\sigma]} := \mathrm{Im}(\sigma)\Delta_X[j] \subset \Pi_X,$$

and we obtain a corresponding finite étale coverings

$$\ldots \to X[j+1,\sigma] \to X[j,\sigma] \to \ldots \to X.$$

Lemma 3.10. ([AbsSect, Lemma 3.1]) *Let X be a hyperbolic curve over a non-Archimedean local field k. Suppose X is defined over a number field. Let $\sigma : G_k \to \Pi_X$ be a section such that* $\mathrm{Im}(\sigma)$ *is not contained in any cuspidal decomposition group of Π_X. Then the following conditions on σ is equivalent:*

(1) $\mathrm{Im}(\sigma)$ *is a decomposition group D_x of a point $x \in X(k)$.*

(2) For any $j \geq 1$, the subgroup $\Pi_{X[j,\sigma]}$ contains a decomposition group of an algebraic closed point of X which surjects onto G_k.

Proof. (1)$\Leftarrow$(2): For $j \geq 1$, take points $x_j \in X[j,\sigma](k)$. Since the topological space $\prod_{j\geq 1} \overline{X}[j,\sigma](k)$ is compact, there exists an infinite set of positive integers J' such that for any $j \geq 1$, the images of $x_{j'}$ in $\overline{X}[j,\sigma](k)$ for $j' \geq j$ with $j' \in J'$ converges to a point $y_j \in \overline{X}[j,\sigma](k)$. By definition of y_j, the point y_{j_1} maps to y_{j_2} in $\overline{X}[j_2](k)$ for any $j_1 > j_2$. We write $y \in \overline{X}(k)$ for the image of y_j in $\overline{X}(k)$. Then we have $\mathrm{Im}(\sigma) \subset D_y$ (up to conjugates), and y is not a cusp by the assumption that $\mathrm{Im}(\sigma)$ is not contained in any cuspidal decomposition group of Π_X.

(1)$\Rightarrow$(2): By using Krasner's lemma, we can approximate $x \in X(k)$ by a point $x' \in X_F(F) \subset X(k)$, where X_F is a model of $X \times_k \overline{k}$ over a number field F, which is sufficiently close to x so that x' lifts to a point $x'_j \in X[j,\sigma](k)$, which is algebraic. $\square$

§ 3.3. Uchida's Lemma.

Let X be a hyperbolic curve over a field k. Let $\overline{k}$ be an algebraic closure of k. Write $G_k := \mathrm{Gal}(\overline{k}/k)$, and $X_{\overline{k}} := X \times_k \overline{k}$. We write $k(X)$ for the function field of X. We write Δ_X and Π_X for the geometric fundamental group (i.e., π_1 of $X_{\overline{k}}$) and the arithmetic fundamental group (i.e., π_1 of X) of X for some basepoint, respectively. Note that we have an exact sequence $1 \to \Delta_X \to \Pi_X \to G_k \to 1$.

We recall that we have $\Gamma(X, \mathcal{O}(D)) = \{f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\}$ for a divisor D on X.

Lemma 3.11. ([AbsTopIII, Proposition 1.2]) *Assume that k be an algebraically closed, and X proper.*

(1) There are distinct points $x, y_1, y_2 \in X(k)$ and a divisor D on X such that $x, y_1, y_2 \notin \mathrm{Supp}(D)$ and $l(D) := \dim_k \Gamma(X, \mathcal{O}(D)) = 2$, and $l(D - E) = 0$ for any $E = e_1 + e_2$ with $e_1, e_2 \in \{x, y_1, y_2\}$, $e_1 \neq e_2$.

(2) Let x, y_1, y_2, D be as in (1). For $i = 1, 2$, and $\lambda \in k^\times$, there exists a unique $f_{\lambda,i} \in k(X)^\times$ such that

$$\mathrm{div}(f_{\lambda,i}) + D \geq 0, \quad f_{\lambda,i}(x) = \lambda, \quad f_{\lambda,i}(y_i) \neq 0, \quad f_{\lambda,i}(y_{3-i}) = 0.$$

(3) Let x, y_1, y_2, D be as in (1), and $\lambda, \mu \in k^\times$ be such that $\frac{\lambda}{\mu} \neq -1$. Let also $f_{\lambda,1}, f_{\mu,2} \in k(X)^\times$ be as in (2). Then $f_{\lambda,1} + f_{\mu,2} \in k(X)^\times$ is characterised as a unique element $g \in k(X)^\times$ such that

$$\mathrm{div}(g) + D \geq 0, \quad g(y_1) = f_{\lambda,1}(y_1), \quad g(y_2) = f_{\mu,2}(y_2).$$

In particular, $\lambda + \mu \in k^\times$ is characterised as $g(x) \in k^\times$.

Proof. (1): For any divisor D of degree $\geq 2g - 2 + 3$ on X, then we have $l(D) = l(K_X - D) + \deg(D) + 1 - g = \deg(D) + 1 - g \geq g + 2 \geq 2$, by the theorem of Riemann-Roch (Here, we write K_X for the canonical divisor of X). For any divisor D on X with $d := l(D) \geq 2$, we write $\Gamma(X, \mathcal{O}(D)) = \langle f_1, \ldots, f_d \rangle_k$, and take a point P in the locus "$f_1 f_2 \cdots f_d \neq 0$" in X of non-vanishing of the section $f_1 f_2 \cdots f_d$ such that $P \notin \mathrm{Supp}(D)$ (Note that this locus is non-empty since there is a non-constant function in $\Gamma(X, \mathcal{O}(D))$ by $l(D) \geq 2$). Then we have $l(D - P) < l(D)$. On the other hand, we have $l(D) - l(D - P) = l(K_X - D) - l(K_X - D + P) + 1 \leq 1$. Thus, we have $l(D - P) = l(D) - 1$. Therefore, by substracting a suitable divisor from a divisor of degree $\geq 2g - 2 + 3$, there is a divisor D on X with $l(D) = 2$. In the same way, take $x \in X(k) \setminus \mathrm{Supp}(D)$ such that there is $f \in \Gamma(X, \mathcal{O}_X(D))$ with $f(x) \neq 0$ (this implies that $l(D - x) = l(D) - 1 = 1$). Let $y_1 \in X(k) \setminus (\mathrm{Supp}(D) \cup \{x\})$ be such that there is $g \in \Gamma(X, \mathcal{O}_X(D - x))$ with $g(y_1) \neq 0$ (this implies that $l(D - x - y_1) = l(D - x) - 1 = 0$), and $y_2 \in X(k) \setminus (\mathrm{Supp}(D) \cup \{x, y_1\})$ such that there are $h_1 \in \Gamma(X, \mathcal{O}_X(D - x))$ and $h_2 \in \Gamma(X, \mathcal{O}_X(D - y_1))$ with $h_1(y_2) \neq 0$ and $h_2(y_2) \neq 0$ (this implies that $l(D - x - y_2) = l(D - y_1 - y_2) = 0$). The first claim (1) is proved. The claims (2) and (3) trivially follow from (1). $\square$

Proposition 3.12. (Uchida's Lemma, [AbsTopIII, Proposition 1.3]) *Assume that k be an algebraically closed, and X proper. There exists a functorial (with respect to isomorphisms of the following triples) algorithm for constructing the additive structure on $k(X)^\times \cup \{0\}$ from the following data:*

(a) the (abstract) group $k(X)^\times$,

(b) the set of surjective homomorphisms $\mathcal{V}_X := \{\mathrm{ord}_x : k(X)^\times \twoheadrightarrow \mathbb{Z}\}_{x \in X(k)}$ *of the valuation maps at* $x \in X(k)$, *and*

(c) the set of the subgroups $\{\mathcal{U}_v := \{f \in k(X)^\times \mid f(x) = 1\} \subset k(X)^\times\}_{v=\mathrm{ord}_x \in \mathcal{V}_X}$ *of* $k(X)^\times$.

Proof. From the above data (a), (b), and (c), we reconstruct the additive structure on $k(X)^\times$ as follows:

(Step 1): We reconstruct $k^\times \subset k(X)^\times$ as $k^\times := \bigcap_{v \in \mathcal{V}_X} \ker(v)$. We also reconstruct the set $X(k)$ as $\mathcal{V}_X$.

(Step 2): For each $v = \mathrm{ord}_x \in \mathcal{V}_X$, we have inclusions $k^\times \subset \ker(v)$ and $\mathcal{U}_v \subset \ker(v)$ with $k^\times \cap \mathcal{U}_v = \{1\}$, thus we obtain a direct product decomposition $\ker(v) = \mathcal{U}_v \times k^\times$. We write pr_v for the projection $\ker(v) \to k^\times$ Then we reconstruct the evaluation map $\ker(v) \ni f \mapsto f(x) \in k^\times$ as $f(x) := \mathrm{pr}_v(f)$ for $f \in \ker(v)$.

(Step 3): We reconstruct divisors (resp. effective divisors) on X as formal finite sums of $v \in \mathcal{V}_X$ with coefficient $\mathbb{Z}$ (resp. $\mathbb{Z}_{\geq 0}$). By using $\mathrm{ord}_x \in \mathcal{V}_X$, we reconstruct the divisor $\mathrm{div}(f)$ for an element f in an abstract group $k(X)^\times$.

(Step 4): We reconstruct a (multiplicative) $k^\times$-module $\Gamma(X, \mathcal{O}(D)) \setminus \{0\}$ for a divisor D as $\{f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0\}$. We also reconstruct $l(D) \geq 0$ for a divisor D as the smallest non-negative integer d such that there is an effective divisor E of degree d on X such that $\Gamma(X, \mathcal{O}(D - E)) \setminus \{0\} = \emptyset$ (cf. also the proof of Lemma 3.11 (1)). Note that $\dim_k$ of $\Gamma(X, \mathcal{O}(D))$ is *not* available yet here, since we *do not* have the additive structure on $\{f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\}$ yet.

(Step 5): For $\lambda, \mu \in k^\times$, $\frac{\lambda}{\mu} \neq -1$ (Here, -1 is the unique element of order 2 in $k^\times$), we take $\mathrm{ord}_x, \mathrm{ord}_{y_1}, \mathrm{ord}_{y_2} \in \mathcal{V}_X$ corresponding to x, y_1, y_2 in Lemma 3.11 (1). Then we obtain unique $f_{\lambda,1}, f_{\mu,2}, g \in k(X)^\times$ as in Lemma 3.11 (2), (3) from abstract data (a), (b), and (c). Then we reconstruct the addition $\lambda + \mu \in k^\times$ of λ and μ as $g(x)$. We also reconstruct the addition $\lambda + \mu := 0$ for $\frac{\lambda}{\mu} = -1$, and $\lambda + 0 = 0 + \lambda := \lambda$ for $\lambda \in k^\times \cup \{0\}$. These reconstruct the additive structure on $k^\times \cup \{0\}$.

(Step 6): We reconstruct the addition $f + g$ of $f, g \in k(X)^\times \cup \{0\}$ as the unique element $h \in k(X)^\times \cup \{0\}$ such that $h(x) = f(x) + g(x)$ for any $\mathrm{ord}_x \in \mathcal{V}_X$ with $f, g \in \ker(\mathrm{ord}_x)$ (Here, we write $f(x) := 0$ for $f = 0$). This reconstructs the additive structure on $k(X)^\times \cup \{0\}$. □

§ 3.4. Mono-anabelian Reconstruction of the Base Field and Function Field.

We continue the notation in Section 3.3 in this subsection. Furthermore, we assume that k is of characteristic 0.

Definition 3.13.

(1) We assume that X has genus ≥ 1. Let $(X \subset)\overline{X}$ be the canonical smooth compactification of X. We define

$$\mu_{\widehat{\mathbb{Z}}}(\Pi_X) := \mathrm{Hom}(H^2(\Delta_{\overline{X}}, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}}).$$

We shall refer to $\mu_{\widehat{\mathbb{Z}}}(\Pi_X)$ as the **cyclotome of Π_X as orientation**.

(2) In the case where the genus of X is not necessarily greater than or equal to 2, we take a finite étale covering $Y \twoheadrightarrow X$ such that Y has genus ≥ 2, and we define the **cyclotome of Π_X as orientation** to be $\mu_{\widehat{\mathbb{Z}}}(\Pi_X) := [\Delta_X : \Delta_Y]\mu_{\widehat{\mathbb{Z}}}(\Pi_Y)$. It does not depend on the choice of Y in the functorial sense, i.e., For any such coverings $Y \twoheadrightarrow X$, $Y' \twoheadrightarrow X$, take $Y'' \twoheadrightarrow X$ which factors through $Y'' \twoheadrightarrow Y \twoheadrightarrow X$ and $Y'' \twoheadrightarrow Y' \twoheadrightarrow X$. Then the restrictions $H^2(\Delta_{\overline{Y}}, \widehat{\mathbb{Z}}) \to H^2(\Delta_{\overline{Y''}}, \widehat{\mathbb{Z}})$, $H^2(\Delta_{\overline{Y'}}, \widehat{\mathbb{Z}}) \to H^2(\Delta_{\overline{Y''}}, \widehat{\mathbb{Z}})$ (where $\overline{Y}$, $\overline{Y'}$, and $\overline{Y''}$ are the canonical compactifications of Y, Y', and Y'' respectively), and taking $\mathrm{Hom}(-, \widehat{\mathbb{Z}})$ induce natural isomorphisms $[\Delta_X : \Delta_Y]\mu_{\widehat{\mathbb{Z}}}(\Pi_Y) \xleftarrow{\sim} [\Delta_X : \Delta_Y][\Delta_Y : \Delta_{Y''}]\mu_{\widehat{\mathbb{Z}}}(\Pi_{Y''}) = [\Delta_X : \Delta_{Y''}]\mu_{\widehat{\mathbb{Z}}}(\Pi_{Y''}) = [\Delta_X : \Delta_{Y'}][\Delta_{Y'} : \Delta_{Y''}]\mu_{\widehat{\mathbb{Z}}}(\Pi_{Y''}) \xrightarrow{\sim} [\Delta_X : \Delta_{Y'}]\mu_{\widehat{\mathbb{Z}}}(\Pi_{Y'})$ (cf. [AbsTopIII, Remark 1.10.1 (i), (ii)]).

(3) For an open subscheme $\emptyset \neq U \subset X$, let $\Delta_U \twoheadrightarrow \Delta_U^{\text{cusp-cent}} (\twoheadrightarrow \Delta_X)$ be the maximal intermediate quotient $\Delta_U \twoheadrightarrow Q \twoheadrightarrow \Delta_X$ such that $\ker(Q \twoheadrightarrow \Delta_X)$ is in the center of Q, and $\Pi_U \twoheadrightarrow \Pi_U^{\text{cusp-cent}}$ the push-out of $\Delta_U \twoheadrightarrow \Delta_U^{\text{cusp-cent}}$ with respect to $\Delta_U \subset \Pi_U$. We shall refer to them as the **maximal cuspidally central quotient** of Δ_U and Π_U respectively.

Remark 3.13.1. In this subsection, by the functoriality of cohomology with $\mu_{\widehat{\mathbb{Z}}}(\Pi_{(-)})$-coefficients for an open injective homomorphism of profinite groups $\Delta_Z \subset \Delta_Y$, we always mean multiplying $\frac{1}{[\Delta_Y : \Delta_Z]}$ on the homomorphism between the cyclotomes Π_Y and Π_Z (cf. also [AbsTopIII, Remark 1.10.1 (i), (ii)]).

Proposition 3.14. (Cyclotomic Rigidity for Inertia Subgroups, [AbsTopIII, Proposition 1.4]) *Assume that X has genus ≥ 2. Let $(X \subset)\overline{X}$ be the canonical smooth compactification of X. Let $U \subset X$ be a non-empty open subscheme. We have an exact sequence $1 \to \Delta_U \to \Pi_U \to G_k \to 1$. For $x \in X(k) \setminus U(k)$, write $U_x := \overline{X} \setminus \{x\}$. We write I_x for the inertia subgroup of x in Δ_U (it is well-defined up to inner automorphism of Δ_U), which is naturally isomorphic to $\widehat{\mathbb{Z}}(1)$.*

(1) $\ker(\Delta_U \twoheadrightarrow \Delta_{U_x})$ *and* $\ker(\Pi_U \twoheadrightarrow \Pi_{U_x})$ *are topologically normally generated by the inertia subgroups of the points of $U_x \setminus U$.*

(2) We have an exact sequence

$$1 \to I_x \to \Delta_{U_x}^{\text{cusp-cent}} \to \Delta_{\overline{X}} \to 1,$$

which induces the Leray spectral sequence $E_2^{p,q} = H^p(\Delta_{\overline{X}}, H^q(I_x, I_x)) \Rightarrow H^{p+q}(\Delta_{U_x}^{\text{cusp-cent}}, I_x)$ *(Here,* I_x *and* $\Delta_{U_x}^{\text{cusp-cent}}$ *act on* I_x *by the conjugates). Then the composite*

$$\begin{aligned}\widehat{\mathbb{Z}} = \mathrm{Hom}(I_x, I_x) \cong H^0(\Delta_X, H^1(I_x, I_x)) = E_2^{0,1}\\ \to E_2^{2,0} = H^2(\Delta_{\overline{X}}, H^0(I_x, I_x)) \cong \mathrm{Hom}(\mu_{\widehat{\mathbb{Z}}}(\Pi_X), I_x)\end{aligned}$$

sends $1 \in \widehat{\mathbb{Z}}$ *to the natural isomorphism*

$$\text{(Cyc. Rig. Iner.)} \qquad \mu_{\widehat{\mathbb{Z}}}(\Pi_X) \xrightarrow{\sim} I_x.$$

(this is a natural identification between "$\widehat{\mathbb{Z}}(1)$*" arising from* H^2 *and "*$\widehat{\mathbb{Z}}(1)$*" arising from* I_x*.) Therefere, we obtain a group-theoretic reconstruction of the isomorphism* (Cyc. Rig. Iner.) *from the surjection* $\Delta_{U_x} \twoheadrightarrow \Delta_{\overline{X}}$ *(Note that the intermediate quotient* $\Delta_{U_x} \twoheadrightarrow \Delta_{U_x}^{\text{cusp-cent}} \twoheadrightarrow \Delta_{\overline{X}}$ *is group-theoretically characterised). We shall refer to the isomorphism* (Cyc. Rig. Iner.) *as the* **cyclotomic rigidity for inertia subgroup**.

Proof. (1) is trivial. (2): By the definitions, for any intermediate quotient $\Delta_{U_x} \twoheadrightarrow Q \twoheadrightarrow \Delta_{\overline{X}}$ such that $\ker(Q \twoheadrightarrow \Delta_{\overline{X}})$ is in the center of Q, the kernel $\ker(Q \twoheadrightarrow \Delta_{\overline{X}})$ is generated by the image of I_x. Thus, we have the exact sequence $1 \to I_x \to \Delta_{U_x}^{\text{cusp-cent}} \to \Delta_{\overline{X}} \to 1$ (cf. also [Cusp, Proposition 1.8 (iii)]). The rest is trivial. □

Remark 3.14.1. In the case where the genus of X is not necessarily greater than or equal to 2, we take a finite étale covering $Y \twoheadrightarrow X$ such that Y has genus ≥ 2, and a point $y \in Y(k')$ lying over $x \in X(k)$ for a finite extension k' of k. Then we have the cyclotomic rigidity $\mu_{\widehat{\mathbb{Z}}}(\Pi_Y) \cong I_y$ by Proposition 3.14. This induces isomorphisms

$$\mu_{\widehat{\mathbb{Z}}}(\Pi_X) = [\Delta_X : \Delta_Y]\mu_{\widehat{\mathbb{Z}}}(\Pi_Y) \overset{\frac{1}{[\Delta_X:\Delta_Y]}}{\xrightarrow{\sim}} \mu_{\widehat{\mathbb{Z}}}(\Pi_Y) \cong I_y = I_x.$$

We shall also refer to this as the **cyclotomic rigidity for inertia subgroup**. It does not depend on the choice of Y and y in the functorial sense of Definition 3.13 (2), i.e., For such $Y \twoheadrightarrow X$, $Y' \twoheadrightarrow X$ with $y \in Y(k_Y)$, $y' \in Y'(k_{Y'})$, take $Y'' \twoheadrightarrow X$ with $y'' \in Y''(k_{Y''})$ lying over Y, Y' and y, y', then we have the following commutative diagram (cf. also

Remark 3.13.1)

$$\begin{array}{ccc}
\widehat{\mathbb{Z}} = \mathrm{Hom}(I_y, I_y) & \longrightarrow & \mathrm{Hom}(\mu_{\widehat{\mathbb{Z}}}(\Pi_Y), I_y) \\
\Big\downarrow = & & \cong \Big\downarrow \frac{1}{[\Delta_Y : \Delta_{Y''}]} \\
\widehat{\mathbb{Z}} = \mathrm{Hom}(I_{y''}, I_{y''}) & \longrightarrow & \mathrm{Hom}(\mu_{\widehat{\mathbb{Z}}}(\Pi_{Y''}), I_{y''}) \\
\Big\uparrow = & & \cong \Big\uparrow \frac{1}{[\Delta_{Y'} : \Delta_{Y''}]} \\
\widehat{\mathbb{Z}} = \mathrm{Hom}(I_{y'}, I_{y'}) & \longrightarrow & \mathrm{Hom}(\mu_{\widehat{\mathbb{Z}}}(\Pi_{Y'}), I_{y'}).
\end{array}$$

For a proper hyperbolic curve X over k, we write J^d for the Picard scheme parametrising line bundles of degree d on X (Note that J^d is a $J := J^0$-torsor). We have a natural map $X \to J^1$ $(P \mapsto \mathcal{O}(P))$, which induces $\Pi_X \to \Pi_{J^1}$ (for some basepoint). For $x \in X(k)$, let $t_x : G_k \to \Pi_{J^1}$ be the composite of the section $G_k \to \Pi_X$ determined by x and the natural map $\Pi_X \to \Pi_{J^1}$. The group structure of Picard schemes also determines a morphism $\Pi_{J^1} \times \cdots (d\text{-times}) \cdots \times \Pi_{J^1} \to \Pi_{J^d}$ for $d \geq 1$. For any divisor D of degree d on X such that $\mathrm{Supp}(D) \subset X(k)$, by forming a $\mathbb{Z}$-linear combination of t_x's, we have a section $t_D : G_k \to \Pi_{J^d}$.

Lemma 3.15. ([AbsTopIII, Proposition 1.6]) *Assume that k is Kummer-faithful, and that X is proper. Let $\emptyset \neq U \subset X$ be an open subscheme, and we write*

$$\kappa_U : \Gamma(U, \mathcal{O}_U^\times) \to H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\overline{k(X)})) = H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\overline{k})) \cong H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))$$

for the composite of the Kummer map (for an algebraic closure $\overline{k(X)}$ of $k(X)$) and the natural isomorphism $\mu_{\widehat{\mathbb{Z}}}(\overline{k}) \cong \mu_{\widehat{\mathbb{Z}}}(\Pi_X)(\cong \widehat{\mathbb{Z}}(1))$ (which comes from the scheme theory).

(1) κ_U is injective.

(2) (cf. also [Cusp, Proposition 2.3 (i)]) For any divisor D of degree 0 on X such that $\mathrm{Supp}(D) \subset X(k)$, the section $t_D : G_k \to \Pi_J$ is equal to (up to conjugates by Δ_X) the section determined by the origin O of $J(k)$ if and only if the divisor D is principal.

(3) (cf. also [Cusp, Proposition 2.1 (i)]) We assume that $U = X \setminus S$, where $S \subset X(k)$ is a finite set. Then the quotient $\Pi_U \twoheadrightarrow \Pi_U^{\text{cusp-cent}}$ induces an isomorphism

$$H^1(\Pi_U^{\text{cusp-cent}}, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \xrightarrow{\sim} H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)).$$

(4) (cf. also [Cusp, Proposition 1.4 (ii)]) We have an isomorphism

$$H^1(\Pi_X, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \cong (k^\times)^\wedge,$$

where we write $(k^\times)^\wedge$ for the profinite completion of $k^\times$.

(5) *(cf. also [Cusp, Proposition 2.1 (ii)]) We have a natural exact sequence induced by the restrictions to* I_x *(*$x \in S$*):*

$$0 \to H^1(\Pi_X, H^0(\prod_{x\in S} I_x, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))) \to H^1(\Pi_U^{\text{cusp-cent}}, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))) \to \bigoplus_{x\in S} H^0(\Pi_X, H^1(I_x, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))).$$

The cyclotomic rigidity isomorphism (Cyc. Rig. Iner.) $\mu_{\widehat{\mathbb{Z}}}(\Pi_X) \cong I_x$ **in Proposition 3.14** *induces an isomorphism*

$$H^0(\Pi_X, H^1(I_x, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))) = \mathrm{Hom}_{\Pi_X}(I_x, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \cong \widehat{\mathbb{Z}}$$

(Hence note that we can use the above isomorphism for a group-theoretic reconstruction later). Then by the isomorphisms in (3) and (4) and the above cyclotomic rigidity isomorphism, the above exact sequence is identified with

$$1 \to (k^\times)^\wedge \to H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x\in S} \widehat{\mathbb{Z}}.$$

(6) *The image of* $\Gamma(U, \mathcal{O}_U^\times)$ *in* $H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))/(k^\times)^\wedge$ *via* κ_U *is equal to the inverse image in* $H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))/(k^\times)^\wedge$ *of the submodule* $\mathcal{P}'_U$ *of* $\bigoplus_{x\in S} \mathbb{Z}\,(\subset \bigoplus_{x\in S} \widehat{\mathbb{Z}})$ *determined by the principal divisors with support in* S*.*

Remark 3.15.1. (A general remark to the readers who are not familiar with the culture of anabelian geometers) In the above lemma, note that we are currently studying in a scheme theory here, and that the natural isomorphism $\mu_{\widehat{\mathbb{Z}}}(\overline{k}) \cong \mu_{\widehat{\mathbb{Z}}}(\Pi_X)$ comes from the scheme theory. A kind of "general principle" of studying anabelian geometry is like this:

(1) First, we study some objects in a scheme theory to obtain group-theoretic properties or group-theoretic characterisations.

(2) Next, by using the group-theoretic properties or group-theoretic characterisations obtained in the first step, we formulate group-theoretic reconstruction algorithms, and we cannot use a scheme theory in this situation.

When we consider cyclotomes as abstract abelian groups with Galois action (i.e., when we are working in the group theory), we only know *a priori* that two cyclotomes are abstractly isomorphic (this is the definition of the cyclotomes), the way to identify them is not given, and there are $\widehat{\mathbb{Z}}^\times$-ways (or we have a $\widehat{\mathbb{Z}}^\times$-torsor) for the identification (i.e., we have $\widehat{\mathbb{Z}}^\times$-indeterminacy for the choice). It is important to note that the cylotomic rigidity isomorphism (Cyc. Rig. Iner.) is constructed *in a purely group-theoretic manner*, and we can reconstruct the identification even when we are working in the group theory. cf. also the (Step 3) in Theorem 3.17.

Proof. (1): By the assumption that k is Kummer-faithful, $k(X)$ is also Kummer-faithful by Lemma 3.2 (3).

(2): The origin $O \in J$ determines a section $s_O : G_k \to \Pi_J$, and, by taking (in the additive expression) the substraction $\eta_D := t_D - s_O : G_k \to \Delta_J\,(\subset \Pi_J)$ (i.e., the quotient $\eta_D := t_D/s_O$ in the multiplicative expression), which is a 1-cocycle, of two sections $t_D, s_O : G_k \to \Pi_J$, we obtain a cohomology class $[\eta_D] \in H^1(G_k, \Delta_J)$. On the other hand, the Kummer map for $J(\overline{k})$ induces an injection $(J(k) \subset) J(k)^\wedge \subset H^1(k, \Delta_J)$, since k is Kummer-faithful (Here, we write $J(k)^\wedge$ for the profinite completion of $J(k)$). Then we claim that $[D] = [\mathcal{O}(D)] \in J(k)$ is sent to $\eta_D \in H^1(G_k, \Delta_J)$ (cf. also [NTs, Lemma 4.14] and [Naka, Claim (2.2)]). We write $\alpha_D : J \to J$ for the morphism which sends x to $x - [D]$, and for a positive integer N, let $J_{D,N} \to J$ be the pull-back of $\alpha_D : J \to J$ via the morphism $[N] : J \to J$ of multiplication by N:

$$\begin{tikzcd} & J_{D,N} \arrow[r] \arrow[d] & J \arrow[d, "{[N]}"] \\ J \setminus \{O\} \arrow[r, hook] & J \arrow[r, "\alpha_D"] & J. \end{tikzcd}$$

The origin $O \in J(\overset{[N]}{\to} J)$ corresponds to a k-rational point $\frac{1}{N}[D] \in J_{D,N}(k)$ lying over $[D] \in J(k)$. By the k-rationality of $\frac{1}{N}[D]$, we have $t_D(\sigma) \in \Pi_{J_{D,N}}\,(\subset \Pi_J)$ for $\sigma \in G_k$. The inertia subgroup $I_O\,(\subset \Delta_{J\setminus\{O\}})$ of the origin $O \in J(\twoheadleftarrow J_{D,N})$ determines a system of geometric points $Q_{D,N} \in J_{D,N}(\overline{k})$ corresponding to the divisor $\frac{1}{N}(-[D])$ for $N \geq 1$ such that I_O always lies over $Q_{D,N}$. The conjugation $\mathrm{conj}(t_D(\sigma)) \in \mathrm{Aut}(\Delta_{J\setminus\{O\}})$ by $t_D(\sigma)$ coincides with the automorphism induced by $\sigma_N^* := \mathrm{id} \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,(\sigma^{-1}) \in \mathrm{Aut}((J \setminus \{O\}) \otimes_k \overline{k})$ (Note that a fundamental group and the corresponding covering transformation group are opposite groups to each other). Thus, $t_D(\sigma) I_O t_D(\sigma)^{-1}$ gives an inertia subgroup over $\sigma_N^*(Q_{D,N}) = \sigma(Q_{D,N})$. On the other hand, by definition, we have $t_D(\sigma) z_O t_D(\sigma)^{-1} = t_D(\sigma) s_O(\sigma)^{-1} s_O(\sigma) z_O s_O(\sigma)^{-1} s_O(\sigma) t_D(\sigma)^{-1} = \eta_D(\sigma) z_O^{\chi_{\mathrm{cyc}}(\sigma)} \eta_D(\sigma)^{-1}$ for a generator z_O of I_O, hence $t_D(\sigma) I_O t_D(\sigma)^{-1}$ is an inertia subgroup over $\nu_N(\eta_D(\sigma)^{-1})(Q_{D,N})$, where $\nu_N : \Delta_J \twoheadrightarrow \mathrm{Aut}((J \setminus J[N]) \otimes_k \overline{k} \overset{[N]}{\to} (J \setminus \{O\}) \otimes_k \overline{k})^{\mathrm{opp}}$ (Here, we write $(-)^{\mathrm{opp}}$ for the opposite group. Note that a fundamental group and the corresponding covering transformation group are opposite groups to each other). Therefore, we have $\sigma(Q_{D,N}) = \nu_N(\eta_D(\sigma)^{-1})(Q_{D,N})$. By noting the natural isomorphism $\mathrm{Aut}\left((J \setminus J[N]) \otimes_k \overline{k} \overset{[N]}{\to} (J \setminus \{O\}) \otimes_k \overline{k}\right) \cong J[N]$ given by $\gamma \mapsto \gamma(O)$, we obtain that

$$\sigma\left(\frac{1}{N}(-[D])\right) = -\nu_N(\eta_D(\sigma))(O) + \frac{1}{N}(-[D]).$$

Hence we have $\sigma\left(\frac{1}{N}[D]\right) - \frac{1}{N}[D] = \nu_N(\eta_D(\sigma))(O)$. This gives us the claim. The assertion (2) follows from this claim.

(3): We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(G_k, H^0(\Delta_U^{\text{cusp-cent}})) & \longrightarrow & H^1(\Pi_U^{\text{cusp-cent}}) & \longrightarrow & H^0(G_k, H^1(\Delta_U^{\text{cusp-cent}})) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(G_k, H^0(\Delta_U)) & \longrightarrow & H^1(\Pi_U) & \longrightarrow & H^0(G_k, H^1(\Delta_U)), \end{array}$$

where the horizontal sequences are exact, and we abbreviate the coefficient $\mu_{\widehat{\mathbb{Z}}}(\Pi_U)$ by the typological reason. Here, we have

$$H^1(G_k, H^0(\Delta_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))) = H^1(G_k, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) = H^1(G_k, H^0(\Delta_U^{\text{cusp-cent}}, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))),$$

and

$$H^0(G_k, H^1(\Delta_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))) = H^0(G_k, \Delta_U^{\text{ab}}) = H^0(G_k, H^1(\Delta_U^{\text{cusp-cent}}, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))).$$

Thus by combining these, the assertion (3) is proved.

(4): By the exact sequence

$$0 \to H^1(G_k, H^0(\Delta_X, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))) \to H^1(\Pi_X, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \to H^0(G_k, H^1(\Delta_X, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)))\,(\cong H^0(G_k, \Delta_X^{\text{ab}})),$$

and $H^1(G_k, H^0(\Delta_X, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))) = H^1(G_k, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \cong (k^\times)^\wedge$, it suffices to show that $H^0(G_k, \Delta_X^{\text{ab}}) = 0$. This follows from $(\Delta_X^{\text{ab}})^{G_k} \cong T(J)^{G_k} = 0$, since $\cap_N NJ(k) = 0$ by the assumption that k is Kummer-faithful (Here, we write $T(J)$ for the Tate module of J, and $J[N]$ is the group of N-torsion points of J).

(5) is trivial by noting $H^1(\Pi_X, H^0(\prod_{x\in S} I_x, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))) = H^1(\Pi_X, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \cong (k^\times)^\wedge$ by (4).

(6) is trivial. □

We write $\overline{k}_{\text{NF}}$ for the algebraic closure of $\mathbb{Q}$ in $\overline{k}$ (Here, NF stands for "number field"). If $X_{\overline{k}}$ is defined over $\overline{k}_{\text{NF}}$, we say that X is an NF**-curve**. For an NF-curve X, points of $X(\overline{k})$ (resp. rational functions on $X_{\overline{k}}$, constant rational functions (i.e., $\overline{k} \subset \overline{k}(X)$)) which descend to $\overline{k}_{\text{NF}}$, we shall refer to them as NF**-points** (resp. NF**-rational functions,** NF**-constants**) on $X_{\overline{k}}$.

Lemma 3.16. ([AbsTopIII, Proposition 1.8]) *Assume that k is Kummer-faithful. Let $\emptyset \neq U \subset X$ be an open subscheme, and write $S := X \setminus U$. We also assume that U is an* NF*-curve (hence X is also an* NF*-curve). We write $\mathcal{P}_U \subset H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))$ for the inverse image of $\mathcal{P}'_U \subset \bigoplus_{x\in S}\mathbb{Z}\,(\subset \bigoplus_{x\in S}\widehat{\mathbb{Z}})$ via the homomorphism $H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x\in S}\widehat{\mathbb{Z}}$ constructed in Lemma 3.15.*

(1) an element $\eta \in \mathcal{P}_U$ is the Kummer class of a non-constant NF*-rational function if and only if there exist a positive integer n and two* NF*-points $x_1, x_2 \in U(k')$*

with a finite extension k' of k such that the restrictions $(n\eta)|_{x_i} := s^*_{x_i}(n\eta) \in H^1(G_{k'}, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))$, where $s_{x_i} : G_{k'} \to \Pi_U$ is the section corresponding to x_i for $i = 1, 2$, satisfy (in the additive expression) $(n\eta)|_{x_1} = 0$ and $(n\eta)|_{x_2} \neq 0$ (i.e., $= 1$ and $\neq 1$ in the multiplicative expression).

(2) *Assume that there exist non-constant* NF*-rational functions in $\Gamma(U, \mathcal{O}_U^\times)$. Then an element $\eta \in \mathcal{P}_U \cap H^1(G_k, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \cong (k^\times)^\wedge$ is the Kummer class of an* NF*-constant in $k^\times$ if and only if there exist a non-constant* NF*-rational function $f \in \Gamma(U, \mathcal{O}_U^\times)$ and an* NF*-point $x \in U(k')$ with a finite extension k' of k such that $\kappa_U(f)|_x = \eta|_x$ in $H^1(G_{k'}, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))$.*

Proof. Let $X_{\rm NF}$ be a model of $X_{\overline{k}}$ over $\overline{k}_{\rm NF}$. Then any non-constant rational function on $X_{\rm NF}$ determines a morphism $X_{\rm NF} \to \mathbb{P}^1_{\overline{k}_{\rm NF}}$, which is non-constant i.e., $X_{\rm NF}(\overline{k}_{\rm NF}) \to \mathbb{P}^1_{\overline{k}_{\rm NF}}(\overline{k}_{\rm NF})$ is surjective. Then the lemma follows from the definitions. □

Theorem 3.17. (Mono-anabelian Reconstruction of NF-Portion, [AbsTopIII, Theorem 1.9]) *Assume that k is sub-p-adic, and that X is a hyperbolic orbicurve of strictly Belyi type. Let $\overline{X}$ be the canonical smooth compactification of X. From the extension $1 \to \Delta_X \to \Pi_X \to G_k \to 1$ of profinite groups, we can functorially group-theoretically reconstruct the* NF*-rational function field $\overline{k}_{\rm NF}(X)$ and* NF*-constant field $\overline{k}_{\rm NF}$ as in the following. Here, the functoriality is with respect to open injective homomorphisms of extension of profinite groups (cf. Remark 3.13.1), as well as with respect to homomorphisms of extension of profinite groups arising from a base change of the base field.*

(Step 1) *By Belyi cuspidalisation (Theorem 3.8), we group-theoretically reconstruct the set of surjections $\{\Pi_U \twoheadrightarrow \Pi_X\}_U$ for open sub-NF-curves $\emptyset \neq U \subset X$ and the decomposition groups D_x in Π_X of NF-points x. We also group-theoretically reconstruct the inertia subgroup $I_x := D_x \cap \Delta_U$.*

(Step 2) *By cyclotomic rigidity for inertia subgroups (Proposition 3.14 and Remark 3.14.1), we group-theoretically obtain isomorphism $I_x \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(\Pi_X)$ for any $x \in X(k)$, where I_x is group-theoretically reconstructed in (Step 1).*

(Step 3) *By the inertia subgroups I_x reconstructed in (Step 1), we group-theoretically reconstruct the restriction homomorphism $H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \to H^1(I_x, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))$. By the cyclotomic rigidity isomorphisms in (Step 2), we have an isomorphism $H^1(I_x, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \cong \widehat{\mathbb{Z}}$. Therefore, we group-theoretically obtain an exact sequence*

$$1 \to (k^\times)^\wedge \to H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x \in S} \widehat{\mathbb{Z}}$$

in Lemma 3.15 (5) (Note that, without the cyclotomic rigidity Proposition 3.14, we would have $\widehat{\mathbb{Z}}^\times$-indeterminacies on each direct summand of $\bigoplus_{x \in S} \widehat{\mathbb{Z}}$, and that the reconstruction algorithm in this theorem would not work). By the characterisation of principal cuspidal divisors (Lemma 3.15 (2), and the decomposition groups in (Step 1)), we group-theoretically reconstruct the subgroup

$$\mathcal{P}_U \subset H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_U))$$

of principal cuspidal divisors.

(Step 4) *Note that we already group-theoretically reconstructed the restriction map $\eta|_{x_i}$ in Lemma 3.16 by the decomposition group D_{x_i} reconstructed in (Step 1). By the characterisations of non-constant* NF*-rational functions and* NF*-constants in Lemma 3.16 (1), (2) in $\mathcal{P}_U$ reconstructed in (Step 3), we group-theoretically reconstruct the subgroups (via Kummer maps κ_U's in Lemma 3.15)*

$$\overline{k}_{\mathrm{NF}}^\times \subset \overline{k}_{\mathrm{NF}}(X)^\times \subset \varinjlim_{U} H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)),$$

where U runs through the open sub-NF-curves of $\overline{X} \times_k k'$ for a finite extension k' of k.

(Step 5) *In (Step 4), we group-theoretically reconstructed the datum $\overline{k}_{\mathrm{NF}}(X)^\times$ in Proposition 3.12 (a). Note that we already reconstructed the data* ord_x*'s in Proposition 3.12 (b) as the component at x of the homomorphism $H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x \in S} \widehat{\mathbb{Z}}$ reconstructed in (Step 3). Note also that we already group-theoretically reconstructed the evaluation map $f \mapsto f(x)$ in Proposition 3.12 as the restriction map to the decomposition group D_x reconstructed in (Step 1). Thus, we group-theoretically obtain the data $\mathcal{U}_v$'s in Proposition 3.12 (c). Therefore, we can apply Uchida's Lemma (Proposition 3.12), and we group-theoretically reconstruct the additive structures on*

$$\overline{k}_{\mathrm{NF}}^\times \cup \{0\}, \quad \overline{k}_{\mathrm{NF}}(X)^\times \cup \{0\}.$$

Proof. The theorem immediately follows from the group-theoretic algorithms referred in the statement of the theorem. The functoriality immediately follows from the described constructions. □

Remark 3.17.1. The input data of Theorem 3.17 is the extension $1 \to \Delta_X \to \Pi_X \to G_k \to 1$ of profinite groups. If k is a number field or a non-Archimedean local field, then we need only the profinite group Π_X as an input datum by Proposition 2.2 (1), and Corollary 2.4. (Note that we have a group-theoretic characterisation of cuspidal decomposition groups for the number field case as well by Remark 2.9.2.)

Remark 3.17.2. (Elementary Birational Analogue, [AbsTopIII, Theorem 1.11]) We write η_X for the generic point of X. If k is l-cyclotomically full for some l, then we have the characterisation of the cuspidal decomposition groups in Π_{η_X} at (not only NF-points but also) all closed points of X (cf. Remark 2.9.2). Therefere, under the assumption that k is Kummer-faithful (cf. also Lemma 3.2 (2)), if we start not from the extension $1 \to \Delta_X \to \Pi_X \to G_k \to 1$, but from the extension $1 \to \Delta_{\eta_X} \to \Pi_{\eta_X} \to G_k \to 1$, then the same group-theoretic algorithm (Step 2)-(Step 5) works without using Belyi cuspidalisation (Theorem 3.8) or (GC) (cf. Theorem B.1), and we can obtain (not only the NF-rational function field $\overline{k}_{\mathrm{NF}}(X)$ but also) the rational function field $\overline{k}(X)$ and (not only the NF-constant field $\overline{k}_{\mathrm{NF}}$ but also) the constant field $\overline{k}$ (Note also that we *do not* use the results in Section 3.2, hence we have no circular arguments here).

Remark 3.17.3. (Slimness of G_k for Kummer-Faithful k, [AbsTopIII, a part of Theorem 1.11]) By using the above Remark 3.17.2 (Note that we *do not* use the results in Section 3.2 to show Remark 3.17.2, hence we have no circular arguments here), we can show that $G_k := \mathrm{Gal}(\overline{k}/k)$ is slim for any Kummer-faithful field k as follows (cf. also [pGC, Lemma 15.8]): Let $G_{k'} \subset G_k$ be an open subgroup, and take $g \in Z_{G_k}(G_{k'})$. Assume that $g \neq 1$. Then we have a finite Galois extension K of k' such that $g : K \xrightarrow{\sim} K$ is not an identity on K. We have $K = k'(\alpha)$ for some $\alpha \in K$. Let E be an elliptic curve over K with j-invariant α. Write $X := E \setminus \{O\}$, where O is the origin of E. Write also $X^g := X \times_{K,g} K$ i.e., the base change by $g : K \xrightarrow{\sim} K$. The conjugate by g defines an isomorphism $\Pi_X \xrightarrow{\sim} \Pi_{X^g}$. This isomorphism is compatible to the quotients to G_K, since g is in $Z_{G_k}(G_{k'})$. Thus, by the *functoriality* of the algorithm in Remark 3.17.2, this isomorphism induces an K-isomorphism $K(X) \xrightarrow{\sim} K(X^g)(= K(X) \otimes_{K,g} K)$ of function fields. Therefore, we have $g(\alpha) = \alpha$ by considering the j-invariants. This is a contradiction.

Remark 3.17.4. (cf. also [AbsTopIII, Remark 1.9.5 (ii)], and [IUTchI, Remark 4.3.2]) The theorem of Neukirch-Uchida (which is a bi-anabelian theorem) uses the data of the decomposition of primes in extensions of number fields. Hence it has no functoriality with respect to the base change from a number field to non-Archimedean local fields. On the other hand, (mono-anabelian) Theorem 3.17 has the functoriality with respect to the base change of the base fields, especially from a number field to non-Archimedean local fields. This is *crucial* for the applications to inter-universal Teichmüller theory (For example, see the beginning of 10, Example 8.12 etc.). cf. also [IUTchI, Remark 4.3.2 requirements (a), (b), and (c)].

In inter-universal Teichmüller theory, we will treat local objects (i.e., objects over local fields) which *a priori* do not come from a global object (i.e., an object over a number field), in fact, we completely destroy the above data of "the decomposition of primes" (Recall also the "analytic section" of $\mathrm{Spec}\, O_K \twoheadrightarrow \mathrm{Spec}\, O_{F_{\mathrm{mod}}}$). Therefore, it is *crucial* to have a mono-anabelian reconstruction algorithm (Theorem 3.17) in a *purely local situation* for the applications to inter-universal Teichmüller theory. It also seems worthwhile to give a remark that such a mono-anabelian reconstruction algorithm in a *purely local situation* got available by the fact that the bi-anabelian theorem in [pGC] was proved for a purely local situation, unexpectedly at that time to many people from a point of view of analogy with Tate conjecture!

Definition 3.18. Let k be a finite extension of $\mathbb{Q}_p$. We define

$$\mu_{\mathbb{Q}/\mathbb{Z}}(G_k) := \varinjlim_{H \subset G_k:\text{ open}} (H^{\mathrm{ab}})_{\mathrm{tors}}, \quad \mu_{\widehat{\mathbb{Z}}}(G_k) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_{\mathbb{Q}/\mathbb{Z}}(G_k)),$$

where the transition maps are given by Verlangerung (or transfer) maps (cf. also the proof of Proposition 2.1 (6) for the definition of Verlangerung map). We shall refer to them as the **cyclotomes of G_k**.

Remark 3.18.1. Similarly as Remark 3.13.1, in this subsection, by the functoriality of cohomology with $\mu_{\mathbb{Q}/\mathbb{Z}}(G_{(-)})$-coefficients for an open injective homomorphism of profinite groups $G_{k'} \subset G_k$, we always mean multiplying $\frac{1}{[G_k:G_{k'}]}$ on the homomorphism between the cyclotomes of G_k and $G_{k'}$ (cf. also [AbsTopIII, Remark 3.2.2]). Note that we have a commutative diagram

$$\begin{array}{ccc} H^2(G_k, \mu_{\mathbb{Q}/\mathbb{Z}}(G_k)) & \xrightarrow{\cong} & \mathbb{Q}/\mathbb{Z} \\ {\scriptstyle \frac{1}{[G_k:G_{k'}]}\cdot\text{restriction}}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle =} \\ H^2(G_{k'}, \mu_{\mathbb{Q}/\mathbb{Z}}(G_{k'})) & \xrightarrow{\cong} & \mathbb{Q}/\mathbb{Z}, \end{array}$$

where the horizontal arrows are the isomorphisms given in Proposition 2.1 (7).

Corollary 3.19. (Mono-anabelian Reconstruction over an MLF, [AbsTopIII, Corollary 1.10, Proposition 3.2 (i), Remark 3.2.1]) *Assume that k is a non-Archimedean local field, and that X is a hyperbolic orbicurve of strictly Belyi type. From the profinite group Π_X, we can group-theoretically reconstruct the following in a functorial manner with respect to open injections of profinite groups:*

(1) the set of the decomposition groups of all closed points in X,

(2) the function field $\overline{k}(X)$ and the constant field $\overline{k}$, and

(3) a natural isomorphism

(Cyc. Rig. LCFT) $$\mu_{\widehat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(O^{\triangleright}(\Pi_X)),$$

where we put $\mu_{\widehat{\mathbb{Z}}}(O^{\triangleright}(\Pi_X)) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \kappa(\overline{k}^{\times}_{\mathrm{NF}}))$ *for* $\kappa : \overline{k}^{\times}_{\mathrm{NF}} \hookrightarrow \varinjlim_U H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))$.

We shall refer to the isomorphism (Cyc. Rig. LCFT) *as the* **cyclotomic rigidity via LCFT** *or* **classical cyclotomic rigidity** *(LCFT stands for "local class field theory").*

Proof. (1) is just a restatement of Corollary 3.9.

(2): By Theorem 3.17 and Corollary 2.4, we can group-theoretically reconstruct the fields $\overline{k}_{\mathrm{NF}}(X)$ and $\overline{k}_{\mathrm{NF}}$. On the other hand, by the natural isomorphism $H^2(G_k, \mu_{\widehat{\mathbb{Z}}}(G_k)) \xrightarrow{\sim} \widehat{\mathbb{Z}}$ group-theoretically constructed in Proposition 2.1 (7) (with $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, -)$) and the cup product, we group-theoretically construct isomorphisms $H^1(G_k, \mu_{\widehat{\mathbb{Z}}}(G_k)) \xrightarrow{\sim} \mathrm{Hom}(H^1(G_k, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}}) \cong G_k^{\mathrm{ab}}$. We also have group-theoretic constructions of a surjection $G_k^{\mathrm{ab}} \twoheadrightarrow G_k^{\mathrm{ab}}/\mathrm{Im}(I_k \to G_k^{\mathrm{ab}})$ and an isormorphism $G_k^{\mathrm{ab}}/\mathrm{Im}(I_k \to G_k^{\mathrm{ab}}) \cong \widehat{\mathbb{Z}}$ by Proposition 2.1 (4a) and Proposition 2.1 (5) respectively (cf. also Remark 2.1.1). Hence we group-theoretically obtain a surjection $H^1(G_k, \mu_{\widehat{\mathbb{Z}}}(G_k)) \twoheadrightarrow \widehat{\mathbb{Z}}$. We have an isomorphism $\mu_{\widehat{\mathbb{Z}}}(G_k) \cong \mu_{\widehat{\mathbb{Z}}}(\Pi_X)$ well-defined up to multiplication by $\widehat{\mathbb{Z}}^\times$. Then this induces a surjection $H^1(G_k, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \twoheadrightarrow \widehat{\mathbb{Z}}$ well-defined up to multiplication by $\widehat{\mathbb{Z}}^\times$. We group-theoretically reconstruct the field k as the completion of the field $(H^1(G_k, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \cap \overline{k}_{\mathrm{NF}}^\times) \cup \{0\}$ (induced by the field structure of $\overline{k}_{\mathrm{NF}}^\times \cup \{0\}$) with respect to the valuation determined by the subring of $(H^1(G_k, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \cap \overline{k}_{\mathrm{NF}}^\times) \cup \{0\}$ generated by $\ker \left\{ H^1(G_k, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \twoheadrightarrow \widehat{\mathbb{Z}} \right\} \cap \overline{k}_{\mathrm{NF}}^\times$. The reconstructed object is independent of the choice of an isomorphism $\mu_{\widehat{\mathbb{Z}}}(G_k) \cong \mu_{\widehat{\mathbb{Z}}}(\Pi_X)$. By taking the inductive limit of this construction with respect to open subgroups of G_k, we group-theoretically reconstruct $\overline{k}$. Finally, we group-theoretically reconstruct $\overline{k}(X)$ by $\overline{k}(X) := \overline{k} \otimes_{\overline{k}_{\mathrm{NF}}} \overline{k}_{\mathrm{NF}}(X)$.

(3): We put $\mu_{\mathbb{Q}/\mathbb{Z}}(O^{\rhd}(\Pi_X)) := \mu_{\widehat{\mathbb{Z}}}(O^{\rhd}(\Pi_X)) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}/\mathbb{Z}$. We group-theoretically reconstruct $G^{\mathrm{ur}} = \mathrm{Gal}(k^{\mathrm{ur}}/k)$ by Proposition 2.1 (4a). Then by the same way as Proposition 2.1 (7), we have group-theoretic constructions of isomorphisms:

$$\begin{aligned} &H^2(G_k, \mu_{\mathbb{Q}/\mathbb{Z}}(O^{\rhd}(\Pi_X))) \xrightarrow{\sim} H^2(G_k, \kappa(\overline{k}^\times)) \xleftarrow{\sim} H^2(G^{\mathrm{ur}}, \kappa((k^{\mathrm{ur}})^\times)) \\ &\xrightarrow{\sim} H^2(G^{\mathrm{ur}}, \mathbb{Z}) \xleftarrow{\sim} H^1(G^{\mathrm{ur}}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G^{\mathrm{ur}}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

Thus, by taking $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, -)$, we obtain a natural isomorphism $H^2(G_k, \mu_{\widehat{\mathbb{Z}}}(O^{\rhd}(\Pi_X))) \xrightarrow{\sim} \widehat{\mathbb{Z}}$. By imposing the compatibility of this isomorphism with the group-theoretically constructed isomorphism $H^2(G_k, \mu_{\widehat{\mathbb{Z}}}(G_k)) \xrightarrow{\sim} \widehat{\mathbb{Z}}$ in (2), we obtain a natural isomorphism $\mu_{\widehat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(O^{\rhd}(\Pi_X))$. □

Remark 3.19.1. ([AbsTopIII, Corollary 1.10 (c)]) Without assuming that X is of strictly Belyi type, we can construct an isomorphism $\mu_{\widehat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(\Pi_X)$ (cf. Corollary 3.19 (3)). However, the construction needs technically lengthy reconstruction algorithms of the graph of special fiber ([profGC, §1–5], [AbsAnab, Lemma 2.3]. cf. also [SemiAnbd, Theorem 3.7, Corollary 3.9] Proposition 6.6 for the reconstruction without Galois action in the case where a tempered structure is available) and the "rational positive structure" of H^2 (cf. also [AbsAnab, Lemma 2.5 (i)]), where we need Raynaud's theory on "ordinary new part" of Jacobians (cf. also [AbsAnab, Lemma 2.4]), though it has an advantage of no need of [pGC]. cf. also Remark 6.12.2.

Remark 3.19.2. ([AbsTopIII, Proposition 3.2, Proposition 3.3]) For a topological monoid (resp. topological group) M with continuous G_k-action, which is isomorphic to $O^{\rhd}_{\overline{k}}$ (resp. $\overline{k}^{\times}$) compatible with the G_k-action, we write $\mu_{\widehat{\mathbb{Z}}}(M) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, M^{\times}))$ and $\mu_{\mathbb{Q}/\mathbb{Z}}(M) := \mu_{\widehat{\mathbb{Z}}}(M) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}/\mathbb{Z}$. We shall refer to them as the **cyclotome of a topological monoid** M. We also write $M^{\mathrm{ur}} := M^{\ker(G \to G^{\mathrm{ur}})}$. We can canonically take the generator of $M^{\mathrm{ur}}/M^{\times} \cong \mathbb{N}$ (resp. the generator of $M^{\mathrm{ur}}/M^{\times}$ up to $\{\pm 1\}$) to obtain an isomorphism $(M^{\mathrm{ur}})^{\mathrm{gp}}/(M^{\mathrm{ur}})^{\times} \cong \mathbb{Z}$ (resp. an isomorphism $(M^{\mathrm{ur}})^{\mathrm{gp}}/(M^{\mathrm{ur}})^{\times} \cong \mathbb{Z}$ well-defined up to $\{\pm 1\}$). Then by the same way as Corollary 3.19 (3), we have

$$\begin{aligned} &H^2(G_k, \mu_{\mathbb{Q}/\mathbb{Z}}(M)) \xrightarrow{\sim} H^2(G_k, M^{\mathrm{gp}}) \xleftarrow{\sim} H^2(G^{\mathrm{ur}}, (M^{\mathrm{ur}})^{\mathrm{gp}}) \\ &\xrightarrow{\sim} H^2(G^{\mathrm{ur}}, (M^{\mathrm{ur}})^{\mathrm{gp}}/(M^{\mathrm{ur}})^{\times}) \underset{(*)}{\xrightarrow{\sim}} H^2(G^{\mathrm{ur}}, \mathbb{Z}) \xleftarrow{\sim} H^1(G^{\mathrm{ur}}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G^{\mathrm{ur}}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}, \end{aligned}$$

where the isomorphism $H^2(G^{\mathrm{ur}}, (M^{\mathrm{ur}})^{\mathrm{gp}}/(M^{\mathrm{ur}})^{\times}) \underset{(*)}{\xrightarrow{\sim}} H^2(G^{\mathrm{ur}}, \mathbb{Z})$ is canonically defined (resp. well-defined up to $\{\pm 1\}$), as noted above. Then we have a canonical isomorphism (resp. an isomorphism well-defined up to $\{\pm 1\}$)

$$\text{(Cyc. Rig. LCFT2)} \qquad \mu_{\widehat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(M),$$

by the same way as in Corollary 3.19 (3). We shall also refer to the isomorphism (Cyc. Rig. LCFT2) as the **cyclotomic rigidity via LCFT** or **classical cyclotomic rigidity**. We also obtain a canonical homomorphism (resp. a homomorphism well-defined up to $\{\pm 1\}$)

$$M \hookrightarrow \varinjlim_{J \subset G\text{: open}} H^1(J, \mu_{\widehat{\mathbb{Z}}}(M)) \cong \varinjlim_{J \subset G\text{: open}} H^1(J, \mu_{\widehat{\mathbb{Z}}}(G_k)),$$

by the above isomorphism, where the first injection is the canonical injection (The notation $\rhd$ in $O^{\rhd}_{\overline{k}} = O^{\times}_{\overline{k}} \cdot (\text{uniformiser})^{\mathbb{N}}$ indicates that the "direction" $\mathbb{N}\,(\cong (\text{uniformiser})^{\mathbb{N}})$ of $\mathbb{Z}\,(\cong (\text{uniformiser})^{\mathbb{Z}})$ (or a generator of $\mathbb{Z}$) is chosen, compared to $\overline{k}^{\times} = O^{\times}_{\overline{k}} \cdot (\text{uniformiser})^{\mathbb{Z}}$, which has $\{\pm 1\}$-indeterminacy of choosing a "direction" or a generator of $\mathbb{Z}\,(\cong (\text{uniformiser})^{\mathbb{Z}})$. In the non-resp'd case (i.e., the $O^{\rhd}$-case), the above canonical injection induces an isomorphism

$$M \overset{\text{Kum}}{\xrightarrow{\sim}} O^{\rhd}_{\overline{k}}(\Pi_X),$$

where we write $O^{\rhd}_{\overline{k}}(\Pi_X)$ for the ind-topological monoid determined by the ind-topological field reconstructed by Corollay 3.19. We shall refer to this isomprhism as the **Kummer isomorphism** for M.

We can also consider the case where M is an topological group with G_k-action, which is isomorphic to $O^{\times}_{\overline{k}}$ compatible with the G_k-action. Then in this case, we have an isomorphism $\mu_{\widehat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(M)$ and an injection $M \hookrightarrow \varinjlim_{J \subset G:\ \text{open}} H^1(J, \mu_{\widehat{\mathbb{Z}}}(G_k))$, which are only well-defined up to $\widehat{\mathbb{Z}}^{\times}$-multiple (i.e., there is *no* rigidity).

It seems important to give a remark that *we use the value group portion* (i.e., we use $O^{\triangleright}$, not $O^{\times}$) in the construction of the cyclotomic rigidity via LCFT. In inter-universal Teichmüller theory, not only the existence of reconstruction algorithms, but also the *contents* of reconstruction algorithms are important, and whether or not we use the value group portion in the algorithm is crucial for the constructions in the final multiradial algorithm in inter-universal Teichmüller theory. cf. also Remark 9.6.2, Remark 11.4.1, Propositin 11.5, and Remark 11.11.1.

§ 3.5. On the Philosophy of Mono-analyticity and Arithmetic Holomorphicity.

In this subsection, we explain Mochizuki's *philosophy of mono-analyticity and arithmetic holomorphicity*, which is closely related to *inter-universality.*

Let k be a finite extension of $\mathbb{Q}_p$, $\overline{k}$ an algebraic closure of k, and $k'(\subset \overline{k})$ a finite extension of $\mathbb{Q}_p$. It is well-known that, at least for $p \neq 2$, the natural map

(nonGC for MLF)
$$\underset{\text{(scheme theory)}}{\mathrm{Isom}_{\text{topological fields}}(\overline{k}/k, \overline{k}/k')} \hookrightarrow \underset{\text{(group theory)}}{\mathrm{Isom}_{\text{profinite groups}}(\mathrm{Gal}(\overline{k}/k'), \mathrm{Gal}(\overline{k}/k))}$$

is *not* bijective (cf. [NSW, Chap. VII, §5, p.420–423]. cf. also [AbsTopI, Corollary 3.7]). This means that there exists an automorphism of $G_k := \mathrm{Gal}(\overline{k}/k)$ which does not come from an isomorphism of topological fields (i.e., does not come from a scheme theory). In this sense, *by treating G_k as an abstract topological group*, we can go outside of a scheme theory. (A part of) Mochizuki's philosophy of arithmetically holomorphicity and mono-analyiticity is to consider the image of the map (nonGC for MLF) as **arithmetically holomorphic**, and the right-hand side of (nonGC for MLF) as **mono-analytic** (Note that this is a bi-anabelian explanation, not a mono-anabelian explanation (cf. Remark 3.4.4) for the purpose of the reader's easy getting the feeling. We will see mono-anabelian one a little bit later). The arithmetic holomorphicity versus mono-analyticity is an arithmetic analougue of holomorphic structure of $\mathbb{C}$ versus the undeyling analytic strucutre of $\mathbb{R}^2(\cong \mathbb{C})$.

Note that G_k has cohomological dimension 2 like $\mathbb{C}$ is two-dimensional as a topological manifold. It is well-known that this two-dimensionality comes from the exact sequence $1 \to I_k \to G_k \to \widehat{\mathbb{Z}}\mathrm{Frob}_k \to 1$ and that both I_k and $\widehat{\mathbb{Z}}\mathrm{Frob}_k$ have cohomological dimension 1. In the abelianisation, these groups correspond to the unit group and the value group respectively via the local class field theory. Proposition 2.1 (2d) says that we can group-theoretically reconstruct the multiplicative group $k^{\times}$ from the abstract topological group G_k. This means that we can see the multiplicative structure of k in any scheme theory, in other words, the multiplicative structure of k is inter-universally *rigid*. However, we cannot group-theoretically reconstruct the field k from the abstract topological group G_k, since there exists a non-scheme-theoretic automorphism of G_k as mentioned above. In other words, the additive structure of k is inter-universally *non-rigid*. Proposition 2.1 (5) also says that we can group-theoretically reconstruct Frobenius element Frob_k in $\widehat{\mathbb{Z}}\mathrm{Frob}_k(\twoheadleftarrow G_k)$ from the abstract topological group G_k, and the unramified quotient $\widehat{\mathbb{Z}}\mathrm{Frob}_k$ corresponds to the value group via the local class field theory. This means that we can detect the Frobenius element in any scheme theory. In other words, the unramified quotient $\widehat{\mathbb{Z}}\mathrm{Frob}_k$ and the value group $\mathbb{Z}(\twoheadleftarrow k^{\times})$ are inter-universally *rigid*. However, there exists automorphisms of the topological group G_k which do not preserve the ramification filtrations (cf. also [AbsTopIII, Remark 1.9.4]), and the ramification filtration (with upper numberings) corresponds to the filtration $(1+\mathfrak{m}_k^n)_n$ of the unit group via the local class field theory, where we write $\mathfrak{m}_k$ for the maximal ideal of O_k. In other words, the inertia subgroup I_k and the unit group $O_k^{\times}$ are inter-universally *non-rigid* (We can also directly see that the unit group $O_k^{\times}$ is non-rigid under the automorphism of topological group $k^{\times}$ without the class field theory). In summary, one dimension of G_k or $k^{\times}$ (i.e., the unramified quotient and the value group) is inter-universally rigid, and the other dimension (i.e., the inertia subgroup and the unit group) is not. Thus, Mochizuki's philosophy of arithmetic holomorphicity and mono-analyticity regards a non-scheme-theoretic automorphism of G_k as a kind of an **arithmetic analogue of the Teichmüller dilation** of the undeyling analytic strucutre of $\mathbb{R}^2(\cong\mathbb{C})$ (cf. also [Pano, Fig. 2.1] instead of the poor picture below):

$$\begin{array}{ccc} \uparrow & \rightsquigarrow & \uparrow \\ \to & & \longrightarrow \end{array}$$

Note that it is a **theatre of encounter of the anabelian geometry, the Teichmüller point of view, the differential over** $\mathbb{F}_1$ (cf. Remark 1.6.1 and Lemma 1.9) **and Hodge-Arakelov theory** (cf. Appendix A), **which gives rise a Diophantine consequence!**

Note also that [$\mathbb{Q}_p$GC, Theorem 4.2] says that if an automorphisms of G_k preserves the ramification filtration, then the automorphism arises from an automorphism of $\overline{k}/k$. This means that when we rigidify the portion corresponding to the unit group (i.e., non-rigid dimension of G_k), then it becomes arithmetically holomorphic i.e., [$\mathbb{Q}_p$GC, Theorem 4.2] supports the philosophy. Note also that we have $\mathbb{C}^{\times}\cong\mathbb{S}^1\times\mathbb{R}_{>0}$, where we write $\mathbb{S}^1:=O_{\mathbb{C}}^{\times}\subset\mathbb{C}^{\times}$ (cf. Section 0.2), and that the unit group $\mathbb{S}^1$ is rigid and the "value group" $\mathbb{R}_{>0}$ is non-rigid under the automorphisms of the topological group $\mathbb{C}^{\times}$

(Thus, the rigidity and non-rigidity for unit group and "value group" in the Archimedean case are opposite to the non-Archimedean case).

Let X be a hyperbolic orbicurve of strictly Belyi type over a non-Archimedean local field k. Corollary 3.19 says that we can group-theoretically reconstruct the field k from the abstract topological group Π_X. From this mono-anabelian reconstruction theorem, we obtain one of the fundamental observations of Mochizuki: Π_X *or equivalently the outer action* $G_k \to \mathrm{Out}(\Delta_X)$ *(and the actions* $\Pi_X \curvearrowright \overline{k}, O_{\overline{k}}, O^{\rhd}_{\overline{k}}, O^{\times}_{\overline{k}}$*) is arithmetically holomorphic, and* G_k *(and the actions* $G_k \curvearrowright O^{\rhd}_{\overline{k}}, O^{\times}_{\overline{k}}$ *on multiplicative monoid and multiplicative group) is mono-analytic* (thus, taking the quotient $\Pi_X \mapsto G_k$ is a "mono-analyticisation") (cf. Section 0.2 for the notation $O^{\rhd}_{\overline{k}}$). In other words, the outer action of G_k on Δ_X rigidifies the "non-rigid dimension" of $k^{\times}$. We can also regard X as a kind of "tangent space" of k, and it rigidifies $k^{\times}$. Note also that, in the p-adic Teichmüller theory (cf. [pOrd] and [pTeich]), a nilpotent ordinary indigenous bundle over a hyperbolic curve in positive characteristic rigidifies the non-rigid p-adic deformations. In the next section, we study an *Archimedean analogue* of this rigidifying action. In inter-universal Teichmüller theory, we study number field case by putting together the local ones. In the analogy between p-adic Teichmüller theory and inter-universal Teichmüller theory, a number field corresponds to a hyperbolic curve over a perfect field of positive characteristic, and a once-punctured elliptic curve over a number field corresponds to a nilpotent ordinary indigenous bundle over a hyperbolic curve over a perfect field of positive characteristic. We will deepen this analogy later such that $\mathfrak{log}$-link corresponds to a Frobenius endomorphism in positive characteristic, a vertical line of log-theta-lattice corresponds to a scheme theory in positive characteristic, Θ-link corresponds to a mixed characteristic lifting of ring of Witt vectors $p^n/p^{n+1} \rightsquigarrow p^{n+1}/p^{n+2}$, a horizontal line of log-theta-lattice corresponds to a deformation to mixed characteristic, and a log-theta-lattice corresponds to a canonical lifting of Frobenius (cf. Section 12.1).

In short, we obtain the following useful dictionaries:

rigid	$\widehat{\mathbb{Z}}\mathrm{Frob}_k$	value group	multiplicative structure of k	$\mathbb{S}^1(\subset \mathbb{C}^{\times})$
non-rigid	I_k	unit group	additive structure of k	$\mathbb{R}_{>0}(\subset \mathbb{C}^{\times})$

$\mathbb{C}$	field k	Π_X	$\Pi_X \curvearrowright \overline{k}, O_{\overline{k}}, O^{\rhd}_{\overline{k}}, O^{\times}_{\overline{k}}$	arith. hol.
$\mathbb{R}^2(\cong \mathbb{C})$	multiplicative group $k^{\times}$	G_k	$G_k \curvearrowright O^{\rhd}_{\overline{k}}, O^{\times}_{\overline{k}}$	mono-an.

inter-universal Teich.	p-adic Teich.
number field	hyperbolic curve of pos. char.
onece-punctured ell. curve	nilp. ord. indigenous bundle
$\mathfrak{log}$-link	Frobenius in pos. char.
vertical line of log-theta-lattice	scheme theory in pos. char.
Θ-link	lifting $p^n/p^{n+1} \rightsquigarrow p^{n+1}/p^{n+2}$
horizontal line of log-theta-lattice	deformation to mixed. char.
log-theta-lattice	canonical lift of Frobenius

cf. also [AbsTopIII, §I.3] and [Pano, Fig. 2.5]. Finally, we give a remark that *separating additive and multiplicative structures* is also one of the main themes of inter-universal Teichmüller theory (cf. Section 10.4 and Section 10.5).

§ 4. The Archimedean Theory — Formulated Without Reference to a Specific Model $\mathbb{C}$.

In this section, we introduce a notion of Aut-holomorphic space to avoid a specific fixed local referred model of $\mathbb{C}$ (i.e., "the $\mathbb{C}$") for the formulation of holomorphicity, i.e., "model-implicit" approach. Then we study an Archimedean analogue mono-anabelian reconstruction algorithms of Section 3, including elliptic cuspidalisation, and an Archimedean analogue of Kummer theory.

§ 4.1. Aut-Holomorphic Spaces.

Definition 4.1. ([AbsTopIII, Definition 2.1])

(1) Let X, Y be Riemann surfaces.

(a) We write $\mathcal{A}_X$ for the assignment, which assigns to any connected open subset $U \subset X$ the group $\mathcal{A}_X(U) := \mathrm{Aut}^{\mathrm{hol}}(U) := \{f : U \xrightarrow{\sim} U \ \text{ holomorphic}\} \subset \mathrm{Aut}(U^{\mathrm{top}}) := \{f : U \xrightarrow{\sim} U \ \text{ homeomorphic}\}$.

(b) Let $\mathcal{U}$ be a set of connected open subset of X such that $\mathcal{U}$ is a basis of the topology of X and that for any connected open subset $V \subset X$, if $V \subset U \in \mathcal{U}$, then $V \in \mathcal{U}$. We shall refer to $\mathcal{U}$ as a **local structure** on the underlying topological space X^{top}.

(c) We shall refer to a map $f : X \to Y$ between Riemann surfaces as an **RC-holomorphic morphism** if f is holormophic or anti-holomorphic at any point $x \in X$ (Here, RC stands for "real complex").

(2) Let X be a Riemann surface, and $\mathcal{U}$ a local structure on X^{top}.

(a) The **Aut-holomorphic space** associated to X is a pair $\mathbb{X} = (\mathbb{X}^{\mathrm{top}}, \mathcal{A}_{\mathbb{X}})$, where $\mathbb{X}^{\mathrm{top}} := X^{\mathrm{top}}$ the underlying topological space of X, and $\mathcal{A}_{\mathbb{X}} := \mathcal{A}_X$.

(b) We shall refer to $\mathcal{A}_{\mathbb{X}}$ as the **Aut-holomorphic structure** on X^{top}.

(c) We shall refer to $\mathcal{A}_{\mathbb{X}}|_{\mathcal{U}}$ as a **$\mathcal{U}$-local pre-Aut-holomorphic structure** on X^{top}.

(d) If X is biholomorphic to an open unit disc, then we shall refer to $\mathbb{X}$ as an **Aut-holomorphic disc**.

(e) If X is a hyperbolic Riemann surface of finite type, then we shall refer to $\mathbb{X}$ as **hyperbolic of finite type**.

(f) If X is a hyperbolic Riemann surface of finite type associated to an elliptically admissible hyperbolic curve over $\mathbb{C}$, then we shall refer to $\mathbb{X}$ as **elliptically admissible**.

(3) Let $\mathbb{X}$, $\mathbb{Y}$ be Aut-holomorphic spaces arising from Riemann surfaces X, Y respectively. Let $\mathcal{U}$, $\mathcal{V}$ be local structures of X^{top}, Y^{top} respectively.

(a) A **$(\mathcal{U}, \mathcal{V})$-local morphism** $\phi : \mathbb{X} \to \mathbb{Y}$ of Aut-holomorphic spaces is a local isomorphism $\phi^{\mathrm{top}} : \mathbb{X}^{\mathrm{top}} \to \mathbb{Y}^{\mathrm{top}}$ of topological spaces suth that, for any $U \in \mathcal{U}$ with $\phi^{\mathrm{top}} : U \xrightarrow{\sim} V \in \mathcal{V}$ (homeomorphism), the map $\mathcal{A}_{\mathbb{X}}(U) \to \mathcal{A}_{\mathbb{Y}}(V)$ obtained by the conjugate by ϕ^{top} is bijective.

(b) If $\mathcal{U}$, $\mathcal{V}$ are the set of all connected open subset of X^{top}, Y^{top} respectively, then we shall refer to ϕ as a **local morphism** of Aut-holomorphic spaces.

(c) If ϕ^{top} is a finite covering space map, then we shall refer to ϕ as **finite étale**.

(4) Let Z, Z' be orientable topological surfaces.

(a) Let $p \in Z$, and we define $\mathrm{Orn}(Z, p) := \varprojlim_{p \in W \subset Z:\ \text{connected, open}} \pi_1(W \setminus \{p\})^{\mathrm{ab}}$, which is non-canonically isomorphic to $\mathbb{Z}$. Note that after taking the abelianisation, there is no indeterminacy of inner automorphisms arising from the choice of a basepoint in (the usual topological) fundamental group $\pi_1(W \setminus \{p\})$.

(b) The assignment $p \mapsto \mathrm{Orn}(Z, p)$ is a trivial local system, since Z is orientable. We write $\mathrm{Orn}(Z)$ for the abelian group of global sections of this trivial local system, which is non-canonically isomorphic to $\mathbb{Z}^{\pi_0(Z)}$.

(c) Let $\alpha, \beta : Z \to Z'$ be local isomorphisms. We say that α and β are **co-oriented** if the induced homomorphisms $\alpha_*, \beta_* : \mathrm{Orn}(Z) \to \mathrm{Orn}(Z')$ of abelian groups coincide.

(d) A **pre-co-orientation** $\zeta : Z \to Z'$ is an equivalence class of local isomorphisms $Z \to Z'$ of orientable topological surfaces with respect to being co-oriented.

(e) The assignment which assigns to the open sets U in Z the sets of pre-co-orientations $U \to Z'$ is a presheaf. We shall refer to a global section $\zeta : Z \to Z'$ of the sheafification of this presheaf as a **co-orientation**.

(5) Let $\mathbb{X}$, $\mathbb{Y}$ be Aut-holomorphic spaces arising from Riemann surfaces X, Y respectively. Let $\mathcal{U}$, $\mathcal{V}$ be local structures of X^{top}, Y^{top} respectively.

(a) $(\mathcal{U}, \mathcal{V})$-local morphisms $\phi_1, \phi_2 : \mathbb{X} \to \mathbb{Y}$ of Aut-holomorphic spaces is called **co-holomorphic**, if ϕ_1^{top} and ϕ_2^{top} are co-oriented.

(b) A **pre-co-holomorphicisation** $\zeta : \mathbb{X} \to \mathbb{Y}$ is an equivalence class of $(\mathcal{U}, \mathcal{V})$-local morphisms $\mathbb{X} \to \mathbb{Y}$ of Aut-holomorphic spaces with respect to being co-holomorphic.

(c) The assignment which assigns to the open sets U in $\mathbb{X}^{\mathrm{top}}$ the sets of pre-co-holomorphicisation $U \to \mathbb{Y}$ is a presheaf. We shall refer to a global section $\zeta : \mathbb{X} \to \mathbb{Y}$ of the sheafification of this presheaf as a **co-holomorphicisation**.

By replacing "Riemann surface" by "one-dimensional complex orbifold", we can easily extend the notion of Aut-holomorphic space to **Aut-holomorphic orbispace**.

Proposition 4.2. ([AbsTopIII, Proposition 2.2]) *Let $\mathbb{X}, \mathbb{Y}$ be Aut-holomorphic discs arising from Riemann surfaces X, Y respectively. We equip the group $\mathrm{Aut}(X^{\mathrm{top}})$ of homeomorphisms with the compact-open topology. We write $\mathrm{Aut}^{\text{RC-hol}}(X)$ $(\subset \mathrm{Aut}(X^{\mathrm{top}}))$ for the subgroup of RC-holomorphic automorphisms of X. We regard $\mathrm{Aut}^{\mathrm{hol}}(X)$ and $\mathrm{Aut}^{\text{RC-hol}}(X)$ as equipped with the induced topology by the inclusions*

$$\mathrm{Aut}^{\mathrm{hol}}(X) \subset \mathrm{Aut}^{\text{RC-hol}}(X) \subset \mathrm{Aut}(X^{\mathrm{top}}).$$

(1) We have isomorphisms

$$\mathrm{Aut}^{\mathrm{hol}}(X) \cong \mathrm{PSL}_2(\mathbb{R}), \quad \mathrm{Aut}^{\text{RC-hol}}(X) \cong \mathrm{PGL}_2(\mathbb{R})$$

as topological groups, $\mathrm{Aut}^{\mathrm{hol}}(X)$ is a subgroup in $\mathrm{Aut}^{\text{RC-hol}}(X)$ of index 2, and $\mathrm{Aut}^{\text{RC-hol}}(X)$ is a closed subgroup of $\mathrm{Aut}(X^{\mathrm{top}})$.

(2) $\mathrm{Aut}^{\text{RC-hol}}(X)$ *is commensurably terminal (cf. Section 0.2) in* $\mathrm{Aut}(X^{\mathrm{top}})$.

(3) Any isomorphism $\mathbb{X} \xrightarrow{\sim} \mathbb{Y}$ *of Aut-holomorphic spaces arises from an RC-holomorphic isomorphism* $X \xrightarrow{\sim} Y$.

Proof. (1) is well-known (the last assertion follows from the fact of complex analysis that the limit of a sequence of holomorphic functions which uniformly converges on compact subsets is also holomorphic).

(2) It suffices to show that $C_{\mathrm{Aut}(X^{\mathrm{top}})}(\mathrm{Aut}^{\mathrm{hol}}(X)) = \mathrm{Aut}^{\text{RC-hol}}(X)$ (cf. Section 0.2). Let $\alpha \in C_{\mathrm{Aut}(X^{\mathrm{top}})}(\mathrm{Aut}^{\mathrm{hol}}(X))$. Then $\mathrm{Aut}^{\mathrm{hol}}(X) \cap \alpha \mathrm{Aut}^{\mathrm{hol}}(X)\alpha^{-1}$ is a closed subgroup of finite index in $\mathrm{Aut}^{\mathrm{hol}}(X)$, hence an open subgroup in $\mathrm{Aut}^{\mathrm{hol}}(X)$. Since $\mathrm{Aut}^{\mathrm{hol}}(X)$ is connected, we have $\mathrm{Aut}^{\mathrm{hol}}(X) \cap \alpha \mathrm{Aut}^{\mathrm{hol}}(X)\alpha^{-1} = \mathrm{Aut}^{\mathrm{hol}}(X)$. Thus, $\alpha \in N_{\mathrm{Aut}(X^{\mathrm{top}})}(\mathrm{Aut}^{\mathrm{hol}}(X))$ (cf. Section 0.2). Then by the conjugation, α gives an automorphism of $\mathrm{Aut}^{\mathrm{hol}}(X)$. The theorem of Schreier-van der Waerden ([SvdW]) says that $\mathrm{Aut}(\mathrm{PSL}_2(\mathbb{R})) \cong \mathrm{PGL}_2(\mathbb{R})$ by the conjugation. Hence we have $\alpha \in \mathrm{Aut}^{\text{RC-hol}}(X)$. (Without using the theorem of Schreier-van der Waerden, we can directly show it as follows: By Cartan's theorem (a homomorphism as topological groups between Lie groups is automatically a homomorphism as Lie groups, cf. [Serre1, Chapter V, §9, Theorem 2]), the automorphism of $\mathrm{Aut}^{\mathrm{hol}}(X)$ given by the conjugate of α is an automorphism of Lie groups. This induces an automorphism of Lie algebra $sl_2(\mathbb{C})$ with $sl_2(\mathbb{R})$ stabilised. Hence α is given by an element of $PGL_2(\mathbb{R})$. cf. also [AbsTopIII, proo of Proposition 2.2 (ii)], [QuConf, the proof of Lemma1.10].)

(3) follows from (2) since (2) implies that $\mathrm{Aut}^{\text{RC-hol}}(X)$ is normally terminal. $\square$

The followoing corollary says that the notions of "holomorphic structure", "Aut-holomorphic structure", and "pre-Aut-holomorphic structure" are equivalent.

Corollary 4.3. (a sort of Bi-Anabelian Grothendieck Conjecture in the Archimedean Theory, [AbsTopIII, Corollary 2.3]) *Let* $\mathbb{X}$, $\mathbb{Y}$ *be Aut-holomorphic spaces arising from Riemann surfaces* X, Y *respectively. Let* $\mathcal{U}$, $\mathcal{V}$ *be local structures of* X^{top}, Y^{top} *respectively.*

(1) Any $(\mathcal{U}, \mathcal{V})$*-local isomorphism* $\phi : \mathbb{X} \to \mathbb{Y}$ *of Aut-holomorphic spaces arises from a unique étale RC-holomorphic morphism* $\psi : X \to Y$. *If* $\mathbb{X}$ *and* $\mathbb{Y}$ *are connected, then there exist precisely* 2 *co-holomorphicisations* $\mathbb{X} \to \mathbb{Y}$, *corresponding to the holomorphic and anti-holomorphic local isomorphisms.*

(2) Any pre-Aut-holomorphic structure on $\mathbb{X}^{\mathrm{top}}$ *extends to a unique Aut-holomorphic structure on* $\mathbb{X}^{\mathrm{top}}$.

Proof. (1) follows from Proposition 4.2 (3).

(2) follows by applying (1) to automorphisms of the Aut-holomorphic spaces determined by the connected open subsets of $\mathbb{X}^{\mathrm{top}}$ which determine the same co-holomorphicisation as the identity automorphism. $\square$

§ 4.2. Elliptic Cuspidalisation and Kummer Theory in the Archimedean Theory.

Lemma 4.4. ([AbsTopIII, Corollary 2.4]) *Let $\mathbb{X}$ be a hyperbolic Aut-holomorphic orbispace of finite type, arising from a hyperbolic orbicurve X over $\mathbb{C}$. Only from the Aut-holomorphic orbispace $\mathbb{X}$, we can determine whether or not X admits $\mathbb{C}$-core, and in the case where X admits $\mathbb{C}$-core, we can construct the Aut-holomorphic orbispace associated to the $\mathbb{C}$-core in a functorial manner with respect to finite étale morphisms by the following algorithms:*

(1) Let $\mathbb{U}^{\mathrm{top}} \to \mathbb{X}^{\mathrm{top}}$ be any universal covering of $\mathbb{X}^{\mathrm{top}}$. Then we reconstruct the topological fundamental group $\pi_1(\mathbb{X}^{\mathrm{top}})$ as the opposite group $\mathrm{Aut}(\mathbb{U}^{\mathrm{top}}/\mathbb{X}^{\mathrm{top}})^{\mathrm{opp}}$ of $\mathrm{Aut}(\mathbb{U}^{\mathrm{top}}/\mathbb{X}^{\mathrm{top}})$.

(2) Let $\mathcal{U}$ be the local structure of $\mathbb{U}^{\mathrm{top}}$ consisting of connected open subsets of $\mathbb{U}^{\mathrm{top}}$ which map isomorphically onto open sub-orbispaces of $\mathbb{X}^{\mathrm{top}}$. We construct a natural $\mathcal{U}$-local pre-Aut-holomorphic structure on $\mathbb{U}^{\mathrm{top}}$ by restricting Aut-holomorphic structure of $\mathbb{X}$ on $\mathbb{X}^{\mathrm{top}}$ and by transporting it to $\mathbb{U}^{\mathrm{top}}$. By Corollary 4.3 (2), this gives us a natural Aut-holomorphic structure $\mathcal{A}_{\mathbb{U}}$ on $\mathbb{U}^{\mathrm{top}}$. We write $\mathbb{U} := (\mathbb{U}^{\mathrm{top}}, \mathcal{A}_{\mathbb{U}})$. Thus, we obtain a natural injection $\pi_1(\mathbb{X}^{\mathrm{top}})^{\mathrm{opp}} = \mathrm{Aut}(\mathbb{U}^{\mathrm{top}}/\mathbb{X}^{\mathrm{top}}) \hookrightarrow \mathrm{Aut}^0(\mathbb{U}) \subset \mathrm{Aut}(\mathbb{U}) \cong \mathrm{PGL}_2(\mathbb{R})$, where we write $\mathrm{Aut}^0(\mathbb{U})$ for the connected component of the identity of $\mathrm{Aut}(\mathbb{U})$, and the last isomorphism is an isomorphism as topological groups (Here, we regard $\mathrm{Aut}(\mathbb{U})$ as a topological space by the compact-open topology).

(3) X admits $\mathbb{C}$-core if and only if $\mathrm{Im}(\pi_1(\mathbb{X}^{\mathrm{top}})^{\mathrm{opp}}) := \mathrm{Im}(\pi_1(\mathbb{X}^{\mathrm{top}})^{\mathrm{opp}} \subset \mathrm{Aut}^0(\mathbb{U}))$ is of finite index in $\Pi_{\mathrm{core}} := C_{\mathrm{Aut}^0(\mathbb{U})}(\mathrm{Im}(\pi_1(\mathbb{X}^{\mathrm{top}})^{\mathrm{opp}}))$. If $\mathbb{X}$ admits $\mathbb{C}$-core, then the quotient $X^{\mathrm{top}} \twoheadrightarrow X_{\mathrm{core}} := \mathbb{U}^{\mathrm{top}}//\Pi_{\mathrm{core}}$ in the sense of stacks is the $\mathbb{C}$-core of X. The restriction of the Aut-holomorphic structure of $\mathbb{U}$ to an appropriate local structure on $\mathbb{U}$ and transporting it to X_{core} give us a natural Aut-holomorphic structure $\mathcal{A}_{X_{\mathrm{core}}}$ of X_{core}, hence the desired Aut-holomorphic orbispace $(\mathbb{X} \twoheadrightarrow)\mathbb{X}_{\mathrm{core}} := (X_{\mathrm{core}}, \mathcal{A}_{X_{\mathrm{core}}})$.

Proof. Assertions follow from the described algorithms. cf. also [CanLift, Remark 2.1.2]. $\square$

Proposition 4.5. (Elliptic Cuspidalisation in the Archimedean Theory, [AbsTopIII, Corollary 2.7], cf. also [AbsTopIII, Proposition 2.5, Proposition 2.6]) *Let $\mathbb{X}$ be an elliptically admissible Aut-holomorphic orbispace arising from a Riemann orbisurface X.*

By the following algorithms, only from the holomorphic space $\mathbb{X}$, we can reconstruct the system of local linear holomorphic structures on $\mathbb{X}^{\mathrm{top}}$ in the sense of (Step 10) below in a functorial manner with respect to finite étale morphisms:

(Step 1) By the definition of elliptical admissibility and Lemma 4.4 (2), we construct $\mathbb{X} \to \mathbb{X}_{\mathrm{core}}$, where $\mathbb{X}_{\mathrm{core}}$ arises from the $\mathbb{C}$-core X_{core} of X, and X_{core} is semi-elliptic (cf. Section 3.1). There is a unique double covering $\mathbb{E} \to \mathbb{X}_{\mathrm{core}}$ by an Aut-holomorphic space (not orbispace), i.e., the covering corresponding to the unique torsion-free subgroup of index 2 of the group Π_{core} of Lemma 4.4. Here, $\mathbb{E}$ is the Aut-holomorphic space associated to a onec-punctured elliptic curve $E \setminus \{O\}$ over $\mathbb{C}$.

(Step 2) We consider elliptic cuspidalisation diagrams $\mathbb{E} \twoheadleftarrow \mathbb{E}^N \hookrightarrow \mathbb{E}$ (cf. also the portion of "$E \setminus \{O\} \twoheadleftarrow E \setminus E[N] \hookrightarrow E \setminus \{O\}$" in the diagram (EllCusp) *of Section 3.2), where $\mathbb{E}^N \twoheadrightarrow \mathbb{E}$ is an abelian finite étale coveing which is also unramified at the unique punctured point, $\mathbb{E}^{\mathrm{top}} \hookrightarrow (\mathbb{E}^N)^{\mathrm{top}}$ is an open immersion, and $\mathbb{E}^N \hookrightarrow \mathbb{E}$, $\mathbb{E}^N \twoheadrightarrow \mathbb{E}$ are co-holomorphic. By these diagrams, we can reconstruct the* **torsion points** *of the elliptic curve E as the points in $\mathbb{E} \setminus \mathbb{E}^N$. We also reconstruct the* **group structure** *on the torsion points induced by the group structure of the Galois group* $\mathrm{Gal}(\mathbb{E}^N/\mathbb{E})$*, i.e., $\sigma \in \mathrm{Gal}(\mathbb{E}^N/\mathbb{E})$ corresponds to "$+[P]$" for some $P \in E[N]$.*

(Step 3) Since the torsion points constructed in (Step 2) are dense in $\mathbb{E}^{\mathrm{top}}$, we reconstruct the **group structure** *on $\mathbb{E}^{\mathrm{top}}$ as the unique topological group structure extending the group structure on the torsion points constructed in (Step 2). In the subsequent steps, we take a simply connected open non-empty subset U in $\mathbb{E}^{\mathrm{top}}$.*

(Step 4) Let $p \in U$. The group structure constructed in (Step 3) induces a **local additive structure** *of U at p, i.e., $a +_p b := (a - p) + (b - p) + p \in U$ for $a, b \in U$, whenever it is defined.*

(Step 5) We reconstruct the **line segments** *of U by one-parameter subgroups relative to the local additive structures constructed in (Step 4). We also reconstruct the pairs of* **parallel line segments** *of U by translations of line segments relative to the local additive structures constructed in (Step 4). For a line segment L, write ∂L to be the subset of L consisting of points whose complements are connected, we shall refer to an element of ∂L as an* **endpoint** *of L.*

(Step 6) We reconstruct the **parallelograms** *of U as follows: We define a* **pre-∂-parallelogram** *A of U to be $L_1 \cup L_2 \cup L_3 \cup L_4$, where L_i ($i \in \mathbb{Z}/4\mathbb{Z}$) are line segments (constructed in (Step 5)) such that (a) for any $p_1 \neq p_2 \in A$, there exists a line segment L constructed in (Step 5) with $\partial L = \{p_1, p_2\}$, (b) L_i and L_{i+2} are parallel line segments constructed in (Step 5) and non-intersecting for any $i \in \mathbb{Z}/4\mathbb{Z}$, and (c)*

$L_i \cap L_{i+1} = (\partial L_i) \cap (\partial L_{i+1})$ with $\#(L_i \cap L_{i+1}) = 1$. We reconstruct the **parallelograms** *of U as the interiors of the unions of the line segments L of U such that $\partial L \subset A$ for a pre-∂-parallelogram A. We define a* **side** *of a parallelogram in U to be a maximal line segment contained in $\overline{P} \setminus P$ for a parallelogram P of U, where we write $\overline{P}$ for the closure of P in U.*

(Step 7) Let $p \in U$. We define a **frame** *$F = (S_1, S_2)$ to be an ordered pair of intersecting sides $S_1 \neq S_2$ of a parallelogram P of U constructed in (Step 6), such that $S_1 \cap S_2 = \{p\}$. If a line segment L of U have an infinite intersection with P, then we shall refer to L as being* **framed** *by F. We reconstruct an* **orientation** *of U at p (of which there are precisely 2) as an equivalence class of frames of U^{top} at p relative to the equivalence relation of frames $F = (S_1, S_2)$, $F = (S_1', S_2')$ of U at p generated by the relation that S_1' is framed by F and S_2 is framed by F'.*

(Step 8) Let $\mathbb{V}$ be the Aut-holomorphic space determined by a parallelogram $\mathbb{V}^{\mathrm{top}} \subset U$ constructed in (Step 7). Let $p \in \mathbb{V}^{\mathrm{top}}$. Let S be a one-parameter subgroup of the topological group $\mathcal{A}_{\mathbb{V}}(\mathbb{V}^{\mathrm{top}})(\cong \mathrm{PSL}_2(\mathbb{R}))$ and a line segment L in U constructed in (Step 5) such that one of the endpoints (cf. (Step 5)) of L is equal to p. Note that one-parameter subgroups are characterised by using topological (not differentiable) group structure as the closed connected subgroups for which the complement of some connected open neighbourhood of the identity element is not connected. We say that L is **tangent** *to $S\cdot p$ at p if any pairs of sequences of points of $L\setminus\{p\}$, $(S\cdot p)\setminus\{p\}$ converge to the same element of the quotient space $\mathbb{V}^{\mathrm{top}} \setminus \{p\} \twoheadrightarrow \mathbb{P}(\mathbb{V}, p)$ determined by identifying positive real multiples of points of $\mathbb{V}^{\mathrm{top}} \setminus \{p\}$ relative to the local additive structure constructed in (Step 4) at p (i.e., projectivification). We can reconstruct the* **orthogonal frames** *of U as the frames consisting of pairs of line segments L_1, L_2 having $p \in U$ as an endpoint that are tangent to the orbits $S_1 \cdot p$, $S_2 \cdot p$ of one-parameter subgroups $S_1, S_2 \subset \mathcal{A}_{\mathbb{V}}(\mathbb{V}^{\mathrm{top}})$ such that S_2 is obtained from S_1 by conjugating S_1 by an element of order 4 (i.e., "$\pm i$") of a compact one-parameter subgroup of $\mathcal{A}_{\mathbb{V}}(\mathbb{V}^{\mathrm{top}})$.*

(Step 9) For $p \in U$, let $(V)_{p\in V\subset U}$ be the projective system of connected open neighbourhoods of p in U, and put

$$\mathcal{A}_p := \left\{ f \in \mathrm{Aut}((V)_{p\in V\subset U}) \;\middle|\; f \text{ satisfies (LAS), (Orth), and (Ori)} \right\},$$

where

(LAS): compatibility with the local additive structures of $V(\subset U)$ at p constructed in (Step 4),

(Orth): preservation of the orthogonal frames of $V(\subset U)$ at p constructed in (Step 8), and

(Ori): preservation of the orientations of $V(\subset U)$ *at* p *constructed in (Step 7)*

(cf. also Section 0.2 for the Hom *for a projective system). We equip* $\mathcal{A}_p$ *with the topology induced by the topologies of the open neighbourhoods of* p *that* $\mathcal{A}_p$ *acts on. The local additive structures of (Step 4) induce an additive structure on* $\overline{\mathcal{A}_p} := \mathcal{A}_p \cup \{0\}$*. Hence we have a natural topological field structure on* $\overline{\mathcal{A}_p}$*. Tha tautological action of* $\mathbb{C}^\times$ *on* $\mathbb{C} \supset U$ *induces a natural isomorphism* $\mathbb{C}^\times \xrightarrow{\sim} \mathcal{A}_p$ *of topological groups, hence a natural isomorphism* $\mathbb{C} \xrightarrow{\sim} \overline{\mathcal{A}_p}$ *of topological fields. In this manner, we reconstruct the* **local linear holomorphic structure** *"*$\mathbb{C}^\times$ *at* p*"* **of** U **at** p *as the topological field* $\overline{\mathcal{A}_p}$ *with the tautological action of* $\mathcal{A}_p(\subset \overline{\mathcal{A}_p})$ *on* $(V)_{p\in V\subset U}$*.*

(Step 10) For $p, p' \in U$*, we construct a natural isomorphism* $\overline{\mathcal{A}_p} \xrightarrow{\sim} \overline{\mathcal{A}_{p'}}$ *of topological fields as follows: If* p' *is sufficiently close to* p*, then the local additive structures constructed in (Step 4) induce homeomorphism from sufficiently small neighbourhoods of* p *onto sufficiently small neighbourhoods of* p' *by the translation (=the addition). These homeomorphisms induce the desired isomorphism* $\overline{\mathcal{A}_p} \xrightarrow{\sim} \overline{\mathcal{A}_{p'}}$*. For general* $p, p' \in U$*, we can obtain the desired isomorphism* $\overline{\mathcal{A}_p} \xrightarrow{\sim} \overline{\mathcal{A}_{p'}}$ *by joining* p' *to* p *via a chain of sufficiently small open neighbourhoods and composing the isomorphisms on local linear holomorphic structures. This isomorphism is independent of the choice of such a chain. We shall refer to* $((\mathcal{A}_p)_p, (\overline{\mathcal{A}_p} \xrightarrow{\sim} \overline{\mathcal{A}_{p'}})_{p,p'})$ *as the* **system of local linear holomorphic structures** *on* $\mathbb{E}^{\mathrm{top}}$ *or* $\mathbb{X}^{\mathrm{top}}$*. We identify* $(\mathcal{A}_p \subset \overline{\mathcal{A}_p})$*'s for* p*'s via the above natural isomorphisms and we write* $\mathcal{A}^{\mathbb{X}} \subset \overline{\mathcal{A}^{\mathbb{X}}}$ *for the identified ones.*

Proof. The assertions immedeately follow from the described algorithms. □

Hence the formulation of "Aut-holomorphic structure" succeeds to avoid a specific fixed local referred model of $\mathbb{C}$ (i.e., "*the* $\mathbb{C}$") in the above sense too, unlike the usual notion of "holomorphic structure". This is also a part of "**mono-anabelian philosophy**" of Mochizuki. cf. also Remark 3.4.4 (3), and [AbsTopIII, Remark 2.1.2, Remark 2.7.4].

Let k be a CAF (cf. Section 0.2). We recall (cf. Section 0.2) that we write $O_k \subset \mathbb{C}$ for the *subset* of elements with $|-| \leq 1$ in k, $O_k^\times \subset O_k$ for the group of units i.e., elements with $|-| = 1$, and $O_k^\triangleright := O_k \setminus \{0\} \subset O_k$ for the multiplicative monoid.

Definition 4.6. ([AbsTopIII, Definition 4.1])

(1) Let $\mathbb{X}$ be an elliptically admissible Aut-holomorphic orbispace. A **model Kummer structure** $\kappa_k : k \xrightarrow{\sim} \overline{\mathcal{A}^{\mathbb{X}}}$ (resp. $\kappa_{O_k^\times} : O_k^\times \hookrightarrow \mathcal{A}^{\mathbb{X}}$, resp. $\kappa_{k^\times} : k^\times \hookrightarrow \mathcal{A}^{\mathbb{X}}$, resp. $\kappa_{O_k^\triangleright} : O_k^\triangleright \hookrightarrow \mathcal{A}^{\mathbb{X}}$) on $\mathbb{X}$ is an isomorphism of topological fields (resp. its restriction to $O_k^\times$, resp. its restriction to $k^\times$, resp. its restriction to $O_k^\triangleright$). An isomorphism

$\kappa_M : M \xrightarrow{\sim} \overline{\mathcal{A}^{\mathbb{X}}}$ of topological fields (resp. an inclusion $\kappa_M : O_k^\times \hookrightarrow \mathcal{A}^{\mathbb{X}}$ of topological groups, resp. an inclusion $\kappa_M : k^\times \hookrightarrow \mathcal{A}^{\mathbb{X}}$ of topological groups, resp. an inclusion $\kappa_M : O_k^\triangleright \hookrightarrow \mathcal{A}^{\mathbb{X}}$ of topological monoids) is called a **Kummer structure** on $\mathbb{X}$ if there exist an automorphicm $f : \mathbb{X} \xrightarrow{\sim} \mathbb{X}$ of Auto-holomorphic spaces, and an isomorphism $g : M \xrightarrow{\sim} k$ of topological fields (resp. an isomorphism $g : M \xrightarrow{\sim} O_k^\times$ of topological groups, resp. an isomorphism $g : M \xrightarrow{\sim} k^\times$ of topological groups, resp. an isomorphism $g : M \xrightarrow{\sim} O_k^\triangleright$ of topological monoids) such that $f^* \circ \kappa_k = \kappa_M \circ g$ (resp. $f^* \circ \kappa_{O_k^\times} = \kappa_M \circ g$ resp. $f^* \circ \kappa_{k^\times} = \kappa_M \circ g$ resp. $f^* \circ \kappa_{O_k^\triangleright} = \kappa_M \circ g$), where $f^* : \overline{\mathcal{A}^{\mathbb{X}}} \xrightarrow{\sim} \overline{\mathcal{A}^{\mathbb{X}}}$ (resp. $f^* : \mathcal{A}^{\mathbb{X}} \xrightarrow{\sim} \mathcal{A}^{\mathbb{X}}$, resp. $f^* : \mathcal{A}^{\mathbb{X}} \xrightarrow{\sim} \mathcal{A}^{\mathbb{X}}$, resp. $f^* : \mathcal{A}^{\mathbb{X}} \xrightarrow{\sim} \mathcal{A}^{\mathbb{X}}$) is the automorphism induced by f. We often abbreviate it as $\mathbb{X} \overset{\kappa}{\curvearrowleft} M$.

(2) A **morphism** $\phi : (\mathbb{X}_1 \overset{\kappa_1}{\curvearrowleft} M_1) \to (\mathbb{X}_2 \overset{\kappa_2}{\curvearrowleft} M_2)$ **of elliptically admissible Aut-holomorphic orbispaces with Kummer structures** is a pair $\phi = (\phi_{\mathbb{X}}, \phi_M)$ of a finite étale morphism $\phi_{\mathbb{X}} : \mathbb{X}_1 \to \mathbb{X}_2$ and a homomorphism $\phi_M : M_1 \to M_2$ of topological monoids, such that the Kummer structures κ_1 and κ_2 are compatible with $\phi_M : M_1 \to M_2$ and the homomorphism $(\phi_{\mathbb{X}})_* : \overline{\mathcal{A}^{\mathbb{X}_1}} \to \overline{\mathcal{A}^{\mathbb{X}_2}}$ arising from the functoriality of the algorithms in Proposition 4.5.

The reconstruction

$$\mathbb{X} \mapsto \left(\mathbb{X}, \mathbb{X} \curvearrowleft \mathcal{A}^{\mathbb{X}} \subset \overline{\mathcal{A}^{\mathbb{X}}} \text{ (with field str.) tautological Kummer structure}\right)$$

described in Proposition 4.5 is an Archimedean analogue of the reconstruction

$$\Pi \mapsto \left(\Pi, \Pi \curvearrowright \overline{k} \text{ (with field str.)} \supset \overline{k}^\times \overset{\text{Kummer map}}{\hookrightarrow} \varinjlim_{J \subset \Pi:\,\text{open}} H^1(J, \mu_{\widehat{\mathbb{Z}}}(\Pi))\right),$$

described in Corollary 3.19 for non-Archimedean local field k. Namely, the reconstruction in Corollary 3.19 relates the base field k to Π_X via the Kummer theory, and the reconstruction in Proposition 4.5 relates the base field $\overline{\mathcal{A}^{\mathbb{X}}}$ ($\cong \mathbb{C}$) to $\mathbb{X}$, hence it is a kind of Archimedean Kummer theory.

Definition 4.7. (cf. also [AbsTopIII, Definition 5.6 (i), (iv)])

(1) We say that a pair $G = (C, \overrightarrow{C})$ of a topological monoid C and a topological submonoid $\overrightarrow{C} \subset C$ is a **split monoid**, if C is isomorphic to $O_{\mathbb{C}}^\triangleright$, and $\overrightarrow{C} \hookrightarrow C$ determines an isomorphism $C^\times \times \overrightarrow{C} \xrightarrow{\sim} C$ of topological monoids (Note that $C^\times$ and $\overrightarrow{C}$ are necessarily isomorphic to $\mathbb{S}^1$ and $(0,1] \overset{\log}{\cong} \mathbb{R}_{\geq 0}$ respectively). A **morphism of split monoids** $G_1 = (C_1, \overrightarrow{C}_1) \to G_2 = (C_2, \overrightarrow{C}_2)$ is an isomorphism $C_1 \xrightarrow{\sim} C_2$ of topological monoids which induce an isomorphism $\overrightarrow{C}_1 \xrightarrow{\sim} \overrightarrow{C}_2$ of the topological submonoids.

Remark 4.7.1. We omit the definition of *Kummer structure of split monoids* ([AbsTopIII, Definition 5.6 (i), (iv)]), since we do not use them in inter-universal Teichmüller theory (Instead, we consider split monoids for mono-analytic Frobenius-like objects). In [AbsTopIII], we consider a split monoid $G = (C, \overrightarrow{C})$ arising from arith-holomorphic "$O^{\rhd}_{\mathbb{C}}$" via the mono-analyticisation, and consider a Frobenius-like object M and $k^{\sim}(G) = C^{\sim} \times C^{\sim}$ (cf. Proposition 5.4 below) for $G = (C, \overrightarrow{C})$. On the other hand, in inter-universal Teichmüller theory, we consider $k^{\sim}(G) = C^{\sim} \times C^{\sim}$ directly from "$O^{\rhd}_{\mathbb{C}}$" (cf. Proposition 12.2 (4)). When we consider $k^{\sim}(G)$ directly from "$O^{\rhd}_{\mathbb{C}}$", then the indeterminacies are only $\{\pm 1\} \times \{\pm 1\}$ (i.e., Archimedean (Indet $\to$)); however, when we consider a Frobenius-like object for $G = (C, \overrightarrow{C})$, then we need to consider the synchronisation of k_1 and k_2 via group-germs, and need to consider $\overrightarrow{C}$ up to $\mathbb{R}_{>0}$ (i.e., we need to consider the category $\mathbb{TB}\boxplus$ in [AbsTopIII, Definition 5.6 (i)]). cf. also [AbsTopIII, Remark 5.8.1 (i)].

We write $G_{\mathbb{X}} = (O^{\rhd}_{\mathcal{A}^{\mathbb{X}}}, \overrightarrow{O}_{\mathcal{A}^{\mathbb{X}}})$ for the split monoid associated to the topological field $\overline{\mathcal{A}^{\mathbb{X}}}$, i.e., the topological monoid $O^{\rhd}_{\mathcal{A}^{\mathbb{X}}}$, and the splitting $O^{\rhd}_{\mathcal{A}^{\mathbb{X}}} \hookleftarrow O^{\rhd}_{\mathcal{A}^{\mathbb{X}}} \cap \mathbb{R}_{>0} =: \overrightarrow{O}_{\mathcal{A}^{\mathbb{X}}}$ of $O^{\rhd}_{\mathcal{A}^{\mathbb{X}}} \twoheadrightarrow O^{\rhd}_{\mathcal{A}^{\mathbb{X}}}/O^{\times}_{\mathcal{A}^{\mathbb{X}}}$ and $\mathbb{X} \curvearrowleft O^{\rhd}_{k}$. For a Kummer structure $\mathbb{X} \overset{\kappa}{\curvearrowleft} O^{\rhd}_{k}$ of an elliptically admissible Aut-holomorphic orbispace, we pull back $\overrightarrow{O_{\mathcal{A}^{\mathbb{X}}}}$ via the Kummer structure $O^{\rhd}_{k} \hookrightarrow \overline{\mathcal{A}^{\mathbb{X}}}$, we obtain a decomposition of $O^{\rhd}_{k}$ as $O^{\times}_{k} \times \overrightarrow{O_k}$, where $\overrightarrow{O_k} \cong O^{\rhd}_{k}/O^{\times}_{k}$. We consider this assignment

$$(\mathbb{X} \curvearrowleft O^{\rhd}_{k}) \mapsto (G_{\mathbb{X}} \curvearrowleft O^{\times}_{k} \times \overrightarrow{O_k})$$

as a mono-analytification.

§ 4.3. On the Philosophy of Étale-like and Frobenius-like Objects.

We further consider the similarities between the reconstruction algorithms in Corollary 3.19 and Proposition 4.5, and then, we explain Mochizuki's philosophy of **the dichotomy of étale-like objects and Frobenius-like objects**.

Note also that the tautological Kummer structure $\mathbb{X} \curvearrowleft \mathcal{A}^{\mathbb{X}}$ rigidifies the non-rigid "$\mathbb{R}_{>0}$" (cf. Secton 3.5) in $\mathcal{A}^{\mathbb{X}}$ ($\cong \mathbb{C}^{\times}$) in the exact sequence $0 \to \mathbb{S}^1 \to \mathbb{C}^{\times} \to \mathbb{R}_{>0} \to 0$ (cf. also [AbsTopIII, Remark 2.7.3]). In short, we have the following dictionary:

	Arith. Hol.	Mono-analytic
non-Arch. $k/\mathbb{Q}_p$: fin.	Π_X, $\Pi_X \curvearrowright O^{\rhd}_{\overline{k}}$ rigidifies $O^{\times}_{k}$	G_k, $G_k \curvearrowright O^{\times}_{\overline{k}} \times \overrightarrow{O_{\overline{k}}}$
$0 \to O^{\times}_{k} \to k^{\times} \to \widehat{\mathbb{Z}}(\text{rigid}) \to 0$	"k" can be reconstructed	$O^{\times}_{k}$: non-rigid
Arch. $k\,(\cong \mathbb{C})$	$\mathbb{X}$, $\mathbb{X} \curvearrowleft O^{\rhd}_{k}$ rigidifies "$\mathbb{R}_{>0}$"	$G_{\mathbb{X}}$, $G_{\mathbb{X}} \curvearrowleft O^{\times}_{k} \times \overrightarrow{O_{k}}$
$0 \to \mathbb{S}^1(\text{rigid}) \to \mathbb{C}^{\times} \to \mathbb{R}_{>0} \to 0$	"$\mathbb{C}$" can be reconstructed	"$\mathbb{R}_{>0}$": non-rigid

We consider profinite groups Π_X, G_k, categories of the finite étale coverings over hyperbolic curves or spectra of fields, and the objects reconstructed from these as **étale-like objects**, and we consider, on the other hand, abstract topological monoids (with actions of Π_X, G_k), the categories of line bundles on finite étale coverings over hyperbolic curves, the categories of arithmetic line bundles on finite étale coverings over spectra of number fields, as **Frobenius-like objects**, i.e., when we reconstruct $\Pi_X \curvearrowright O^{\rhd}_{\overline{k}}$ or $\mathbb{X} \curvearrowleft O^{\rhd}_{k}$, then these are regarded as étale-like objects whenever we *remember* that the relations with Π_X and $\mathbb{X}$ via the reconstruction algorithms; however, if we *forget* the relations with Π_X and $\mathbb{X}$ via the reconstruction algorithms, and we consider them as an abstract topological monoid with an action of Π_X, and an abstract topological monoid with Kummer structure on $\mathbb{X}$, then these objects are regarded as Frobenius-like objects (cf. also [AbsTopIII, Remark 3.7.5 (iii), (iv), Remark 3.7.7], [FrdI, §I4], [IUTchI, §I1]). Note that if we forget the relations with Π_X and $\mathbb{X}$ via the reconstruction algorithms, then we *cannot* obtain the functoriality with respect to Π_X or $\mathbb{X}$ for the abstract objects.

We have the dichotomy of étale-like objects and Frobenius-like objects both on arithmetically holomorphic objects and mono-analytic objects, i.e., we can consider 4 kinds of objects – arithmetically holomorphic étale-like objects (indicated by $\mathcal{D}$), arithmetically holomorphic Frobenius-like objects (indicated by $\mathcal{F}$), mono-analytic étale-like objects (indicated by $\mathcal{D}^{\vdash}$), and mono-analytic Frobenius-like objects (indicated by $\mathcal{F}^{\vdash}$) (Here, as we can easily guess, the symbol $\vdash$ means "mono-analytic"). The types and structures of prime-strips (cf. Section 10.3) and Hodge theatres reflect this classification of objects (cf. Section 10).

Note that the above table also exhibits these 4 kinds of objects. Here, we consider $G_k \curvearrowright O^{\times}_{\overline{k}} \times (O^{\rhd}_{\overline{k}}/O^{\times}_{\overline{k}})$ and $G_{\mathbb{X}} \curvearrowleft O^{\times}_{k} \times (O^{\rhd}_{k}/O^{\times}_{k})$ as the mono-analyticisations of arithmetically holomorphic objects $\Pi_k \curvearrowright O^{\rhd}_{\overline{k}}$, and $\mathbb{X} \curvearrowleft O^{\rhd}_{k}$ respectively. cf. the following diagrams:

$$
\begin{array}{ccccc}
 & \text{Frobenius-like} \xrightarrow{\text{forget}} \text{étale-like} & & \text{Frobenius-like} \xrightarrow{\text{forget}} \text{étale-like} \\
 & \text{(base with line bundle)} \qquad \text{(base)} & & \text{(base with line bundle)} \qquad \text{(base)}
\end{array}
$$

$$
\begin{tikzcd}
\text{arith. hol.} \arrow[d, "\text{mono-anlyticisation}"] & \Pi_X \curvearrowright O^{\rhd}_{\overline{k}} \arrow[r, mapsto] \arrow[d, mapsto] & \Pi_X \arrow[d, mapsto] & \mathbb{X} \curvearrowleft O^{\rhd}_{k} \arrow[r, mapsto] \arrow[d, mapsto] & \mathbb{X} \arrow[d, mapsto] \\
\text{mono-an.} & G_k \curvearrowright O^{\times}_{\overline{k}} \times \overrightarrow{O_{\overline{k}}} \arrow[r, mapsto] & G_k, & G_{\mathbb{X}} \curvearrowleft O^{\times}_{k} \times \overrightarrow{O_{k}} \arrow[r, mapsto] & G_{\mathbb{X}}.
\end{tikzcd}
$$

The composite of the reconstruction algorithms Theorem 3.17 and Proposition 4.5 with "forgetting the relations with the input data via the reconstruction algorithms" are the canonical "sections" of the corresponding functors Frobenius-like $\overset{\text{forget}}{\longrightarrow}$ étale-like (Note also that, by Proposition 2.1 (2c), the topological monoid $O^{\rhd}_{\overline{k}}$ can be group-theoretically reconstructed from G_k; however, we cannot reconstruct $O^{\rhd}_{\overline{k}}$ as a submonoid of a topological field k, which needs an arithmetically holomorphic structure).

In inter-universal Teichmüller theory, the Frobenius-like objects are used *to construct links* (i.e., $\mathfrak{log}$-links and Θ-links). On the other hand, some of étale-like objects are used (a) *to construct shared objects* (i.e., vertically coric, horizontally coric, and bi-coric objects) in both sides of the links, and (b) *to exchange (!)* both sides of a Θ-link (which is called **étale-transport**. cf. also Remark 9.6.1, Remark 11.1.1, and Theorem 13.12 (1)), after going from Frobenius-like picture to étale-like picture, which is called **Kummer-detachment** (cf. also Section 13.2), by Kummer theory and by admitting indeterminacies (Indet $\to$), (Indet $\uparrow$), and (Indet $\curvearrowright$). (More precisely, étale-like Π_X and G_k are shared in $\mathfrak{log}$-links. The mono-analytic G_k is also (as an abstract topological group) shared in Θ-links; however, arithmetically holomorphic Π_X cannot be shared in Θ-links, and even though $O^{\times}_{\overline{k}}$/tors's are Frobenius-like objects, $O^{\times}_{\overline{k}}$/tors's (*not* $O^{\rhd}_{\overline{k}}$'s because the portion of the value group is dramatically dilated) are shared after admitting $\widehat{\mathbb{Z}}^{\times}$-indeterminacies.) cf. also Theorem 12.5.

étale -like	objects reconstructed from Π_X, G_k, $\mathbb{X}$, $G_{\mathbb{X}}$	Galois category coverings	indifferent to order can be shared, can be exchanged
Frobenius -like	abstract $\Pi_X \curvearrowright O^{\rhd}_{\overline{k}}$, $G_k \curvearrowright O^{\times}_{\overline{k}} \times \overrightarrow{O_{\overline{k}}}$, $\mathbb{X} \curvearrowleft O^{\rhd}_{\mathbb{C}}$, $G_{\mathbb{X}} \curvearrowleft O^{\times}_{\mathbb{C}} \times \overrightarrow{O_{\mathbb{C}}}$	Frobenioids line bundles	order-conscious can make links

§ 4.4. Mono-anabelian Reconstruction Algorithms in the Archimedean Theory.

The following theorem is an Archimedean analogue of Theorem 3.17.

Proposition 4.8. (Mono-anabelian Reconstruction, [AbsTopIII, Corollary 2.8]) *Let X be a hyperbolic curve of strictly Belyi type over a number field k. Let $\overline{k}$ be an algebraic closure of k, and Π_X the arithmetic fundamental group of X for some basepoint. From the topological group Π_X, we group-theoretically reconstruct the field $\overline{k} = \overline{k}_{\mathrm{NF}}$ by the algorithm in Theorem 3.17 (cf. Remark 3.17.1). Let $\overline{v}$ be an Archimedean place of $\overline{k}$. By the following group-theoretic algorithm, from the topological group Π_X and the Archimedean place $\overline{v}$, we can reconstruct the Aut-holomorphic space $\mathbb{X}_{\overline{v}}$ associated to $X_{\overline{v}} := X \times_k k_{\overline{v}}$ in a functorial manner with respect to open injective homomorphisms of profinite groups which are compatible with the respective choices of Archimedean valuations:*

(Step 1) We reconstruct NF-points of $X_{\overline{v}}$ as conjugacy classes of decomposition groups of NF-points in Π_X by in Theorem 3.17. We also reconstruct non-constant NF-rational functions on $X_{\overline{k}}$ by Theorem 3.17 (Step 4) (or Lemma 3.16). Note that we also group-theoretically obtain the evaluation map $f \mapsto f(x)$ at NF-point x as the restriction to the decomposition group of x (cf. Theorem 3.17 (Step 4), (Step 5)), and that the order function ord_x at NF-point x as the component at x of the homomorphism $H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \to \oplus_{y \in S}\widehat{\mathbb{Z}}$ in Theorem 3.17 (Step 3) (cf. Theorem 3.17 (Step 5)).

(Step 2) Define a **Cauchy sequence** *$\{x_j\}_{j\in\mathbb{N}}$ of NF-points to be a sequence of NF-points x_j such that there exists an exceptional finite set of NF-points S satisfying the following conditions:*

- *$x_j \notin S$ for all but finitely many $j \in \mathbb{N}$, and*
- *For any non-constant NF-rational function f on $X_{\overline{k}}$, whose divisor of poles avoids S, the sequence of values $\{f(x_j) \in k_{\overline{v}}\}_{j\in\mathbb{N}}$ forms a Cauchy sequence (in the usual sense) in $k_{\overline{v}}$.*

For two Cauchy sequences $\{x_j\}_{j\in\mathbb{N}}$, $\{y_j\}_{j\in\mathbb{N}}$ of NF-points with common exceptional set S, we call that these are **equivalent***, if for any non-constant NF-rational function f on $X_{\overline{k}}$, whose divisor of poles avoids S, the Cauchy sequences $\{f(x_j) \in k_{\overline{v}}\}_{j\in\mathbb{N}}$, $\{f(y_j) \in k_{\overline{v}}\}_{j\in\mathbb{N}}$ in $k_{\overline{v}}$ converge to the same element of $k_{\overline{v}}$.*

(Step 3) *For an open subset $U \subset k_{\overline{v}}$ and a non-constant NF-rational function f on $X_{\overline{v}}$, write $N(U, f)$ to be the set of Cauchy sequences of NF-points $\{x_j\}_{j \in \mathbb{N}}$ such that $f(x_j) \in U$ for all $j \in \mathbb{N}$. We reconstruct the topological space $\mathbb{X}^{\mathrm{top}} = X_{\overline{v}}(k_{\overline{v}})$ as the set of equivalence classes of Cauchy sequences of NF-points, equipped with the topology defined by the sets $N(U, f)$. A non-contant NF-rational function extends to a function on $\mathbb{X}^{\mathrm{top}}$, by taking the limit of the values.*

(Step 4) *Let $U_{\mathbb{X}} \subset \mathbb{X}^{\mathrm{top}}$, $U_{\overline{v}} \subset k_{\overline{v}}$ be connected open subsets, and f a non-constant NF-rational function on $X_{\overline{k}}$, such that the function defined by f on $U_{\mathbb{X}}$ gives us a homeomorphism $f_U : U_{\mathbb{X}} \xrightarrow{\sim} U_{\overline{v}}$. We write $\mathrm{Aut}^{\mathrm{hol}}(U_{\overline{v}})$ for the group of homeomorphisms $f : U_{\overline{v}} \xrightarrow{\sim} U_{\overline{v}} (\subset k_{\overline{v}})$, which can locally be expressed as a convergent power series with coefficients in $k_{\overline{v}}$ with respect to the topological field structure of $k_{\overline{v}}$.*

(Step 5) *Write $\mathcal{A}_{\mathbb{X}}(U_{\mathbb{X}}) := f_U^{-1} \circ \mathrm{Aut}^{\mathrm{hol}}(U_{\overline{v}}) \circ f_U \subset \mathrm{Aut}(U_{\mathbb{X}})$. By Corollary 4.3, we reconstruct the Aut-holomorphic structure $\mathcal{A}_{\mathbb{X}}$ on $\mathbb{X}^{\mathrm{top}}$ as the unique Aut-holomorphic structure which extends the pre-Aut-holomorphic structure defined by the groups $\mathcal{A}_{\mathbb{X}}(U_{\mathbb{X}})$ in (Step 4).*

Proof. The assertions immediately follow from the described algorithms. □

We can easily generalise the above theorem to hyperbolic *orbi*curves of strictly Belyi type over number fields.

Lemma 4.9. (Compatibility of Elliptic Cuspidalisation in Archimedean Place with Galois Theoretic Belyi Cuspidalisation, [AbsTopIII, Corollary 2.9]) *In the situation of Proposition 4.8, suppose further that X is elliptically admissible. From the topological group Π_X, we group-theoretically reconstruct the field $\overline{k} = \overline{k}_{\mathrm{NF}}$ by Theorem 3.17 (cf. Remark 3.17.1), i.e., via Belyi cuspidalisation. Let $\overline{v}$ be an Archimedean place of $\overline{k}(\Pi_X)$. Let $\mathbb{X} = (\mathbb{X}^{\mathrm{top}}, \mathcal{A}_{\mathbb{X}})$ be the Aut-holomorphic space constructed from the topological group Π_X and the Archimedean valuation $\overline{v}$ in Proposition 4.8, i.e., via Cauchy sequences. Let $\overline{\mathcal{A}^{\mathbb{X}}}$ be the field constructed in Proposition 4.5, i.e., via elliptic cuspidalisation. By the following group-theoretically algorithm, from the topological group Π_X and the Archimedean valuation $\overline{v}$, we can construct an isomorphism $\overline{\mathcal{A}^{\mathbb{X}}} \xrightarrow{\sim} k_{\overline{v}}$ of topological fields in a functorial manner with respect to open injective homomorphisms of profinite groups which are compatible with the respective choices of Archimedean valuations:*

(Step 1) *As in Proposition 4.8, we reconstruct NF-points of $X_{\overline{v}}$, non-constant NF-rational functions on $X_{\overline{k}}$, the evaluation map $f \mapsto f(x)$ at NF-point x, and the order function ord_x at NF-point x. We also reconstruct $\mathbb{E}^{\mathrm{top}}$ and the local additive structures on it in Proposition 4.5.*

(Step 2) The local additive structures of $\mathbb{E}^{\text{top}}$ determines the local additive structures of $\mathbb{X}^{\text{top}}$. Let x be an NF-point of $X_{\overline{v}}(k_{\overline{v}})$, $\vec{v}$ an element of a sufficiently small neighbourhood $U_{\mathbb{X}} \subset \mathbb{X}^{\text{top}}$ of x in $\mathbb{X}^{\text{top}}$ which admits such a local additive structure. For each NF-rational function f which vanishes at x, the assignment $(\vec{v}, f) \mapsto \lim_{n\to\infty} nf\,(n \cdot_x \vec{v}) \in k_{\overline{v}}$, where " $\cdot_x$ " is the operation induced by the local additive structure at x, depends only on the image $df|_x \in \omega_x$ of f in the Zariski cotangent space ω_x to $X_{\overline{v}}$. It determines an embedding $U_{\mathbb{X}} \hookrightarrow \mathrm{Hom}_{k_{\overline{v}}}(\omega_x, k_{\overline{v}})$ of topological spaces, which is compatible with the local additive structures.

(Step 3) Varying the neighbourhood $U_{\mathbb{X}}$ of x, the embeddings in (Step 2) give us an isomorphism $\overline{\mathcal{A}_x} \xrightarrow{\sim} k_{\overline{v}}$ of topological fields by the compatibility with the natural actions of $\mathcal{A}_x$, $k_{\overline{v}}^{\times}$ respectively. As x varies, the isomorphisms in (Step 3) are compatible with the isomorphisms $\overline{\mathcal{A}_x} \xrightarrow{\sim} \overline{\mathcal{A}_y}$ in Proposition 4.5. This gives us the desired isomorphism $\overline{\mathcal{A}^{\mathbb{X}}} \xrightarrow{\sim} k_{\overline{v}}$.

Remark 4.9.1. An importance of Proposition 4.5 lies in the fact that the algorithm starts in a *purely local situation*, since we will treat local objects (i.e., objects over local fields) which *a priori* do not come from a global object (i.e., an object over a number field) in inter-universal Teichmüller theory. cf. also Remark 3.17.4.

Proof. The assertions immediately follow from the described algorithms. □

§ 5. Log-volumes and Log-shells.

In this section, we construct a kind of "rigid containers" called log-shells both for non-Archimedean and Archimedean local fields. We also reconstruct the local log-volume functions. By putting them together, we reconstruct the degree functions of arithmetic line bundles.

§ 5.1. Non-Archimedean Places.

Let k be a finite extension of $\mathbb{Q}_p$, and $\overline{k}$ an algebraic closure of k. Let X be a hyperbolic orbicurve over k of strictly Belyi type. Write $k^{\sim} := (O_{\overline{k}}^{\times})^{\text{pf}}\,(\twoheadleftarrow O_{\overline{k}}^{\times})$ the perfection of $O_{\overline{k}}^{\times}$ (cf. Section 0.2). The p-adic logarithm $\log_{\overline{k}}$ induces an isomorphism

$$\log_{\overline{k}} : k^{\sim} \xrightarrow{\sim} \overline{k}$$

of topological monoids, which is compatible with the actions of Π_X. We equip $k^{\sim}$ with the topological field structure by transporting it from $\overline{k}$ via the above isomorphism $\log_{\overline{k}}$. Then we have the following diagram, which is called a **log-link**:

$$\text{(Log-link (non-Arch))} \qquad O_{\overline{k}}^{\rhd} \supset O_{\overline{k}}^{\times} \twoheadrightarrow k^{\sim} = \overline{\left(O_{k^{\sim}}^{\rhd}\right)^{\text{gp}}} := \left(O_{k^{\sim}}^{\rhd}\right)^{\text{gp}} \cup \{0\} \leftarrow O_{k^{\sim}}^{\rhd},$$

which is compatible with the action of Π_X (this will mean that Π_X is *vertically core*. cf. Proposition 12.2 (1), Remark 12.3.1, and Theorem 12.5 (1)). Note that we can construct the sub-diagram $O^{\triangleright}_{\overline{k}} \supset O^{\times}_{\overline{k}} \to k^{\sim}$, which is compatible with the action of G_k, only from the topological monoid $O^{\triangleright}_{\overline{k}}$ (i.e., only from the mono-analytic structure); however, we need the topological field $\overline{k}$ (i.e., need the arithmetically holomorphic structure) to equip $k^{\sim}$ a topological field structure and to construct the remaining diagram $k^{\sim} = (O^{\triangleright}_{k^{\sim}})^{\mathrm{gp}} \leftarrow O^{\triangleright}_{k^{\sim}}$.

Definition 5.1. We put

$$\left(O^{\Pi_X}_{k^{\sim}} \subset\right) \mathcal{I}_k := \frac{1}{2p}\mathcal{I}^*_k \ \left(\subset (k^{\sim})^{\Pi_X}\right), \qquad \text{where } \mathcal{I}^*_k := \mathrm{Im}\left\{O^{\times}_k \to \left(O^{\times}_{\overline{k}}\right)^{\mathrm{pf}} = k^{\sim}\right\}$$

where we write $(-)^{\Pi_X}$ for the fixed part of the action of Π_X, and we shall refer to $\mathcal{I}_k$ as a **Frobenius-like holomorphic log-shell**.

On the other hand, from Π_X, we can group-theoretically reconstruct an isomorph $\overline{k}(\Pi_X)$ of the ind-topological field $\overline{k}$ by Theorem 3.19, and we can construct a log-shell $\mathcal{I}(\Pi_X)$ by using $\overline{k}(\Pi_X)$, instead of $\overline{k}$. Then we shall refer to $\mathcal{I}(\Pi_X)$ as the **étale-like holomorphic log-shell for $\boldsymbol{\Pi_X}$**. By the cyclotomic rigidity isomorphism (Cyc. Rig. LCFT2), the Kummer homomorphism gives us a **Kummer isomorphism**

$$(\Pi_X \curvearrowright \overline{k}^{\times}) \xrightarrow{\sim} (\Pi_X \curvearrowright \overline{k}^{\times}(\Pi_X)) \ (\subset \varinjlim_{U} H^1(\Pi_U), \mu_{\widehat{\mathbb{Z}}}(\Pi_X))$$

for $\overline{k}^{\times}(\Pi_X)$ (cf. (Step 4) of Theorem 3.17, and Remark 3.19.2), hence obtain a **Kummer isomorphism**

$$\text{(Kum (non-Arch))} \qquad \mathcal{I}_k \xrightarrow{\sim} \mathcal{I}(\Pi_X)$$

for $\mathcal{I}_k$. In inter-universal Teichmüller theory, we will also use the Kummer isomorphism of log-shells via the cyclotomic rigidity of mono-theta environments in Theorem 7.23 (1) cf. Proposition 12.2.

Note that we have important natural inclusions
(Upper Semi-Compat. (non-Arch))

$$O^{\times}_k,\ \log_{\overline{k}}(O^{\times}_k) \ \subset \mathcal{I}_k \ \text{ and } \ O^{\times}_k(\Pi_X),\ \log_{\overline{k}(\Pi_X)}(O^{\times}_k(\Pi_X)) \ \subset \mathcal{I}(\Pi_X),$$

which will be used for **the upper semi-compatibility** of $\mathfrak{log}$-Kummer correspondence (cf. Proposition 13.7 (2)). Here, we write $O^{\times}_k(\Pi_X) := O_k(\Pi_X)^{\times}$, $O_k(\Pi_X) := O_{\overline{k}}(\Pi)^{\Pi_X}$, and $O_{\overline{k}}(\Pi_X)$ is the ring of integers of the ind-topological field $\overline{k}(\Pi)$.

Proposition 5.2. (Mono-analytic Reconstruction of Log-shell and Local Log-volume in non-Archimedean Places, [AbsTopIII, Proposition 5.8 (i), (ii), (iii)]) *Let G* *be a topological group, which is isomorphic to G_k. By the following algorithm, from G, we can group-theoretically reconstruct the log-shell "$\mathcal{I}_k$" and the (non-normalised) local log-volume function "$\mu_k^{\log}$" (cf. Section 1.3) in a functorial manner with respect to open homomorphisms of topological groups:*

(Step 1) *We reconstruct p, $f(k)$, $e(k)$, $\overline{k}^{\times}$, $O_{\overline{k}}^{\rhd}$, and $O_{\overline{k}}^{\times}$ by Proposition 2.1 (1), (3b), (3c), (2a), (2c), and (2b) respectively. To indicate that these are reconstructed from G, we write p_G, f_G, e_G, $\overline{k}^{\times}(G)$, $O_{\overline{k}}^{\rhd}(G)$ and $O_{\overline{k}}^{\times}(G)$ for them respectively (From now on, we use the notation $(-)(G)$ in this sense). Let $p_G^{m_G}$ be the number of elements of $\overline{k}^{\times}(G)^G$ of p_G-power orders, where we write $(-)^G$ for the fixed part of the action of G.*

(Step 2) *We reconstruct the* **log-shell** *"$\mathcal{I}_k$" as $\mathcal{I}(G) := \frac{1}{2p_G}\mathrm{Im}\left\{O_{\overline{k}}^{\times}(G)^G \to k^{\sim}(G) := O_{\overline{k}}^{\times}(G)^{\mathrm{pf}}\right\}$. Note that, by the canonical injection $\mathbb{Q} \hookrightarrow \mathrm{End}(k^{\sim}(G))$ (Here,* End *means the endomorphisms as (additive) topological groups), the multiplication by $\frac{1}{2p_G}$ canonically makes sense. We shall refer to $\mathcal{I}(G)$ as the* **étale-like mono-analytic log-shell**.

(Step 3) *Write $\mathbb{R}_{\mathrm{non}}(G) := (\overline{k}^{\times}(G)/O_{\overline{k}}^{\times}(G))^{\wedge}$, where we write $(-)^{\wedge}$ for the completion with respect to the order structure determined by the image of $O_{\overline{k}}^{\rhd}(G)/O_{\overline{k}}^{\times}(G)$. By the canonical isomorphism $\mathbb{R} \cong \mathrm{End}(\mathbb{R}_{\mathrm{non}}(G))$, we consider $\mathbb{R}_{\mathrm{non}}(G)$ as an $\mathbb{R}$-module. It is also equipped with a distinguished element, i.e., the image $\mathbb{F}(G) \in \mathbb{R}_{\mathrm{non}}(G)$ of the Frobenius element (constructed in Proposition 2.1 (5)) of $O_{\overline{k}}^{\rhd}(G)^G/O_{\overline{k}}^{\times}(G)^G$ via the composite $O_{\overline{k}}^{\rhd}(G)^G/O_{\overline{k}}^{\times}(G)^G \subset O_{\overline{k}}^{\rhd}(G)/O_{\overline{k}}^{\times}(G) \subset \mathbb{R}_{\mathrm{non}}(G)$. By sending $f_G \log p_G \in \mathbb{R}$ to $\mathbb{F}(G) \in \mathbb{R}_{\mathrm{non}}(G)$, we have an isomorphism $\mathbb{R} \xrightarrow{\sim} \mathbb{R}_{\mathrm{non}}(G)$ of $\mathbb{R}$-modules. By transporting the topological field structure from $\mathbb{R}$ to $\mathbb{R}_{\mathrm{non}}(G)$ via this bijection, we consider $\mathbb{R}_{\mathrm{non}}(G)$ as a topological field, which is isomorphic to $\mathbb{R}$.*

(Step 4) *We write $\mathbb{M}(k^{\sim}(G)^G)$ for the set of compact open non-empty subsets of the topological additive group $k^{\sim}(G)^G$. We can reconstruct the* **local log-volume function** *$\mu^{\log}(G) : \mathbb{M}(k^{\sim}(G)^G) \to \mathbb{R}_{\mathrm{non}}(G)$ by using the following characterisation properties:*

(a) *(additivity) For $A, B \in \mathbb{M}(k^{\sim}(G)^G)$ with $A \cap B = \emptyset$, we have $\exp(\mu^{\log}(G)(A \cup B)) = \exp(\mu^{\log}(G)(A)) + \exp(\mu^{\log}(G)(B))$, where we use the topological field structure of $\mathbb{R}_{\mathrm{non}}(G)$ to define $\exp(-)$,*

(b) *(+-translation invariance) For $A \in \mathbb{M}(k^{\sim}(G)^G)$ and $a \in k^{\sim}(G)^G$, we have $\mu^{\log}(G)(A + a) = \mu^{\log}(G)(A)$,*

(c) *(normalisation)*

$$\mu^{\log}(G)(\mathcal{I}(G)) = \left(-1 - \frac{m_G}{f_G} + \epsilon_G e_G f_G\right)\mathbb{F}(G),$$

where we write ϵ_G to be 1 if $p_G \neq 2$, and to be 2 if $p_G = 2$.

Moreover, if a field structure on $k := k^{\sim}(G)^G$ *is given, then we have the p-adic logarithm* $\log_k : O_k^{\times} \to k$ *on* k *(where we can see* k *both on the domain and the codomain), and we have*

$$\mu^{\log}(G)(A) = \mu^{\log}(G)(\log_k(A)) \tag{5.1}$$

for an open subset $A \subset O_k^{\times}$ *such that* $\log_k$ *induces a bijection* $A \xrightarrow{\sim} \log_k(A)$.

Remark 5.2.1. Note that, we cannot normalise $\mu^{\log}(G)$ by "$\mu^{\log}(G)(O_{k^{\sim}}^{G}) = 0$", since "$O_{k^{\sim}}^{G}$" needs arithmetically holomorphic structure to reconstruct (cf. [$\mathbb{Q}_p$GC]).

Remark 5.2.2. The formula (5.1) will be used for **the compatibility of $\mathfrak{log}$-links with log-volume functions** (cf. Proposition 13.10 (4)).

Proof. To lighten the notation, write $p := p_G$, $e := e_G$, $f := f_G$, $m := m_G$, $\epsilon := \epsilon_G$. Then we have $\mu_k^{\log}(\mathcal{I}_k) = \epsilon e f \log p + \mu_k^{\log}(\log(O_k^{\times})) = (\epsilon e f - m)\log p - \log(p^f - 1) + \mu_k^{\log}(O_k^{\times}) = (\epsilon e f - m)\log p - \log(p^f - 1) + \log\left(1 - \frac{1}{p^f}\right) + \mu_k^{\log}(O_k) = (\epsilon e f - m - f)\log p = \left(-1 + \epsilon e - \frac{m}{f}\right) f \log p$. □

§ 5.2. Archimedean Places.

Let k be a CAF (cf. Section 0.2). Let $\mathbb{X}$ be an elliptically admissible Aut-holomorphic orbispace, and $\kappa_k : k \xrightarrow{\sim} \overline{\mathcal{A}^{\mathbb{X}}}$ a Kummer structure. Note that k (resp. $k^{\times}$, $O_k^{\times}$) and $\overline{\mathcal{A}^{\mathbb{X}}}$ have natural Aut-holomorphic structures, and κ_k determines co-holomorphicisations between k (resp. $k^{\times}$, $O_k^{\times}$) and $\overline{\mathcal{A}^{\mathbb{X}}}$. Let $k^{\sim} \twoheadrightarrow k^{\times}$ be the universal covering of $k^{\times}$, which is uniquely determined up to unique isomorphism, as a pointed topological space (It is well-known that it can be explicitly constructed by the homotopy classes of paths on $k^{\times}$). The topological group structure of $k^{\times}$ induces a natural topological group structure of $k^{\sim}$. The inverse (i.e., the Archimedean logarithm) of the exponential map $k \twoheadrightarrow k^{\times}$ induces an isomorphism

$$\log_k : k^{\sim} \xrightarrow{\sim} k$$

of topological groups. We equip $k^{\sim}$ (resp. $O_{k^{\sim}}^{\triangleright}$) with the topological field structure (resp. the topological multiplicative monoid structure) by transporting it from k via the above isomorphism $\log_k$. Then κ_k determines a Kummer structure $\kappa_{k^{\sim}} : k^{\sim} \xrightarrow{\sim} \overline{\mathcal{A}^{\mathbb{X}}}$ (resp. $\kappa_{O_{k^{\sim}}} : O_{k^{\sim}} \hookrightarrow \overline{\mathcal{A}^{\mathbb{X}}}$) which is uniquely characterised by the property that the co-holomorphicisation determined by $\kappa_{k^{\sim}}$ (resp. $\kappa_{O_{k^{\sim}}}$) coincides with the co-holomorphicisation determined by the composite of $k^{\sim} \xrightarrow{\sim} k$ and the co-holomorphicisation determined by κ_k. By definition, the co-holomorphicisations determined by κ_k, and $\kappa_{k^{\sim}}$ (resp. $\kappa_{O_{k^{\sim}}}$) are compatible with $\log_k$ (This compatibility is an Archimedean analogue of the compatibility of the actions of Π_X in the non-Archimedean situation). We have the following diagram, which is called a **$\mathfrak{log}$-link**:

(Log-link (Arch)) $$O_k^{\triangleright} \subset k^{\times} \twoheadleftarrow k^{\sim} = \left(O_{k^{\sim}}^{\triangleright}\right)^{\underline{\mathrm{gp}}} := \left(O_{k^{\sim}}^{\triangleright}\right)^{\mathrm{gp}} \cup \{0\} \leftarrow O_{k^{\sim}}^{\triangleright},$$

which is compatible with the co-holomorphicisations determined by the Kummer structures (This will mean $\mathbb{X}$ is *vertically core*. cf. Proposition 12.2 (1)). Note that we can construct the sub-diagram $O_k^{\triangleright} \subset k^{\times} \twoheadleftarrow k^{\sim}$ only from the topological monoid $O_k^{\triangleright}$ (i.e., only from the mono-analytic structure); however, we need the topological field k (i.e., need the arithmetically holomorphic structure) to equip $k^{\sim}$ a topological field structure and to construct the remaining diagram $k^{\sim} = (O_{k^{\sim}}^{\triangleright})^{\underline{\mathrm{gp}}} \leftarrow O_{k^{\sim}}^{\triangleright}$.

Definition 5.3. We put

$$\left(O_{k^{\sim}} = \frac{1}{\pi}\mathcal{I}_k \subset\right) \mathcal{I}_k := O_{k^{\sim}}^{\times} \cdot \mathcal{I}_k^{*} \left(\subset k^{\sim}\right),$$

where $\mathcal{I}_k^{*}$ is the the uniquely determined "line segment" (i.e., closure of a connected pre-compact open subset of a one-parameter subgroup) of $k^{\sim}$ which is preserved by multiplication by ± 1 and whose endpoints differ by a generator of $\ker(k^{\sim} \twoheadrightarrow k^{\times})$ (i.e., $\mathcal{I}_k^{*}$ is the interval between "$-\pi i$" and "πi", and $\mathcal{I}_k$ is the closed disk with redius π). Here, a pre-compact subset means a subset contained in a compact subset, and see Section 0.2 for π. We shall refer to $\mathcal{I}_k$ as a **Frobenius-like holomorphic log-shell**.

On the other hand, from $\mathbb{X}$, we can group-theoretically reconstruct an isomorph $k(\mathbb{X}) := \overline{\mathcal{A}^{\mathbb{X}}}$ of the field k by Proposition 4.5, and we can construct a log-shell $\mathcal{I}(\mathbb{X})$ by using $k(\mathbb{X})$, instead of k. Then we shall refer to $\mathcal{I}(\mathbb{X})$ as the **étale-like holomorphic log-shell for** $\mathbb{X}$. The Kummer structure κ_k gives us a **Kummer isomorphism**

(Kum (Arch)) $$\mathcal{I}_k \xrightarrow{\sim} \mathcal{I}(\mathbb{X})$$

for $\mathcal{I}_k$.

Note that we have important natural inclusions
(Upper Semi-Compat. (Arch))
$$O_{k^{\sim}}^{\triangleright} \subset \mathcal{I}_k,\ O_k^{\times} \subset \exp_k(\mathcal{I}_k) \ \text{ and } \ O_{k^{\sim}}^{\triangleright}(\mathbb{X}) \subset \mathcal{I}(\mathbb{X}),\ O_k^{\times}(\mathbb{X}) \subset \exp_{k(\mathbb{X})}(\mathcal{I}(\mathbb{X}))$$

which will be used for **the upper semi-compatibility** of $\mathfrak{log}$-Kummer correspondence (cf. Proposition 13.7 (2)). Here, we write $O_k^{\times}(\mathbb{X}) := O_k(\mathbb{X})^{\times}$, and $O_k(\mathbb{X})$ (cf. also Section 0.2) is the subset of elements of absolute value ≤ 1 for the topological field $k(\mathbb{X})$ (or, if we do not want to use absolute value, the topological closure of the subset of elements x with $\lim_{n\to\infty} x^n = 0$), and $\exp_k$ (resp. $\exp_{k(\mathbb{X})}$) is the exponential function for the topological field k (resp. $k(\Pi)$).

Note also that we use $O^{\times}_{k^{\sim}}$ to define $\mathcal{I}_k$ in the above, and we need the topological field structure of $\overline{k}$ to construct $O^{\times}_{k^{\sim}}$; however, we can construct $\mathcal{I}_k$ as the closure of the union of the images of $\mathcal{I}^*_k$ via the finite order automorphisms of the topological (additive) group $k^{\sim}$, thus, we need only the topological (multiplicative) group structure of $\overline{k}^{\times}$ (*not* the topological field structure of $\overline{k}$) to construct $\mathcal{I}_k$.

Proposition 5.4. (Mono-analytic Reconstruction of Log-shell and Local Log-volumes in Archimedean Places, [AbsTopIII, Proposition 5.8 (iv), (v), (vi)]) *Let* $G = (C, \overrightarrow{C})$ *be a split monoid. By the following algorithm, from* G*, we can group-theoretically reconstruct the log-shell "*$\mathcal{I}_{\mathbb{C}}$*", the (non-normalised) local radial log-volume function "*$\mu^{\log}_{\mathbb{C}}$*" and the (non-normalised) local angular log-volume function "*$\check{\mu}^{\log}_{\mathbb{C}}$*" in a functorial manner with respect to morphisms of split monoids (In fact, the constructions do not depend on* $\overrightarrow{C}$*, which is "non-rigid" portion. cf. also [AbsTopIII, Remark 5.8.1]):*

(Step 1) Let $C^{\sim} \twoheadrightarrow C^{\times}$ *be the (pointed) unversal covering of* $C^{\times}$*. The topological group structure of* $C^{\times}$ *induces a natural topological group structure on* $C^{\sim}$*. We regard* $C^{\sim}$ *as a topological group (Note that* $C^{\times}$ *and* $C^{\sim}$ *are isomorphic to* $\mathbb{S}^1$ *and the additive group* $\mathbb{R}$ *respectively). Write*

$$k^{\sim}(G) := C^{\sim} \times C^{\sim}, \quad k^{\times}(G) := C^{\times} \times C^{\sim}.$$

(Step 2) Let $\mathrm{Seg}(G)$ *be the equivalence classes of compact line segments on* $C^{\sim}$*, i.e., compact subsets which are either equal to the closure of a connected open set or are sets of one element, relative to the equivalence relation determined by translation on* $C^{\sim}$*. Forming the union of two compact line segments whose intersection is a set of one element determines a monoid structure on* $\mathrm{Seg}(G)$ *with respect to which* $\mathrm{Seg}(G) \cong \mathbb{R}_{\geq 0}$ *(non-canonical isomorphism). Thus, this monoid structure determines a topological monoid structure on* $\mathrm{Seg}(G)$ *(Note that the topological monoid structure on* $\mathrm{Seg}(G)$ *is independent of the choice of an isomorphism* $\mathrm{Seg}(G) \cong \mathbb{R}_{\geq 0}$*).*

(Step 3) We have a natural homomorphism $k^{\sim}(G) = C^{\sim} \times C^{\sim} \twoheadrightarrow k^{\times}(G) = C^{\times} \times C^{\sim}$ *of two dimensional Lie groups, where we equip* $C^{\sim}, C^{\times}$ *with the differentiable structure by choosing isomorphisms* $C^{\sim} \cong \mathbb{R}$*,* $C^{\times} \cong \mathbb{R}^{\times}$ *(the differentiable structures do not depend on the choices of isomorphisms). We reconstruct the* **log-shell** *"*$\mathcal{I}_{\mathbb{C}}$*" as*

$$\mathcal{I}(G) := \left\{(ax, bx) \mid x \in \mathcal{I}^*_{C^{\sim}};\ a, b \in \mathbb{R};\ a^2 + b^2 = 1\right\} \subset k^{\sim}(G),$$

where we write $\mathcal{I}^*_{C^{\sim}} \subset C^{\sim}$ *for the unique compact line segment on* $C^{\sim}$ *which is invariant with respect to the action of* $\{\pm 1\}$*, and maps bijectively, except for its endpoints, to* $C^{\times}$*. Note that, by the canonical isomorphism* $\mathbb{R} \cong \mathrm{End}(C^{\sim})$ *(Here,* End *means the endomorphisms as (additive) topological groups),* ax *for* $a \in \mathbb{R}$ *and*

$x \in I^*_{C^\sim}$ canonically makes sense. We shall refer to $\mathcal{I}(G)$ as the **étale-like mono-analytic log-shell**.

(Step 4) We write $\mathbb{R}_{\mathrm{arc}}(G) := \mathrm{Seg}(G)^{\mathrm{gp}}$ (Note that $\mathbb{R}_{\mathrm{arc}}(G) \cong \mathbb{R}$ as (additive) topological groups). By the canonical isomorphism $\mathbb{R} \cong \mathrm{End}(\mathbb{R}_{\mathrm{arc}}(G))$, we consider $\mathbb{R}_{\mathrm{arc}}(G)$ as an $\mathbb{R}$-module. It is also equipped with a distinguished element, i.e., (Archimedean) Frobenius element $\mathbb{F}(G) \in \mathrm{Seg}(G) \subset \mathbb{R}_{\mathrm{arc}}(G)$ determined by $\mathcal{I}^*_{C^\sim}$. By sending $2\pi \in \mathbb{R}$ to $\mathbb{F}(G) \in \mathbb{R}_{\mathrm{arc}}(G)$, we have an isomorphism $\mathbb{R} \xrightarrow{\sim} \mathbb{R}_{\mathrm{arc}}(G)$ of $\mathbb{R}$-modules. By transporting the topological field structure from $\mathbb{R}$ to $\mathbb{R}_{\mathrm{arc}}(G)$ via this bijection, we consider $\mathbb{R}_{\mathrm{arc}}(G)$ as a topological field, which is isomorphic to $\mathbb{R}$.

(Step 5) By the same way as $\mathcal{I}(G)$, we put

$$O^\times_{k^\sim}(G) := \left\{(ax, bx) \mid x \in \partial\mathcal{I}^*_{C^\sim};\ a, b \in \mathbb{R};\ a^2 + b^2 = \pi^{-2}\right\} \subset k^\sim(G),$$

where $\partial\mathcal{I}^*_{C^\sim}$ is the set of endpoints of the line segment $\mathcal{I}^*_{C^\sim}$ (i.e., the points whose complement are connected. cf. Proposition 4.5). Then we have a natural isomorphism $\mathbb{R}_{>0} \times O^\times_{k^\sim}(G) \sim k^\sim(G) \setminus \{(0,0)\}$, where (a, x) is sent to ax (Note that ax makes sense by the canonical isomorphism $\mathbb{R} \cong \mathrm{End}(C^\sim)$ as before). We write $\mathrm{pr}_{\mathrm{rad}} : k^\sim(G) \setminus \{(0,0)\} \twoheadrightarrow \mathbb{R}_{>0}$, $\mathrm{pr}_{\mathrm{ang}} : k^\sim(G) \setminus \{(0,0)\} O^\times_{k^\sim}(G)$ for the first and second projection via the above isomorphism. We extend the map $\mathrm{pr}_{\mathrm{rad}} : k^\sim(G) \setminus \{(0,0)\} \twoheadrightarrow \mathbb{R}_{>0}$ to a map $\mathrm{pr}_{\mathrm{rad}} : k^\sim(G) \to \mathbb{R}$.

(Step 6) Let $\mathbb{M}(k^\sim(G))$ be the set of non-empty compact subsets $A \subset k^\sim(G)$ such that A projects to a (compact) subset $\mathrm{pr}_{\mathrm{rad}}(A)$ of $\mathbb{R}$ which is the closure of its interior in $\mathbb{R}$. For any $A \in \mathbb{M}(k^\sim(G))$, by taking the length $\mu(G)(A)$ of $\mathrm{pr}_{\mathrm{rad}}(A) \subset \mathbb{R}$ with respect to the usual Lebesgues measure on $\mathbb{R}$. By taking the logarithm $\mu^{\log}(G)(A) := \log(\mu(G)(A)) \in \mathbb{R} \cong \mathbb{R}_{\mathrm{arc}}(G)$, where we use the canonical identification $\mathbb{R} \cong \mathbb{R}_{\mathrm{arc}}(G)$, we reconstruct the desired **local radial log-volume function** *$\mu^{\log}(G) : \mathbb{M}(k^\sim(G)) \to \mathbb{R}_{\mathrm{arc}}(G)$. This also satisfies*

$$\mu^{\log}(G)(\mathcal{I}(G)) = \frac{\log \pi}{2\pi}\mathbb{F}(G)$$

by definition.

(Step 7) We write $\breve{\mathbb{M}}(k^\sim(G))$ for the set of non-empty compact subsets $A \subset k^\sim(G) \setminus \{(0,0)\}$ such that A projects to a (compact) subset $\mathrm{pr}_{\mathrm{ang}}(A)$ of $O^\times_{k^\sim}(G)$ which is the closure of its interior in $O^\times_{k^\sim}(G)$. We reconstruct the **local angular log-volume function** *$\breve{\mu}^{\log}(G) : \breve{\mathbb{M}}(k^\sim(G)) \to \mathbb{R}_{\mathrm{arc}}(G)$ by taking the integration $\breve{\mu}(G)(A)$ of $\mathrm{pr}_{\mathrm{ang}}(A) \subset O^\times_{k^\sim}(G)$ on $O^\times_{k^\sim}(G)$ with respect to the differentiable structure induced by the one in (Step 1), taking the logarithm $\breve{\mu}^{\log}(G)(A) := \log(\breve{\mu}(G)(A)) \in \mathbb{R} \cong \mathbb{R}_{\mathrm{arc}}(G)$, where*

we use the canonical identification $\mathbb{R} \cong \mathbb{R}_{\mathrm{arc}}(G)$, *and the normalisation*

$$\breve{\mu}^{\log}(G)(O^{\times}_{k^{\sim}}(G)) = \frac{\log 2\pi}{2\pi}\mathbb{F}(G).$$

Moreover, if a field structure on $k := k^{\sim}(G)$ *is given, then we have the exponential map* $\exp_k : k \to k^{\times}$ *on* k *(where we can see* k *both on the domain and the codomain), and we have*

$$\mu^{\log}(G)(A) = \breve{\mu}^{\log}(G)(\exp_k(A)) \tag{5.2}$$

for a non-empty compact subset $A \subset k$ *with* $\exp_k(A) \subset O^{\times}_k$, *such that* $\mathrm{pr}_{\mathrm{rad}}$ *and* $\exp_k$ *induce bijections* $A \overset{\sim}{\to} \mathrm{pr}_{\mathrm{rad}}(A)$, *and* $A \overset{\sim}{\to} \exp_k(A)$ *respectively.*

Remark 5.4.1. The formula (5.2) will be used for **the compatibility of $\mathfrak{log}$-links with log-volume functions** (cf. Proposition 13.10 (4)).

Proof. Proposition immediately follows from the described algorithms. □

§ 6. Preliminaries on Tempered Fundamental Groups.

In this section, we collect some prelimiraries on tempered fundamental groups, and we show a theorem on "profinite conjugates vs. tempered conjugates", which plays an important role in inter-universal Teichmüller theory.

§ 6.1. Some Definitions.

From this section, we use André's theory of tempered fundamental groups ([A1]) for rigid-analytic spaces (in the sense of Berkovich) over non-Archimedean fields. We give a short review on it here. He introduced the tempered fundamental groups to obtain a fundamental group of "reasonable size" for rigid analytic spaces: On one hand, the topological fundamental groups π_1^{top} for rigid analytic spaces are too small (e.g., $\pi_1^{\mathrm{top}}(\mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, 1, \infty\}, x) = \{1\}$. If X is a proper curve with good reduction, then $\pi_1^{\mathrm{top}}(X^{\mathrm{an}}, x) = \{1\}$). On the other hand, the étale fundamental groups $\pi_1^{\text{ét}}$ for rigid analytic spaces aree too big (e.g., By the Gross-Hopkins period mappings ([GH1], [GH2]), we have a surjection $\pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, x) \twoheadrightarrow \mathrm{SL}_2(\mathbb{Q}_p)$. cf. also [A2, II.6.3.3, and Remark after III Corollary 1.4.7]). André's tempered fundamental group π_1^{temp} is of reasonable size, and it comparatively behaves well at least for curves. An étale covering $Y \twoheadrightarrow X$ of rigid analytic spaces is called **tempered covering** if there exists a commutative diagram

$$\begin{tikzcd} Z \arrow[r, two heads] \arrow[d, two heads] & T \arrow[d, two heads] \\ Y \arrow[r, two heads] & X \end{tikzcd}$$

of étale coverings, where $T \twoheadrightarrow X$ is a finite étale covering, and $Z \twoheadrightarrow T$ is a possibly inifinite topological covering. When we define a class of coverings, then we can define the fundamental group associated to the class. In this case, $\pi_1^{\mathrm{temp}}(X, x)$ classifies all tempered pointed coverings of (X, x). For example, we have $\pi_1^{\mathrm{temp}}(\mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, \infty\}) = \widehat{\mathbb{Z}}$, and for an elliptic curve E over $\mathbb{C}_p$ with j-invariant j_E, we have $\pi_1^{\mathrm{temp}}(E) \cong \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$ if $|j|_p \leq 1$, and $\pi_1^{\mathrm{temp}}(E) \cong \mathbb{Z} \times \widehat{\mathbb{Z}}$ if $|j|_p > 1$ ([A1, §4.6]). Here, $\mathbb{Z}$ corresponds to the universal covering of the graph of the special fiber. The topology of π_1^{temp} is a little bit complicated. In general, it is neither discrete, profinite, nor locally compact; however, it is pro-discrete. For a (log-)orbicurve X over an MLF, we write $\mathcal{B}^{\mathrm{temp}}(X)$ for the category of the (log-)tempered coverings over the rigid analytic space associated with X. For a (log-)orbicurve X over a field, we also write $\mathcal{B}(X)$ for the Galois category of the finite (log-)étale coverings over X.

Definition 6.1. ([SemiAnbd, Definition 3.1 (i), Definition 3.4])

(1) If a topological group Π can be written as an inverse limit of an inverse system of surjections of countable discrete topological groups, then we shall refer to Π as a **tempered group** (Note that any profinite group is a tempered group).

(2) Let Π be a tempered group. We say that Π is **temp-slim** if we have $Z_\Pi(H) = \{1\}$ for any open subgroup $H \subset \Pi$.

(3) Let $f : \Pi_1 \to \Pi_2$ be a continuous homomorphism of tempered groups. We say Π_1 is **relatively temp-slim** over Π_2 (via f) if we have $Z_{\Pi_2}(\mathrm{Im}\{H \to \Pi_2\}) = \{1\}$ for any open subgroup $H \subset \Pi_1$.

(4) ([IUTchI, §0]) For a topological group Π, we write $\mathcal{B}^{\mathrm{temp}}(\Pi)$ (resp. $\mathcal{B}(\Pi)$) for the category whose objects are countable discrete sets (resp. finite sets) with a continuous Π-action, and whose morphisms are morphisms of Π-sets. A category $\mathcal{C}$ is called a **connected temperoid**, (resp. a **connected anabelioid**) if $\mathcal{C}$ is equivalent to $\mathcal{B}^{\mathrm{temp}}(\Pi)$ (resp. $\mathcal{B}(\Pi)$) for a tempered group Π (resp. a profinite group Π). Note that, if $\mathcal{C}$ is a connected temperoid (resp. a connected anabelioid), then $\mathcal{C}$ is naturally equivalent to $(\mathcal{C}^0)^\top$ (resp. $(\mathcal{C}^0)^\perp$) (cf. Section 0.2 for $(-)^0$, $(-)^\top$ and $(-)^\perp$). If a category $\mathcal{C}$ is equivalent to $\mathcal{B}^{\mathrm{temp}}(\Pi)$ (resp. $\mathcal{B}(\Pi)$) for a tempered group Π with countable basis (resp. a profinite group Π), then we can reconstruct the topological group Π, up to inner automorphism, by the same way as Galois category (resp. by the theory of Galois category). (Note that in the anabelioid/profinite case, we have no need of condition like "having countable basis", since "compact set arguments" are available in profinite topology.) We write $\pi_1(\mathcal{C})$ for it. We also write $\pi_1(\mathcal{C}^0) := \pi_1((\mathcal{C}^0)^\top)$ (resp. $\pi_1(\mathcal{C}^0) := \pi_1((\mathcal{C}^0)^\perp)$) for $\mathcal{C}$ a connected temperoid (resp. a connected anabelioid).

(5) For connected temperoids (resp. anabelioids) $\mathcal{C}_1$, $\mathcal{C}_2$, a **morphism $\mathcal{C}_1 \to \mathcal{C}_2$ of temperoids** (resp. a **morphism $\mathcal{C}_1 \to \mathcal{C}_2$ of anabelioids**) is an isomorphism class of functors $\mathcal{C}_2 \to \mathcal{C}_1$ which preserves finite limits and countable colimits (resp. finite colimits) (This is definition in [IUTchI, §0] is slightly different from the one in [SemiAnbd, Definition 3.1 (iii)]). We also define a morphism $\mathcal{C}_1^0 \to \mathcal{C}_2^0$ to be a morphism $(\mathcal{C}_1^0)^\top \to (\mathcal{C}_2^0)^\top$ (resp. $(\mathcal{C}_1^0)^\perp \to (\mathcal{C}_2^0)^\perp$).

Note that if Π_1, Π_2 are tempered groups with countable basis (resp. profinite groups), then there are natural bijections among

- the set of continuous outer homomorphisms $\Pi_1 \to \Pi_2$,
- the set of morphisms $\mathcal{B}^{\mathrm{temp}}(\Pi_1) \to \mathcal{B}^{\mathrm{temp}}(\Pi_2)$ (resp. $\mathcal{B}(\Pi_1) \to \mathcal{B}(\Pi_2)$), and
- the set of morphisms $\mathcal{B}^{\mathrm{temp}}(\Pi_1)^0 \to \mathcal{B}^{\mathrm{temp}}(\Pi_2)^0$ (resp. $\mathcal{B}(\Pi_1)^0 \to \mathcal{B}(\Pi_2)^0$).

(cf. also [IUTchI, Remark 2.5.3].)

Let K be a finite extension of $\mathbb{Q}_p$.

Lemma 6.2. *Let X be a hyperbolic curve over K. We write $\Delta_X^{\mathrm{temp}} \subset \Pi_X^{\mathrm{temp}}$ for the geometric tempered fundamental group $\pi_1^{\mathrm{temp}}(X, \overline{x})$ and the arithmetic tempered fundamental group $\pi_1^{\mathrm{temp}}(X, \overline{x})$ for some basepoint $\overline{x}$, respectively. Then we have a group-theoretic characterisation of the closed subgroup Δ_X^{temp} in Π_X^{temp}.*

Remark 6.2.1. By remark 2.4.1, pro-Σ version of Lemma 6.2 holds as well.

Proof. Note that the homomorphisms $\Delta_X^{\mathrm{temp}} \to \Delta_X := (\Delta_X^{\mathrm{temp}})^\wedge$ and $\Pi_X^{\mathrm{temp}} \to \Pi_X := (\Pi_X^{\mathrm{temp}})^\wedge$ to the profinite completions are injective respectively, since the homomorphism from a (discrete) free group to its profinite completion is injective (Free groups and surface groups are residually finite (cf. also Proposition C.5)). Then by using the group-theoretic characterisation of Δ_X in Π_X (Corollary 2.4), we obtain a group-theoretic characterisation of Δ_X^{temp} as $\Delta_X^{\mathrm{temp}} = \Pi_X^{\mathrm{temp}} \cap \Delta_X$. □

Let $\overline{K}$ be an algebraic closure of K. We write k and $\overline{k}$ for the residue field of K and $\overline{K}$ respectively ($\overline{k}$ is an algebraic closure of k).

Definition 6.3.

(1) Let $\overline{X}$ be a pointed stable curve over $\overline{k}$ with marked points D. Write $X := \overline{X} \setminus D$. Then we associate a **dual semi-graph** (resp. **dual graph**) $\mathbb{G}_X$ to X as follows: We set the set of the vertices of $\mathbb{G}_X$ to be the set of the irreducible components of X, the set of the closed edges of $\mathbb{G}_X$ to be the set of the nodes of X, and the

set of the open edges of $\mathbb{G}_X$ to be the set of the divisor of infinity of X (i.e., the marked points D of $\overline{X}$). To avoid confusion, we write X_v and ν_e for the irreducible component of X and the node of X corresponding to a vertex v and an closed edge e respectively. A closed edge e connects vertices v and v' (we may allow the case of $v = v'$), if and only if the node ν_e is the intersection of two branches corresponding to X_v and $X_{v'}$. An open e connects a vertex v, if and only if the marked point corresponding to e lies in X_v.

(2) (cf. [AbsAnab, Appendix]) We contitue the situation of (1). Let Σ be a set of prime numbers. A finite étale covering of curves is called of Σ-power degree if any prime number dividing the degree is in Σ. We also associate a (pro-Σ) **semi-graph** $\mathcal{G}_X(= \mathcal{G}_X^\Sigma)$ **of anabelioids** to X, such that the underlying semi-graph is $\mathbb{G}_X$ as follows: Write $X' := X \setminus \{\text{nodes}\}$. For each vertex v of $\mathbb{G}_X$, let $\mathcal{G}_v$ be the Galois category (or a connected anabelioid) of the finite étale coverings of Σ-power degree of $X'_v := X_v \times_X X'$ which are tamely ramified along the nodes and the marked points. For the branches $\nu_e(1)$ and $\nu_e(2)$ of the node ν_e corresponding to a closed edge e of $\mathbb{G}_X$, we consider the scheme-theoretic interstion $X'_{\nu_e(i)}$ of the completion along the branch $\nu_e(i)$ at the node ν_e of X' for $i = 1, 2$ (Note that $X'_{\nu_e(i)}$ is non-canonically isomorphic to $\operatorname{Spec} \overline{k}((t))$). We fix a $\overline{k}$-isomorphism $X'_{\nu_e(1)} \cong X'_{\nu_e(2)}$, we identify these, and we write X'_e for the identified object. Let $\mathcal{G}_e$ be the Galois category (or a connected anabelioid) of the finite étale coverings of Σ-power degree of X'_e which are tamely ramified along the node. For each open edge e_x corresponding to a marked point x, write X'_x to be the scheme-theoretic interstion of the completion of $\overline{X}$ at the marked point x with X' (Note that X'_x is non-canonically isomorphic to $\operatorname{Spec} \overline{k}((t))$). Let $\mathcal{G}_{e_x}$ be the Galois category (or a connected anabelioid) of the finite étale coverings of Σ-power degree of X'_x which are tamely ramified along the marked point. For each edge e connecting vertices v_1 and v_2, we have natural functors $\mathcal{G}_{v_1} \to \mathcal{G}_e$, $\mathcal{G}_{v_2} \to \mathcal{G}_e$ by the pull-backs. For an open edge e connected to a vertex v, we have a natural functor $\mathcal{G}_v \to \mathcal{G}_e$ by the pull-backs. Then the data $\mathcal{G}_X(= \mathcal{G}_X^\Sigma) := \{\mathcal{G}_v; \mathcal{G}_e; \mathcal{G}_v \to \mathcal{G}_e\}$ defines a semi-graph of anabelioids.

(3) (cf. [SemiAnbd, Definition 2.1]) For a (pro-Σ) semi-graph $\mathcal{G}(= \mathcal{G}^\Sigma) = \{\mathcal{G}_v; \mathcal{G}_e; \mathcal{G}_v \to \mathcal{G}_e\}$ of anabelioids with connected underlying semi-graph $\mathbb{G}$, we define a category $\mathcal{B}(\mathcal{G})(= \mathcal{B}(\mathcal{G}^\Sigma))$ as follows: An object of $\mathcal{B}(\mathcal{G})(= \mathcal{B}(\mathcal{G}^\Sigma))$ is data $\{S_v, \phi_e\}_{v,e}$, where v (resp. e) runs over the vertices (resp. the edges) of $\mathbb{G}$, such that S_v is an object of $\mathcal{G}_v$, and $\phi_e : e(1)^* S_{v_1} \overset{\sim}{\to} e(2)^* S_{v_2}$ is an isomorphism in $\mathcal{G}_e$, where $e(1)$ and $e(2)$ are the branches of e connecting v_1 and v_2 respectively (Here, $e(i)^* : \mathcal{G}_{v_i} \to \mathcal{G}_e$ is a given datum of $\mathcal{G}$). We define a morphism of $\mathcal{B}(\mathcal{G})$ in the evident manner. Then $\mathcal{B}(\mathcal{G})$ itself is a Galois category (or a connected anabelioid). In the case of $\mathcal{G} = \mathcal{G}_X$ in (2), the fundamental group associated to $\mathcal{B}(\mathcal{G})(= \mathcal{B}(\mathcal{G}^\Sigma))$ is called the (pro-Σ)

admissible fundamental group of X.

(4) (cf. [SemiAnbd, paragraph before Definition 3.5 and Definition 3.5]) Let $\mathcal{G}(=\mathcal{G}^{\Sigma})=\{\mathcal{G}_v;\mathcal{G}_e;\mathcal{G}_v\to\mathcal{G}_e\}$ be a (pro-Σ) semi-graph of anabelioids such that the underlying semi-graph $\mathbb{G}$ is connected and countable. We define a category $\mathcal{B}^{\mathrm{cov}}(\mathcal{G})(=\mathcal{B}^{\mathrm{cov}}(\mathcal{G}^{\Sigma}))$ as follows: An object of $\mathcal{B}^{\mathrm{cov}}(\mathcal{G})(=\mathcal{B}^{\Sigma,\mathrm{cov}}(\mathcal{G}))$ is data $\{S_v,\phi_e\}_{v,e}$, where v (resp. e) runs over the vertices (resp. the edges) of $\mathbb{G}$, such that S_v is an object of $(\mathcal{G}_v^0)^{\top}$ (cf. Section 0.2 for $(-)^0$ and $(-)^{\top}$), and $\phi_e: e(1)^*S_{v_1}\xrightarrow{\sim}e(2)^*S_{v_2}$ is an isomorphism in $(\mathcal{G}_e^0)^{\top}$, where $e(1)$ and $e(2)$ are the branches of e connecting v_1 and v_2 respectively (Here, $e(i)^*:\mathcal{G}_v\to\mathcal{G}_e$ is a given datum of $\mathcal{G}$). We define a morphism of $\mathcal{B}^{\mathrm{cov}}(\mathcal{G})$ in the evident manner. We can extend the definition of $\mathcal{B}^{\mathrm{cov}}(\mathcal{G})$ to a semi-graph of anabelioids such that the underlying semi-graph $\mathbb{G}$ is countable; however, is not connected. We have a natural full embedding $\mathcal{B}(\mathcal{G})\hookrightarrow\mathcal{B}^{\mathrm{cov}}(\mathcal{G})$. We write $(\mathcal{B}(\mathcal{G})\subset)\,\mathcal{B}^{\mathrm{temp}}(\mathcal{G})(=\mathcal{B}^{\mathrm{temp}}(\mathcal{G}^{\Sigma}))\subset\mathcal{B}^{\mathrm{cov}}(\mathcal{G})$ for the full subcategory whose objects $\{S_v,\phi_e\}_{v,e}$ are as follows: There exists an object $\{S'_v,\phi'_e\}$ of $\mathcal{B}(\mathcal{G})$ such that for any vertex or edge c, the restriction of $\{S'_v,\phi'_e\}$ to $\mathcal{G}_c$ splits the restriction of $\{S_v,\phi_e\}$ to $\mathcal{G}_c$ i.e., the fiber product of S'_v (resp. ϕ'_e) with S_v (resp. ϕ_e) over the terminal object (resp. over the identity morphism of the terminal object) in $(\mathcal{G}_v^0)^{\top}$ (resp. $(\mathcal{G}_e^0)^{\top}$) is isomorphic to the coproduct of a countable number of copies of S'_v (resp. ϕ'_e) for any vertex v and any edge e. We shall refer to $\mathcal{B}^{\mathrm{temp}}(\mathcal{G})(=\mathcal{B}^{\mathrm{temp}}(\mathcal{G}^{\Sigma}))$ as the (pro-Σ) (connected) **temperoid** associated with $\mathcal{G}(=\mathcal{G}^{\Sigma})$.

We can associate the fundamental group $\Delta_{\mathcal{G}}^{\mathrm{temp}}(=\Delta_{\mathcal{G}}^{(\Sigma),\mathrm{temp}}):=\pi_1(\mathcal{B}^{\mathrm{temp}}(\mathcal{G}))$ $(=\pi_1(\mathcal{B}^{\mathrm{temp}}(\mathcal{G}^{\Sigma})))$ of $\mathcal{B}^{\mathrm{temp}}(\mathcal{G})(=\mathcal{B}^{\mathrm{temp}}(\mathcal{G}^{\Sigma}))$ (after taking a fiber functor) by the same way as a Galois category. We write $\Delta_{\mathcal{G}}(=\Delta_{\mathcal{G}}^{(\Sigma)})$ for the profinite completion of $\Delta_{\mathcal{G}}^{(\Sigma),\mathrm{temp}}$. (Note that $\Delta_{\mathcal{G}}(=\Delta_{\mathcal{G}}^{(\Sigma)})$ is *not* the maximal pro-Σ quotient of $\pi_1(\mathcal{B}(\mathcal{G}^{\Sigma}))$ since the profinite completion of the "graph covering portion" is not pro-Σ). By definition, $\Delta_{\mathcal{G}}^{\mathrm{temp}}(=\Delta_{\mathcal{G}}^{(\Sigma),\mathrm{temp}})$ and $\Delta_{\mathcal{G}}^{(\Sigma)}$ are tempered groups (Definition 6.1 (1), cf. also [SemiAnbd, Proposition 3.1 (i)]).

Remark 6.3.1. (cf. [SemiAnbd, Example 3.10]) Let X be a smooth log-curve over $\overline{K}$. The special fiber of the stable model of X determines a semi-graph $\mathcal{G}$ of anabelioids. We can relate the tempered fundamental group $\Delta_X^{\mathrm{temp}}:=\pi_1^{\mathrm{temp}}(X)$ of X with a system of admissible fundamental groups of the special fibers of the stable models of coverings of X as follows: Let $\cdots\subset N_i\subset\cdots\subset\Delta_X^{\mathrm{temp}}$ $(i\geq 1)$ be an exhausitive sequence of open characteristic subgroups of finite index of Δ_X^{temp}. Then N_i determines a finite log-étale covering of X whose special fiber of the stable model gives us a semi-graph $\mathcal{G}_i$ of anabelioids, on which $\Delta_X^{\mathrm{temp}}/N_i$ acts faithfully. Then we obtain a natural sequence of functors $\cdots\leftarrow\mathcal{B}^{\mathrm{temp}}(\mathcal{G}_i)\leftarrow\cdots\leftarrow\mathcal{B}^{\mathrm{temp}}(\mathcal{G})$ which are compatible with the actions of $\Delta_X^{\mathrm{temp}}/N_i$. Hence this gives us a sequence of surjections of tempered groups $\Delta_X^{\mathrm{temp}}\twoheadrightarrow$

$\cdots \twoheadrightarrow \pi_1(\mathcal{B}^{\mathrm{temp}}(\mathcal{G}_i)) \overset{\mathrm{out}}{\rtimes} (\Delta_X^{\mathrm{temp}}/N_i) \twoheadrightarrow \cdots \twoheadrightarrow \pi_1(\mathcal{B}^{\mathrm{temp}}(\mathcal{G}_j)) \overset{\mathrm{out}}{\rtimes} (\Delta_X^{\mathrm{temp}}/N_j) \twoheadrightarrow \cdots \twoheadrightarrow \pi_1(\mathcal{B}^{\mathrm{temp}}(\mathcal{G}))$. Then by construction, we have

$$\Delta_X^{\mathrm{temp}} \cong \varprojlim_i \left(\Delta_{\mathcal{G}_i}^{\mathrm{temp}} \overset{\mathrm{out}}{\rtimes} (\Delta_X^{\mathrm{temp}}/N_i)\right) = \varprojlim_i \Delta_X^{\mathrm{temp}}/\ker(N_i \twoheadrightarrow \Delta_{\mathcal{G}_i}^{\mathrm{temp}}). \tag{6.1}$$

We also have

$$\Delta_X \cong \varprojlim_i \left(\Delta_{\mathcal{G}_i} \overset{\mathrm{out}}{\rtimes} (\Delta_X/\widehat{N_i})\right) = \varprojlim_i \Delta_X/\ker(\widehat{N_i} \twoheadrightarrow \Delta_{\mathcal{G}_i}), \tag{6.2}$$

where we write $\widehat{N_i}$ for the closure of N_i in Δ_X. By these expressions of Δ_X^{temp} and Δ_X in terms of $\Delta_{\mathcal{G}_i}^{\mathrm{temp}}$'s and $\Delta_{\mathcal{G}_i}$'s, we can reduce some properties of the tempered fundamental group Δ_X^{temp} of the generic fiber to some properties of the admissible fundamental groups of the special fibers (cf. Lemma 6.4 (5), and Corollary 6.10 (1)). We write $\Delta_X^{(\Sigma),\mathrm{temp}}$ for the fundamental group associated to the category of the tempered coverings dominated by coverings which arise as a graph covering of a finite étale Galois covering of X over $\overline{K}$ of Σ-power degree, and $\Delta_X^{(\Sigma)}$ its profinite completion (Note that $\Delta_X^{(\Sigma)}$ is *not* the maximal pro-Σ quotient of Δ_X^{temp} or Δ_X since the profinite completion of the "graph covering portion" is not pro-Σ). If $p \notin \Sigma$, then we have

$$\Delta_X^{(\Sigma),\mathrm{temp}} \cong \Delta_{\mathcal{G}}^{(\Sigma),\mathrm{temp}} \quad \text{and} \quad \Delta_X^{(\Sigma)} \cong \Delta_{\mathcal{G}}^{(\Sigma)},$$

since Galois coverings of Σ-power degree are necessarily admissible (cf. [Hur, §3], [SemiAnbd, Corollary 3.11]).

§ 6.2. Profinite Conjugates vs. Tempered Conjugates.

Lemma 6.4. (special case of [SemiAnbd, Proposition 2.6, Corollary 2.7 (i), (ii), Proposition 3.6 (iv)] and [SemiAnbd, Example 3.10]) *Let X be a smooth hyperbolic log-curve over K. Write $\Delta_X^{\mathrm{temp}} := \pi_1^{\mathrm{temp}}(X \times_K \overline{K})$ and $\Pi_X^{\mathrm{temp}} := \pi_1^{\mathrm{temp}}(X)$. We write $\mathcal{G}^{\mathrm{temp}}(= \mathcal{G}^{\Sigma,\mathrm{temp}})$ for the temperoid determined by the special fiber of the stable model of $X \times_K \overline{K}$ and a set Σ of prime numbers, and write $\Delta_{\mathcal{G}}^{\mathrm{temp}} := \pi_1(\mathcal{G}^{\mathrm{temp}})$ (for some base point). Let $\mathbb{H}$ be a connected sub-semi-graph containing a vertex of the underling semi-graph $\mathbb{G}$ of $\mathcal{G}^{\mathrm{temp}}$. We assume that $\mathbb{H}$ is stabilised by the natural action of G_K on $\mathbb{G}$. We write $\mathcal{H}^{\mathrm{temp}}$ for the temperoid over $\mathbb{H}$ obtained by the restriction of $\mathcal{G}^{\mathrm{temp}}$ to $\mathbb{H}$. Write $\Delta_{\mathcal{H}}^{\mathrm{temp}} := \pi_1(\mathcal{H}^{\mathrm{temp}})(\subset \Delta_{\mathcal{G}}^{\mathrm{temp}})$. We write $\Delta_{\mathcal{G}}$ and $\Delta_{\mathcal{H}}$ for the profinite completion of $\Delta_{\mathcal{G}}^{\mathrm{temp}}$ and $\Delta_{\mathcal{H}}^{\mathrm{temp}}$ respectively.*

(1) $\Delta_{\mathcal{H}} \subset \Delta_{\mathcal{G}}$ is commensurably terminal,

(2) $\Delta_{\mathcal{H}} \subset \Delta_{\mathcal{G}}$ is relatively slim (resp. $\Delta_{\mathcal{H}}^{\mathrm{temp}} \subset \Delta_{\mathcal{G}}^{\mathrm{temp}}$ is relatively temp-slim),

(3) $\Delta_{\mathcal{H}}$ and $\Delta_{\mathcal{G}}$ are slim (resp. $\Delta_{\mathcal{H}}^{\mathrm{temp}}$ and $\Delta_{\mathcal{G}}^{\mathrm{temp}}$ are temp-slim),

(4) inertia subgroups in $\Delta_{\mathcal{G}}^{\mathrm{temp}}$ of cusps are commensurably terminal, and

(5) Δ_X^{temp} and Π_X^{temp} are temp-slim.

Proof. (1) can be shown by the same manner as in Proposition 2.7 (1a) (i.e., consider coverings which are connected over $\mathbb{H}$ and totally split over a vertex outside $\mathbb{H}$). (3) for Δ: We can show that $\Delta_{\mathcal{H}}$ and $\Delta_{\mathcal{G}}$ are slim in the same way as in Proposition 2.7. (2): $\Delta_{\mathcal{H}} \subset \Delta_{\mathcal{G}}$ is relatively slim, by (1), (3) for Δ and Lemma 2.6 (2). Then the injectivity (which comes from the residual finiteness of free groups and surface groups (cf. also Proposition C.5)) of $\Delta_{\mathcal{H}}^{\mathrm{temp}} \hookrightarrow \Delta_{\mathcal{H}}$ and $\Delta_{\mathcal{G}}^{\mathrm{temp}} \hookrightarrow \Delta_{\mathcal{G}}$ implies that $\Delta_{\mathcal{H}}^{\mathrm{temp}} \subset \Delta_{\mathcal{G}}^{\mathrm{temp}}$ is relatively temp-slim. (3) for Δ^{temp}: It follows from (2) for Δ^{temp} in the same way as in Proposition 2.6 (2). (4) can also be shown by the same manner as in Proposition 2.7 (2c). (5): By the isomorphism (6.1) in Remark 6.3.1 and (3) for Δ^{temp}, it follows that Δ_X^{temp} is temp-slim (cf. [SemiAnbd, Example 3.10]). Hence Π_X^{temp} is also temp-slim by Proposition 2.7 (1c). □

Definition 6.5. Let $\mathcal{G}$ be a semi-graph of anabelioids.

(1) We shall refer to a subgroup of the form $\Delta_v := \pi_1(\mathcal{G}_v)\,(\subset \Delta_{\mathcal{G}}^{\mathrm{temp}})$ for a vertex v as a **verticial subgroup**.

(2) We shall refer to a subgroup of the form $\Delta_e := \pi_1(\mathcal{G}_e)\,(\cong \widehat{\mathbb{Z}}^{\Sigma\setminus\{p\}} := \prod_{l\in\Sigma\setminus\{p\}} \mathbb{Z}_l)(\subset \Delta_{\mathcal{G}}^{\mathrm{temp}})$ for a closed edge e as an **edge-like subgroup**.

Proposition 6.6. ([SemiAnbd, Theorem 3.7 (iv)]) *Let X be a smooth hyperbolic log-curve over $\overline{K}$. We write $\mathcal{G}^{\mathrm{temp}}(= \mathcal{G}^{\Sigma,\mathrm{temp}})$ for the temperoid determined by the special fiber of the stable model of X and a set Σ of prime numbers, and write $\Delta_{\mathcal{G}}^{\mathrm{temp}} := \pi_1(\mathcal{G}^{\mathrm{temp}})$ (for some base point). For a vertex v (resp. an edge e) of the underlying sub-semi-graph $\mathbb{G}$ of $\mathcal{G}^{\mathrm{temp}}$, we write $\Delta_v := \pi_1(\mathcal{G}_v)(\subset \Delta_{\mathcal{G}}^{\mathrm{temp}})$ (resp. $\Delta_e := \pi_1(\mathcal{G}_e)(\subset \Delta_{\mathcal{G}}^{\mathrm{temp}})$) to be the profinite group corresponding to $\mathcal{G}_v$ (resp. $\mathcal{G}_e$) (Note that we are not considering open edges here). Then we have the followng group-theoretic characterisations of Δ_v's and Δ_e's.*

(1) The maximal compact subgroups of $\Delta_{\mathcal{G}}^{\mathrm{temp}}$ are precisely the verticial subgroups of $\Delta_{\mathcal{G}}^{\mathrm{temp}}$.

(2) The nontrivial intersection of two maximal compact subgroups of $\Delta_{\mathcal{G}}^{\mathrm{temp}}$ are precisely the edge-like subgroups of $\Delta_{\mathcal{G}}^{\mathrm{temp}}$.

Remark 6.6.1. Proposition 6.6 reconstructs the dual graph (not the dual *semi*-graph) of the special fiber from the *tempered* fundamental group *without* using the action of the Galois group of the base field. In Corollary 6.12 below, we reconstruct the inertia subgroups, hence open edges as well, *using* the Galois action. However, we can reconstruct the open edges *without* Galois action, by more delicate method in [SemiAnbd, Corollary 3.11] (i.e., by constructing a covering whose fiber at a cusp under consideration contains a node).

We can also reconstruct the dual semi-graph of the special fiber from the *profinite* fundamental group by *using* the action of the Galois group of the base field (cf. [profGC]).

Proof. We write $\Delta_{\mathcal{G}}$ for the profinite completion of $\Delta_{\mathcal{G}}^{\mathrm{temp}}$. First, note that it follows that $\Delta_v \cap \Delta_{v'}$ has infinite index in Δ_v for any vertices $v \neq v'$ by the commensurable terminality of Δ_v^{temp} (Lemma 6.4 (1)). Next, we take an exhausitive sequence of open characteristic subgroups $\cdots \subset N_i \subset \cdots \subset \Delta_{\mathcal{G}}^{\mathrm{temp}}$ of finite index, and let $\mathcal{G}_i (\to \mathcal{G})$ be the covering corresponding to $N_i (\subset \Delta_{\mathcal{G}}^{\mathrm{temp}})$. We write $\mathbb{G}_i^{\infty}$ for the universal graph covering of the underlying semi-graph $\mathbb{G}_i$ of $\mathcal{G}_i$.

Let $H \subset \Delta_{\mathcal{G}}^{\mathrm{temp}}$ be a compact subgroup, then H acts continuously on $\mathbb{G}_i^{\infty}$ for each $i \in I$, thus its action factors through a finite quotient. Hence H fixes a vertex or an edge of $\mathbb{G}_i^{\infty}$ (cf. also [SemiAnbd, Lemma 1.8 (ii)]), since *an action of a finite group on a tree has a fixed point* by [Serre2, Chapter I, §6.5, Proposition 27] (Note that a graph in [Serre2] is an oriented graph; however, if we split each edge of $\mathbb{G}_i^{\infty}$ into two edges, then the argument works). Since the action of H is over $\mathbb{G}$, if H fixes an edge, then it does not change the branches of an edge. Therefore, H fixes at least one vertex. If, for some cofinal subset $J \subset I$, H fixes more than or equal to three vertices of $\mathbb{G}_j^{\infty}$ for each $j \in J$, then by considering paths connecting these vertices (cf. [Serre2, Chapter I, §2.2, Proposition 8]), it follows that there exists a vertex having (at least) two closed edges in which H fixes the vertex and the closed edges (cf. also [SemiAnbd, Lemma 1.8 (ii)]). Since each $\mathbb{G}_j$ is finite semi-graph, we can choose a compatible system of such a vertex having (at least) two closed edges on which H acts trivially. This implies that H is contained in (some conjugate in $\Delta_{\mathcal{G}}$ of) the intersection of Δ_e and $\Delta_{e'}$, where e and e' are distinct closed edges. Hence H should be trivial. By the above arguments also show that any compact subgroup in $\Delta_{\mathcal{G}}^{\mathrm{temp}}$ is contained in Δ_v for precisely one vertex v or in $\Delta_v, \Delta_{v'}$ for precisely two vertices v, v', and, in the latter case, it is contained in Δ_e for precisely one closed edge e. $\square$

Proposition 6.7. ([IUTchI, Proposition 2.1]) *Let X be a smooth hyperbolic log-curve over $\overline{K}$. We write $\mathcal{G}^{\mathrm{temp}} (= \mathcal{G}^{\Sigma,\mathrm{temp}})$ for the temperoid determined by the special fiber of the stable model of X and a set Σ of prime numbers. Write $\Delta_{\mathcal{G}}^{\mathrm{temp}} := \pi_1(\mathcal{G}^{\mathrm{temp}})$, and we write $\Delta_{\mathcal{G}}$ for the profinite completion of $\Delta_{\mathcal{G}}^{\mathrm{temp}}$ (Note that the "profinite portion" remains pro-Σ, and the "combinatorial portion" changes from discrete to profinite). Let $\Lambda \subset \Delta_{\mathcal{G}}^{\mathrm{temp}}$ be a nontrivial compact subgroup, $\gamma \in \Delta_{\mathcal{G}}$ an element such that $\gamma \Lambda \gamma^{-1} \subset \Delta_{\mathcal{G}}^{\mathrm{temp}}$. Then $\gamma \in \Delta_{\mathcal{G}}^{\mathrm{temp}}$.*

Proof. Let $\widehat{\Gamma}$ (resp. $\Gamma^{\rm temp}$) be the "profinite semi-graph" (resp. "pro-semi-graph") associated with the universal profinite étale (resp. tempered) covering of $\mathcal{G}^{\rm temp}$. Then we have a natural inclusion $\Gamma^{\rm temp} \hookrightarrow \widehat{\Gamma}$. We shall refer to a pro-vertex in $\widehat{\Gamma}$ in the image of this inclusion as a tempered vertex. Since Λ and $\gamma\Lambda\gamma^{-1}$ are compact subgroups of $\Delta_{\mathcal{G}}^{\rm temp}$, there exists vertices v, v' of $\mathbb{G}$ (here we write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}^{\rm temp}$) such that $\Lambda \subset \Delta_v^{\rm temp}$ and $\gamma\Lambda\gamma^{-1} \subset \Delta_{v'}^{\rm temp}$ by Proposition 6.6 (1) for some base points. Here, $\Delta_v^{\rm temp}$ and $\Delta_{v'}^{\rm temp}$ for this base points correspond to tempered vertices $\widetilde{v}, \widetilde{v'} \in \Gamma^{\rm temp}$. Now, $\{1\} \neq \gamma\Lambda\gamma^{-1} \subset \gamma\Delta_v^{\rm temp}\gamma^{-1} \cap \Delta_{v'}^{\rm temp}$, and $\gamma\Delta_v^{\rm temp}\gamma^{-1}$ is also a fundamental group of $\mathcal{G}_v^{\rm temp}$ with the base point obtained by conjugating the base point under consideration above by γ. This correponding to a tempered vertex $\widetilde{v}^\gamma \in \Gamma^{\rm temp}$. Hence for the tempered vertices $\widetilde{v}^\gamma$ and $\widetilde{v'}$, the associated fundamental group has nontrivial intersection.

By replacing $\Pi_{\mathcal{G}}^{\rm temp}$ by an open covering, we may assume that each irreducible component has genus ≥ 2, any edge of $\mathbb{G}$ abuts to two distinct vertices, and that, for any two (not necessarily distinct) vertices w, w', the set of edges e in $\mathbb{G}$ such that e abuts to a vertex w'' if and only if $w'' \in \{w, w'\}$ is either empty of of cardinality ≥ 2. In the case where $\Sigma = \{2\}$, then by replacing $\Pi_{\mathcal{G}}^{\rm temp}$ by an open covering, we may assume that the last condition "cardinality ≥ 2" is strongthen ot the condition "even cardinality".

If $\widetilde{v}^\gamma$ is not equal to $\widetilde{v'}$ nor $\widetilde{v}^\gamma$ is adjacent to $\widetilde{v'}$, then we can construct the covering over X_v (here X_v is the irreducible component corresponding to v), such that the ramification indices at the nodes and cusps of X_v are all equal (Note that such a covering exists by the assumed condition on $\mathbb{G}$ in the last paragraph), then we extend this covering over the irreducible components which adjacent to X_v, finally we extend the covering to a split covering over the rest of X (cf. also [AbsTopII, Proposition 1.3 (iv)] or [NodNon, Proposition 3.9 (i)]). This implies that there exist open subgroups $J \subset \Delta_{\mathcal{G}}^{\rm temp}$ which contain $\Delta_{v'}^{\rm temp}$ and determine arbitrarily small neighbourhoods $\gamma\Delta_v^{\rm temp}\gamma^{-1} \cap J$ of $\{1\}$. This is a contradiction. Therefore, $\widetilde{v}^\gamma$ is equal to $\widetilde{v'}$, or $\widetilde{v}^\gamma$ is adjacent to $\widetilde{v'}$. In particular, $\widetilde{v}^\gamma$ is tempered since $\widetilde{v'}$ is tempered. Hence both $\widetilde{v}$ and $\widetilde{v}^\gamma$ are tempered. Thus, we have $\gamma \in \Delta_{\mathcal{G}}^{\rm temp}$, as desired. $\square$

Corollary 6.8. ([IUTchI, Proposition 2.2]) *Let $\Delta_{\mathcal{G}}^{\rm temp}$ and $\Delta_{\mathcal{H}}^{\rm temp}$ be as in Lemma 6.4.*

(1) $\Delta_{\mathcal{G}}^{\rm temp} \subset \Delta_{\mathcal{G}}$ is commensurably terminal, and

(2) $\Delta_{\mathcal{H}}^{\rm temp} \subset \Delta_{\mathcal{G}}$ is commensurably terminal. In particular, $\Delta_{\mathcal{H}}^{\rm temp} \subset \Delta_{\mathcal{G}}^{\rm temp}$ is also commensurably terminal as well.

Proof. (1): Let $\gamma \in \Delta_{\mathcal{G}}$ be an element such that $\Delta_{\mathcal{G}}^{\rm temp} \cap \gamma\Delta_{\mathcal{G}}^{\rm temp}\gamma^{-1}$ is finite index in $\Delta_{\mathcal{G}}^{\rm temp}$. Let $\Delta_v \subset \Delta_{\mathcal{G}}^{\rm temp}$ be a verticial subgroup, and write $\Lambda := \Delta_v \cap \gamma\Delta_{\mathcal{G}}^{\rm temp}\gamma^{-1} \subset \Delta_v \subset \Delta_{\mathcal{G}}^{\rm temp}$. Since $[\Delta_v : \Lambda] = [\Delta_{\mathcal{G}}^{\rm temp} : \Delta_{\mathcal{G}}^{\rm temp} \cap \gamma\Delta_{\mathcal{G}}^{\rm temp}\gamma^{-1}] < \infty$, the subgroup Λ is open in the compact subgroup Δ_v, so, it is a nontrivial compact subgroup of $\Delta_{\mathcal{G}}^{\text{temp}}$. Now, $\gamma^{-1}\Lambda\gamma = \gamma^{-1}\Delta_v\gamma \cap \Delta_{\mathcal{G}}^{\text{temp}} \subset \Delta_{\mathcal{G}}^{\text{temp}}$. Since $\Lambda, \gamma^{-1}\Lambda\gamma \subset \Delta_{\mathcal{G}}^{\text{temp}}$ and Λ is a nontrivial compact subgroup, we have $\gamma^{-1} \in \Delta_{\mathcal{G}}^{\text{temp}}$ by Proposition 6.7. Thus $\gamma \in \Delta_{\mathcal{G}}^{\text{temp}}$, as desired.

(2): We have $\Delta_{\mathcal{H}}^{\text{temp}} \subset C_{\Delta_{\mathcal{G}}^{\text{temp}}}(\Delta_{\mathcal{H}}^{\text{temp}}) \subset C_{\Delta_{\mathcal{G}}}(\Delta_{\mathcal{H}}^{\text{temp}}) \subset C_{\Delta_{\mathcal{G}}}(\Delta_{\mathcal{H}})$ by definition. By Lemma 6.4 (1), we have $C_{\Delta_{\mathcal{G}}}(\Delta_{\mathcal{H}}) = \Delta_{\mathcal{H}}$. Thus, we have $C_{\Delta_{\mathcal{G}}}(\Delta_{\mathcal{H}}^{\text{temp}}) = C_{\Delta_{\mathcal{H}}}(\Delta_{\mathcal{H}}^{\text{temp}})$ combining these. On the other hand, by (1) for $\Delta_{\mathcal{H}}^{\text{temp}}$, we have $C_{\Delta_{\mathcal{H}}}(\Delta_{\mathcal{H}}^{\text{temp}}) = \Delta_{\mathcal{H}}^{\text{temp}}$. By combining these, we have $\Delta_{\mathcal{H}}^{\text{temp}} \subset C_{\Delta_{\mathcal{G}}}(\Delta_{\mathcal{H}}^{\text{temp}}) = C_{\Delta_{\mathcal{H}}}(\Delta_{\mathcal{H}}^{\text{temp}}) = \Delta_{\mathcal{H}}^{\text{temp}}$, as desired. $\square$

Corollary 6.9. ([IUTchI, Corollary 2.3]) *Let Δ_X, $\Delta_{\mathcal{G}}^{\text{temp}}$, $\Delta_{\mathcal{H}}^{\text{temp}}$, $\mathbb{H}$, $\Delta_{\mathcal{G}}$, $\Delta_{\mathcal{H}}$ be as in Lemma 6.4. Write $\Delta_{X,\mathbb{H}}^{\text{temp}} := \Delta_X^{\text{temp}} \times_{\Delta_{\mathcal{G}}^{\text{temp}}} \Delta_{\mathcal{H}}^{\text{temp}} (\subset \Delta_X^{\text{temp}})$, and $\Delta_{X,\mathbb{H}} := \Delta_X \times_{\Delta_{\mathcal{G}}} \Delta_{\mathcal{H}} (\subset \Delta_X)$.*

(1) $\Delta_{X,\mathbb{H}}^{\text{temp}} \subset \Delta_X^{\text{temp}}$ (resp. $\Delta_{X,\mathbb{H}} \subset \Delta_X$) is commensurably terminal.

(2) The closure of $\Delta_{X,\mathbb{H}}^{\text{temp}}$ in Δ_X is equal to $\Delta_{X,\mathbb{H}}$.

(3) We have $\Delta_{X,\mathbb{H}} \cap \Delta_X^{\text{temp}} = \Delta_{X,\mathbb{H}}^{\text{temp}} (\subset \Delta_X)$.

(4) Let $I_x \subset \Delta_X^{\text{temp}}$ (resp. $I_x \subset \Delta_X$) be a cusp x of X. Write $\widetilde{x}$ for the cusp in the stable model corresponding to x. Then I_x lies in a Δ_X^{temp}-(resp. Δ_X-)conjugate of $\Delta_{X,\mathbb{H}}^{\text{temp}}$ (resp. $\Delta_{X,\mathbb{H}}$) if and only if $\widetilde{x}$ meets an irreducible component of the special fiber of the stable model which is contained in $\mathbb{H}$.

(5) Suppose that $p \notin \Sigma$, and there is a prime number $l \notin \Sigma \cup \{p\}$. Then $\Delta_{X,\mathbb{H}}$ is slim. In particular, we can define

$$\Pi_{X,\mathbb{H}}^{\text{temp}} := \Delta_{X,\mathbb{H}}^{\text{temp}} \overset{\text{out}}{\rtimes} G_K, \quad \Pi_{X,\mathbb{H}} := \Delta_{X,\mathbb{H}} \overset{\text{out}}{\rtimes} G_K$$

by the natural outer actions of G_K on $\Delta_{X,\mathbb{H}}^{\text{temp}}$ and $\Delta_{X,\mathbb{H}}$ respectively.

(6) Suppose that $p \notin \Sigma$, and there is a prime number $l \notin \Sigma \cup \{p\}$. $\Pi_{X,\mathbb{H}}^{\text{temp}} \subset \Pi_X^{\text{temp}}$ and $\Pi_{X,\mathbb{H}} \subset \Pi_X$ are commensurably terminal.

Proof. (1) follows from Lemma 6.4 (1) and Corollary 6.8 (2). Next, (2) and (3) are trivial. (4) follows by noting that an inertia subgroup of a cusp is contained in precisely one verticial subgroup. We can show this, (possibly after replacing $\mathcal{G}$ by a finite étale covering) for any vertex v which is not abuted by the open edge e corresponding to the inertia subgroup, by constructing a covering which is trivial over $\mathcal{G}_v$ and nontrivial over $\mathcal{G}_e$ ([CombGC, Proposition 1.5 (i)]). (6) follows from (5) and (1). We show (5) (The following proof is a variant of the proof of Proposition 2.7 (2a)). Let $J \subset \Delta_X$ be an open normal subgroup, and write $J_{\mathbb{H}} := J \cap \Delta_{X,\mathbb{H}}$. We write $J \twoheadrightarrow J^{\Sigma\cup\{l\}}$ for the maximal pro-$\Sigma \cup \{l\}$ quotient, and $J_{\mathbb{H}}^{\Sigma\cup\{l\}} := \mathrm{Im}(J_{\mathbb{H}} \to J^{\Sigma\cup\{l\}})$. Suppose $\alpha \in \Delta_{X,\mathbb{H}}$ commutes with $J_{\mathbb{H}}$. Let v be a vertex of the dual graph of the geometric special fiber of a stable model $\mathcal{X}_J$ of the covering X_J of $X_{\overline{K}}$ corresponding to J. We write $J_v \subset J$ for the decomposition group of v, (which is well-defined up to conjugation in J), and we write $J_v^{\Sigma\cup\{l\}} := \mathrm{Im}(J_v \to J^{\Sigma\cup\{l\}})$. First, we show a claim that $J_v^{\Sigma\cup\{l\}} \cap J_{\mathbb{H}}^{\Sigma\cup\{l\}}$ is infinite and non-abelian. Note that $J_v \cap J_{\mathbb{H}}$, hence also $J_v^{\Sigma\cup\{l\}} \cap J_{\mathbb{H}}^{\Sigma\cup\{l\}}$, surjects onto the maxmal pro-l quotient J_v^l of J_v, since the image of the homomorphism $J_v \subset J \subset \Delta_X \twoheadrightarrow \Delta_{\mathcal{G}}$ is pro-Σ, and we have $\ker(J_v \subset J \subset \Delta_X \twoheadrightarrow \Delta_{\mathcal{G}}) \subset J_v \cap J_{\mathbb{H}}$, and $l \notin \Sigma$. Now, J_v^l is the pro-l completion of the fundamental group of hyperbolic Riemann surface, hence is infinite and non-abelian. Therefore, the claim is proved. Next, we show (5) from the claim. We consider various Δ_X-conjugates of $J_v^{\Sigma\cup\{l\}} \cap J_{\mathbb{H}}^{\Sigma\cup\{l\}}$ in $J^{\Sigma\cup\{l\}}$. Then by Proposition 6.6, it follows that α fixes v, since α commutes with $J_v^{\Sigma\cup\{l\}} \cap J_{\mathbb{H}}^{\Sigma\cup\{l\}}$. Moreover, since the conjugation by α on $J_v^l (\twoheadleftarrow J_v^{\Sigma\cup\{l\}} \cap J_{\mathbb{H}}^{\Sigma\cup\{l\}})$ is trivial, it follows that α not only fixes v, but also acts trivially on the irreducible component of the special fiber of $\mathcal{X}_J$ corresponding to v (Note that any nontrivial automorphism of an irreducible component of the special fiber induces a nontrivial outer automorphism of the tame pro-l fundamental group of the open subscheme of this irreducible component given by taking the complement of the nodes and cusps). Then α acts on $(J^{\Sigma\cup\{l\}})^{\mathrm{ab}}$ as a unipotent automorphism of finite order, since v is arbitrary, hence α acts trivially on $(J^{\Sigma\cup\{l\}})^{\mathrm{ab}}$. Then we have $\alpha = 1$, as desired since J is arbitrary. □

Corollary 6.10. ([IUTchI, Proposition 2.4 (i), (iii)]) *We continue to use the same notation as above. We assume that* $p \notin \Sigma$ *(which implies that* $\Delta_X^{\mathrm{temp}} \twoheadrightarrow \Delta_X^{(\Sigma),\mathrm{temp}} \cong \Delta_{\mathcal{G}}^{(\Sigma),\mathrm{temp}} = \Delta_{\mathcal{G}}^{\mathrm{temp}}$ *and* $\Delta_X \twoheadrightarrow \Delta_X^{(\Sigma)} \cong \Delta_{\mathcal{G}}^{(\Sigma)} = \Delta_{\mathcal{G}}$*).*

(1) Let $\Lambda \subset \Delta_X^{\mathrm{temp}}$ *be a nontrivial pro-*Σ *compact group,* $\gamma \in \Pi_X$ *an element such that* $\gamma\Lambda\gamma^{-1} \subset \Delta_X^{\mathrm{temp}}$*. Then we have* $\gamma \in \Pi_X^{\mathrm{temp}}$*.*

(2) ([A1, Corollary 6.2.2]) $\Delta_X^{\mathrm{temp}} \subset \Delta_X$ *(resp.* $\Pi_X^{\mathrm{temp}} \subset \Pi_X$*) is commensurably terminal.*

Remark 6.10.1. By Corollary 6.10 (2) and Theorem B.1, we can show a tempered version of Theorem B.1:

$$\mathrm{Hom}_K^{\mathrm{dom}}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_{G_K}^{\text{dense in an open subgp. of fin. index}}(\Pi_X^{\mathrm{temp}}, \Pi_Y^{\mathrm{temp}})/\mathrm{Inn}(\Delta_Y^{\mathrm{temp}})$$

(For a homomorphism, up to inner automorphisms of Δ_Y^{temp}, in the right-hand side, consider the induced homomorphism on the profinite completions. Then it comes from a morphism in the left-hand side by Theorem B.1, and we can reduce the ambiguity of inner automorphisms of the profinite completion of Δ_Y^{temp} to the one of inner automorphisms of Δ_Y^{temp} by Corollary 6.10 (2)). cf. also [SemiAnbd, Theorem 6.4].

Proof. (1): Let $\widetilde{\gamma} \in \Pi_X^{\mathrm{temp}} \twoheadrightarrow G_K$ be a lift of the image of $\gamma \in \Pi_X \twoheadrightarrow G_K$. By replacing γ by $\gamma(\widetilde{\gamma})^{-1} \in \Delta_X$, we may assume that $\gamma \in \Delta_X$. For an open characteristic sugroup $N \subset \Delta_X^{\mathrm{temp}}$, we write $\widehat{N}$ for the closure of N in Δ_X, and we write $\mathcal{G}_N$ for the (pro-Σ) semi-graph of anabelioids determined by the stable model of the covering of $X \times_K \overline{K}$ corresponding to N. By the isomorphisms (6.1) and (6.2) in Remark 6.3.1, it suffices to show that for any open characteristic subgroup $N \subset \Delta_X^{\mathrm{temp}}$, the image of $\gamma \in \Delta_X \twoheadrightarrow \Delta_X/\ker(\widehat{N} \twoheadrightarrow \Delta_{\mathcal{G}_N})$ comes from $\Delta_X^{\mathrm{temp}}/\ker(N \twoheadrightarrow \Delta_{\mathcal{G}_N}^{\mathrm{temp}}) \hookrightarrow \Delta_X/\ker(\widehat{N} \twoheadrightarrow \Delta_{\mathcal{G}_N})$. Let N be such an open characteristic subgroup of Δ_X^{temp}. Since N is of finite index in Δ_X^{temp}, we have $\Delta_X^{\mathrm{temp}}/N \cong \Delta_X/\widehat{N}$. We take a lift $\widetilde{\gamma} \in \Delta_X^{\mathrm{temp}} \twoheadrightarrow \Delta_X^{\mathrm{temp}}/N \cong \Delta_X/\widehat{N}$ of the image $\gamma \in \Delta_X \twoheadrightarrow \Delta_X/\widehat{N}$. By replacing γ by $\gamma(\widetilde{\gamma})^{-1} \in \widehat{N}$, we may assume that $\gamma \in \widehat{N}$. Note that $\Lambda_N := \Lambda \cap N (\subset N \subset \Delta_X^{\mathrm{temp}})$ is a nontrivial open compact subgroup, since N is of finite index in Δ_X^{temp}. Since Λ_N is a pro-Σ subgroup in Δ_X^{temp}, it is sent isomorphically to the image by $\Delta_X^{\mathrm{temp}} \twoheadrightarrow \Delta_X^{(\Sigma),\mathrm{temp}}$. Hence the image $\overline{\Lambda_N} \subset \Delta_{\mathcal{G}}^{\mathrm{temp}}$ of Λ_N by $\Delta_X^{\mathrm{temp}} \twoheadrightarrow \Delta_X^{(\Sigma),\mathrm{temp}} \cong \Delta_{\mathcal{G}}^{(\Sigma),\mathrm{temp}} = \Delta_{\mathcal{G}}^{\mathrm{temp}}$ is also nontrivial open compact subgroup (Here we need the assumption $p \notin \Sigma$. If $p \in \Sigma$, then we only have a surjection $\Delta_X^{(\Sigma),\mathrm{temp}} \twoheadrightarrow \Delta_{\mathcal{G}}^{(\Sigma),\mathrm{temp}}$, and the image of Λ_N might be trivial). Note that $\overline{\Lambda_N}$ is in $\Delta_{\mathcal{G}_N}^{\mathrm{temp}} = \mathrm{Im}(N \subset \Delta_X^{\mathrm{temp}} \twoheadrightarrow \Delta_{\mathcal{G}}^{\mathrm{temp}})$. Consider the following diagram, where the horizontal sequences are exact:

$$\begin{tikzcd}
1 \arrow[r] & \Delta_{\mathcal{G}_N}^{\mathrm{temp}} \arrow[r] \arrow[d, hook] & \Delta_X^{\mathrm{temp}}/\ker(N \twoheadrightarrow \Delta_{\mathcal{G}_N}^{\mathrm{temp}}) \arrow[r] \arrow[d, hook] & \Delta_X^{\mathrm{temp}}/N \arrow[r] \arrow[d, "\cong"] & 1 \\
1 \arrow[r] & \Delta_{\mathcal{G}_N} \arrow[r] & \Delta_X/\ker(\widehat{N} \twoheadrightarrow \Delta_{\mathcal{G}_N}^{\mathrm{temp}}) \arrow[r] & \Delta_X/\widehat{N} \arrow[r] & 1
\end{tikzcd}$$

Since γ is in $\widehat{N}$, the image $\overline{\gamma}$ of $\gamma \in \Delta_X \twoheadrightarrow \Delta_X/\ker(\widehat{N} \twoheadrightarrow \Delta_{\mathcal{G}_N})$ lands in $\Delta_{\mathcal{G}_N}$. Since $\overline{\Lambda_N} (\subset \Delta_{\mathcal{G}_N}^{\mathrm{temp}})$ is a nontrivial open compact subgroup, and $\overline{\gamma}\overline{\Lambda_N}\overline{\gamma}^{-1} \subset \Delta_{\mathcal{G}_N}^{\mathrm{temp}}$ by assumption, we conclude $\overline{\gamma} \in \Delta_{\mathcal{G}_N}^{\mathrm{temp}}$ by Proposition 6.7, as desired. (2) follows from (1) by the same way as in Corollary 6.8 (1). □

The following theorem is technically important for inter-universal Teichmüller theory :

Theorem 6.11. (Profinite Conjugates vs. Tempered Conjugates, [IUTchI, Corollary 2.5]) *We continue to use the same notation as above. We assume that $p \notin \Sigma$. Then*

(1) Any inertia subgroup in Π_X of a cusp of X is contained in Π_X^{temp} if and only if it is an inertia subgroup in Π_X^{temp} of a cusp of X, and

(2) A Π_X-conjugete of Π_X^{temp} contains an inertia subgroup in Π_X^{temp} of a cusp of X if and only if it is equal to Π_X^{temp}.

Remark 6.11.1. In inter-universal Teichmüller theory,

(1) we need to use tempered fundamental groups, because the theory of the étale theta function (cf. Section 7) plays a crucial role, and

(2) we also need to use profinite fundamental groups, because we need hyperbolic orbicurve *over a number field* for the purpose of putting "labels" for each places in a consistent manner (cf. Proposition 10.19 and Proposition 10.33). Note also that tempered fundamental groups are available only over non-Archimedean local fields, and we need to use profinite fundamental groups for hyperbolic orbicurve over a number field.

Then in this way, the "Profinite Conjugates vs. Tempered Conjugates" situation as in Theorem 6.11 naturally arises (cf. Lemma 11.9). The theorem says that *the profinite conjugacy indeterminacy is reduced to the harmless tempered conjugacy indeterminacy.*

Proof. Let $I_x(\cong \widehat{\mathbb{Z}})$ be an inertia subgroup of a cusp x. By applying Corollary 6.10 to the unique pro-Σ subgroup of I_x, it follows that a Π_X-conjugate of I_x is contained in Π_X^{temp} if and only if it is a Π_X^{temp}-conjugate of I_x, and that a Π_X-conjugate of Π_X^{temp} containes I_x if and only if it is equal to Π_X^{temp} □

Corollary 6.12. *Let X be a smooth hyperbolic log-curve over K, an algebraic closure $\overline{K}$ of K. Then we can group-theoretically reconstruct the inertia subgroups and the decomposition groups of cusps in $\Pi_X^{\mathrm{temp}} := \pi_1^{\mathrm{temp}}(X)$.*

Remark 6.12.1. By combining Corollary 6.12 with Proposition 6.6, we can group-theoretically reconstruct the dual semi-graph of the special fiber (cf. also Remark 6.6.1).

Proof. By Lemma 6.2 (with Remark 6.2.1) we have a group-theoretic reconstruction of the quotient $\Pi_X^{\mathrm{temp}} \twoheadrightarrow G_K$ from Π_X^{temp}. We write Δ_X and Π_X for the profinite completions of Δ_X^{temp} and Π_X^{temp} respectively. By using the injectivity of $\Delta_X^{\mathrm{temp}} \hookrightarrow \Delta_X$ and $\Pi_X^{\mathrm{temp}} \hookrightarrow \Pi_X$ (i.e., residual finiteness (cf. also Proposition C.5)), we can reconstruct inertia subgroups I of cusps by using Corollary 2.9, Remark 2.9.2, and Theorem 6.11 (Note that the reconstruction of the inertia subgroups in Δ_X has Δ_X-conjugacy indeterminacy; however, by using Theorem 6.11, this indeterminacy is reduced to Δ_X^{temp}-conjugacy indeterminacy, and it is harmless). Then we can group-theoretically reconstruct the decomposition groups of cusps, by taking the normaliser $N_{\Pi_X^{\mathrm{temp}}}(I)$, since I is normally terminal in Δ_X^{temp} by Lemma 6.4 (4). □

Remark 6.12.2. (a little bit sketchy here, cf. [AbsAnab, Lemma 2.5], [AbsTopIII, Theorem 1.10 (c)]) By using the reconstruction of the dual semi-graph of the special fiber (Remark 6.12.1), we can reconstruct

(1) a **positive rational structure** on $H^2(\Delta_X, \mu_{\widehat{\mathbb{Z}}}(G_K))^\vee := \mathrm{Hom}(H^2(\Delta_X, \mu_{\widehat{\mathbb{Z}}}(G_K)), \widehat{\mathbb{Z}})$,

(2) hence a cyclotomic rigidity isomorphism:

$$\text{(Cyc. Rig. via Pos. Rat. Str.)} \qquad \mu_{\widehat{\mathbb{Z}}}(G_K) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(\Pi_X)$$

(We shall refer to this as the **cyclotomic rigidity isomorphism via positive rational structure and LCFT**.)

as follows (cf. also Remark 3.19.1):

(1) By taking finite étale covering of X, it is easy to see that we may assume that the normalisation of each irreducible component of the special fiber of the stable model $\mathcal{X}$ of X has genus ≥ 2, and that the dual semi-graph Γ_X of the special fiber is non-contractible (cf. [profGC, Lemma 2.9, the first two paragraphs of the proof o Theorem 9.2]). By Remark 6.12.1, we can group-theoretically reconstruct the quotient $\Delta_X^{\mathrm{temp}} \twoheadrightarrow \Delta_X^{\mathrm{comb}}$ corresponding to the coverings of graphs (Note that, in [AbsAnab], we reconstruct the dual semi-graph of the special fiber from *profinite* fundamental group, i.e., *without* using tempered structure, via the reconstruction algorithms in [profGC]. cf. also Remark 6.6.1). We write Δ_X for the profinite completion of Δ_X^{temp}, and write $V := \Delta_X^{\mathrm{ab}}$. Note that the abelianisation $V^{\mathrm{comb}} := (\Delta_X^{\mathrm{comb}})^{\mathrm{ab}} \cong H_1^{\mathrm{sing}}(\Gamma_X, \mathbb{Z}) (\neq 0)$ is a free $\mathbb{Z}$-module. By using a theorem of Raynaud (cf. [AbsAnab, Lemma 2.4], [Tam, Lemma 1.9], [Ray, Théorème 4.3.1]), after replacing X by a finite étale covering (whose degree depends only on p and the genus of X), and K by a finite unramified extension, we may assume that the "new parts" of the Jacobians of the irreducible components of the special fiber are all ordinary, hence we obtain a G_K-equivariant quotient $V \twoheadrightarrow V^{\mathrm{new}}$, such that we have an exact sequence

$$0 \to V^{\mathrm{mult}} \to V^{\mathrm{new}}_{\mathbb{Z}_p} := V^{\mathrm{new}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_p \to V^{\text{ét}} \to 0,$$

where $V^{\text{ét}}$ is an unramified G_K-module, and V^{mult} is the Cartier dual of an unramified G_K-module, and that $V^{\mathrm{new}} \twoheadrightarrow V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}} := V^{\mathrm{comb}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} (\neq 0)$. We write $(-)_-$ (like $V^{\mathrm{new}}_{\mathbb{Z}_p}$, $V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}}$) for the tensor product in this proof. Then the restriction of the non-degenerate group-theoretic cup product

$$V^\vee \otimes_{\widehat{\mathbb{Z}}} V^\vee \otimes_{\widehat{\mathbb{Z}}} \mu_{\widehat{\mathbb{Z}}}(G_K) \to M := H^2(\Delta, \mu_{\widehat{\mathbb{Z}}}(G_K)) \, (\cong \widehat{\mathbb{Z}}),$$

where $(-)^\vee := \mathrm{Hom}(-, \widehat{\mathbb{Z}})$, to $(V^{\mathrm{new}})^\vee$

$$(V^{\mathrm{new}})^\vee \otimes_{\widehat{\mathbb{Z}}} (V^{\mathrm{new}})^\vee \otimes_{\widehat{\mathbb{Z}}} \mu_{\widehat{\mathbb{Z}}}(G_K) \to M \, (\cong \widehat{\mathbb{Z}})$$

is still non-degenerate since it arises from the restriction of the polarisation given by the theta divisor on the Jacobian of X to the "new part" of X (i.e., it gives us an ample divisor). Then we obtain an inclusion

$$(V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}})^\vee \otimes_{\widehat{\mathbb{Z}}} \mu_{\widehat{\mathbb{Z}}}(G_K) \otimes_{\widehat{\mathbb{Z}}} M^\vee \hookrightarrow (V^{\mathrm{new}})^\vee \otimes_{\widehat{\mathbb{Z}}} \mu_{\widehat{\mathbb{Z}}}(G_K) \otimes_{\widehat{\mathbb{Z}}} M^\vee \hookrightarrow \ker(V^{\mathrm{new}} \twoheadrightarrow V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}}) \subset V^{\mathrm{new}},$$

where the second last inclusion comes from $\mu_{\widehat{\mathbb{Z}}}(G_K)^{G_K} = 0$.

By the Riemann hypothesis for abelian varieties over finite fields, the $(\ker(V^{\text{ét}} \twoheadrightarrow V^{\mathrm{comb}}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{G_K} = ((\ker(V^{\text{ét}} \twoheadrightarrow V^{\mathrm{comb}}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)_{G_K} = 0$, where we write $(-)_{G_K}$ for the G_K-coinvariant quotient (Note that $\ker(V^{\text{ét}} \twoheadrightarrow V^{\mathrm{comb}}_{\mathbb{Z}_p})$ arises from the p-divisible group of an abelian variety over the residue field). Thus, the surjection $V^{\text{ét}} \twoheadrightarrow V^{\mathrm{comb}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_p$ has a unique G_K-splitting $V^{\mathrm{comb}}_{\mathbb{Z}_p} \hookrightarrow V^{\text{ét}} \otimes_{\mathbb{Q}_p}$. Similarly, by taking Cartier duals, the injection $(V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}})^\vee \otimes_{\widehat{\mathbb{Z}}} \mu_{\widehat{\mathbb{Z}}}(G_K) \otimes M^\vee \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_p \hookrightarrow V^{\mathrm{mult}}$ also has a unique G_K-splitting $V^{\mathrm{mult}} \twoheadrightarrow (V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}})^\vee \otimes_{\widehat{\mathbb{Z}}} \mu_{\widehat{\mathbb{Z}}}(G_K) \otimes M^\vee \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p$. By these splittings, the G_K-action on $V^{\mathrm{new}} \otimes_{\mathbb{Z}_p}$ gives us a p-adic extension class

$$\eta_{\mathbb{Z}_p} \in ((V^{\mathrm{comb}}_{\mathbb{Q}_p})^\vee)^{\otimes 2} \otimes M^\vee \otimes H^1(K, \mu_{\widehat{\mathbb{Z}}}(G_K))/H^1_f(K, \mu_{\widehat{\mathbb{Z}}}(G_K)) = ((V^{\mathrm{comb}}_{\mathbb{Q}_p})^\vee)^{\otimes 2} \otimes M^\vee :$$

$$\begin{tikzcd}
0 \arrow[r] & V^{\mathrm{mult}}_{\mathbb{Q}_p} \arrow[r] \arrow[d, dotted, bend left] & V^{\mathrm{new}}_{\mathbb{Q}_p} \arrow[r] & V^{\text{ét}}_{\mathbb{Q}_p} \arrow[r] \arrow[d] & 0 \\
& (V^{\mathrm{comb}}_{\mathbb{Q}_p})^\vee \otimes \mu_{\widehat{\mathbb{Z}}}(G_K) \otimes M^\vee \arrow[u, hook] & & V^{\mathrm{comb}}_{\mathbb{Q}_p}. \arrow[u, dotted, hook, bend right] &
\end{tikzcd}$$

Next, $\ker(V^{\mathrm{new}}_{\widehat{\mathbb{Z}}'} \twoheadrightarrow V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}'})$ is an unramified G_K-module, since it arises from $l(\neq p)$-divisible group of a semi-abelian variety over the residue field, where we write $\widehat{\mathbb{Z}}' := \prod_{l \neq p} \mathbb{Z}_l$. Again by the Riemann hypothesis for abelian varieties over finite fields, the injection $(V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}'})^\vee \otimes \mu_{\widehat{\mathbb{Z}}}(G_K) \otimes M^\vee \hookrightarrow \ker(V^{\mathrm{new}}_{\widehat{\mathbb{Z}}'} \twoheadrightarrow V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}'})$ of unramified G_K-modules splits uniquely over $\mathbb{Q}$. Then we can construct a prime-to-p-adic extension class

$$\eta_{\widehat{\mathbb{Z}}'} \in ((V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}'})^\vee)^{\otimes 2} \otimes M^\vee \otimes H^1(K, \mu_{\widehat{\mathbb{Z}}}(G_K))/H^1_f(K, \mu_{\widehat{\mathbb{Z}}}(G_K)) \otimes \mathbb{Q} = ((V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}'})^\vee)^{\otimes 2} \otimes M^\vee \otimes \mathbb{Q} :$$

$$\begin{tikzcd}
0 \arrow[r] & \ker(V^{\mathrm{new}}_{\widehat{\mathbb{Z}}' \otimes \mathbb{Q}} \twoheadrightarrow V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}' \otimes \mathbb{Q}}) \arrow[r] \arrow[d, dotted, bend left] & V^{\mathrm{new}}_{\widehat{\mathbb{Z}}' \otimes \mathbb{Q}} \arrow[r] & V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}' \otimes \mathbb{Q}} \arrow[r] & 0 \\
& (V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}' \otimes \mathbb{Q}})^\vee \otimes \mu_{\widehat{\mathbb{Z}}}(G_K) \otimes M^\vee. \arrow[u, hook] & & &
\end{tikzcd}$$

Then combining p-adic extension class and prime-to-p-adic extension class, we obtain an extension class

$$\eta_{\widehat{\mathbb{Z}}} \in ((V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}})^\vee)^{\otimes 2} \otimes M^\vee \otimes H^1(K, \mu_{\widehat{\mathbb{Z}}}(G_K))/H^1_f(K, \mu_{\widehat{\mathbb{Z}}}(G_K)) \otimes \mathbb{Q} = ((V^{\mathrm{comb}}_{\widehat{\mathbb{Z}}})^\vee)^{\otimes 2} \otimes M^\vee \otimes \mathbb{Q}.$$

Therefore, we obtain a bilinear form

$$(V_{\widehat{\mathbb{Z}}}^{\mathrm{comb}})^{\otimes 2} \to M^{\vee} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q},$$

and the image of $(V^{\mathrm{comb}})^{\otimes 2} \subset (V_{\widehat{\mathbb{Z}}}^{\mathrm{comb}})^{\otimes 2}$ gives us a **positive rational structure** (i.e., $\mathbb{Q}_{>0}$-structure) on $M^{\vee} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}$ (cf. [AbsAnab, Lemma 2.5]).

(2) By the group-theoretically reconstructed homomorphisms

$$H^1(G_K, \mu_{\widehat{\mathbb{Z}}}(G_K)) \stackrel{\sim}{\to} \mathrm{Hom}(H^1(G_K, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}}) \cong G_K^{\mathrm{ab}} \twoheadrightarrow G_K^{\mathrm{ab}}/\mathrm{Im}(I_K \to G_K^{\mathrm{ab}}) \cong \widehat{\mathbb{Z}}$$

in the proof of Corollary 3.19 (2), we obtain a natural surjection

$$H^1(G_K \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \twoheadrightarrow \mathrm{Hom}(\mu_{\widehat{\mathbb{Z}}}(G_K), \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \cong H^2(\Delta_X, \mu_{\widehat{\mathbb{Z}}}(G_K))^{\vee}$$

(Recall the definition of $\mu_{\widehat{\mathbb{Z}}}(\Pi_X)$). Then by taking the unique topological generator of $\mathrm{Hom}(\mu_{\widehat{\mathbb{Z}}}(G_K), \mu_{\widehat{\mathbb{Z}}}(\Pi_X))$ which is contained in the positive rational structue of $H^2(\Delta_X, \mu_{\widehat{\mathbb{Z}}}(G_K))^{\vee}$, we obtain the cyclotomic rigidity isomorphism $\mu_{\widehat{\mathbb{Z}}}(G_K) \stackrel{\sim}{\to} \mu_{\widehat{\mathbb{Z}}}(\Pi_X)$.

It seems important to give a remark that *we use the value group portion* (i.e., we use $O^{\triangleright}$, not $O^{\times}$) in the construction of the above surjection $H^1(G_K, \mu_{\widehat{\mathbb{Z}}}(G_K)) \stackrel{\sim}{\to} \mathrm{Hom}(H^1(G_K, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}}) \cong G_K^{\mathrm{ab}} \twoheadrightarrow G_K^{\mathrm{ab}}/\mathrm{Im}(I_K \to G_K^{\mathrm{ab}}) \cong \widehat{\mathbb{Z}}$, hence in the construction of the cyclotomic rigidity via positive rational structure and LCFT as well. In inter-universal Teichmüller theory, not only the existence of reconstruction algorithms, but also the *contents* of reconstruction algorithms are important, and whether or not we use the value group portion in the algorithm is crucial for the constructions in the final multiradial algorithm in inter-universal Teichmüller theory. cf. also Remark 9.6.2, Remark 11.4.1, Propositin 11.5, and Remark 11.11.1.

§ 7. Étale Theta Functions — Three Fundamental Rigidities.

In this sectin, we introduce another (probably the most) important ingredient of inter-universal Teichmüller theory, that is, the theory of the étale theta functions. In Section 7.1, we introduce some varieties related to the étale theta function. In Section 7.4, we introduce the notion of mono-theta environment, which plays important roles in inter-universal Teichmüller theory.

§ 7.1. Theta-related Varieties.

We introduce some varieties and study them in this subsection. Let K be a finite extension of $\mathbb{Q}_p$, and $\overline{K}$ an algebraic closure of K. Write $G_K := \mathrm{Gal}(\overline{K}/K)$. Let

$\mathfrak{X} \to \operatorname{Spf} O_K$ be a stable curve of type $(1,1)$ such that the special fiber is singular and geometrically irreducible, the node is rational, and the Raynaud generic fiber X (which is a rigid-analytic space) is smooth. For the varieties and rigid-analytic spaces in this Section, we also call marked points cusps, we always write log-structure on them, and we always consider the fundamental groups for the log-schemes and log-rigid-analytic spaces. We write Π_X^{temp}, Δ_X^{temp} for the tempered fundamental group of X (with log-structure on the marked point) for some basepoint. We have an exact sequence $1 \to \Delta_X \to \Pi_X \to G_K \to 1$. Write $\Pi_X := (\Pi_X^{\mathrm{temp}})^\wedge$, $\Delta_X := (\Delta_X^{\mathrm{temp}})^\wedge$ to be the profinite completions of Π_X^{temp}, Δ_X^{temp} respectively. We have the natural surjection $\Delta_X^{\mathrm{temp}} \twoheadrightarrow \mathbb{Z}$ corresponding to the universal graph-covering of the dual-graph of the configuration of the irreducible components of $\mathfrak{X}$. We write $\underline{\mathbb{Z}}$ for this quotient for the purpose of distinguish it from other $\mathbb{Z}$'s. We also write $\Delta_X \twoheadrightarrow \widehat{\underline{\mathbb{Z}}}$ for the profinite completion of $\Delta_X^{\mathrm{temp}} \twoheadrightarrow \underline{\mathbb{Z}}$.

Write $\Delta_X^\Theta := \Delta_X/[\Delta_X, [\Delta_X, \Delta_X]]$, and we shall refer to it as the **theta quotient** of Δ_X. We also write $\Delta_\Theta := \bigwedge^2 \Delta_X^{\mathrm{ab}} \,(\cong \widehat{\mathbb{Z}}(1))$, and $\Delta_X^{\mathrm{ell}} := \Delta_X^{\mathrm{ab}}$. We have the following exact sequences:

$$1 \to \Delta_\Theta \to \Delta_X^\Theta \to \Delta_X^{\mathrm{ell}} \to 1,$$

$$1 \to \widehat{\mathbb{Z}}(1) \to \Delta_X^{\mathrm{ell}} \to \widehat{\underline{\mathbb{Z}}} \to 1.$$

We write $(\Delta_X^{\mathrm{temp}})^\Theta$ and $(\Delta_X^{\mathrm{temp}})^{\mathrm{ell}}$ for the image of Δ_X^{temp} via the surjections $\Delta_X \twoheadrightarrow \Delta_X^\Theta$ and $\Delta_X \twoheadrightarrow (\Delta_X^\Theta \twoheadrightarrow)\Delta_X^{\mathrm{ell}}$ respectively:

$$\begin{tikzcd}
\Delta_X \arrow[r, two heads] & \Delta_X^\Theta \arrow[r, two heads] & \Delta_X^{\mathrm{ell}} \\
\Delta_X^{\mathrm{temp}} \arrow[u, hook] \arrow[r, two heads] & (\Delta_X^{\mathrm{temp}})^\Theta \arrow[u, hook] \arrow[r, two heads] & (\Delta_X^{\mathrm{temp}})^{\mathrm{ell}}. \arrow[u, hook]
\end{tikzcd}$$

We write $(\Pi_X^{\mathrm{temp}})^\Theta$ and $(\Pi_X^{\mathrm{temp}})^{\mathrm{ell}}$ for the push-out of Π_X^{temp} via the surjections $\Delta_X^{\mathrm{temp}} \twoheadrightarrow (\Delta_X^{\mathrm{temp}})^\Theta$ and $\Delta_X^{\mathrm{temp}} \twoheadrightarrow ((\Delta_X^{\mathrm{temp}})^\Theta \twoheadrightarrow)(\Delta_X^{\mathrm{temp}})^{\mathrm{ell}}$ respectively:

$$\begin{tikzcd}
\Pi_X^{\mathrm{temp}} \arrow[r, two heads] & (\Pi_X^{\mathrm{temp}})^\Theta \arrow[r, two heads] & (\Pi_X^{\mathrm{temp}})^{\mathrm{ell}} \\
\Delta_X^{\mathrm{temp}} \arrow[u, hook] \arrow[r, two heads] & (\Delta_X^{\mathrm{temp}})^\Theta \arrow[u, hook] \arrow[r, two heads] & (\Delta_X^{\mathrm{temp}})^{\mathrm{ell}}. \arrow[u, hook]
\end{tikzcd}$$

We have the following exact sequences:

$$1 \to \Delta_\Theta \to (\Delta_X^{\mathrm{temp}})^\Theta \to (\Delta_X^{\mathrm{temp}})^{\mathrm{ell}} \to 1,$$

$$1 \to \widehat{\mathbb{Z}}(1) \to (\Delta_X^{\mathrm{temp}})^{\mathrm{ell}} \to \underline{\mathbb{Z}} \to 1.$$

Let $Y \twoheadrightarrow X$ (resp. $\mathfrak{Y} \twoheadrightarrow \mathfrak{X}$) be the infinite étale covering correspoinding to the kernel Π_Y^{temp} of $\Pi_X^{\mathrm{temp}} \twoheadrightarrow \underline{\mathbb{Z}}$. We have $\mathrm{Gal}(Y/X) = \underline{\mathbb{Z}}$. Here, $\mathfrak{Y}$ is an infinite chain of copies of the projective line with a marked point $\neq 0, \infty$ (which we call a cusp), joined at 0 and ∞, and each of these points "0" and "∞"is a node in $\mathfrak{Y}$. We write $(\Delta_Y^{\mathrm{temp}})^{\Theta}$, $(\Delta_Y^{\mathrm{temp}})^{\mathrm{ell}}$ (resp. $(\Pi_Y^{\mathrm{temp}})^{\Theta}$, $(\Pi_Y^{\mathrm{temp}})^{\mathrm{ell}}$) for the image of Δ_Y^{temp} (resp. Π_Y^{temp}) via the surjections $\Delta_X^{\mathrm{temp}} \twoheadrightarrow (\Delta_X^{\mathrm{temp}})^{\Theta}$ and $\Delta_X^{\mathrm{temp}} \twoheadrightarrow ((\Delta_X^{\mathrm{temp}})^{\Theta} \twoheadrightarrow)(\Delta_X^{\mathrm{temp}})^{\mathrm{ell}}$ (resp. $\Pi_X^{\mathrm{temp}} \twoheadrightarrow (\Pi_X^{\mathrm{temp}})^{\Theta}$ and $\Pi_X^{\mathrm{temp}} \twoheadrightarrow ((\Pi_X^{\mathrm{temp}})^{\Theta} \twoheadrightarrow)(\Pi_X^{\mathrm{temp}})^{\mathrm{ell}}$) respectively:

$$\begin{tikzcd}
\Delta_X^{\mathrm{temp}} \arrow[r, two heads] & (\Delta_X^{\mathrm{temp}})^{\Theta} \arrow[r, two heads] & (\Delta_X^{\mathrm{temp}})^{\mathrm{ell}} \\
\Delta_Y^{\mathrm{temp}} \arrow[r, two heads] \arrow[u, hook] & (\Delta_Y^{\mathrm{temp}})^{\Theta} \arrow[r, two heads] \arrow[u, hook] & (\Delta_Y^{\mathrm{temp}})^{\mathrm{ell}}, \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Pi_X^{\mathrm{temp}} \arrow[r, two heads] & (\Pi_X^{\mathrm{temp}})^{\Theta} \arrow[r, two heads] & (\Pi_X^{\mathrm{temp}})^{\mathrm{ell}} \\
\Pi_Y^{\mathrm{temp}} \arrow[r, two heads] \arrow[u, hook] & (\Pi_Y^{\mathrm{temp}})^{\Theta} \arrow[r, two heads] \arrow[u, hook] & (\Pi_Y^{\mathrm{temp}})^{\mathrm{ell}}. \arrow[u, hook]
\end{tikzcd}$$

We also have a natural exact sequence

$$1 \to \Delta_{\Theta} \to (\Delta_Y^{\mathrm{temp}})^{\Theta} \to (\Delta_Y^{\mathrm{temp}})^{\mathrm{ell}} \to 1.$$

Note that $(\Delta_Y^{\mathrm{temp}})^{\mathrm{ell}} \cong \widehat{\mathbb{Z}}(1)$ and that $(\Delta_Y^{\mathrm{temp}})^{\Theta} (\cong \widehat{\mathbb{Z}}(1)^{\oplus 2})$ is abelian.

Let $q_X \in O_K$ be the q-parameter of X. For an integer $N \geq 1$, set $K_N := K(\mu_N, q_X^{1/N}) \subset \overline{K}$. Any decomposition group of a cusp of Y gives us a section $G_K \to (\Pi_Y^{\mathrm{temp}})^{\mathrm{ell}}$ of the natural surjection $(\Pi_Y^{\mathrm{temp}})^{\mathrm{ell}} \twoheadrightarrow G_K$ (Note that the inertia subgroup of cusps are killed in the quotient $(-)^{\mathrm{ell}}$). This section is well-defined up to conjugate by $(\Delta_Y^{\mathrm{temp}})^{\mathrm{ell}}$. The composite $G_{K_N} \hookrightarrow G_K \to (\Pi_Y^{\mathrm{temp}})^{\mathrm{ell}} \twoheadrightarrow (\Pi_Y^{\mathrm{temp}})^{\mathrm{ell}}/N(\Delta_Y^{\mathrm{temp}})^{\mathrm{ell}}$ is injective by the definition of K_N, and the image is stable under the conjugate by Π_X^{temp}, since G_{K_N} acts trivially on $1 \to \mathbb{Z}/N\mathbb{Z}(1) \to (\Delta_X^{\mathrm{temp}})^{\mathrm{ell}}/N(\Delta_Y^{\mathrm{temp}})^{\mathrm{ell}} \to \underline{\mathbb{Z}} \to 1$ (whose extension class is given by $q_X^{1/N}$), by the definition of K_N. Thus, the image $G_{K_N} \hookrightarrow (\Pi_Y^{\mathrm{temp}})^{\mathrm{ell}}/N(\Delta_Y^{\mathrm{temp}})^{\mathrm{ell}}$ determines a Galois covering $Y_N \twoheadrightarrow Y$. We have natural exact sequences:

$$1 \to \Pi_{Y_N}^{\mathrm{temp}} \to \Pi_Y^{\mathrm{temp}} \to \mathrm{Gal}(Y_N/Y) \to 1,$$

$$1 \to (\Delta_Y^{\mathrm{temp}})^{\mathrm{ell}} \otimes \mathbb{Z}/N\mathbb{Z} \, (\cong \mathbb{Z}/N\mathbb{Z}(1)) \to \mathrm{Gal}(Y_N/Y) \to \mathrm{Gal}(K_N/K) \to 1.$$

We write $(\Delta_{Y_N}^{\mathrm{temp}})^{\Theta}$, $(\Delta_{Y_N}^{\mathrm{temp}})^{\mathrm{ell}}$ (resp. $(\Pi_{Y_N}^{\mathrm{temp}})^{\Theta}$, $(\Pi_{Y_N}^{\mathrm{temp}})^{\mathrm{ell}}$) for the image of $\Delta_{Y_N}^{\mathrm{temp}}$ (resp. $\Pi_{Y_N}^{\mathrm{temp}}$) via the surjections $\Delta_Y^{\mathrm{temp}} \twoheadrightarrow (\Delta_Y^{\mathrm{temp}})^{\Theta}$ and $\Delta_Y^{\mathrm{temp}} \twoheadrightarrow ((\Delta_Y^{\mathrm{temp}})^{\Theta} \twoheadrightarrow)(\Delta_Y^{\mathrm{temp}})^{\mathrm{ell}}$ (resp. $\Pi_Y^{\mathrm{temp}} \twoheadrightarrow (\Pi_Y^{\mathrm{temp}})^{\Theta}$ and $\Pi_Y^{\mathrm{temp}} \twoheadrightarrow ((\Pi_Y^{\mathrm{temp}})^{\Theta} \twoheadrightarrow)(\Pi_Y^{\mathrm{temp}})^{\mathrm{ell}}$) respectively:

$$\begin{tikzcd}
\Delta_Y^{\mathrm{temp}} \arrow[r, two heads] & (\Delta_Y^{\mathrm{temp}})^{\Theta} \arrow[r, two heads] & (\Delta_Y^{\mathrm{temp}})^{\mathrm{ell}} \\
\Delta_{Y_N}^{\mathrm{temp}} \arrow[r, two heads] \arrow[u, hook] & (\Delta_{Y_N}^{\mathrm{temp}})^{\Theta} \arrow[r, two heads] \arrow[u, hook] & (\Delta_{Y_N}^{\mathrm{temp}})^{\mathrm{ell}}, \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Pi_Y^{\mathrm{temp}} \arrow[r, two heads] & (\Pi_Y^{\mathrm{temp}})^{\Theta} \arrow[r, two heads] & (\Pi_Y^{\mathrm{temp}})^{\mathrm{ell}} \\
\Pi_{Y_N}^{\mathrm{temp}} \arrow[r, two heads] \arrow[u, hook] & (\Pi_{Y_N}^{\mathrm{temp}})^{\Theta} \arrow[r, two heads] \arrow[u, hook] & (\Pi_{Y_N}^{\mathrm{temp}})^{\mathrm{ell}}. \arrow[u, hook]
\end{tikzcd}$$

We also have a natural exact sequence

$$1 \to \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z}\,(\cong \mathbb{Z}/N\mathbb{Z}(1)) \to (\Pi_{Y_N}^{\mathrm{temp}})^\Theta/N(\Delta_Y^{\mathrm{temp}})^\Theta \to G_{K_N} \to 1.$$

Let $\mathfrak{Y}_N \twoheadrightarrow \mathfrak{Y}$ be the normalisation of $\mathfrak{Y}$ in Y_N, i.e., write $\mathfrak{Y}$ and Y_N as the formal scheme and the rigid-analytic space associated to O_K-algebra A and K-algebra B_N respectively, and take the normalisation A_N of A in B_N, then $\mathfrak{Y}_N = \mathrm{Spf}\, A_N$. Here, $\mathfrak{Y}_N$ is also an infinite chain of copies of the projective line with N marked points $\neq 0, \infty$ (which we call cusps), joined at 0 and ∞, and each of these points "0" and "∞"is a node in $\mathfrak{Y}$. The covering $\mathfrak{Y}_N \twoheadrightarrow \mathfrak{Y}$ is the covering of N-th power map on the each copy of $\mathbb{G}_m$ obtained by removing the nodes, and the cusps correspond to "1", since we take a section $G_K \to (\Pi_Y^{\mathrm{temp}})^{\mathrm{ell}}$ corresponding to a cusp in the construction of Y_N. Note also that if N is divisible by p, then $\mathfrak{Y}_N$ is not a stable model over $\mathrm{Spf}\, O_{K_N}$.

We choose some irreducible component of $\mathfrak{Y}$ *as a "basepoint"*, then by the natural action of $\underline{\mathbb{Z}} = \mathrm{Gal}(Y/X)$ on $\mathfrak{Y}$, the projective lines in $\mathfrak{Y}$ are labelled by elements of $\underline{\mathbb{Z}}$. The isomorphism class of a line bundle on $\mathfrak{Y}_N$ is completely determined by the degree of the restriction of the line bundle to each of these copies of the projective line. Thus, these degrees give us an isomorphism

$$\mathrm{Pic}(\mathfrak{Y}_N) \xrightarrow{\sim} \mathbb{Z}^{\underline{\mathbb{Z}}},$$

i.e., the abelian group of the functions $\underline{\mathbb{Z}} \to \mathbb{Z}$. In the following, we consider Cartier divisors on $\mathfrak{Y}_N$, i.e., invertible sheaves for the structure sheaf $\mathcal{O}_{\mathfrak{Y}_N}$ of $\mathfrak{Y}_N$. Note that we can also consider an irreducible component of $\mathfrak{Y}_N$ as a $\mathbb{Q}$-Cartier divisor of $\mathfrak{Y}_N$ (cf. also the proof of [EtTh, Proposition 3.2 (i)]) although it has codimension 0 as underlying topological space in the formal scheme $\mathfrak{Y}_N$. We write $\mathfrak{L}_N$ for the line bundle on $\mathfrak{Y}_N$ correspoinding to the function $\underline{\mathbb{Z}} \to \mathbb{Z} : a \mapsto 1$ for any $a \in \underline{\mathbb{Z}}$, i.e., it has degree 1 on any irreducible component. Note also that we have $\Gamma(\mathfrak{Y}_N, \mathcal{O}_{\mathfrak{Y}_N}) = O_{K_N}$. In this section, we naturally identify a line bundle as a locally free sheaf with a geometric object (i.e., a (log-)(formal) scheme) defined by it.

Write $J_N := K_N(a^{1/N} \mid a \in K_N) \subset \overline{K}$, which is a finite Galois extension of K_N, since $K_N^\times/(K_N^\times)^N$ is finite. Two splitting of the exact sequence

$$1 \to \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z} \to (\Pi_{Y_N}^{\mathrm{temp}})^\Theta/N(\Delta_Y^{\mathrm{temp}})^\Theta \to G_{K_N} \to 1$$

determines an element of $H^1(G_{K_N}, \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z})$. By the definition of J_N, the restriction of this element to G_{J_N} is trivial. Thus, the splittings coincide over G_{J_N}, and the image $G_{J_N} \hookrightarrow (\Pi_{Y_N}^{\mathrm{temp}})^\Theta/N(\Delta_Y^{\mathrm{temp}})^\Theta$ is stable under the conjugate by Π_X^{temp}. Hence the image $G_{J_N} \hookrightarrow (\Pi_{Y_N}^{\mathrm{temp}})^\Theta/N(\Delta_Y^{\mathrm{temp}})^\Theta$ determines a finite Galois covering $Z_N \twoheadrightarrow Y_N$. We have the natural exact sequences

$$1 \to \Pi_{Z_N}^{\mathrm{temp}} \to \Pi_{Y_N}^{\mathrm{temp}} \to \mathrm{Gal}(Z_N/Y_N) \to 1,$$

$$(7.1) \qquad 1 \to \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z} \to \mathrm{Gal}(Z_N/Y_N) \to \mathrm{Gal}(J_N/K_N) \to 1.$$

We write $(\Delta_{Z_N}^{\mathrm{temp}})^\Theta$, $(\Delta_{Z_N}^{\mathrm{temp}})^{\mathrm{ell}}$ (resp. $(\Pi_{Z_N}^{\mathrm{temp}})^\Theta$, $(\Pi_{Z_N}^{\mathrm{temp}})^{\mathrm{ell}}$) for the image of $\Delta_{Z_N}^{\mathrm{temp}}$ (resp. $\Pi_{Z_N}^{\mathrm{temp}}$) via the surjections $\Delta_{Y_N}^{\mathrm{temp}} \twoheadrightarrow (\Delta_{Y_N}^{\mathrm{temp}})^\Theta$ and $\Delta_{Y_N}^{\mathrm{temp}} \twoheadrightarrow ((\Delta_{Y_N}^{\mathrm{temp}})^\Theta \twoheadrightarrow)(\Delta_{Y_N}^{\mathrm{temp}})^{\mathrm{ell}}$ (resp. $\Pi_{Y_N}^{\mathrm{temp}} \twoheadrightarrow (\Pi_{Y_N}^{\mathrm{temp}})^\Theta$ and $\Pi_{Y_N}^{\mathrm{temp}} \twoheadrightarrow ((\Pi_{Y_N}^{\mathrm{temp}})^\Theta \twoheadrightarrow)(\Pi_{Y_N}^{\mathrm{temp}})^{\mathrm{ell}}$) respectively:

$$\begin{tikzcd}
\Delta_{Y_N}^{\mathrm{temp}} \arrow[r, two heads] & (\Delta_{Y_N}^{\mathrm{temp}})^\Theta \arrow[r, two heads] & (\Delta_{Y_N}^{\mathrm{temp}})^{\mathrm{ell}} \\
\Delta_{Z_N}^{\mathrm{temp}} \arrow[r, two heads] \arrow[u, hook] & (\Delta_{Z_N}^{\mathrm{temp}})^\Theta \arrow[r, two heads] \arrow[u, hook] & (\Delta_{Z_N}^{\mathrm{temp}})^{\mathrm{ell}}, \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Pi_{Y_N}^{\mathrm{temp}} \arrow[r, two heads] & (\Pi_{Y_N}^{\mathrm{temp}})^\Theta \arrow[r, two heads] & (\Pi_{Y_N}^{\mathrm{temp}})^{\mathrm{ell}} \\
\Pi_{Z_N}^{\mathrm{temp}} \arrow[r, two heads] \arrow[u, hook] & (\Pi_{Z_N}^{\mathrm{temp}})^\Theta \arrow[r, two heads] \arrow[u, hook] & (\Pi_{Z_N}^{\mathrm{temp}})^{\mathrm{ell}}. \arrow[u, hook]
\end{tikzcd}$$

Let $\mathfrak{Z}_N \twoheadrightarrow \mathfrak{Y}_N$ be the normalisation of $\mathfrak{Y}$ in Z_N in the same sense as in the definition of $\mathfrak{Y}_N$. Note that the irreducible components of $\mathfrak{Z}_N$ are not isomorphic to the projective line in general.

A section $s_1 \in \Gamma(\mathfrak{Y}, \mathfrak{L}_1)$ whose zero locus is the cusps is well-defined up to an $O_K^\times$-multiple, since we have $\Gamma(\mathfrak{Y}, \mathcal{O}_\mathfrak{Y}) = O_K$. Fix an isomorphism $\mathfrak{L}_N^{\otimes N} \xrightarrow{\sim} \mathfrak{L}_1|_{\mathfrak{Y}_N}$ and we identify them. A natural action of $\mathrm{Gal}(Y/X) \,(\cong \underline{\mathbb{Z}})$ on $\mathfrak{L}_1$ is uniquely determined by the condition that it preserves s_1. This induces a natural action of $\mathrm{Gal}(Y_N/X)$ on $\mathfrak{L}_1|_{\mathfrak{Y}_N}$.

Lemma 7.1. ([EtTh, Proposition 1.1])

(1) The section $s_1|_{\mathfrak{Y}_N} \in \Gamma(\mathfrak{Y}_N, \mathfrak{L}_1|_{\mathfrak{Y}_N}) = \Gamma(\mathfrak{Y}_N, \mathfrak{L}_N^{\otimes N})$ has an N-th root $s_N \in \Gamma(\mathfrak{Z}_N, \mathfrak{L}_N|_{\mathfrak{Z}_N})$ over $\mathfrak{Z}_N$.

(2) There is a unique action of Π_X^{temp} on the line bundle $\mathfrak{L}_N \otimes_{O_{K_N}} O_{J_N}$ over $\mathfrak{Y}_N \times_{O_{K_N}} O_{J_N}$ which is compatible with the section $s_N : \mathfrak{Z}_N \to \mathfrak{L}_N \otimes_{O_{K_N}} O_{J_N}$. Furthermore, this action factors through $\Pi_X^{\mathrm{temp}} \twoheadrightarrow \Pi_X^{\mathrm{temp}}/\Pi_{Z_N}^{\mathrm{temp}} = \mathrm{Gal}(Z_N/X)$, and the action of $\Delta_X^{\mathrm{temp}}/\Delta_{Z_N}^{\mathrm{temp}}$ on $\mathfrak{L}_N \otimes_{O_{K_N}} O_{J_N}$ is faithful.

Proof. Write $(Y_N)_{J_N} := Y_N \times_{K_N} J_N$, and $\mathcal{G}_N$ to be the group of automorphisms of $\mathfrak{L}_N|_{(Y_N)_{J_N}}$ which is lying over the J_N-automorphisms of $(Y_N)_{J_N}$ induced by elements of $\Delta_X^{\mathrm{temp}}/\Delta_{Y_N}^{\mathrm{temp}} \subset \mathrm{Gal}(Y_N/X)$ and whose N-th tensor power fixes the $s_1|_{(Y_N)_{J_N}}$. Then by definition, we have a natural exact sequence

$$1 \to \mu_N(J_N) \to \mathcal{G}_N \to \Delta_X^{\mathrm{temp}}/\Delta_{Y_N}^{\mathrm{temp}} \to 1.$$

We claim that

$$\mathcal{H}_N := \ker(\mathcal{G}_N \twoheadrightarrow \Delta_X^{\mathrm{temp}}/\Delta_{Y_N}^{\mathrm{temp}} \twoheadrightarrow \Delta_X^{\mathrm{temp}}/\Delta_Y^{\mathrm{temp}} \cong \underline{\mathbb{Z}})$$

is an abelian group killed by N, where the above two surjections are natural ones, and the kernels are $\mu_N(J_N)$ and $(\Delta_X^{\mathrm{temp}})^{\mathrm{ell}} \otimes \mathbb{Z}/N\mathbb{Z} \,(\cong \mathbb{Z}/N\mathbb{Z}(1))$ respectively. Proof of the claim (This immediate follows from the structure of the theta group (=Heisenberg group); however, we include a proof here): Note that we have a natural commutative diagram

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mu_N(J_N) \arrow[r] \arrow[d, "="] & \mathcal{H}_N \arrow[r] \arrow[d] & (\Delta_Y^{\mathrm{temp}})^{\mathrm{ell}} \otimes \mathbb{Z}/N\mathbb{Z}\,(\cong \mathbb{Z}/N\mathbb{Z}(1)) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_N(J_N) \arrow[r] & \mathcal{G}_N \arrow[r] \arrow[d] & \Delta_X^{\mathrm{temp}}/\Delta_{Y_N}^{\mathrm{temp}} \arrow[r] \arrow[d] & 1 \\
 & & \Delta_X^{\mathrm{temp}}/\Delta_Y^{\mathrm{temp}} \arrow[r, "="] \arrow[d] & \Delta_X^{\mathrm{temp}}/\Delta_Y^{\mathrm{temp}}\,(\cong \underline{\mathbb{Z}}), \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}$$

whose rows and columns are exact. Let ζ be a primitive N-th root of unity. The function whose restriction to every irreducible component minus nodes $\widehat{\mathbb{G}}_m = \operatorname{Spf} O_K[[U]]$ of $\mathfrak{Y}_N$ is equal $f(U) := \frac{U-1}{U-\zeta}$ represents an element of $\mathcal{H}$ which maps to a generator of $\Delta_Y^{\mathrm{temp}}/\Delta_{Y_N}^{\mathrm{temp}}$, since it changes the pole divisor from 1 to ζ. Then the claim follows from the identity $\prod_{0\leq j\leq N-1} f(\zeta^{-j}U) = \frac{U-1}{U-\zeta}\frac{U-\zeta}{U-\zeta^2}\cdots\frac{U-\zeta^N}{U-\zeta^{N-1}} = 1$. The claim is shown.

Let $\mathfrak{N}_N$ be the tautological $\mathbb{Z}/N\mathbb{Z}(1)$-torsor $\mathfrak{N}_N \to \mathfrak{Y}_N$ obtained by taking an N-th root of s_1, i.e., the finite $\mathfrak{Y}_N$-formal scheme $\mathbf{Spf}\left(\oplus_{0\leq j\leq N-1}\mathfrak{L}_N^{\otimes(-j)}\right)$, where the algebra structure is defined by the multiplication $\mathfrak{L}_N^{\otimes(-N)} \to \mathcal{O}_{\mathfrak{Y}_N}$ by $s_1|_{\mathfrak{Y}_N}$. Then $\mathcal{G}_N$ naturally acts on $(\mathfrak{N}_N)_{J_N} := \mathfrak{N}_N \times_{O_{K_N}} J_N$ by the definition of $\mathcal{G}_N$. Since $s_1|_{Y_N}$ has zero of order 1 at each cusp, $(\mathfrak{N}_N)_{J_N}$ is connected and Galois over $X_{J_N} := X \times_K J_N$, and $\mathcal{G}_N \xrightarrow{\sim} \mathrm{Gal}((\mathfrak{N}_N)_{J_N}/X_{J_N})$. Since (i) $\Delta_X^{\mathrm{temp}}/\Delta_{Y_N}^{\mathrm{temp}}$ acts trivially on $\mu_N(J_N)$, and (ii) $\mathcal{H}_N$ is killed by N by the above claim, we have a morphism $\mathfrak{Z}_N \times_{O_{J_N}} \overline{K} \to \mathfrak{N}_N \times_{O_{K_N}} O_{J_N}$ over $\mathfrak{Y}_N \times_{O_{K_N}} O_{J_N}$ by the definitions of $\Delta_X^{\Theta} = \Delta_X/[\Delta_X, [\Delta_X, \Delta_X]]$ and Z_N, i.e., geometrically, $\mathfrak{Z}_N \times_{O_{J_N}} \overline{K} (\twoheadrightarrow \mathfrak{Y}_N \times_{O_{K_N}} \overline{K})$ has the universality having properties (i) and (ii) (Note that the domain of the morphism is $\mathfrak{Z}_N \times_{O_{J_N}} \overline{K}$, not $\mathfrak{Z}_N$ since we are considering $\Delta_{(-)}$, not $\Pi_{(-)}$). Since we used the open immersion $G_{J_N} \hookrightarrow (\Pi_{Y_N}^{\mathrm{temp}})^{\Theta}/N(\Delta_Y^{\mathrm{temp}})^{\Theta}$, whose image is stable under conjugate by Π_X^{temp}, to define the morphism $\mathfrak{Z}_N \twoheadrightarrow \mathfrak{Y}_N$, and $s_1|_{Y_N}$ is defined over K_N, the above morphism $\mathfrak{Z}_N \times_{O_{J_N}} \overline{K} \to \mathfrak{N}_N \times_{O_{K_N}} O_{J_N}$ factors through $\mathfrak{Z}_N$, and induces an isomorphism $\mathfrak{Z}_N \xrightarrow{\sim} \mathfrak{N}_N \times_{O_{K_N}} O_{J_N}$ by considering the degrees over $\mathfrak{Y}_N \times_{O_{K_N}} O_{J_N}$ on both sides (i.e., this isomorphism means that the covering determined by $\Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z}$ coincides with the covering determined by an N-th root of $s_1|_{Y_N}$). This proves the claim (1) of the lemma.

Next, we show the claim (2) of the lemma. We have a unique action of Π_X^{temp} on $\mathfrak{L}_N \otimes_{O_{K_N}} O_{J_N}$ over $\mathfrak{Y}_N \times_{O_{K_N}} O_{J_N}$ which is compatible with the section $s_N : \mathfrak{Z}_N \to \mathfrak{L}_N \otimes_{O_{K_N}} O_{J_N}$, since the action of $\Pi_X^{\mathrm{temp}}(\twoheadrightarrow \mathrm{Gal}(Y_N/X))$ on $\mathfrak{L}_1|_{\mathfrak{Y}_N} \cong \mathfrak{L}_N^{\otimes N}$ preserves $s_1|_{\mathfrak{Y}_N}$, and the action of Π_X^{temp} on $\mathfrak{Y}_N$ preserves the isomorphism class of $\mathfrak{L}_N$. This action factors through $\Pi_X^{\mathrm{temp}}/\Pi_{Z_N}^{\mathrm{temp}}$, since s_N is defined over Z_N. Finally, the action of $\Pi_X^{\mathrm{temp}}/\Pi_{Z_N}^{\mathrm{temp}}$ is faithful since s_1 has zeroes of order 1 at the cusps of Y_N, and the action of $\Delta_X^{\mathrm{temp}}/\Delta_{Y_N}^{\mathrm{temp}}$ on Y_N is tautologically faithful. □

We set

$$\ddot{K}_N := K_{2N}, \quad \ddot{J}_N := \ddot{K}_N(a^{1/N} \mid a \in \ddot{K}_N) \quad \subset \overline{K},$$

$$\ddot{\mathfrak{Y}}_N := \mathfrak{Y}_{2N} \times_{O_{\ddot{K}_N}} O_{\ddot{J}_N},\ \ddot{Y}_N := Y_{2N} \times_{\ddot{K}_N} \ddot{J}_N,\ \ddot{\mathfrak{L}}_N := \mathfrak{L}_N|_{\ddot{\mathfrak{Y}}_N} \cong \mathfrak{L}_{2N}^{\otimes 2} \times_{O_{\ddot{K}_N}} O_{\ddot{J}_N}.$$

(The symbol $(\ddot{-})$ roughly expresses "double covering". Note that we need to consider double coverings of the rigid analytic spaces under consideration to consider a theta function below.) Let $\ddot{Z}_N$ be the composite of the coverings $\ddot{Y}_N \twoheadrightarrow Y_N$ and $Z_N \twoheadrightarrow Y_N$, and $\ddot{\mathfrak{Z}}_N$ the normalisation of $\mathfrak{Z}_N$ in $\ddot{Z}_N$ in the same sense as in the definition of $\mathfrak{Y}_N$. Write also

$$\ddot{Y} := \ddot{Y}_1 = Y_2, \quad \ddot{\mathfrak{Y}} := \ddot{\mathfrak{Y}}_1 = \mathfrak{Y}_2, \quad \ddot{K} := \ddot{K}_1 = \ddot{J}_1 = K_2.$$

Since Π_X^{temp} acts compatibly on $\ddot{\mathfrak{Y}}_N$ and $\mathfrak{Y}_N$, and on $\mathfrak{L}_N \otimes_{O_{K_N}} O_{J_N}$, and the natural commutative diagram

$$\begin{tikzcd}
\ddot{\mathfrak{L}}_N \arrow[r] \arrow[d] & \mathfrak{L}_N \arrow[d] \\
\ddot{\mathfrak{Y}}_N \arrow[r] & \mathfrak{Y}_N
\end{tikzcd}$$

is cartesian, we have a natural action of Π_X^{temp} on $\ddot{\mathfrak{L}}_N$, which factors through $\Pi_X^{\mathrm{temp}}/\Pi_{\ddot{Z}_N}^{\mathrm{temp}}$.

Next, *we choose an orientation on the dual graph of the configuration of the irreducible components of* $\mathfrak{Y}$. Such an orientation gives us an isomorphism $\underline{\mathbb{Z}} \xrightarrow{\sim} \mathbb{Z}$. We give a label $\in \mathbb{Z}$ for each irreducible component of $\mathfrak{Y}$. This choice of labels also determines a label $\in \mathbb{Z}$ for each irreducible component of $\mathfrak{Y}_N$, $\ddot{\mathfrak{Y}}_N$. Recall that we can also consider the irreducible component $(\ddot{\mathfrak{Y}}_N)_j$ of $\ddot{\mathfrak{Y}}_N$ labelled j as a $\mathbb{Q}$-Cartier divisor of $\ddot{\mathfrak{Y}}_N$ (cf. also the proof of [EtTh, Proposition 3.2 (i)]) although it has codimension 0 as underlying topological space in the formal scheme $\ddot{\mathfrak{Y}}_N$ (Note that $(\ddot{\mathfrak{Y}}_N)_j$ is Cartier, since the completion of $\ddot{\mathfrak{Y}}_N$ at each node is isomorphic to $\mathrm{Spf}\, O_{\ddot{J}_N}[[u,v]]/(uv - q_X^{1/2N})$). Write $\mathfrak{D}_N := \sum_{j\in\mathbb{Z}} j^2 (\ddot{\mathfrak{Y}}_N)_j$ (i.e., the divisor defined by the summation of "$q_X^{j^2/2N} = 0$" on the irreducible component labelled j with respect to $j \in \mathbb{Z}$). We claim that

$$\mathcal{O}_{\ddot{\mathfrak{Y}}_N}(\mathfrak{D}_N) \cong \ddot{\mathfrak{L}}_N \, (\cong \mathfrak{L}_{2N}^{\otimes 2} \otimes_{O_{\ddot{K}_N}} O_{\ddot{J}_N}). \tag{7.2}$$

Proof of the claim: Since $\mathrm{Pic}(\ddot{\mathfrak{Y}}_N) \cong \mathbb{Z}^{\mathbb{Z}}$, it suffices to show that $\mathfrak{D}_N.(\ddot{\mathfrak{Y}}_N)_i = 2$ for any $i \in \mathbb{Z}$, where we write $\mathfrak{D}_N.(\ddot{\mathfrak{Y}}_N)_i$ for the intersection product of $\mathfrak{D}_N$ and $(\ddot{\mathfrak{Y}}_N)_i$, i.e., the degree of $\mathcal{O}_{\ddot{\mathfrak{Y}}_N}(\mathfrak{D}_N)|_{(\ddot{\mathfrak{Y}}_N)_i}$. We have $0 = \ddot{\mathfrak{Y}}_N.(\ddot{\mathfrak{Y}}_N)_i = \sum_{j\in\mathbb{Z}}(\ddot{\mathfrak{Y}}_N)_j.(\ddot{\mathfrak{Y}}_N)_i = 2 + ((\ddot{\mathfrak{Y}}_N)_i)^2$ by the configuration of the irreducible components of $\ddot{\mathfrak{Y}}_N$ (i.e., an infinite chain of copies of the projective line joined at 0 and ∞). Thus, we obtain $((\ddot{\mathfrak{Y}}_N)_i)^2 = -2$. Then we have $\mathfrak{D}_N.(\ddot{\mathfrak{Y}}_N)_i = \sum_{j\in\mathbb{Z}} j^2(\ddot{\mathfrak{Y}}_N)_j.(\ddot{\mathfrak{Y}}_N)_i = (i-1)^2 - 2i^2 + (i+1)^2 = 2$. This proves the claim.

By the claim, there exists a section

$$\tau_N : \ddot{\mathfrak{Y}}_N \to \ddot{\mathfrak{L}}_N,$$

well-defined up to an $O^{\times}_{\ddot{J}_N}$-multiple, whose zero locus is equal to $\mathfrak{D}_N$. We shall refer to τ_N as a **theta trivialisation**. Note that the action of Π_Y^{temp} on $\ddot{\mathfrak{Y}}_N$, $\ddot{\mathfrak{L}}_N$ preserves τ_N up to an $O^{\times}_{\ddot{J}_N}$-multiple, since the action of Π_Y^{temp} on $\ddot{\mathfrak{Y}}_N$ fixes $\mathfrak{D}_N$.

Let $M \geq 1$ be an integer which divides N. Then we have natural morphisms $\mathfrak{Y}_N \twoheadrightarrow \mathfrak{Y}_M \twoheadrightarrow \mathfrak{Y}$, $\ddot{\mathfrak{Y}}_N \twoheadrightarrow \ddot{\mathfrak{Y}}_M \twoheadrightarrow \mathfrak{Y}$, $\mathfrak{Z}_N \twoheadrightarrow \mathfrak{Z}_M \twoheadrightarrow \mathfrak{Y}$, and natural isomorphisms $\mathfrak{L}_M|_{\mathfrak{Y}_N} \cong \mathfrak{L}_N^{\otimes(N/M)}$, $\ddot{\mathfrak{L}}_M|_{\mathfrak{Y}_N} \cong \ddot{\mathfrak{L}}_N^{\otimes(N/M)}$. By the definition of $\ddot{J}_N(= K_{2N}(a^{1/N} \mid a \in K_{2N}))$, we also have a natural diagram

$$\begin{tikzcd} \ddot{\mathfrak{L}}_N \arrow[r, two heads] & \ddot{\mathfrak{L}}_M \\ \ddot{\mathfrak{Y}}_N \arrow[u, "\tau_N"] \arrow[r, two heads] & \ddot{\mathfrak{Y}}_M, \arrow[u, "\tau_M"'] \end{tikzcd}$$

which is commutative up to an $O^{\times}_{\ddot{J}_N}$-multiple at $\ddot{\mathfrak{L}}_N$, and an $O^{\times}_{\ddot{J}_M}$-multiple at $\ddot{\mathfrak{L}}_M$, since τ_N and τ_M are defined over $\mathfrak{Y}_{2N}$ and $\mathfrak{Y}_{2M}$ respectively (Recall that $\ddot{\mathfrak{Y}}_N := \mathfrak{Y}_{2N} \times_{O_{\ddot{K}_N}} O_{\ddot{J}_N}$). By the relation $\ddot{\Theta}(-\ddot{U}) = -\ddot{\Theta}(\ddot{U})$ given in Lemma 7.4 (2), (3) below (Note that we have no circular argument here), we can choose τ_1 so that the natural action of $\Pi_{\ddot{Y}}^{\mathrm{temp}}$ on $\ddot{\mathfrak{L}}_1$ preserves $\pm\tau_1$. In summary, by the definition of $\ddot{J}_N$, we have the following:

- By modifying τ_N's by $O^{\times}_{\ddot{J}_N}$-multiples, we can assume that $\tau_N^{N/M} = \tau_M$ for any positive integers N and M such that $M \mid N$.

- In particular, we have a compatible system of actions of $\Pi_{\ddot{Y}}^{\mathrm{temp}}$ on $\{\ddot{\mathfrak{Y}}_N\}_{N\geq 1}$, $\{\ddot{\mathfrak{L}}_N\}_{N\geq 1}$ which preserve $\{\tau_N\}_{N\geq 1}$.

- Each of the above actions of $\Pi_{\ddot{Y}}^{\mathrm{temp}}$ on $\ddot{\mathfrak{Y}}_N$, $\ddot{\mathfrak{L}}_N$ differs from the action determined by the action of Π_X^{temp} on $\mathfrak{Y}_N$, $\mathfrak{L}_N \otimes_{O_{K_N}} O_{J_N}$ in Lemma 7.1 (2) by an element of $\mu_N(\ddot{J}_N)$.

Definition 7.2. We take τ_N's as above. By taking the difference of the compatible system of the action of $\Pi_{\ddot{Y}}^{\mathrm{temp}}$ on $\{\ddot{\mathfrak{Y}}_N\}_{N\geq 1}$, $\{\ddot{\mathfrak{L}}_N\}_{N\geq 1}$ in Lemma 7.1 determined by $\{s_N\}_{N\geq 1}$ and the compatible system of the action of $\Pi_{\ddot{Y}}^{\mathrm{temp}}$ on $\{\ddot{\mathfrak{Y}}_N\}_{N\geq 1}$, $\{\ddot{\mathfrak{L}}_N\}_{N\geq 1}$ in the above determined by $\{\tau_N\}_{N\geq 1}$ (Note also that the former actions, i.e., the one determined by $\{s_N\}_{N\geq 1}$ in Lemma 7.1 come from the actions of Π_X^{temp}; however, the latter actions, i.e., the one determined by $\{\tau_N\}_{N\geq 1}$ in the above do not come from the actions of Π_X^{temp}), we obtain a cohomology class

$$\ddot{\eta}^{\Theta} \in H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}, \Delta_\Theta),$$

via the isomorphism $\mu_N(\ddot{J}_N) \cong \mathbb{Z}/N\mathbb{Z}(1) \cong \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z}$ (Note that we are currently studying in a scheme theory here, and that the natural isomorphism $\mu_N(\ddot{J}_N) \cong \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z}$ comes from the scheme theory (cf. also Remark 3.15.1).

Remark 7.2.1. (cf. also [EtTh, Proposition 1.3])

(1) Note that $\ddot{\eta}^{\Theta}$ arises from a cohomology class in $\varprojlim_{N\geq 1} H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}/\Pi_{\ddot{Z}_N}^{\mathrm{temp}}, \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z})$, and that the restriction

$$\begin{aligned}&\varprojlim_{N\geq 1} H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}/\Pi_{\ddot{Z}_N}^{\mathrm{temp}}, \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z}) \to \varprojlim_{N\geq 1} H^1(\Delta_{\ddot{Y}_N}^{\mathrm{temp}}/\Delta_{\ddot{Z}_N}^{\mathrm{temp}}, \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z})\\ &\cong \varprojlim_{N\geq 1} \mathrm{Hom}(\Delta_{\ddot{Y}_N}^{\mathrm{temp}}/\Delta_{\ddot{Z}_N}^{\mathrm{temp}}, \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z})\end{aligned}$$

sends $\ddot{\eta}^{\Theta}$ to the system of the natural isomorphisms $\{\Delta_{\ddot{Y}_N}^{\mathrm{temp}}/\Delta_{\ddot{Z}_N}^{\mathrm{temp}} \xrightarrow{\sim} \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z}\}_{N\geq 1}$.

(2) Note also that $s_2 : \ddot{\mathfrak{Y}} \to \ddot{\mathfrak{L}}_1$ is well-defined up to an $O_{\ddot{K}}^{\times}$-multiple, $s_{2N} : \ddot{\mathfrak{Z}}_N \to \ddot{\mathfrak{L}}_N$ is an N-th root of s_2, $\tau_1 : \ddot{\mathfrak{Y}} \to \ddot{\mathfrak{L}}_1$ is well-defined up to an $O_{\ddot{K}}^{\times}$-multiple, and $\tau_N : \ddot{\mathfrak{Y}}_N \to \ddot{\mathfrak{L}}_N$ is an N-th root of τ_1. Thus, $\ddot{\eta}^{\Theta} \in H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}, \Delta_\Theta)$ is well-defined up to an $O_{\ddot{K}}^{\times}$-multiple. Hence the set of cohomology classes

$$O_{\ddot{K}}^{\times} \cdot \ddot{\eta}^{\Theta} \subset H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}, \Delta_\Theta)$$

is independent of the choices of s_N's and τ_N's, where $O_{\ddot{K}}^{\times}$ acts on $H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}, \Delta_\Theta)$ via the composite of the Kummer map $O_{\ddot{K}}^{\times} \to H^1(G_{\ddot{K}}, \Delta_\Theta)$ and the natural homomorphism $H^1(G_{\ddot{K}}, \Delta_\Theta) \to H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}, \Delta_\Theta)$. We shall refer to any element in the set $O_{\ddot{K}}^{\times} \cdot \ddot{\eta}^{\Theta}$ as the **étale theta class**.

§ 7.2. The Étale Theta Function.

Let $(\widehat{\mathbb{G}}_m \cong)\, \mathfrak{U} \subset \ddot{\mathfrak{Y}}$ be the irreducible component labelled $0 \in \mathbb{Z}$ minus nodes. We take the unique cusp of $\mathfrak{U}$ as the origin. The group structure of the underlying elliptic curve X, determines a group structure on $\mathfrak{U}$. By the orientation on the dual graph of the configuration of the irreducible components of $\mathfrak{Y}$, we have a unique isomorphism $\mathfrak{U} \cong \widehat{\mathbb{G}}_m$ over O_K. This gives us a multiplicative coordinate $U \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{U}}^{\times})$. This has a square root $\ddot{U} \in \Gamma(\ddot{\mathfrak{U}}, \mathcal{O}_{\ddot{\mathfrak{U}}}^{\times})$ on $\ddot{\mathfrak{U}} := \mathfrak{U} \times_{\mathfrak{Y}} \ddot{\mathfrak{Y}}$ (Note that the theta function lives in the double covering. cf. also Lemma 7.4 below).

We recall the section associated with a tangential basepoint. (cf. also [AbsSect, Definition 4.1 (iii), and the terminology before Definition 4.1]): For a cusp $y \in \ddot{Y}(L)$ with a finite extension L of $\ddot{K}$, let $D_y \subset \Pi_{\ddot{Y}}$ be a cuspidal decomposition group of y (which is well-defined up to conjugates). We have an exact sequence

$$1 \to I_y\,(\cong \widehat{\mathbb{Z}}(1)) \to D_y \to G_L \to 1,$$

and the set $\mathrm{Sect}(D_y \twoheadrightarrow G_L)$ of splittings of this short exact sequnece up to conjugates by I_y is a torsor over $H^1(G_L, \widehat{\mathbb{Z}}(1)) \cong (L^{\times})^{\wedge}$ by the usual way (the difference of two sections gives us a 1-cocycle, and the conjugates by I_y yield 1-coboundaries), where $(L^{\times})^{\wedge}$ is the profinite completion of L. We write ω_y for the cotangent space to $\ddot{Y}$ at y. For a non-zero element $\theta \in \omega_y$, take a system of N-th roots ($N \geq 1$) of any local coordinate $t \in \mathfrak{m}_{\ddot{Y},y}$ with $dt|_y = \theta$, then, this system gives us a $\widehat{\mathbb{Z}}(1)\,(\cong I_y)$-torsor $(\ddot{Y}|_y^{\wedge}(t^{1/N}))_{N\geq 1} \twoheadrightarrow \ddot{Y}|_y^{\wedge}$ over the formal completion of $\ddot{Y}$ at y. This $\widehat{\mathbb{Z}}(1)\,(\cong I_y)$-covering $(\ddot{Y}|_y^{\wedge}(t^{1/N}))_{N\geq 1} \twoheadrightarrow \ddot{Y}|_y^{\wedge}$ corresponding to the kernel of a surjection $D_y \twoheadrightarrow I_y\,(\cong \widehat{\mathbb{Z}}(1))$, hence it gives us a section of the above short exact sequence. This is called **the (conjugacy class of) section associated with the tangential basepoint** θ. In this manner, the structure group $(L^{\times})^{\wedge}$ of the $(L^{\times})^{\wedge}$-torsor $\mathrm{Sect}(D_y \twoheadrightarrow G_L)$ is canonically reduced to $L^{\times}$, and the $L^{\times}$-torsor obtained in this way is canonically identified with the $L^{\times}$-torsor of the non-zero elements of ω_y. Furthermore, noting also that $\ddot{Y}$ comes from the stable model $\ddot{\mathfrak{Y}}$, which gives us the canonical O_L-submodule $\widehat{\omega}_y(\subset \omega_y)$ of ω_y, the structure group $(L^{\times})^{\wedge}$ of the $(L^{\times})^{\wedge}$-torsor $\mathrm{Sect}(D_y \twoheadrightarrow G_L)$ is canonically reduced to $O_L^{\times}$, and the $O_L^{\times}$-torsor obtained in this way is canonically identified with the $O_L^{\times}$-torsor of the generators of $\widehat{\omega}_y$.

Definition 7.3. We shall refer to this canonical reduction of the $(L^{\times})^{\wedge}$-torsor $\mathrm{Sect}(D_y \twoheadrightarrow G_L)$ to the canonical $O_L^{\times}$-torsor as **the canonical integral structure** of D_y, and we say that a section s in $\mathrm{Sect}(D_y \twoheadrightarrow G_L)$ is **compatible with the canonical integral structure of** D_y, if s comes from a section of the canonical $O_L^{\times}$-torsor. We shall refer to the $L^{\times}$-torsor obtained by the push-out of the canonical $O_L^{\times}$-torsor via $O_L^{\times} \to L^{\times}$ as **the canonical discrete structure** of D_y. We write $\widehat{\mathbb{Z}}'$ for the maximal prime-to-p quotient of $\widehat{\mathbb{Z}}$, and write $(O_L^{\times})' := \mathrm{Im}(O_L^{\times} \to (L^{\times})\otimes\widehat{\mathbb{Z}}')$. We shall refer to the $(O_L^{\times})'$-torsor obtained by the push-out of the canonical $O_L^{\times}$-torsor via $O_L^{\times} \to (O_L^{\times})'$ as **the canonical tame integral structure** of D_y (cf. [AbsSect, Definition 4.1 (ii), (iii)]). We also shall refer to a reduction of the $(L^{\times})^{\wedge}$-torsor $\mathrm{Sect}(D_y \twoheadrightarrow G_L)$ to a $\{\pm 1\}$-torsor (resp. μ_{2l}-torsor) as **$\{\pm 1\}$-structure** of D_y (resp. **μ_{2l}-structure** of D_y). When a $\{\pm 1\}$-

structure (resp. μ_{2l}-structure) of D_y is given, we say that a section s in $\mathrm{Sect}(D_y \twoheadrightarrow G_L)$ is **compatible with the $\{\pm 1\}$-structure of D_y**, (resp. **the μ_{2l}-structure of D_y**, if s comes from a section of the $\{\pm 1\}$-torsor (resp. the μ_{2l}-torsor).

Lemma 7.4. ([EtTh, Proposition 1.4]) *Write*

$$\ddot{\Theta}(\ddot{U}) := q_X^{-\frac{1}{8}} \sum_{n\in\mathbb{Z}} (-1)^n q_X^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \ddot{U}^{2n+1} \in \Gamma(\ddot{\mathfrak{U}}, \mathcal{O}_{\ddot{\mathfrak{U}}}).$$

Note that $\ddot{\Theta}(\ddot{U})$ extends uniquely to a meromorphic function on $\ddot{\mathfrak{Y}}$ (cf. a classical complex theta function

$$\theta_{1,1}(\tau, z) := \sum_{n\in\mathbb{Z}} \exp\left(\pi i\tau\left(n+\frac{1}{2}\right)^2 + 2\pi i\left(z+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\right) = \frac{1}{i}\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \ddot{U}^{2n+1},$$

where $q := e^{2\pi i\tau}$, and $\ddot{U} := e^{\pi i z}$) and that $q_X^{-\frac{1}{8}} q_X^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} = q_X^{\frac{n(n+1)}{2}}$ is in K.

(1) $\ddot{\Theta}(\ddot{U})$ has zeroes of order 1 at the cusps of $\ddot{\mathfrak{Y}}$, and there is no other zeroes. $\ddot{\Theta}(\ddot{U})$ has poles of order j^2 on the irreducible component labelled j, and there is no other poles, i.e., the divisor of poles of $\ddot{\Theta}(\ddot{U})$ is equal to $\mathfrak{D}_1$.

(2) For $a \in \mathbb{Z}$, we have

$$\ddot{\Theta}(\ddot{U}) = -\ddot{\Theta}(\ddot{U}^{-1}), \quad \ddot{\Theta}(-\ddot{U}) = -\ddot{\Theta}(\ddot{U}),$$
$$\ddot{\Theta}\left(q_X^{\frac{a}{2}}\ddot{U}\right) = (-1)^a q_X^{-\frac{a^2}{2}} \ddot{U}^{-2a} \ddot{\Theta}(\ddot{U}).$$

(3) The classes $O_{\ddot{K}}^\times \cdot \ddot{\eta}^\Theta$ are precisely the Kummer classes associated to an $O_{\ddot{K}}^\times$-multiple of the regular function $\ddot{\Theta}(\ddot{U})$ on the Raynaud generic fiber $\ddot{Y}$. In particular, for a non-cuspidal point $y \in \ddot{Y}(L)$ with a finite extension L of $\ddot{K}$, the restriction of the classes

$$O_{\ddot{K}}^\times \cdot \ddot{\eta}^\Theta|_y \in H^1(G_L, \Delta_\Theta) \cong H^1(G_L, \widehat{\mathbb{Z}}(1)) \cong (L^\times)^\wedge$$

lies in $L^\times \subset (L^\times)^\wedge$, and are equal to $O_{\ddot{K}}^\times \cdot \ddot{\Theta}(y)$ (Note that we are currently studying in a scheme theory here, and that the natural isomorphism $\Delta_\Theta \cong \widehat{\mathbb{Z}}(1)$ comes from the scheme theory (cf. also Remark 3.15.1).

(4) For a cusp $y \in \ddot{Y}(L)$ with a finite extension L of $\ddot{K}$, we have a similar statement as in (3) by modifying as below: Let $D_y \subset \Pi_{\ddot{Y}}$ be a cuspidal decomposition group of y (which is well-defined up to conjugates). Let $s : G_L \hookrightarrow D_y$ be a section which is compatible with the canonical integral structure of D_y. Let s comes from a generator $\widehat{\theta} \in \widehat{\omega}_y$. Then the restriction of the classes

$$O_{\ddot{K}}^\times \cdot \ddot{\eta}^\Theta|_{s(G_L)} \in H^1(G_L, \Delta_\Theta) \cong H^1(G_L, \widehat{\mathbb{Z}}(1)) \cong (L^\times)^\wedge,$$

via $G_L \overset{s}{\hookrightarrow} D_y \subset \Pi^{\mathrm{temp}}_{\ddot{Y}}$, lies in $L \subset (L^\times)^\wedge$, and are equal to $O^\times_{\ddot{K}} \cdot \frac{d\ddot{\Theta}}{\widehat{\theta}}(y)$, where $\frac{d\ddot{\Theta}}{\widehat{\theta}}(y)$ is the value at y of the first derivative of $\ddot{\Theta}(\ddot{U})$ at y by $\widehat{\theta}$. In particular, the set of the restriction of the classes $O^\times_{\ddot{K}} \cdot \ddot{\eta}^\Theta|_{s(G_L)}$ is independent of the choice of the generator $\widehat{\theta} \in \widehat{\omega}_y$ (hence the choice of the section s which is compatible with the canonical integral structure of D_y).

We also shall refer to the classes in $O^\times_{\ddot{K}} \cdot \ddot{\eta}^\Theta$ as the **étale theta functions** in light of the above relationship of the values of the theta function and the restrictions of these classes to G_L via points.

Proof. (2):

$$\begin{aligned}\ddot{\Theta}(\ddot{U}^{-1}) &= q_X^{-\frac{1}{8}} \sum_{n\in\mathbb{Z}} (-1)^n q_X^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \ddot{U}^{-2n-1} = q_X^{-\frac{1}{8}} \sum_{n\in\mathbb{Z}} (-1)^{-n-1} q_X^{\frac{1}{2}\left(-n-1+\frac{1}{2}\right)^2} \ddot{U}^{2n+1} \\ &= -q_X^{-\frac{1}{8}} \sum_{n\in\mathbb{Z}} (-1)^n q_X^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \ddot{U}^{2n+1} = -\ddot{\Theta}(\ddot{U}),\end{aligned}$$

$$\ddot{\Theta}(-\ddot{U}) = q_X^{-\frac{1}{8}} \sum_{n\in\mathbb{Z}} (-1)^n q_X^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} (-\ddot{U})^{2n+1} = -q_X^{-\frac{1}{8}} \sum_{n\in\mathbb{Z}} (-1)^n q_X^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \ddot{U}^{2n+1} = -\ddot{\Theta}(\ddot{U}),$$

$$\begin{aligned}\ddot{\Theta}\left(q_X^{\frac{a}{2}}\ddot{U}\right) &= q_X^{-\frac{1}{8}} \sum_{n\in\mathbb{Z}} (-1)^n q_X^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} (q_X^{\frac{a}{2}}\ddot{U})^{2n+1} = q_X^{-\frac{1}{8}} \sum_{n\in\mathbb{Z}} (-1)^n q_X^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2+a\left(n+\frac{1}{2}\right)} \ddot{U}^{2n+1} \\ &= q_X^{-\frac{1}{8}} \sum_{n\in\mathbb{Z}} (-1)^n q_X^{\frac{1}{2}\left(n+a+\frac{1}{2}\right)^2-\frac{a^2}{2}} \ddot{U}^{2n+1} = (-1)^a q_X^{-\frac{a^2}{2}} \ddot{\Theta}(\ddot{U}).\end{aligned}$$

(1): Firstly, note that $q_X^{\frac{a}{2}}\ddot{U}$ is the canonical coordinate of the irreducible component labelled a, and that the last equality of (2) gives us the translation formula for changing the irreducible components. The description of the divisor of poles comes from this translation formula and $\ddot{\Theta}(\ddot{U}) \in \Gamma(\ddot{\mathfrak{U}}, \mathcal{O}_{\ddot{\mathfrak{U}}})$ (i.e., $\ddot{\Theta}(\ddot{U})$ is a regular function on $\ddot{\mathfrak{U}}$). Next, by putting $\ddot{U} = \pm 1$ in the first equality of (1), we obtain $\ddot{\Theta}(\pm 1) = 0$. Then by the last equality of (2) again, it suffices to show that $\ddot{\Theta}(\ddot{U})$ has simple zeroes at $\ddot{U} = \pm 1$ on $\ddot{\mathfrak{U}}$. By taking modulo the maximal ideal of $O_{\ddot{K}}$, we have $\ddot{\Theta}(\ddot{U}) \equiv \ddot{U} - \ddot{U}^{-1}$. This shows the claim.

(3) is a consequence of the construction of the classes $O^\times_{\ddot{K}} \cdot \ddot{\eta}^\Theta$ and (1).

(4): For a generator $\widehat{\theta} \in \widehat{\omega}_y$, the corresponding section $s \in \mathrm{Sect}(D_y \twoheadrightarrow G_L)$ described before this lemma is as follows: Let $t \in \mathfrak{m}_{\ddot{\mathfrak{Y}},y}$ be a system of N-th roots ($N \geq 1$) of any local coordinate with $dt|_y = \widehat{\theta}$, then, this system gives us a $\widehat{\mathbb{Z}}(1)\,(\cong I_y)$-torsor $(\ddot{\mathfrak{Y}}|^\wedge_y(t^{1/N}))_{N\geq 1} \twoheadrightarrow \ddot{\mathfrak{Y}}|^\wedge_y$ over the formal completion of $\ddot{\mathfrak{Y}}$ at y. This $\widehat{\mathbb{Z}}(1)\,(\cong I_y)$-covering $(\ddot{\mathfrak{Y}}|^\wedge_y(t^{1/N}))_{N\geq 1} \twoheadrightarrow \ddot{\mathfrak{Y}}|^\wedge_y$ corresponding to the kernel of a surjection $D_y \twoheadrightarrow I_y\,(\cong \widehat{\mathbb{Z}}(1))$, hence a section $s \in \mathrm{Sect}(D_y \twoheadrightarrow G_L)$. For $g \in G_L$, take any lift $\widetilde{g} \in D_y\,(\Pi_{\ddot{Y}}^{\mathrm{temp}})$ of G_L, then the above description says that $s(g) = (\widetilde{g}(t^{1/N})/t^{1/N})_{N\geq 1}^{-1} \cdot \widetilde{g}$, where $(\widetilde{g}(t^{1/N})/t^{1/N})_{N\geq 1} \in \widehat{\mathbb{Z}}(1) \cong I_y$ (Note that the right-hand side does not depend on the choice of a lift $\widetilde{g}$). The Kummer class of $\ddot{\Theta} := \ddot{\Theta}(\ddot{U})$ is given by $\Pi_{\ddot{Y}}^{\mathrm{temp}} \ni h \mapsto (h(\ddot{\Theta}^{1/N})/\ddot{\Theta}^{1/N})_{N\geq 1} \in \widehat{\mathbb{Z}}(1)$. Hence the restriction to G_L via $G_L \overset{s}{\to} D_y \subset \Pi_{\ddot{Y}}^{\mathrm{temp}}$ is given by $G_L \ni g \mapsto ((\widetilde{g}(t^{1/N})/t^{1/N})^{-1}\widetilde{g}(\ddot{\Theta}^{1/N})/\ddot{\Theta}^{1/N})_{N\geq 1} = (\widetilde{g}((\ddot{\Theta}/t)^{1/N})/(\ddot{\Theta}/t)^{1/N})_{N\geq 1} \in \widehat{\mathbb{Z}}(1)$. Since $\ddot{\Theta}(\ddot{U})$ has a simple zero at y, we have $(\widetilde{g}((\ddot{\Theta}/t)^{1/N})/(\ddot{\Theta}/t)^{1/N})_{N\geq 1} = (g((d\ddot{\Theta}/\widehat{\theta})^{1/N})/(d\ddot{\Theta}/\widehat{\theta})^{1/N})_{N\geq 1}$, where $d\ddot{\Theta}/\widehat{\theta}$ is the first derivative $\frac{d\ddot{\Theta}}{\widehat{\theta}}$ at y by $\widehat{\theta}$. Then $G_L \ni g \mapsto (g((d\ddot{\Theta}/\widehat{\theta})^{1/N})/(d\ddot{\Theta}/\widehat{\theta})^{1/N})_{N\geq 1} \in \widehat{\mathbb{Z}}(1)$ is the Kummer class of the value $\frac{d\ddot{\Theta}}{\widehat{\theta}}(y)$ at y. $\square$

If an automorphism ι_Y of Π_Y^{temp} is lying over the action of "-1" on the underlying elliptic curve of X which fixes the irreducible component of $\mathfrak{Y}$ labelled 0, then we shall refer to ι_Y as an **inversion automorphism** of Π_Y^{temp}.

Lemma 7.5. ([EtTh, Proposition 1.5])

(1) Both of the Leray-Serre spectral sequences

$$E_2^{a,b} = H^a((\Delta_{\ddot{Y}}^{\mathrm{temp}})^{\mathrm{ell}}, H^b(\Delta_\Theta, \Delta_\Theta)) \Longrightarrow H^{a+b}((\Delta_{\ddot{Y}}^{\mathrm{temp}})^\Theta, \Delta_\Theta),$$
$$E_2'^{a,b} = H^a(G_{\ddot{K}}, H^b((\Delta_{\ddot{Y}}^{\mathrm{temp}})^\Theta, \Delta_\Theta)) \Longrightarrow H^{a+b}((\Pi_{\ddot{Y}}^{\mathrm{temp}})^\Theta, \Delta_\Theta)$$

associated to the filtration of closed subgroups

$$\Delta_\Theta \subset (\Delta_{\ddot{Y}}^{\mathrm{temp}})^\Theta \subset (\Pi_{\ddot{Y}}^{\mathrm{temp}})^\Theta$$

degenerate at E_2, and this determines a filtration $0 \subset \mathrm{Fil}^2 \subset \mathrm{Fil}^1 \subset \mathrm{Fil}^0 = H^1((\Pi_{\ddot{Y}}^{\mathrm{temp}})^\Theta, \Delta_\Theta)$ on $H^1((\Pi_{\ddot{Y}}^{\mathrm{temp}})^\Theta, \Delta_\Theta)$ such that we have

$$\begin{aligned}
\mathrm{Fil}^0/\mathrm{Fil}^1 &= \mathrm{Hom}(\Delta_\Theta, \Delta_\Theta) = \widehat{\mathbb{Z}},\\
\mathrm{Fil}^1/\mathrm{Fil}^2 &= \mathrm{Hom}((\Delta_{\ddot{Y}}^{\mathrm{temp}})^\Theta/\Delta_\Theta, \Delta_\Theta) = \widehat{\mathbb{Z}} \cdot \log(\ddot{U}),\\
\mathrm{Fil}^2 &= H^1(G_{\ddot{K}}, \Delta_\Theta) \overset{\sim}{\to} H^1(G_{\ddot{K}}, \widehat{\mathbb{Z}}(1)) \overset{\sim}{\to} (\ddot{K}^\times)^\wedge.
\end{aligned}$$

Here, we write $\log(\ddot{U})$ for the standard isomorphism $(\Delta_{\ddot{Y}}^{\mathrm{temp}})^\Theta/\Delta_\Theta = (\Delta_{\ddot{Y}}^{\mathrm{temp}})^{\mathrm{ell}} \overset{\sim}{\to} \widehat{\mathbb{Z}}(1) \overset{\sim}{\to} \Delta_\Theta$ (given in a scheme theory).

(2) Any theta class $\ddot{\eta}^\Theta \in H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}, \Delta_\Theta)$ arises from a unique class $\ddot{\eta}^\Theta \in H^1((\Pi_{\ddot{Y}}^{\mathrm{temp}})^\Theta, \Delta_\Theta)$ (Here, we use the same symbol $\ddot{\eta}^\Theta$ by abuse of the notation) which maps to the identity homomorphism in the quotient $\mathrm{Fil}^0/\mathrm{Fil}^1 = \mathrm{Hom}(\Delta_\Theta, \Delta_\Theta)$ (i.e., maps to $1 \in \widehat{\mathbb{Z}} = \mathrm{Hom}(\Delta_\Theta, \Delta_\Theta)$). We consider $O_{\ddot{K}}^\times \cdot \ddot{\eta}^\Theta \subset H^1((\Pi_{\ddot{Y}}^{\mathrm{temp}})^\Theta, \Delta_\Theta)$ additively, and

write $\ddot{\eta}^{\Theta} + \log(O^{\times}_{\ddot{K}})$ *for it. Then* $a \in \mathbb{Z} \cong \underline{\mathbb{Z}} = \Pi^{\mathrm{temp}}_{X}/\Pi^{\mathrm{temp}}_{Y}$ *acts on* $\ddot{\eta}^{\Theta} + \log(O^{\times}_{\ddot{K}})$ *as*

$$\ddot{\eta}^{\Theta} + \log(O^{\times}_{\ddot{K}}) \mapsto \ddot{\eta}^{\Theta} - 2a\log(\ddot{U}) - \frac{a^2}{2}\log(q_X) + \log(O^{\times}_{\ddot{K}}).$$

In a similar way, for any inversion automorphism ι_Y *of* Π^{temp}_{Y}*, we have*

$$\iota_Y(\ddot{\eta}^{\Theta} + \log(O^{\times}_{\ddot{K}})) = \ddot{\eta}^{\Theta} + \log(O^{\times}_{\ddot{K}})$$
$$\iota_Y(\log(\ddot{U}) + \log(O^{\times}_{\ddot{K}})) = -\log(\ddot{U}) + \log(O^{\times}_{\ddot{K}}).$$

Proof. (1): Since $\Delta_{\Theta} \cong \widehat{\mathbb{Z}}(1)$ and $(\Delta^{\mathrm{temp}}_{\ddot{Y}})^{\mathrm{ell}} \cong \widehat{\mathbb{Z}}(1)$ and $\widehat{\mathbb{Z}}(1)$ has cohomological dimension 1, the first spectral sequence degenerates at E_2, and this gives us a short exact sequence

$$0 \to H^1((\Delta^{\mathrm{temp}}_{\ddot{Y}})^{\mathrm{ell}}, \Delta_{\Theta}) \to H^1((\Delta^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta}) \to H^1(\Delta_{\Theta}, \Delta_{\Theta}) \to 0.$$

This is equal to

$$0 \to \widehat{\mathbb{Z}} \cdot \log(\ddot{U}) \to H^1((\Delta^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta}) \to \widehat{\mathbb{Z}} \to 0.$$

On the other hand, the second spectral sequence gives us an exact sequnece

$$0 \to H^1(G_{\ddot{K}}, \Delta_{\Theta}) \to H^1((\Pi^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta}) \to H^1((\Delta^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta})^{G_{\ddot{K}}} \to H^2(G_{\ddot{K}}, \Delta_{\Theta}) \to 0.$$

Then by Remark 7.2.1 (1), the composite

$$H^1((\Pi^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta}) \to H^1((\Delta^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta})^{G_{\ddot{K}}}$$
$$\subset H^1((\Delta^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta}) \to H^1(\Delta_{\Theta}, \Delta_{\Theta}) = \widehat{\mathbb{Z}}$$

maps the Kummer class of $\ddot{\Theta}(\ddot{U})$ to 1 (Recall also the definition of Z_N and the short exact sequence (7.1)). Hence the second spectral sequence degenerates at E_2, and we have the description of the graded quotients of the filtration on $H^1((\Pi^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta})$.

(2): The first assertion holds by definition. Next, note that the subgroup $(\Delta^{\mathrm{temp}}_{\ddot{Y}})^{\mathrm{ell}} \subset (\Delta^{\mathrm{temp}}_{X})^{\mathrm{ell}}$ corresponds to the subgroup $2\widehat{\mathbb{Z}}(1) \subset \widehat{\mathbb{Z}}(1) \times \mathbb{Z} \cong (\Delta^{\mathrm{temp}}_{X})^{\mathrm{ell}}$ by the theory of Tate curves, where $\widehat{\mathbb{Z}}(1) \subset (\Delta^{\mathrm{temp}}_{X})^{\mathrm{ell}}$ corresponds to the system of $N(\geq 1)$-th roots of the canonical coordinate U of the Tate curve associated to X, and $2\widehat{\mathbb{Z}}(1) \cong (\Delta^{\mathrm{temp}}_{\ddot{Y}})^{\mathrm{ell}}$ corresponds to the system of $N(\geq 1)$-th roots of the canonical coordinate $\ddot{U}$ introduced before (In this sense, the usage of the symbol $\log(\ddot{U}) \in \mathrm{Hom}((\Delta^{\mathrm{temp}}_{\ddot{Y}})^{\mathrm{ell}}, \Delta_{\Theta})$ is justified). Then the description of the action of $a \in \mathbb{Z} \cong \underline{\mathbb{Z}}$ follows from the last equality of Lemma 7.4 (2), and the first description of the action of an inversion automorphism follows from the first equality of Lemma 7.4 (2). The second description of the action of an inversion automorphism immediately follows from the definition. □

The following proposition says that the étale thete function has an anabelian rigidity, i.e., it is preserved under the changes of scheme theory.

Proposition 7.6. (Anabelian Rigidity of the Étale Theta Function, [EtTh, Theorem 1.6]) *Let X (resp. ${}^{\dagger}X$) be a smooth log-curve of type $(1,1)$ over a finite extension K (resp. ${}^{\dagger}K$) of $\mathbb{Q}_p$ such that X (resp. ${}^{\dagger}X$) has stable reduction over O_K (resp $O_{{}^{\dagger}K}$), and that the special fiber is singular, geometrically irreducible, the node is rational. We use similar notation for objects associated to ${}^{\dagger}X$ to the notation which was used for objects associated to X. Let*

$$\gamma : \Pi_X^{\mathrm{temp}} \xrightarrow{\sim} \Pi_{{}^{\dagger}X}^{\mathrm{temp}}$$

be any isomorphism of abstract topological groups. Then we have the following:

(1) $\gamma(\Pi_{\ddot{Y}}^{\mathrm{temp}}) = \Pi_{{}^{\dagger}\ddot{Y}}^{\mathrm{temp}}$.

(2) γ *induces an isomorphism* $\Delta_\Theta \xrightarrow{\sim} {}^{\dagger}\Delta_\Theta$, *which is compatible with the surjections*

$$H^1(G_{\ddot{K}}, \Delta_\Theta) \xrightarrow{\sim} H^1(G_{\ddot{K}}, \widehat{\mathbb{Z}}(1)) \xrightarrow{\sim} (\ddot{K}^{\times})^{\wedge} \twoheadrightarrow \widehat{\mathbb{Z}}$$
$$H^1(G_{{}^{\dagger}\ddot{K}}, {}^{\dagger}\Delta_\Theta) \xrightarrow{\sim} H^1(G_{{}^{\dagger}\ddot{K}}, \widehat{\mathbb{Z}}(1)) \xrightarrow{\sim} ({}^{\dagger}\ddot{K}^{\times})^{\wedge} \twoheadrightarrow \widehat{\mathbb{Z}}$$

determined the valuations on $\ddot{K}$ *and* ${}^{\dagger}\ddot{K}$ *respectively. In other words,* γ *induces an isomorphism* $H^1(G_{\ddot{K}}, \Delta_\Theta) \xrightarrow{\sim} H^1(G_{{}^{\dagger}\ddot{K}}, {}^{\dagger}\Delta_\Theta)$ *which preserves both the kernel of these surjections and the element* $1 \in \widehat{\mathbb{Z}}$ *in the quotients.*

(3) The isomorphism $\gamma^* : H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}, \Delta_\Theta) \cong H^1(\Pi_{{}^{\dagger}\ddot{Y}}^{\mathrm{temp}}, {}^{\dagger}\Delta_\Theta)$ *induced by* γ *sends* $O_{\ddot{K}}^{\times} \cdot \ddot{\eta}^{\Theta}$ *to some* ${}^{\dagger}\underline{\mathbb{Z}} \cong \Pi_{{}^{\dagger}X}^{\mathrm{temp}}/\Pi_{{}^{\dagger}Y}^{\mathrm{temp}}$*-conjugate of* $O_{{}^{\dagger}\ddot{K}}^{\times} \cdot {}^{\dagger}\ddot{\eta}^{\Theta}$ *(This indeterminacy of* ${}^{\dagger}\underline{\mathbb{Z}}$*-conjugate inevitably arises from the choice of the irreducible component labelled* 0*).*

Remark 7.6.1. ([EtTh, Remark 1.10.3 (i)]) The étale theta function lives in a cohomology group of the theta quotient $(\Pi_X^{\mathrm{temp}})^{\Theta}$, not whole of Π_X^{temp}. However, when we study anabelian properties of the étale theta function as in Proposition 7.6, the theta quotient $(\Pi_X^{\mathrm{temp}})^{\Theta}$ is insufficient, and we *need whole of* Π_X^{temp}.

Remark 7.6.2. ([IUTchIII, Remark 2.1.2]) Related with Remark 7.6.1, then, how about considering $\Pi_X^{\text{partial temp}} := \Pi_X \times_{\widehat{\underline{\mathbb{Z}}}} \underline{\mathbb{Z}}$ instead of Π_X^{temp}? (Here, we write Π_X for the profinite fundamental group, and $\Pi_X \twoheadrightarrow \widehat{\underline{\mathbb{Z}}}$ is the profinite completion of the natural surjection $\Pi_X^{\mathrm{temp}} \twoheadrightarrow \underline{\mathbb{Z}}$.) The answer is that it does *not* work in inter-universal Teichmüller theory since we have $N_{\Pi_X}(\Pi_X^{\text{partial temp}})/\Pi_X^{\text{partial temp}} \xrightarrow{\sim} \widehat{\underline{\mathbb{Z}}}/\underline{\mathbb{Z}}$ (On the other hand, $N_{\Pi_X}(\Pi_X^{\mathrm{temp}}) = \Pi_X^{\mathrm{temp}}$ by Cororally 6.10 (2)). The profinite conjugacy indeterminacy on $\Pi_X^{\text{partial temp}}$ gives rise to $\widehat{\underline{\mathbb{Z}}}$-translation indeterminacies on the coordinates of the evaluation points (cf. Definition 10.17). On the other hand, for Π_X^{temp}, we can reduce the $\widehat{\underline{\mathbb{Z}}}$-translation indeterminacies to $\underline{\mathbb{Z}}$-translation indeterminacies by Theorem 6.11 (cf. also Lemma 11.9).

Remark 7.6.3. The statements in Proposition 7.6 are bi-anabelian ones (cf. Remark 3.4.4). However, we can reconstruct the ${}^{\dagger}\underline{\mathbb{Z}}$-conjugate class of the theta classes

$O^{\times}_{{}^\dagger\ddot{K}} \cdot {}^\dagger\ddot{\eta}^{\Theta}$ in Proposition 7.6 (3) in a *mono-anabelian* manner, by considering the descriptions of the zero-divisor and the pole-divisor of the theta function.

Proof. (1): Firstly, γ sends Δ_X^{temp} to $\Delta_{{}^\dagger X}^{\mathrm{temp}}$, by Lemma 6.2. Next, note that γ sends Δ_Y^{temp} to $\Delta_{{}^\dagger Y}^{\mathrm{temp}}$ by the discreteness (which is a group-theoretic property) of $\underline{\mathbb{Z}}$ and ${}^\dagger\underline{\mathbb{Z}}$. Finally, γ sends the cuspidal decomposition groups to the cuspidal decomposition groups by Corollary 6.12. Hence γ sends $\Pi_{\ddot{Y}}$ to $\Pi_{{}^\dagger\ddot{Y}}$, since the double coverings $\ddot{Y} \twoheadrightarrow Y$ and ${}^\dagger\ddot{Y} \twoheadrightarrow {}^\dagger Y$ are the double covering characterised as the 2-power map $[2] : \widehat{\mathbb{G}}_m \twoheadrightarrow \widehat{\mathbb{G}}_m$ on each irreducible component, where the origin of the target is given by the cusps.

(2): We proved that $\gamma(\Delta_X^{\mathrm{temp}}) = {}^\dagger\Delta_X^{\mathrm{temp}}$. Then $\gamma(\Delta_\Theta) = {}^\dagger\Delta_\Theta$ holds, since Δ_Θ (resp. ${}^\dagger\Delta_\Theta$) is group-theoretically defined from Δ_X^{temp} (resp. ${}^\dagger\Delta_X^{\mathrm{temp}}$). The rest of the claim follows from Corollary 6.12 and Proposition 2.1 (5), (6).

(3): After taking some $\Pi_X^{\mathrm{temp}}/\Pi_Y^{\mathrm{temp}} \cong \underline{\mathbb{Z}}$-conjugate, we may assume that $\gamma : \Pi_{\ddot{Y}}^{\mathrm{temp}} \xrightarrow{\sim} \Pi_{{}^\dagger\ddot{Y}}^{\mathrm{temp}}$ is compatible with suitable inversion automorphisms ι_Y and ${}^\dagger\iota_Y$ by Theorem B.1 (cf. [SemiAnbd, Theorem 6.8 (ii)], [AbsSect, Theorem 2.3]). Next, note that γ tautologically sends $1 \in \widehat{\mathbb{Z}} = \mathrm{Hom}(\Delta_\Theta, \Delta_\Theta) = \mathrm{Fil}^0/\mathrm{Fil}^1$ to $1 \in \widehat{\mathbb{Z}} = \mathrm{Hom}({}^\dagger\Delta_\Theta, {}^\dagger\Delta_\Theta) = {}^\dagger\mathrm{Fil}^0/{}^\dagger\mathrm{Fil}^1$. On the other hand, $\ddot{\eta}^{\Theta}$ (resp. ${}^\dagger\ddot{\eta}^{\Theta}$) is sent to $1 \in \widehat{\mathbb{Z}} = \mathrm{Hom}(\Delta_\Theta, \Delta_\Theta) = \mathrm{Fil}^0/\mathrm{Fil}^1$ (resp. $1 \in \widehat{\mathbb{Z}} = \mathrm{Hom}({}^\dagger\Delta_\Theta, {}^\dagger\Delta_\Theta) = {}^\dagger\mathrm{Fil}^0/{}^\dagger\mathrm{Fil}^1$), and fixed by ι_Y (resp. ${}^\dagger\iota_Y$) up to an $O^{\times}_{\ddot{K}}$-multiple (resp. an $O^{\times}_{{}^\dagger\ddot{K}}$-multiple) by Lemma 7.5 (2). This determines $\ddot{\eta}^{\Theta}$ (resp. ${}^\dagger\ddot{\eta}^{\Theta}$) up to a $(\ddot{K}^{\times})^{\wedge}$-multiple (resp. a $({}^\dagger\ddot{K}^{\times})^{\wedge}$-multiple). Hence it is sufficient to reduce this $(\ddot{K}^{\times})^{\wedge}$-indeterminacy (resp. $({}^\dagger\ddot{K}^{\times})^{\wedge}$-indeterminacy) to an $O^{\times}_{\ddot{K}}$-indeterminacy (resp. an $O^{\times}_{{}^\dagger\ddot{K}}$-indeterminacy). This is done by evaluating the class $\ddot{\eta}^{\Theta}$ (resp. ${}^\dagger\ddot{\eta}^{\Theta}$) at a cusp y of the irreducible component labelled 0 (Note that "labelled 0" is group-theoretically characterised as "fixed by inversion isomorphism ι_Y (resp. ${}^\dagger\iota_Y$)"), if we show that γ preserves the canonical integral structure of D_y.

(cf. also [SemiAnbd, Corollary 6.11] and [AbsSect, Theorem 4.10, Corollary 4.11] for the rest of the proof). To show the preservation of the canonical integral structure of D_y by γ, we may restrict the fundamental group of the irreducible component labelled 0 by Proposition 6.6 and Corollary 6.12 (cf. also Remark 6.12.1). The irreducible component minus nodes $\ddot{\mathfrak{U}}$ is isomorphic to $\widehat{\mathbb{G}}_m$ with marked points (=cusps) $\{\pm 1\} \subset \widehat{\mathbb{G}}_m$. Then the prime-to-$p$-quotient $\Delta_{\mathfrak{U}_{\ddot{K}}}^{\text{prime-to-}p}$ of the geometric fudamental group of the generic fiber is isomorphic to the prime-to-p-quotient $\Delta_{\mathfrak{U}_{\ddot{k}}}^{\text{prime-to-}p}$ of the one of the special fiber, where we write $\ddot{k}$ for the residue field of $\ddot{K}$. This shows that the reduction of the structure group of $(\ddot{K}^{\times})^{\wedge}$-torsor $\mathrm{Sect}(D_y \twoheadrightarrow G_{\ddot{K}})$ to $(O^{\times}_{\ddot{K}})' := \mathrm{Im}(O^{\times}_{\ddot{K}} \to \ddot{K}^{\times} \otimes \widehat{\mathbb{Z}}')$, which is determined the canonical integral strucure (i.e., the canonical tame integral structure), is group-theoretically preserved as follows (cf. [AbsSect, Proposition 4.4 (i)]): The outer action $G_{\ddot{K}} \to \mathrm{Out}(\Delta_{\mathfrak{U}_{\ddot{K}}}^{\text{prime-to-}p})$ canonically factors through $G_{\ddot{k}} \to \mathrm{Out}(\Delta_{\mathfrak{U}_{\ddot{K}}}^{\text{prime-to-}p})$, and the geometrically prime-to-p-quotient $\Pi_{\mathfrak{U}_{\ddot{k}}}^{(\text{prime-to-}p)}$ of the arithmetic fundamental group of the special fiber is group-theoretically constructed as $\Delta^{\text{prime-to-}p}_{\mathfrak{U}_{\ddot{K}}} \overset{\text{out}}{\rtimes} G_{\ddot{k}}$ by using $G_{\ddot{k}} \to \mathrm{Out}(\Delta^{\text{prime-to-}p}_{\mathfrak{U}_{\ddot{K}}})$. Then the decomposition group D'_y in the geometrically prime-to-p-quotient of the arithmetic fundamental group of the integral model fits in a short exact sequence $1 \to (I'_y :=) I_y \otimes \widehat{\mathbb{Z}}' \to D'_y \to G_{\ddot{k}} \to 1$, where I_y is an inertia subgroup at y. The set of the splitting of this short exact sequence forms a torsor over $H^1(G_{\ddot{k}}, I'_y) \cong \ddot{k}^\times$. These splittings can be regarded as elements of $H^1(D'_y, I'_y)$ whose restriction to I'_y is equal to the identity element in $H^1(I'_y, I'_y) = \mathrm{Hom}(I'_y, I'_y)$. Thus, the pull-back to D_y of any such element of $H^1(D'_y, I'_y)$ gives us the reduction of the structure group to $(O^\times_{\ddot{K}})'$ determined by the canonical integral structure.

Then it suffices to show that the reduction of the structure group of $(\ddot{K}^\times)^\wedge$-torsor $\mathrm{Sect}(D_y \twoheadrightarrow G_{\ddot{K}})$ to $\ddot{K}^\times$, which is determined the canonical integral strucure (i.e., the canonical discrete structure), is group-theoretically preserved since the restriction of the projection $\widehat{\mathbb{Z}} \twoheadrightarrow \widehat{\mathbb{Z}}'$ to $\mathbb{Z} \subset \widehat{\mathbb{Z}}$ is injective (cf. [AbsSect, Proposition 4.4 (ii)]).

Finally, we show that the canonical discrete structure of $(\ddot{K}^\times)^\wedge$-torsor $\mathrm{Sect}(D_y \twoheadrightarrow G_{\ddot{K}})$ is group-theoretically preserved. Let $\ddot{U}$ be the canonical cooridnate of $\mathbb{G}_{m\,\ddot{K}}$. For $y = \pm 1$, we consider the unit $\ddot{U} \mp 1 \in \Gamma(\mathbb{G}_{m\,\ddot{K}} \setminus \{\pm 1\}, \mathcal{O}_{\mathbb{G}_{m\,\ddot{K}} \setminus \{\pm 1\}})$, which is invertible at 0, fails to be invertible at y, and has a zero of order 1 at y. We consider the exact sequence

$$1 \to (\ddot{K}^\times)^\wedge \to H^1(\Pi_{\mathbb{P}^1 \setminus \{0,y\}}, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \to \widehat{\mathbb{Z}} \oplus \widehat{\mathbb{Z}}$$

constructed in Lemma 3.15 (5). The image of the Kummer class $\kappa(T \mp 1) \in H^1(\Pi_{\mathbb{P}^1 \setminus \{0,y\}}, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))$ in $\widehat{\mathbb{Z}} \oplus \widehat{\mathbb{Z}}$ (i.e., $(1,0)$) determines the set $(\ddot{K}^\times)^\wedge \cdot \kappa(\ddot{U} \mp 1)$. The restriction of $(\ddot{K}^\times)^\wedge \cdot \kappa(\ddot{U} \mp 1)$ to D_y is the $(\ddot{K}^\times)^\wedge$-torsor $\mathrm{Sect}(D_y \twoheadrightarrow G_{\ddot{K}})$, since the zero of order of $\kappa(\ddot{U} \mp 1)$ at y is 1. On the other hand, $\kappa(\ddot{U} \mp 1)$ is invertible at 0. Thus, the subset $\ddot{K}^\times \cdot \kappa(\ddot{U} \mp 1) \subset (\ddot{K}^\times)^\wedge \cdot \kappa(\ddot{U} \mp 1)$ is characterised as the set of elements of $(\ddot{K}^\times)^\wedge \cdot \kappa(\ddot{U} \mp 1)$ whose restriction to the decomposition group D_0 at 0 (which lies in $(\ddot{K}^\times)^\wedge \cong H^1(G_{\ddot{K}}, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \subset H^1(D_0, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))$ since $\kappa(\ddot{U} \mp 1)$ is invertible at 0) in fact lies in $\ddot{K}^\times \subset (\ddot{K}^\times)^\wedge$. Thus, we are done by Corollary 6.12 (or Corollary 2.9) (cf. the proof of [AbsSect, the proof of Theorem 4.10 (i)]). $\square$

From now on, we assume that

(1) $\ddot{K} = K$,

(2) the hyperbolic curve X minus the marked points admits a K-core $X \twoheadrightarrow C := X/\!/\{\pm 1\}$, where the quotient is taken in the sense of stacks, by the natural action of $\{\pm 1\}$ determined by the multiplication-by-2 map of the underlying elliptic curve of X (Note that this excludes four exceptional j-invariants by Lemma C.3, and

(3) $\sqrt{-1} \in K$.

We write $\ddot{X} \twoheadrightarrow X$ for the Galois covering of degree 4 determined by the multiplication-by-2 map of the underlying elliptic curve of X (i.e., $\mathbb{G}_m^{\mathrm{rig}}/q_X^{\mathbb{Z}} \twoheadrightarrow \mathbb{G}_m^{\mathrm{rig}}/q_X^{\mathbb{Z}}$ sending the coordinate U of the $\mathbb{G}_m^{\mathrm{rig}}$ in the codomain to $\ddot{U}^2$, where $\ddot{U}$ is the coordinate of the ${\mathbb{G}_m}^{\mathrm{rig}}$ in the domain). We write $\ddot{\mathfrak{X}} \twoheadrightarrow \mathfrak{X}$ for its natural integral model. Note that $\ddot{X} \twoheadrightarrow C$ is Galois with $\mathrm{Gal}(\ddot{X}/C) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$.

Choose a square root $\sqrt{-1} \in \overline{K}$ of -1. Note that the 4-torsion points of the underlying elliptic curve of $\ddot{X}$ are $\ddot{U} = \sqrt{-1}^i \sqrt{q_X}^{\frac{j}{4}} \subset \overline{K}$ for $0 \leq i, j \leq 3$, and that, in the irreducible components of $\ddot{\mathfrak{X}}$, the 4-torsion points avoiding nodes are $\pm\sqrt{-1}$. We write τ for the 4-torsion point determined by $\sqrt{-1} \in K$. For an étale theta class $\ddot{\eta}^{\Theta} \in H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}, \Delta_{\Theta})$, we write

$$\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2} \subset H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}, \Delta_{\Theta})$$

for the $\Pi_X^{\mathrm{temp}}/\Pi_{\ddot{Y}}^{\mathrm{temp}} \cong \underline{\mathbb{Z}} \times \mu_2$-orbit of $\ddot{\eta}^{\Theta}$.

Definition 7.7. (cf. [EtTh, Definition 1.9])

(1) We shall refer to each of two sets of values of $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}$

$$\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}|_{\tau}, \quad \ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}|_{\tau^{-1}} \subset K^{\times}$$

as a **standard set of values** of $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}$.

(2) There are two values in $K^{\times}$ of maximal valuations of some standard set of values of $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}$ (Note that $\ddot{\Theta}(q_X^{\frac{a}{2}} \sqrt{-1}) = (-1)^a q_X^{-\frac{a^2}{2}} (\sqrt{-1})^{-2a} \ddot{\Theta}(\sqrt{-1})$ by the third equality of Lemma 7.4 (2), and $\ddot{\Theta}(-q_X^{\frac{a}{2}} \sqrt{-1}) = -\ddot{\Theta}(q_X^{\frac{a}{2}} \sqrt{-1})$ by the second equality of Lemma 7.4 (2)). If they are equal to ± 1, then we say that $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}$ is **of standard type**.

Remark 7.7.1. Double coverings $\dot{X} \twoheadrightarrow X$ and $\dot{C} \twoheadrightarrow C$ are introduced in [EtTh], and they are used to formulate the definitions of a standard set of values and an étale theta class of standard type, ([EtTh, Definition 1.9]), the definition of log-orbicurve of type $(1, \mathbb{Z}/l\mathbb{Z})$, $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$, $(1, \mathbb{Z}/l\mathbb{Z})_{\pm}$, $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})_{\pm}$ ([EtTh, Definition 2.5]), and the constant multiple rigidity of the étale theta function ([EtTh, Theorem 1.10]). However, we avoid them in this survey, since they are not directly used in inter-universal Teichmüller theory, and it is enough to formulate the above things by modifying in a suitable manner.

Lemma 7.8. (cf. [EtTh, Proposition 1.8]) *Let $C = X//\{\pm 1\}$ (resp. ${}^{\dagger}C = {}^{\dagger}X//\{\pm 1\}$) be a smooth log-orbicurve over a finite extension K (resp. ${}^{\dagger}K$) of $\mathbb{Q}_p$ such that $\sqrt{-1} \in K$ (resp. $\sqrt{-1} \in {}^{\dagger}K$). We use the notation ${}^{\dagger}(-)$ for the associated objects with ${}^{\dagger}C$. Let $\gamma : \Pi_C^{\mathrm{temp}} \xrightarrow{\sim} \Pi_{{}^{\dagger}C}^{\mathrm{temp}}$ be an isomorphism of topological groups. Then γ induces isomorphisms $\Pi_X^{\mathrm{temp}} \xrightarrow{\sim} \Pi_{{}^{\dagger}X}^{\mathrm{temp}}$, $\Pi_{\ddot{X}}^{\mathrm{temp}} \xrightarrow{\sim} \Pi_{{}^{\dagger}\ddot{X}}^{\mathrm{temp}}$, and $\Pi_{\ddot{Y}}^{\mathrm{temp}} \xrightarrow{\sim} \Pi_{{}^{\dagger}\ddot{Y}}^{\mathrm{temp}}$.*

Proof. (cf. also the proof of Proposition 7.6 (1)). By Lemma 6.2, the isomorphism γ induces an isomorphism $\gamma_{\Delta_C} : \Delta_C^{\mathrm{temp}} \xrightarrow{\sim} \Delta_{{}^\dagger C}^{\mathrm{temp}}$. Since $\Delta_X^{\mathrm{temp}} \subset \Delta_C^{\mathrm{temp}}$ (resp. $\Delta_{{}^\dagger X}^{\mathrm{temp}} \subset \Delta_{{}^\dagger C}^{\mathrm{temp}}$) is characterised as the open subgroup of index 2 whose profinite completion is torsion-free i.e., corresponds to the geometric fundamental group of a scheme, not a non-scheme-like stack (cf. also [AbsTopI, Lemma 4.1 (iv)]), γ_{Δ_C} induces an isomorphism $\gamma_{\Delta_X} : \Delta_X^{\mathrm{temp}} \xrightarrow{\sim} \Delta_{{}^\dagger X}^{\mathrm{temp}}$. Then γ_{Δ_X} induces an isomorphism $\gamma_{\Delta_X^{\mathrm{ell}}} : (\Delta_X^{\mathrm{temp}})^{\mathrm{ell}} \xrightarrow{\sim} (\Delta_{{}^\dagger X}^{\mathrm{temp}})^{\mathrm{ell}}$, since $(\Delta_X^{\mathrm{temp}})^{\mathrm{ell}}$ (resp. $(\Delta_{{}^\dagger X}^{\mathrm{temp}})^{\mathrm{ell}}$) is group-theoretically constructed from Δ_X^{temp} (resp. $\Delta_{{}^\dagger X}^{\mathrm{temp}}$). By the discreteness of $\mathrm{Gal}(Y/X) \cong \underline{\mathbb{Z}}$ (resp. $\mathrm{Gal}({}^\dagger Y/{}^\dagger X) \cong {}^\dagger\underline{\mathbb{Z}}$), the isomorphism $\gamma_{\Delta_X^{\mathrm{ell}}}$ induces an isomorphism $\gamma_{\underline{\mathbb{Z}}} : \Delta_X^{\mathrm{temp}}/\Delta_Y^{\mathrm{temp}}(\cong \underline{\mathbb{Z}}) \xrightarrow{\sim} \Delta_{{}^\dagger X}^{\mathrm{temp}}/\Delta_{{}^\dagger Y}^{\mathrm{temp}}(\cong {}^\dagger\underline{\mathbb{Z}})$. Thus, by considering the kernel of the action of Π_C^{temp} (resp. $\Pi_{{}^\dagger C}^{\mathrm{temp}}$) on $\Delta_X^{\mathrm{temp}}/\Delta_Y^{\mathrm{temp}}$ (resp. $\Delta_{{}^\dagger X}^{\mathrm{temp}}/\Delta_{{}^\dagger Y}^{\mathrm{temp}}$), the isomorphisms γ and $\gamma_{\underline{\mathbb{Z}}}$ induce an isomorphism $\gamma_{\Pi_X} : \Pi_X^{\mathrm{temp}} \xrightarrow{\sim} \Pi_{{}^\dagger X}^{\mathrm{temp}}$. Since γ_{Π_X} preserves the cuspical decomposition groups by Corollary 6.12, it induces isomorphisms $\Pi_{\ddot{X}}^{\mathrm{temp}} \xrightarrow{\sim} \Pi_{{}^\dagger \ddot{X}}^{\mathrm{temp}}$, and $\Pi_{\ddot{Y}}^{\mathrm{temp}} \xrightarrow{\sim} \Pi_{{}^\dagger \ddot{Y}}^{\mathrm{temp}}$. □

Proposition 7.9. (Constant Multiple Rigidity of the Étale Theta Function, cf. [EtTh, Theorem 1.10]) *Let $C = X//\{\pm 1\}$ (resp. ${}^\dagger C = {}^\dagger X//\{\pm 1\}$) be a smooth log-orbicurve over a finite extension K (resp. ${}^\dagger K$) of $\mathbb{Q}_p$ such that $\sqrt{-1} \in K$ (resp. $\sqrt{-1} \in {}^\dagger K$). We assume that C is a K-core. We use the notation ${}^\dagger(-)$ for the associated objects with ${}^\dagger C$. Let $\gamma : \Pi_C^{\mathrm{temp}} \xrightarrow{\sim} \Pi_{{}^\dagger C}^{\mathrm{temp}}$ be an isomorphism of topological groups. Note that the isomorphism γ induces an isomorphism $\Pi_X^{\mathrm{temp}} \xrightarrow{\sim} \Pi_{{}^\dagger X}^{\mathrm{temp}}$ by Lemma 7.8. Assume that γ maps the subset $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2} \subset H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}, \Delta_\Theta)$ to the subset ${}^\dagger\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2} \subset H^1(\Pi_{{}^\dagger\ddot{Y}}^{\mathrm{temp}}, {}^\dagger\Delta_\Theta)$ (cf. Proposition 7.6 (3)). Then we have the following:*

(1) The isomorphism γ preserves the property that $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ is of standard type, i.e., $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ is of standard type if and only if ${}^\dagger\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ is of standard type. This property uniquely determines this collection of classes.

(2) Note that γ induces an isomorphism $K^\times \xrightarrow{\sim} {}^\dagger K^\times$, where $K^\times$ (resp. ${}^\dagger K^\times$) is regarded a subset of $(K^\times)^\wedge \cong H^1(G_K, \Delta_\Theta) \subset H^1(\Pi_C^{\mathrm{temp}}, \Delta_\Theta))$ (resp. $({}^\dagger K^\times)^\wedge \cong H^1(G_{{}^\dagger K}, {}^\dagger\Delta_\Theta) \subset H^1(\Pi_{{}^\dagger C}^{\mathrm{temp}}, {}^\dagger\Delta_\Theta))$). Then γ maps the standard sets of values of $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ to the standard sets of values of ${}^\dagger\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$.

(3) Assume that $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ (hence ${}^\dagger\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ as well by the claim (1)) is of standard type, and that the residue characteristic of K (hence ${}^\dagger K$ as well) is > 2. Then $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ (resp. ${}^\dagger\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$) determines a $\{\pm 1\}$-structure (cf. Definition 7.3) on $(K^\times)^\wedge$-torsor (resp. $({}^\dagger K^\times)^\wedge$-torsor) at the unique cusp of C (resp. ${}^\dagger C$) which is compatible with the canonical integral structure, and it is preserved by γ.

Remark 7.9.1. The statements in Proposition 7.9 are bi-anabelian ones (cf. Remark 3.4.4). However, we can reconstruct the set ${}^\dagger\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ in Proposition 7.9 (2) and

(3) in a *mono-anabelian* manner, by a similar way as Remark 7.6.3.

Proof. The claims (1) and (3) follows from the claim (2). We show the claim (2). Since γ induces an isomorphism from the dual graph of $\ddot{\mathfrak{Y}}$ to the dual graph of ${}^{\dagger}\ddot{\mathfrak{Y}}$ (Proposition 6.6), by the elliptic cuspidalisation (Theorem 3.7), the isomorphism γ maps the decomposition group of the points of $\ddot{Y}$ lying over τ to the decomposition group of the points of ${}^{\dagger}\ddot{Y}$ lying over $\tau^{\pm 1}$. The claim (2) follows from this. $\square$

§ 7.3. *l*-th Root of the Étale Theta Function.

First, we introduce some log-curves, which are related with l-th root of the étale theta function. Let X be a smooth log-curve of type $(1,1)$ over a field K of characteristic 0 (As before, we always write the log-structure associated to the cusp on X, and consider the log-fundamental group). Note also that we are working in a field of characteristic 0, *not* in a finite extension of $\mathbb{Q}_p$ as in the previous subsections.

Assumption (0): We assume that X admits K-core.

We have a short exact sequence $1 \to \Delta_X \to \Pi_X \to G_K \to 1$, where Π_X and Δ_X are the arithmetic fundamental group and the geometric fundamental group (with respect to some basepoints) respectively, and $G_K = \mathrm{Gal}(\overline{K}/K)$. Write $\Delta_X^{\mathrm{ell}} := \Delta_X^{\mathrm{ab}} = \Delta_X/[\Delta_X, \Delta_X]$, $\Delta_X^{\Theta} := \Delta_X/[\Delta_X, [\Delta_X, \Delta_X]]$, and $\Delta_\Theta := \mathrm{Im}\{\wedge^2 \Delta_X^{\mathrm{ell}} \to \Delta_X^{\Theta}\}$. Then we have a natural exact sequence $1 \to \Delta_\Theta \to \Delta_X^{\Theta} \to \Delta_X^{\mathrm{ell}} \to 1$. Write also $\Pi_X^{\Theta} := \Pi_X/\ker(\Delta_X \twoheadrightarrow \Delta_X^{\Theta})$.

Let $l > 2$ be a prime number. Note that the subgroup of Δ_X^{Θ} generated by l-th powers of elements of Δ_X^{Θ} is normal (Here we use $l \neq 2$). We write $\Delta_X^{\Theta} \twoheadrightarrow \overline{\Delta}_X$ for the quotient of Δ_X^{Θ} by this normal subgroup. Write $\overline{\Delta}_\Theta := \mathrm{Im}\{\Delta_\Theta \to \overline{\Delta}_X\}$, $\overline{\Delta}_X^{\mathrm{ell}} := \overline{\Delta}_X/\overline{\Delta}_\Theta$, $\overline{\Pi}_X := \Pi_X/\ker(\Delta_X \twoheadrightarrow \overline{\Delta}_X)$, and $\overline{\Pi}_X^{\mathrm{ell}} := \overline{\Pi}_X/\overline{\Delta}_\Theta$. Note that $\overline{\Delta}_\Theta \cong (\mathbb{Z}/l\mathbb{Z})(1)$ and $\overline{\Delta}_X^{\mathrm{ell}}$ is a free $\mathbb{Z}/l\mathbb{Z}$-module of rank 2.

Let x be the unique cusp of X, and we write $I_x \subset D_x$ for the inertia subgroup and the decomposition subgroup at x respectively. Then we have a natural injective homomorphism $D_x \hookrightarrow \Pi_X^{\Theta}$ such that the restriction to I_x gives us an isomorphism $I_x \xrightarrow{\sim} \Delta_\Theta (\subset \Pi_X^{\Theta})$. Write also $\overline{D}_x := \mathrm{Im}\{D_x \to \overline{\Pi}_X\}$. Then we have a short exact sequence

$$1 \to \overline{\Delta}_\Theta \to \overline{D}_x \to G_K \to 1.$$

Assumption (1): We choose a quotient $\overline{\Pi}_X^{\mathrm{ell}} \twoheadrightarrow Q$ onto a free $\mathbb{Z}/l\mathbb{Z}$-module of rank 1 such that the restriction $\overline{\Delta}_X^{\mathrm{ell}} \to Q$ to $\overline{\Delta}_X^{\mathrm{ell}}$ remains surjective, and the restriction $D_x \to Q$ to D_x is trivial.

We write

$$\underline{X} \twoheadrightarrow X$$

for the corresponding covering (Note that every cusp of $\underline{X}$ is K-rational, since the restriction $D_x \to Q$ to D_x is trivial) with $\mathrm{Gal}(\underline{X}/X) \cong Q$, and we write $\overline{\Pi}_{\underline{X}} \subset \overline{\Pi}_X$, $\overline{\Delta}_{\underline{X}} \subset \overline{\Delta}_X$, and $\overline{\Delta}^{\mathrm{ell}}_{\underline{X}} \subset \overline{\Delta}^{\mathrm{ell}}_X$ for the corresponding open subgroups. We write ι_X (resp. $\iota_{\underline{X}}$) for the automorphism of X (resp. $\underline{X}$) given by the multiplication by -1 on the underlying elliptic curve, where the origin is given by the unique cusp of X (resp. a choice of a cusp of $\underline{X}$). Write $C := X//\iota_X$, $\underline{C} := \underline{X}//\iota_{\underline{X}}$ (Here, $//$'s mean the quotients in the sense of stacks). We shall refer to a cusp of $\underline{C}$, which arises from the zero (resp. a non-zero) element of Q, as the **zero cusp** (resp. a **non-zero cusp**) of $\underline{C}$. We shall refer to ι_X and $\iota_{\underline{X}}$ as **inversion automorphisms**. We also shall refer to the unique cusp of $\underline{X}$ over the zero cusp of $\underline{C}$ as the **zero cusp** of $\underline{X}$. This $\underline{X}$ (resp. $\underline{C}$) is the main actor for the *global additive* ($\boxplus$) *portion* (resp. *global multiplicative* ($\boxtimes$) *portion*) in inter-universal Teichmüller theory.

Definition 7.10. ([EtTh, Definition 2.1]) A smooth log-orbicurve over K is called **of type $(1, l$-tors$)$** (resp. **of type $(1, l$-tors$)_\pm$**) if it is isomorphic to $\underline{X}$ (resp. $\underline{C}$) for some choice of $\overline{\Pi}^{\mathrm{ell}}_X \twoheadrightarrow Q$ (satisfying Assumption (0), (1)).

Note that $\underline{X} \to X$ is Galois with $\mathrm{Gal}(\underline{X}/X) \cong Q$; however, $\underline{C} \to C$ is *not* Galois, since $\iota_{\underline{X}}$ acts on Q by the multiplication by -1, and any generator of $\mathrm{Gal}(\underline{X}/X)$ does not descend to an automorphism of $\underline{C}$ over C (Here we use $l \neq 2$. cf. [EtTh, Remark 2.1.1]). We write $\Delta_C \subset \Pi_C$ (resp. $\Delta_{\underline{C}} \subset \Pi_{\underline{C}}$) for the geometric fundamental group and the arithmetic fundamental group of C (resp. $\underline{C}$) respectively. Write also $\overline{\Pi}_C := \Pi_C/\ker(\Pi_X \twoheadrightarrow \overline{\Pi}_X)$, (resp. $\overline{\Pi}_{\underline{C}} := \Pi_{\underline{C}}/\ker(\Pi_{\underline{X}} \twoheadrightarrow \overline{\Pi}_{\underline{X}})$,) $\overline{\Delta}_C := \Delta_C/\ker(\Delta_X \twoheadrightarrow \overline{\Delta}_X)$, (resp. $\overline{\Delta}_{\underline{C}} := \Delta_{\underline{C}}/\ker(\Delta_{\underline{X}} \twoheadrightarrow \overline{\Delta}_{\underline{X}})$,), and $\overline{\Delta}^{\mathrm{ell}}_C := \overline{\Delta}_C/\ker(\overline{\Delta}_X \twoheadrightarrow \overline{\Delta}^{\mathrm{ell}}_X)$.

<ins>Assumption (2)</ins>: We choose $\epsilon_{\iota_{\underline{X}}} \in \overline{\Delta}_{\underline{C}}$ an element which lifts the nontrivial element of $\mathrm{Gal}(\underline{X}/\underline{C}) \cong \mathbb{Z}/2\mathbb{Z}$.

We consider the conjugate action of $\epsilon_{\iota_{\underline{X}}}$ on $\overline{\Delta}_{\underline{X}}$, which is a free $\mathbb{Z}/l\mathbb{Z}$-module of rank 2. Then the eigenspace of $\overline{\Delta}_{\underline{X}}$ with eigenvalue -1 (resp. $+1$) is equal to $\overline{\Delta}^{\mathrm{ell}}_{\underline{X}}$ (resp. $\overline{\Delta}_\Theta$). Hence we obtain a direct product decomposition

$$\overline{\Delta}_{\underline{X}} \cong \overline{\Delta}^{\mathrm{ell}}_{\underline{X}} \times \overline{\Delta}_\Theta$$

([EtTh, Proposition 2.2 (i)]) which is compatible with the conjugate action of $\overline{\Pi}_{\underline{X}}$ (since the conjugate action of $\epsilon_{\iota_{\underline{X}}}$ commutes with the conjugate action of $\overline{\Pi}_{\underline{X}}$). We write

$s_\iota : \overline{\Delta}^{\mathrm{ell}}_{\underline{X}} \hookrightarrow \overline{\Delta}_{\underline{X}}$ for the splitting of $\overline{\Delta}_{\underline{X}} \twoheadrightarrow \overline{\Delta}^{\mathrm{ell}}_{\underline{X}}$ given by the above direct product decomposition. Then the normal subgroup $\mathrm{Im}(s_\iota) \subset \overline{\Pi}_{\underline{X}}$ induces an isomorphism

$$\overline{D}_x \xrightarrow{\sim} \overline{\Pi}_{\underline{X}}/\mathrm{Im}(s_\iota)$$

over G_K.

Assumption (3): We choose any element $s^{\mathrm{A}(3)}$ of the $H^1(G_K, \overline{\Delta}_\Theta)(\cong K^\times/(K^\times)^l)$-torsor $\mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K)$, where we write $\mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K)$ for the set of sections of the surjection $\overline{D}_x \twoheadrightarrow G_K$.

Then we obtain a quotient $\Pi_{\underline{X}} \twoheadrightarrow \overline{\Pi}_{\underline{X}} \twoheadrightarrow \overline{\Pi}_{\underline{X}}/\mathrm{Im}(s_\iota) \xrightarrow{\sim} \overline{D}_x \twoheadrightarrow \overline{D}_x/s^{\mathrm{A}(3)}(G_K) \cong \overline{\Delta}_\Theta$. This quotient gives us a covering

$$\underline{\underline{X}} \twoheadrightarrow \underline{X}$$

with $\mathrm{Gal}(\underline{\underline{X}}/\underline{X}) \cong \overline{\Delta}_\Theta$. We write $\overline{\Delta}_{\underline{\underline{X}}} \subset \overline{\Delta}_{\underline{X}}$, $\overline{\Pi}_{\underline{\underline{X}}} \subset \overline{\Pi}_{\underline{X}}$ for the open subgroups determined by $\underline{\underline{X}}$. Note that the composition $\overline{\Delta}_{\underline{\underline{X}}} \hookrightarrow \overline{\Delta}_{\underline{X}} \twoheadrightarrow \overline{\Delta}^{\mathrm{ell}}_{\underline{X}}$ is an isomorphism, and that $\overline{\Delta}_{\underline{\underline{X}}} = \mathrm{Im}(s_\iota)$, $\overline{\Delta}_{\underline{X}} = \overline{\Delta}_{\underline{\underline{X}}} \cdot \overline{\Delta}_\Theta$. Since $\mathrm{Gal}(\underline{\underline{X}}/\underline{X}) = \overline{\Delta}_{\underline{X}}/\overline{\Delta}_{\underline{\underline{X}}} = \overline{\Delta}_\Theta$, and $I_x \cong \Delta_\Theta \twoheadrightarrow \overline{\Delta}_\Theta$, the covering $\underline{\underline{X}} \twoheadrightarrow \underline{X}$ is *totally ramified at the cusps* (Note also that the irreducible components of the special fiber of the stable model of $\underline{X}$ are isomorprhic to $\mathbb{P}^1$; however, the irreducible components of the special fiber of the stable model of $\underline{\underline{X}}$ are *not* isomorphic to $\mathbb{P}^1$). Note also that the image of $\epsilon_{\iota_{\underline{X}}}$ in $\overline{\Delta}_C/\overline{\Delta}_{\underline{\underline{X}}}$ is characterised as the unique coset of $\overline{\Delta}_C/\overline{\Delta}_{\underline{\underline{X}}}$ which lifts the nontrivial element of $\overline{\Delta}_C/\overline{\Delta}_X$ and normalises the subgroup $\overline{\Delta}_{\underline{\underline{X}}} \subset \overline{\Delta}_C$, since the eigenspace of $\overline{\Delta}_{\underline{X}}/\overline{\Delta}_{\underline{\underline{X}}} \cong \overline{\Delta}_\Theta$ with eigenvalue 1 is equal to $\overline{\Delta}_\Theta$ ([EtTh, Proposition 2.2 (ii)]). We omit the construction of "$\underline{\underline{C}}$" (cf. [EtTh, Proposition 2.2 (iii)]), since we do not use it. This $\underline{\underline{X}}$ plays the central role in the theory of mono-theta environment, and it also plays the central role in inter-universal Teichmüller theory for places in $\underline{\mathbb{V}}^{\mathrm{bad}}$.

Definition 7.11. ([EtTh, Definition 2.3]) A smooth log-orbicurve over K is called **of type $(1, l\text{-tors}^\Theta)$** if it is isomorphic to $\underline{\underline{X}}$ (which is constructed under Assumptions (0), (1), (2), and (3)).

The underlines in the notation of $\underline{X}$ and $\underline{C}$ indicate "extracting a copy of $\mathbb{Z}/l\mathbb{Z}$", and the double underlines in the notation of $\underline{\underline{X}}$ and $\underline{\underline{C}}$ indicate "extracting two copy of $\mathbb{Z}/l\mathbb{Z}$" ([EtTh, Remark 2.3.1]).

Lemma 7.12. (cf. [EtTh, Proposition 2.4]) *Let $\underline{\underline{X}}$ (resp. ${}^\dagger\underline{\underline{X}}$) be a smooth log-curve of type* $(1, l\text{-tors}^\Theta)$ *over a finite extension K (resp. ${}^\dagger K$) of $\mathbb{Q}_p$. We use the notation ${}^\dagger(-)$ for the associated objects with ${}^\dagger\underline{\underline{X}}$. Assume that X (resp. ${}^\dagger X$) has stable reduction over O_K (resp. $O_{{}^\dagger K}$) whose special fiber is singular and geometrically* *irreducible, and the node is rational. Let* $\gamma : \Pi^{\mathrm{temp}}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{{}^\dagger\underline{\underline{X}}}$ *be an isomorphism of topological groups. Then* γ *induces isomorphisms* $\Pi^{\mathrm{temp}}_{C} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{{}^\dagger C}$, $\Pi^{\mathrm{temp}}_{\underline{C}} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{{}^\dagger\underline{C}}$ $\Pi^{\mathrm{temp}}_{X} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{{}^\dagger X}$, $\Pi^{\mathrm{temp}}_{\underline{X}} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{{}^\dagger\underline{X}}$, *and* $\Pi^{\mathrm{temp}}_{\ddot{Y}} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{{}^\dagger\ddot{Y}}$.

Proof. By Lemma 6.2, γ induces an isomorphism $\Delta^{\mathrm{temp}}_{\underline{\underline{X}}} \xrightarrow{\sim} \Delta^{\mathrm{temp}}_{{}^\dagger\underline{\underline{X}}}$. By the K-coricity, the isomorphism γ induces an isomorphism $\Pi^{\mathrm{temp}}_{C} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{{}^\dagger C}$, which induces an isomorphism $\Delta^{\mathrm{temp}}_{C} \xrightarrow{\sim} \Delta^{\mathrm{temp}}_{{}^\dagger C}$. Then by the same way as in Lemma 7.8, this induces isomorphisms $\Delta^{\mathrm{temp}}_{X} \xrightarrow{\sim} \Delta^{\mathrm{temp}}_{{}^\dagger X}$, $\Pi^{\mathrm{temp}}_{X} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{{}^\dagger X}$, and $\Pi^{\mathrm{temp}}_{\ddot{Y}} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{{}^\dagger\ddot{Y}}$. Note that $\overline{\Delta}_X$ (resp. $\overline{\Delta}_{{}^\dagger X}$) and $\overline{\Delta}_\Theta$ (resp. ${}^\dagger\overline{\Delta}_\Theta$) are group-theoretically constructed from $\Delta^{\mathrm{temp}}_{X}$ (resp. $\Delta^{\mathrm{temp}}_{{}^\dagger X}$), and that we can group-theoretically reconstruct $\overline{\Delta}_{\underline{\underline{X}}} \subset \Delta^{\mathrm{temp}}_{X}$ (resp. $\overline{\Delta}_{{}^\dagger\underline{\underline{X}}} \subset \Delta^{\mathrm{temp}}_{{}^\dagger X}$) by the image of $\Delta^{\mathrm{temp}}_{\underline{\underline{X}}}$ (resp. $\Delta^{\mathrm{temp}}_{{}^\dagger\underline{\underline{X}}}$). Hence the above isomorphisms induce an isomorphism $\overline{\Delta}_{\underline{X}} \xrightarrow{\sim} \overline{\Delta}_{{}^\dagger\underline{X}}$, since $\overline{\Delta}_{\underline{X}} = \overline{\Delta}_{\underline{\underline{X}}} \cdot \overline{\Delta}_\Theta$ (resp. $\overline{\Delta}_{{}^\dagger\underline{X}} = \overline{\Delta}_{{}^\dagger\underline{\underline{X}}} \cdot {}^\dagger\overline{\Delta}_\Theta$). This isomorphism induces an isomorphism $\Delta^{\mathrm{temp}}_{\underline{X}} \xrightarrow{\sim} \Delta^{\mathrm{temp}}_{{}^\dagger\underline{X}}$, since $\Delta^{\mathrm{temp}}_{\underline{X}}$ (reps. $\Delta^{\mathrm{temp}}_{{}^\dagger\underline{X}}$) is the inverse image of $\overline{\Delta}_{\underline{X}} \subset \Delta^{\mathrm{temp}}_{X}$ (resp. $\overline{\Delta}_{{}^\dagger\underline{X}} \subset \Delta^{\mathrm{temp}}_{{}^\dagger X}$) under the natural quotient $\Delta^{\mathrm{temp}}_{X} \twoheadrightarrow \overline{\Delta}_X$ (resp. $\Delta^{\mathrm{temp}}_{{}^\dagger X} \twoheadrightarrow \overline{\Delta}_{{}^\dagger X}$). The isomorphism $\Delta^{\mathrm{temp}}_{\underline{X}} \xrightarrow{\sim} \Delta^{\mathrm{temp}}_{{}^\dagger\underline{X}}$ induces an isomprhism $\Pi^{\mathrm{temp}}_{\underline{X}} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{{}^\dagger\underline{X}}$, since $\Pi^{\mathrm{temp}}_{\underline{X}}$ (resp. $\Pi^{\mathrm{temp}}_{{}^\dagger\underline{X}}$) is reconstructed as the outer semi-direct product $\Delta_{\underline{X}} \overset{\mathrm{out}}{\rtimes} G_K$ (resp. $\Delta_{{}^\dagger\underline{X}} \overset{\mathrm{out}}{\rtimes} G_{{}^\dagger K}$), where the homomorphism $G_K \to \mathrm{Out}(\Delta_{\underline{X}})$ (resp. $G_{{}^\dagger K} \to \mathrm{Out}(\Delta_{{}^\dagger\underline{X}})$) is given by the above constructions induced by the action of G_K (resp. $G_{{}^\dagger K}$). $\square$

Remark 7.12.1. ([EtTh, Remark 2.6.1]) Suppose $\mu_l \subset K$. By Lemma 7.12, we obtain

$$\mathrm{Aut}_K(\underline{\underline{X}}) = \mu_l \times \{\pm 1\}, \quad \mathrm{Aut}_K(\underline{X}) = \mathbb{Z}/l\mathbb{Z} \rtimes \{\pm 1\}, \quad \mathrm{Aut}_K(\underline{C}) = \{1\},$$

where $\rtimes$ is given by the natural multiplicative action of $\{\pm 1\}$ on $\mathbb{Z}/l\mathbb{Z}$ (Note that $\underline{C} \to C$ is *not* Galois, as already remarked after Definition 7.10 (cf. [EtTh, Remark 2.1.1])).

Now, we return to the situation where K is a finite extension of $\mathbb{Q}_p$.

Definition 7.13. ([EtTh, Definition 2.5]) Assume that the residue characteristic of K is odd, and that $K = \ddot{K}$. We also make the following two assumptions:

Assumption (4): We assume that the quotient $\overline{\Pi}^{\mathrm{ell}}_X \twoheadrightarrow Q$ factors through the natural quotient $\Pi_X \twoheadrightarrow \widehat{\mathbb{Z}}$ determined by the quotient $\Pi^{\mathrm{temp}}_X \twoheadrightarrow \mathbb{Z}$ discussed when we defined Y.

Assumption (5): We assume that the choice of an element of $\mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K)$ in Assumption (3) is compatible with the $\{\pm 1\}$-structure (cf. Definition 7.3) of Proposition 7.9 (3).

A smooth log-orbicurve over K is called **of type $(1, \mathbb{Z}/l\mathbb{Z})$** (resp. **of type $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$**, resp. **of type $(1, \mathbb{Z}/l\mathbb{Z})_{\pm}$**), if it is isomorphic to $\underline{X}$ (resp. $\underline{\underline{X}}$, resp. $\underline{C}$) (which is constructed under the Assumptions (0), (1), (2), (3), (4), and (5)).

Note also that the definitions of smooth log-(orbi)curves of type $(1, l\text{-tors})$, of type $(1, l\text{-tors})_{\pm}$, and of type $(1, l\text{-tors}^{\Theta})$ are made over any field of characteristic 0, and that the definitions of smooth log-(orbi)curves of type $(1, \mathbb{Z}/l\mathbb{Z})$, of type $(1, \mathbb{Z}/l\mathbb{Z})_{\pm}$ and of type $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$ are made only over finite extensions of $\mathbb{Q}_p$.

Let $\underline{\underline{Y}} \twoheadrightarrow X$ (resp. $\underline{\underline{\ddot{Y}}} \twoheadrightarrow X$) be the composite of the covering $Y \twoheadrightarrow X$ (resp. $\ddot{Y} \twoheadrightarrow X$) with $\underline{\underline{X}} \twoheadrightarrow X$. Note that the coverings $\underline{\underline{\ddot{Y}}} \twoheadrightarrow \ddot{Y}$ and $\underline{\underline{Y}} \twoheadrightarrow Y$ are of degree l.

We have the following diagram

$$\begin{tikzcd}
 & & \underline{\underline{\ddot{Y}}} \arrow[dll, "\mu_2"'] \arrow[dr, "\overline{\Delta}_{\Theta}(\cong \mathbb{Z}/l\mathbb{Z})"] & \\
\underline{\underline{Y}} \arrow[rr, "\overline{\Delta}_{\Theta}(\cong \mathbb{Z}/l\mathbb{Z})"] \arrow[d, "l\underline{\mathbb{Z}}"] & & Y \arrow[d, "\underline{\mathbb{Z}}"] & \ddot{Y} \arrow[l, "\mu_2"'] \arrow[d, "2\underline{\mathbb{Z}}"] \\
\underline{\underline{X}} \arrow[r, "\overline{\Delta}_{\Theta}(\cong \mathbb{Z}/l\mathbb{Z})"] & \underline{X} \arrow[r, "Q(\cong \mathbb{Z}/l\mathbb{Z})"] \arrow[d, "\{\pm 1\}"] & X \arrow[d, "\{\pm 1\}"] & \ddot{X} \arrow[l, "\text{ext of } \mathbb{Z}/2\mathbb{Z}"', "\text{by } \mu_2"] \\
 & \underline{C} \arrow[r, "\text{non-Galois}", "\deg = l"'] & C, &
\end{tikzcd}$$

and note that the irreducible components and cusps in the special fibers of X, $\ddot{X}$, $\underline{X}$, $\underline{\underline{X}}$, Y, $\ddot{Y}$, $\underline{\underline{Y}}$, and $\underline{\underline{\ddot{Y}}}$ are described as follows (Note that $\underline{\underline{X}} \twoheadrightarrow \underline{X}$ and $\underline{\underline{Y}} \twoheadrightarrow Y$ are *totally ramified at each cusp*):

- X: 1 irreducible component (whose noramalisation $\cong \mathbb{P}^1$) and 1 cusp on it.

- $\ddot{X}$: 2 irreducible components ($\cong \mathbb{P}^1$) and 2 cusps on each,

- $\underline{X}$: l irreducible components ($\cong \mathbb{P}^1$) and 1 cusp on each,

- $\underline{\underline{X}}$: l irreducible components ($\ncong \mathbb{P}^1$) and 1 cusp on each,

- Y: the irreducible components ($\cong \mathbb{P}^1$) are parametrised by $\underline{\mathbb{Z}}$, and 1 cusp on each,

- $\ddot{Y}$: the irreducible components ($\cong \mathbb{P}^1$) are parametrised by $\underline{\mathbb{Z}}$, and 2 cusps on each,

- $\underline{\underline{Y}}$: the irreducible components ($\ncong \mathbb{P}^1$) are parametrised by $l\underline{\mathbb{Z}}$, and 1 cusp on each,

- $\underline{\underline{\ddot{Y}}}$: the irreducible components ($\ncong \mathbb{P}^1$) are parametrised by $l\underline{\mathbb{Z}}$, and 2 cusps on each.

We have introduced the needed log-curves. Now, we consider the étale theta functions. By Assumption (4), the covering $\ddot{Y} \twoheadrightarrow X$ factors through $\underline{X}$. Hence the class $\ddot{\eta}^{\Theta} \in H^1(\Pi^{\mathrm{temp}}_{\ddot{Y}}, \Delta_{\Theta})$, which is well-defined up to an $O_K^{\times}$-multiple, and its $\Pi^{\mathrm{temp}}_{X}/\Pi^{\mathrm{temp}}_{\ddot{Y}} \cong \underline{\mathbb{Z}} \times \mu_2$-orbit can be regarded as objects associated to $\Pi^{\mathrm{temp}}_{\underline{X}}$.

We recall that the element $\ddot{\eta}^{\Theta} \in H^1(\Pi^{\mathrm{temp}}_{\ddot{Y}}, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z})$ arises froma an element $\ddot{\eta}^{\Theta} \in H^1((\Pi^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z})$ by the first claim of Lemma 7.5 (2), where we use the same symbol $\ddot{\eta}^{\Theta}$ by abuse of notation. The natural map $D_x \to \Pi^{\mathrm{temp}}_{\ddot{Y}} \to (\Pi^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}$ induces a homomorphism $H^1((\Pi^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}) \to H^1(D_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z})$, and the image of $\ddot{\eta}^{\Theta} \in H^1((\Pi^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z})$ in $H^1(D_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z})$ comes from an element $\ddot{\eta}^{\Theta} \in H^1(\overline{D}_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z})$, where we use the same symbol $\ddot{\eta}^{\Theta}$ by abuse of notation again, via the natural map $H^1(\overline{D}_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}) \to H^1(D_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z})$, since we have an exact sequence

$$0 \to H^1(\overline{D}_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}) \to H^1(D_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}) \to H^1(l\Delta_{\Theta}, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}),$$

and the image of $\ddot{\eta}^{\Theta}$ in $H^1(l\Delta_{\Theta}, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}) = \mathrm{Hom}(l\Delta_{\Theta}, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z})$ vanishes by the first claim of Lemma 7.5 (2). On the other hand, for any element $s \in \mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K)$, the map $\overline{D}_x \ni g \mapsto g(s(\overline{g}))^{-1}$ gives us a 1-cocycle, hence a cohomology class in $H^1(\overline{D}_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z})$, where we write $\overline{g}$ for the image of g via the natural map $\overline{D}_x \twoheadrightarrow G_K$. In this way, we obtain a map $\mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K) \to H^1(\overline{D}_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z})$. (cf. the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^1(\overline{D}_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}) \arrow[r] & H^1(D_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}) \arrow[r] & \mathrm{Hom}(l\Delta_{\Theta}, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}) \\
& \mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K) \arrow[u] & H^1((\Pi^{\mathrm{temp}}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}), \arrow[u] &
\end{tikzcd}$$

where the horizontal sequence is exact.) We also have a natural exact sequence

$$0 \to H^1(G_K, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}) \to H^1(\overline{D}_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}) \to H^1(\Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}).$$

The image of $\ddot{\eta}^{\Theta} \in H^1(\overline{D}_x, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z})$ in $H^1(\Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}) = \mathrm{Hom}(\Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z}, \Delta_{\Theta} \otimes \mathbb{Z}/l\mathbb{Z})$ is the identity homomorphism by the first claim of Lemma 7.5 (2) again. The image $\mathrm{Im}(s) \in H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$ of any element $s \in \mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K)$ via the above map $\mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K) \to H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$ in $H^1(\Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z}, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z}) = \mathrm{Hom}(\Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z}, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$ is also the identity homomorphism by the calculation $\Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z} \ni g \mapsto g(s(\overline{g}))^{-1} = g(s(1))^{-1} = g \cdot 1^{-1} = g$. Hence any element in $\mathrm{Im}\{\mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K) \to H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})\}$ differs from $\ddot{\eta}^\Theta \in H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$ by an $H^1(G_K, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z}) \cong K^\times/(K^\times)^l$-mutiple. Now, we consider the element $s^{\mathrm{A}(3)} \in \mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K)$ which is chosen in Assumption (3), and we write $\mathrm{Im}(s^{\mathrm{A}(3)}) \in H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$ for its image in $H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$. By the above discussions, we can modify $\ddot{\eta}^\Theta \in H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$ by a $K^\times$-multiple, which is well-defined up to a $(K^\times)^l$-multiple, to make it coincide with $\mathrm{Im}(s^{\mathrm{A}(3)}) \in H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$. Note that stronger claim also holds, i.e., we can modify $\ddot{\eta}^\Theta$ by an $O_K^\times$-multiple, which is well-defined up to an $(O_K^\times)^l$-multiple, to make it coincide with $\mathrm{Im}(s^{\mathrm{A}(3)})$, since $s^{\mathrm{A}(3)} \in \mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K)$, is compatible with the canonical integral structure of D_x by Assumption (5) (Note that now we do not assume that $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}$ is of standard type; however, the assumption that $s^{\mathrm{A}(3)}$ is compatible with the $\{\pm 1\}$-structure in the case where $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}$ is of standard type implies that $s^{\mathrm{A}(3)}$ is compatible with the canonical integral structure of D_x even we do not assume that $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}$ is of standard type). As a conclusion, by modifying $\ddot{\eta}^\Theta \in H^1((\Pi_{\ddot{Y}}^{\mathrm{temp}})^\Theta, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$ by an $O_K^\times$-multiple, which is well-defined up to an $(O_K^\times)^l$-multiple, we can and we shall assume that $\ddot{\eta}^\Theta = \mathrm{Im}(s^{\mathrm{A}(3)}) \in H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$, and we obtain an element $\ddot{\eta}^\Theta \in H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$, which is well-defined up to an $(O_K^\times)^l$-multiple (not an $O_K^\times$-multiple), i.e., by the choice of $\underline{\underline{X}}$, the indeterminacy on the ratio of s_l and τ_l in the definition of $\ddot{\eta}^\Theta$ disappeared. In the above construction, an element $\mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K)$ can be considered as "modulo l tangential basepoint" at the cusp x, the theta function $\ddot{\Theta}$ has a simple zero at the cusps (i.e., it is a uniformiser at the cusps), and we made choices in such a way that $\ddot{\eta}^\Theta = \mathrm{Im}(s^{\mathrm{A}(3)})$ holds. Hence the covering $\underline{\underline{X}} \twoheadrightarrow \underline{X}$ can be regarded as a covering of "taking a l-th root of the theta function".

Note that we have the following diagram

$$\begin{tikzcd}
 & H^1(s^{\mathrm{A}(3)}(G_K), \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z}) & & \\
 & H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z}) \arrow[u] & & \\
0 \arrow[r] & H^1(\overline{D}_x / s^{\mathrm{A}(3)}(G_K), \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z}) \arrow[r] \arrow[u] & H^1(\Pi_{\ddot{Y}}^{\mathrm{temp}}, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z}) \arrow[r] & H^1(\Pi_{\underline{\underline{\ddot{Y}}}}^{\mathrm{temp}}, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z}) \\
 & 0, \arrow[u] & &
\end{tikzcd}$$

where the horizontal sequence and the vertical sequence are exact. Now, the image of $\ddot{\eta}^{\Theta} = \mathrm{Im}(s^{\mathrm{A}(3)}) \in H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$ in $H^1(s^{\mathrm{A}(3)}(G_K), \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$ vanishes by the calculation $s^{\mathrm{A}(3)}(G_K) \ni s^{\mathrm{A}(3)}(g) \mapsto s^{\mathrm{A}(3)}(g)(s^{\mathrm{A}(3)}(\overline{s^{\mathrm{A}(3)}(g)}))^{-1} = s^{\mathrm{A}(3)}(g)(s^{\mathrm{A}(3)}(g))^{-1} = 1$ and the above vertical sequence. Thus, $\ddot{\eta}^{\Theta} = \mathrm{Im}(s^{\mathrm{A}(3)})$ comes from an element of $H^1(\overline{D}_x/s^{\mathrm{A}(3)}(G_K), \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$. Therefore, the image of $\ddot{\eta}^{\Theta} \in H^1(\Pi^{\mathrm{temp}}_{\ddot{Y}}, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$ in $H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$ vanishes since it arises from the element of $H^1(\overline{D}_x/s^{\mathrm{A}(3)}(G_K), \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$ and the above horizontal sequence. As a conclusion, the image of $\ddot{\eta}^{\Theta} \in H^1(\Pi^{\mathrm{temp}}_{\ddot{Y}}, \Delta_\Theta)$ in $H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}, \Delta_\Theta)$ arises from an element $\underline{\underline{\ddot{\eta}}}^{\Theta} \in H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}, l\Delta_\Theta)$, which is well-defined up to $O_K^{\times}$. In some sense, $\underline{\underline{\ddot{\eta}}}^{\Theta}$ can be considered as an "l-th root of the étale theta function". We write $\underline{\underline{\ddot{\eta}}}^{\Theta, l\underline{\mathbb{Z}} \times \mu_2}$ for the $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}/\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}} \cong (l\underline{\mathbb{Z}} \times \mu_2)$-orbits of $\underline{\underline{\ddot{\eta}}}^{\Theta}$.

Definition 7.14. ([EtTh, Definition 2.7]) We shall refer to $\underline{\underline{\ddot{\eta}}}^{\Theta, l\underline{\mathbb{Z}} \times \mu_2}$ as **of standard type**, if $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}$ is of standard type.

By combining Proposition 7.9 Lemma 7.12, and definitions, we obtain the following:

Corollary 7.15. (Constant Multiple Rigidity of l-th Roots of the Étale Theta Function, cf. [EtTh, Corollary 2.8]) *Let $\underline{\underline{X}}$ (resp. ${}^{\dagger}\underline{\underline{X}}$) be a smooth log-curve of type $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$ over a finite extension K (resp. ${}^{\dagger}K$) of $\mathbb{Q}_p$. We use the notation ${}^{\dagger}(-)$ for the associated objects with ${}^{\dagger}\underline{\underline{X}}$. Let $\gamma : \Pi^{\mathrm{temp}}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{{}^{\dagger}\underline{\underline{X}}}$ be an isomorphism of topological groups.*

(1) The isomorphism γ preserves the property that $\underline{\underline{\ddot{\eta}}}^{\Theta, l\underline{\mathbb{Z}} \times \mu_2}$ is of standard type. Moreover, this property determines this collection of classes up to a μ_l-multiple.

(2) Assume that the cusps of $\underline{\underline{X}}$ are rational over K, the residue characteristic of K is prime to l, and that $\mu_l \subset K$. Then the $\{\pm 1\}$-structure of Proposition 7.9 (3) determinesa μ_{2l}-structure (cf. Definition 7.3) at the decomposition groups of the cusps of $\underline{\underline{X}}$. Moreover, this μ_{2l}-structure is compatible with the canonical integral structure (cf. Definition 7.3) at the decomposition groups of the cusps of $\underline{\underline{X}}$, and is preserved by γ.

Remark 7.15.1. The statements in Corollary 7.15 are bi-anabelian ones (cf. Remark 3.4.4). However, we can reconstruct the set $\underline{\underline{\ddot{\eta}}}^{\Theta, l\underline{\mathbb{Z}} \times \mu_2}$ in Corollary 7.15 (1) in a *mono-anabelian* manner, by a similar way as Remark 7.6.3 and Remark 7.9.1.

Lemma 7.16. ([EtTh, Corollary 2.9]) *Assume that $\mu_l \subset K$. We make a labelling on the cusps of $\underline{\underline{X}}$, which is induced by the labelling of the irreducible components of $\mathfrak{Y}$ by $\mathbb{Z}$. Then this determines a bijection*

$$\left\{ \textit{Cusps of } \underline{\underline{X}} \right\} / \mathrm{Aut}_K(\underline{\underline{X}}) \cong |\mathbb{F}_l|$$

(cf. Section 0.2 for $|\mathbb{F}_l|$), and this bijection is preserved by any isomorphism $\gamma : \Pi^{\mathrm{temp}}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$ *of topological groups.*

Proof. The first claim is trivial (cf. also Remark 7.12.1). The second claim follows from Remark 6.12.1. $\square$

§ 7.4. Three Fundamental Rigidities of Mono-theta Environments.

In this subsection, we introduce the notion of mono-theta environment, and show important three rigidities of mono-theta environment, that is, the constant multiple rigidity, the cyclotomic rigidity, and the discrete rigidity.

Definition 7.17. For an integer $N \geq 1$, we put

$$\Pi_{\mu_N,K} := \mu_N \rtimes G_K.$$

For a topological group Π with a surjective continuous homomorphism $\Pi \twoheadrightarrow G_K$, we put

$$\Pi[\mu_N] := \Pi \times_{G_K} \Pi_{\mu_N,K}, \quad \Delta[\mu_N] := \ker(\Pi[\mu_N] \twoheadrightarrow G_K) = \Delta \times \mu_N,$$

where $\Delta := \ker(\Pi \twoheadrightarrow G_K)$, and we shall refer to $\Pi[\mu_N]$ as **cyclotomic envelope** of $\Pi \twoheadrightarrow G_K$. We also write

$$\mu_N(\Pi[\mu_N]) := \ker(\Pi[\mu_N] \twoheadrightarrow \Pi).$$

and we shall refer to $\mu_N(\Pi[\mu_N])$ as the (mod N) **cyclotome of the cyclotomic envelope** $\Pi[\mu_N]$. Note that we have a tautological section $G_K \to \Pi_{\mu_N,K}$ of $\Pi_{\mu_N,K} \twoheadrightarrow G_K$, and that it determines a section

$$s^{\mathrm{alg}}_{\Pi} : \Pi \to \Pi[\mu_N],$$

and we shall refer to it as a **mod N tautological section**. For any object with $\Pi[\mu_N]$-conjugate action, we shall refer to a μ_N-orbit as a **μ_N-conjugacy class**.

Here, the μ_N in $\Pi[\mu_N]$ plays a roll of "μ_N" which comes from line bundles.

Lemma 7.18. ([EtTh, Proposition 2.11]) *Let $\Pi \twoheadrightarrow G_K$ (resp. ${}^\dagger\Pi \twoheadrightarrow G_{{}^\dagger K}$) be an open subgroup of the tempered or profinite fundamental group of hyperbolic orbicurve over a finite extension K (resp. ${}^\dagger K$) of $\mathbb{Q}_p$, and write $\Delta := \ker(\Pi \twoheadrightarrow G_K)$ (resp. ${}^\dagger\Delta := \ker({}^\dagger\Pi \twoheadrightarrow G_{{}^\dagger K})$).*

(1) The kernel of the natural surjection $\Delta[\mu_N] \twoheadrightarrow \Delta$ (resp. ${}^\dagger\Delta[\mu_N] \twoheadrightarrow {}^\dagger\Delta$) is equal to the center of $\Delta[\mu_N]$ (resp. ${}^\dagger\Delta[\mu_N]$). In particular, any isomorphism $\Delta[\mu_N] \xrightarrow{\sim} {}^\dagger\Delta[\mu_N]$ is compatible with the surjections $\Delta[\mu_N] \twoheadrightarrow \Delta$, ${}^\dagger\Delta[\mu_N] \twoheadrightarrow {}^\dagger\Delta$.

(2) The kernel of the natural surjection $\Pi[\mu_N] \twoheadrightarrow \Pi$ (resp. ${}^\dagger\Pi[\mu_N] \twoheadrightarrow {}^\dagger\Pi$) is equal to the union of the center of the open subgroups of $\Pi[\mu_N]$ (resp. ${}^\dagger\Pi[\mu_N]$). In particular, any isomorphism $\Pi[\mu_N] \xrightarrow{\sim} {}^\dagger\Pi[\mu_N]$ is compatible with the surjections $\Pi[\mu_N] \twoheadrightarrow \Pi$, ${}^\dagger\Pi[\mu_N] \twoheadrightarrow {}^\dagger\Pi$.

Proof. Lemma follows from the temp-slimness (Lemma 6.4 (5)) or the slimness (Proposition 2.7 (2a), (2b)) of Δ, ${}^\dagger\Delta$, Π, ${}^\dagger\Pi$. □

Proposition 7.19. ([EtTh, Proposition 2.12])

(1) We have an inclusion

$$\ker\left((\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta \twoheadrightarrow (\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^{\mathrm{ell}}\right) = l\Delta_\Theta \subset \left[(\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta, (\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta\right].$$

(2) We have an equality

$$\left[(\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta[\mu_N], (\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta[\mu_N]\right] \bigcap (l\Delta_\Theta)[\mu_N] = \mathrm{Im}\left(s^{\mathrm{alg}}_{(\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta}\big|_{l\Delta_\Theta} : l\Delta_\Theta \to (\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta[\mu_N]\right)$$
$$\left(\subset (l\Delta_\Theta)[\mu_N] \subset (\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta[\mu_N]\right),$$

where we write $s^{\mathrm{alg}}_{(\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta}\big|_{l\Delta_\Theta}$ for the restriction of the mod N tautological section $s^{\mathrm{alg}}_{(\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta} : (\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta \to (\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta[\mu_N]$ to $l\Delta_\Theta$ $(\subset (\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta)$.

Proof. The inclusion of (1) follows from the structure of the theta group (=Heisenberg group) $(\Delta^{\mathrm{temp}}_{X})^\Theta$. The equality of (2) follows from (1). □

Remark 7.19.1. (cf. [EtTh, Remark2.12.1]) As a conclusion of Proposition 7.19 the subgroup $\mathrm{Im}\left(s^{\mathrm{alg}}_{(\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^\Theta}\big|_{l\Delta_\Theta}\right)$, – i.e., the splitting $l\Delta_\Theta \times \mu_N$ –, can be group-theoretically reconstructed, and the cyclotomic rigidity of mono-theta environment (cf. Theorem 7.23 (1)), which plays an important role in inter-universal Teichmüller theory, comes from this fact. Note that the inclusion of Proposition 7.19 (1) does not hold if we use $\underline{X}$ instead of $\underline{\underline{X}}$, i.e., $\ker\left((\Delta^{\mathrm{temp}}_{\underline{X}})^\Theta \twoheadrightarrow (\Delta^{\mathrm{temp}}_{\underline{X}})^{\mathrm{ell}}\right) = \Delta_\Theta \not\subset \left[(\Delta^{\mathrm{temp}}_{\underline{X}})^\Theta, (\Delta^{\mathrm{temp}}_{\underline{X}})^\Theta\right]$.

We write $s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}$ for the composite

$$s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}} : \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}} \xrightarrow{s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}}} \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}[\mu_N] \hookrightarrow \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N],$$

and we shall refer to it as a **mod N algebraic section**. Let $\eta : \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}} \to l\Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z} \cong \mu_N$ denote the composite of the reduction modulo N of any element (i.e., a

1-cocycle) of the collection of classes $\underline{\underline{\ddot{\eta}}}^{\Theta,l\underline{\underline{\mathbb{Z}}}\times\mu_2} \subset H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}, l\Delta_\Theta)$, and the isomorphism $l\Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z} \cong \mu_N$, which comes from a scheme theory (cf. Remark 3.15.1). We put

$$s^{\Theta}_{\underline{\underline{\ddot{Y}}}} := \eta^{-1} \cdot s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}} : \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N].$$

We shall refer to $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$ as a **mod N theta section**. Note that $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$ is a homomorphism, since

$$\begin{aligned} s^{\Theta}_{\underline{\underline{\ddot{Y}}}}(gh) &= \eta(gh)^{-1} s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}(gh) = (g(\eta(h))\eta(g))^{-1} s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}(g) s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}(h) \\ &= (s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}(g)\eta(h) s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}(g)^{-1}\eta(g))^{-1} s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}(g) s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}(h) \\ &= \eta(g)^{-1} s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}(g)\eta(h)^{-1} s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}(h) = s^{\Theta}_{\underline{\underline{\ddot{Y}}}}(g) s^{\Theta}_{\underline{\underline{\ddot{Y}}}}(g). \end{aligned}$$

Note also that the natural outer action

$$\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}}) \cong \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}/\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}} \cong \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}[\mu_N]/\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N] \hookrightarrow \mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$$

of $\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}})$ on $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]$ fixes $\mathrm{Im}(s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}} : \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$ up to a conjugate by μ_N, since the mod N algebraic section $s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}$ extends to a mod N tautological section $s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}} : \Pi^{\mathrm{temp}}_{\underline{\underline{X}}} \to \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}[\mu_N]$. Hence $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$ up to $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}[\mu_N]$-conjugates is independent of the choice of an element of $\underline{\underline{\ddot{\eta}}}^{\Theta,l\underline{\underline{\mathbb{Z}}}\times\mu_2} \subset H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}, l\Delta_\Theta)$ (Recall that $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}} \twoheadrightarrow \mathrm{Gal}(\underline{\underline{\ddot{Y}}}/\underline{\underline{X}}) \cong l\underline{\underline{\mathbb{Z}}} \times \mu_2$). Note also that conjugates by μ_N corresponds to modifying a 1-cocycle by 1-coboundaries.

Note that we have a natural outer action

$$K^\times \twoheadrightarrow K^\times/(K^\times)^N \xrightarrow{\sim} H^1(G_K, \mu_N) \hookrightarrow H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}, \mu_N) \to \mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]),$$

where the isomorphism is the Kummer map, and the last homomorphism is given by sending a 1-cocycle s to an outer homomorphism $s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)a \mapsto s(g)s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)a$ ($g \in \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$, $a \in \mu_N$) (Note that the last homomorphism is well-defined, since $s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)as^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g')a'(= s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(gg')s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g')^{-1}(a)a')$ for $g, g' \in \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$, $a, a' \in \mu_N$ is sent to

$$\begin{aligned} &s(gg')s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(gg')s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g')^{-1}(a)a' = g(s(g'))s(g)s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(gg')s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g')^{-1}as^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g')a' \\ &= s(g)g(s(g'))s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)as^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g')a' = s(g)s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)s(g')as^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g')a' \end{aligned}$$

by s, and since for a 1-coboundary $s(g) = b^{-1}g(b)$ ($b \in \mu_N$) is sent to

$$\begin{aligned} &s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)a \mapsto s(g)s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)a = b^{-1}g(b)s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)a = b^{-1}s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)bs^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)^{-1}s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)a \\ &= b^{-1}s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)ba = b^{-1}s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}}(g)ab, \end{aligned}$$

which is an inner automorphism). Note also any element $\mathrm{Im}(K^\times) := \mathrm{Im}(K^\times \to \mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]))$ lifts to an element of $\mathrm{Aut}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$ which induces the identity automorphisms of both the quotient $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N] \twoheadrightarrow \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$ and the kernel of this quotient. In this natural outer action of $K^\times$, an $O_K^\times$-multiple on $\underline{\underline{\ddot{\eta}}}^{\Theta, l\underline{\underline{\mathbb{Z}}}\times\mu_2}$ corresponds to an $O_K^\times$-conjugate of $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$.

Definition 7.20. (Mono-theta Environment, [EtTh, Definition 2.13]) We write

$$\mathcal{D}_{\underline{\underline{Y}}} := \langle \mathrm{Im}(K^\times), \mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}}) \rangle \subset \mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$$

for the subgroup of $\mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$ generated by $\mathrm{Im}(K^\times)$ and $\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}})$ ($\cong l\underline{\underline{\mathbb{Z}}}$).

(1) We shall refer to the following collection of data as a **mod N model mono-theta environment**:

- the topological group $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]$,
- the subgroup $\mathcal{D}_{\underline{\underline{Y}}}$ ($\subset \mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$), and
- the μ_N-conjugacy class of subgroups in $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]$ determined by the image of the theta section $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$.

(2) We shall refer to any collection $\mathbb{M} = (\Pi, \mathcal{D}_\Pi, s^{\Theta}_\Pi)$ of the following data as a **mod N mono-theta environment**:

- a topological group Π,
- a subgroup $\mathcal{D}_\Pi$ ($\subset \mathrm{Out}(\Pi)$), and
- a collection of subgroups s^{Θ}_Π of Π,

such that there exists an isomorphism $\Pi \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]$ of topological groups which maps $\mathcal{D}_\Pi \subset \mathrm{Out}(\Pi)$ to $\mathcal{D}_{\underline{\underline{Y}}}$, and s^{Θ}_Π to the μ_N-conjugacy class of subgroups in $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]$ determined by the image of the theta section $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$.

(3) For two mod N mono-theta environments $\mathbb{M} = (\Pi, \mathcal{D}_\Pi, s^{\Theta}_\Pi)$, ${}^\dagger\mathbb{M} = ({}^\dagger\Pi, \mathcal{D}_{{}^\dagger\Pi}, s^{\Theta}_{{}^\dagger\Pi})$, we define an **isomorphism of mod N mono-theta environments** $\mathbb{M} \xrightarrow{\sim} {}^\dagger\mathbb{M}$ to be an isomorphism of topological groups $\Pi \xrightarrow{\sim} {}^\dagger\Pi$ which maps $\mathcal{D}_\Pi$ to $\mathcal{D}_{{}^\dagger\Pi}$, and s^{Θ}_Π to $s^{\Theta}_{{}^\dagger\Pi}$. For a mod N mono-theta environment $\mathbb{M}$ and a mod M mono-theta environment ${}^\dagger\mathbb{M}$ with $M \mid N$, we define a **homomorphism of mono-theta environments** $\mathbb{M} \to {}^\dagger\mathbb{M}$ to be an isomorphism $\mathbb{M}_M \xrightarrow{\sim} {}^\dagger\mathbb{M}$, where we write $\mathbb{M}_M$ for the mod M mono-theta environment induced by $\mathbb{M}$.

Remark 7.20.1. We can also consider a **mod N bi-theta environment** $\mathbb{B} = (\Pi, \mathcal{D}_{\Pi}, s^{\Theta}_{\Pi}, s^{\mathrm{alg}}_{\Pi})$, which is a mod N mono-theta environment $(\Pi, \mathcal{D}_{\Pi}, s^{\Theta}_{\Pi})$ with a datum s^{alg}_{Π} corresponding to the μ_N-conjugacy class of the image of mod N algebraic section $s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}$ (cf. [EtTh, Definition 2.13 (iii)]). As shown below in Theorem 7.23, three important rigidities (the cyclotomic reigidity, the discrete rigidity, and the constant multiple rigidity) hold for mono-theta environments. On the other hand, the cyclotomic rigidity, and the constant multiple rigidity trivially holds for bi-theta environments; however, the discrete rigidity does not hold for them (cf. also Remark 7.23.1). We omit the details of bi-theta environments, since we will not use bi-theta environments in inter-universal Teichmüller theory.

Lemma 7.21. ([EtTh, Proposition 2.14])

(1) We have the following group-theoretic chracterisation of the image of the tautological section of $(l\Delta_{\Theta})[\mu_N] \twoheadrightarrow l\Delta_{\Theta}$ as the following subgroup of $(\Delta^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}})^{\Theta}[\mu_N]$:

$$(l\Delta_{\Theta})[\mu_N] \bigcap \left\{ \gamma(a)a^{-1} \in (\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}[\mu_N] \;\middle|\; a \in (\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}[\mu_N],\ \gamma \in \mathrm{Aut}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]) \text{ such that } (*) \right\},$$

where

$(*)$: *the image of γ in $\mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$ belongs to $\mathcal{D}_{\underline{\underline{Y}}}$,*
and γ induces the identity on the quotient $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N] \twoheadrightarrow \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}} \twoheadrightarrow G_K$.

(2) Let $t^{\Theta}_{\underline{\underline{\ddot{Y}}}} : \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]$ be a section obtained as a conjugate of $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$ relative to the actions of $K^{\times}$ and $l\underline{\mathbb{Z}}$. Write $\delta := (s^{\Theta}_{\underline{\underline{\ddot{Y}}}})^{-1} t^{\Theta}_{\underline{\underline{\ddot{Y}}}}$, which is a 1-cocycle of $\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}$ valued in μ_N. We write $\ddot{\alpha}_{\delta} \in \mathrm{Aut}(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}[\mu_N])$ for the automorphism given by $s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}}(g)a \mapsto \delta(g)s^{\mathrm{alg}}_{\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}}(g)a$ ($g \in \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}$, $a \in \mu_N$), which induces the identity homomorphisms on both the quotient $\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}[\mu_N] \twoheadrightarrow \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}$ and the kernel of this quotient. Then $\ddot{\alpha}_{\delta}$ extends to an automorphism $\alpha_{\delta} \in \mathrm{Aut}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$, which induces the identity homomorphisms on both the quotient $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N] \twoheadrightarrow \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$ and the kernel of this quotient. The conjugate by α_{δ} maps $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$ to $t^{\Theta}_{\underline{\underline{\ddot{Y}}}}$, and preserves the subgroup $\mathcal{D}_{\underline{\underline{Y}}} \subset \mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$.

(3) Let $\mathbb{M} = (\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N], \mathcal{D}_{\underline{\underline{Y}}}, s^{\Theta}_{\underline{\underline{\ddot{Y}}}})$ be the mod N model mono-theta environment. Then every automorphism of $\mathbb{M}$ induces an automorphism of $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$ by Lemma 7.18 (2), hence an automorphism of $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}} = \mathrm{Aut}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}) \overset{\mathrm{out}}{\rtimes} \mathrm{Im}(\mathcal{D}_{\underline{\underline{Y}}} \to \mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})) =$

$\mathrm{Aut}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}) \times_{\mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})} \mathrm{Im}(\mathcal{D}_{\underline{\underline{Y}}} \to \mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}))$. *It also induces an automorphism of the set of cusps of* $\underline{\underline{Y}}$. *Relative to the labelling by* $\underline{\mathbb{Z}}$ *on these cusps, this induces an automorphism of* $\underline{\mathbb{Z}}$ *given by* $(l\underline{\mathbb{Z}}) \rtimes \{\pm 1\}$. *This assignment gives us a surjective homomorphism*

$$\mathrm{Aut}(\mathbb{M}) \twoheadrightarrow (l\underline{\mathbb{Z}}) \rtimes \{\pm 1\}.$$

Proof. (1): Let $\gamma \in \mathrm{Aut}((\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})[\mu_N])$ be a lift of an element in $\mathrm{Im}(K^\times) \subset \mathcal{D}_{\underline{\underline{Y}}}$ $(\subset \mathrm{Out}((\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})[\mu_N]))$ such that γ satisfies (*). Then γ can be written as $\gamma = \gamma_1\gamma_2$, where $\gamma_1 \in \mathrm{Inn}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$, $\gamma_2 \in \mathrm{Aut}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$, the image of γ_2 in $\mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$ is in $\mathrm{Im}\{K^\times \to H^1(G_K, \mu_N) \to H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}, \mu_N) \to \mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])\}$, and the automorphism induced by γ_2 of the quotient $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N] \twoheadrightarrow \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$ and the automorphism of its kernel (= μ_N) are trivial. Since the composite $H^1(G_K, \mu_N) \to H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}, \mu_N) \to H^1(\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}}, \mu_N)$ is trivial, the composite $H^1(G_K, \mu_N) \to H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}, \mu_N) \to H^1(\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}}, \mu_N) \to \mathrm{Out}(\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$ is trivial as well. Hence the automorphism induced by γ_2 of $\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]$ is an inner automorphism. On the other hand, the automorphism induced by γ_1 of G_K is trivial since the automorphism induced by γ_2 of G_K is trivial, and the condition (*). Then the center-freeness of G_K (cf. Proposition 2.7 (1c)) implies that $\gamma_1 \in \mathrm{Inn}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$ is in $\mathrm{Inn}(\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$. Hence the automorphism induced by $\gamma = \gamma_1\gamma_2$ of $\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]$ is also an inner automorphism. Since $(\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}[\mu_N] (\cong l\underline{\mathbb{Z}} \times \widehat{\mathbb{Z}}(1) \times \mu_N)$ is abelian, the inner automorphism induced by γ of $(\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}[\mu_N]$ is trivial. Then (1) follows from Proposition 7.19 (2).

(2): By definition, the conjugate by $\ddot{\alpha}_\delta$ maps $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$ to $t^{\Theta}_{\underline{\underline{\ddot{Y}}}}$. Since the outer action of $\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}}) \cong l\underline{\mathbb{Z}}$ on $\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]$ fixes $s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}}$ up to μ_N-conjugacy, the cohomology class of δ in $H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}, \mu_N)$ is in the submodule generated by the Kummer classes of $K^\times$ and $(1/l)2l\log(\ddot{U}) = 2\log(\ddot{U})$ by the first displayed formula of Lemma 7.5 (2) (cf. Lemma 7.5 (1) for the cohomology class $\log(\ddot{U})$). Here, note that the cohomology class of δ is in Fil^1 since both $(s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}})^{-1} \cdot s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$ and $s^{\mathrm{alg}}_{\underline{\underline{\ddot{Y}}}} \cdot t^{\Theta}_{\underline{\underline{\ddot{Y}}}}$ maps to 1 in $\mathrm{Fil}^0/\mathrm{Fil}^1 = \mathrm{Hom}(l\Delta_\Theta, l\Delta_\Theta)$ by Lemma 7.5 (2). Note also that "$1/l$" comes from that we are working with l-th roots of the theta functions $\underline{\underline{\ddot{\eta}}}^{\Theta, l\underline{\mathbb{Z}}\times\mu_2}$ (cf. the proof of Lemma 7.5 (2)), and that "l" comes from $l\underline{\mathbb{Z}}$. Thus, δ descends to a 1-cocycle of $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$ valued in μ_N since the coordinate $\ddot{U}^2$ descends to $\underline{\underline{Y}}$. Hence $\ddot{\alpha}_\delta$ extends to an automorphism $\alpha_\delta \in \mathrm{Aut}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$, which induces identity automorphisms on both the quotient $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N] \to \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$ and the kernel of this quotient. The conjugate by α_δ preserves $\mathcal{D}_{\underline{\underline{Y}}} \subset \mathrm{Out}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N])$, since the action of $\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}})$ maps $2\log(\ddot{U})$ to a $K^\times$-multiple of $2\log(\ddot{U})$.

(3) comes from (2). $\square$

Corollary 7.22. (Group-Theoretic Reconstruction of Mono-theta Environment, [EtTh, Corollary 2.18]) *Let $N \geq 1$ be an integer, l a prime number and $\underline{\underline{X}}$ a smooth log-curve of type $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$ over a finite extension K of $\mathbb{Q}_p$. We assume that l and p are odd, and $K = \ddot{K}$. Let $\mathbb{M}_N$ be the resulting mod N model mono-theta environment, which is independent of the choice of a member of $\underline{\underline{\ddot{\eta}}}^{\Theta, l\mathbb{Z}\times\mu_2}$, up to isomorphism over the identity of $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$ by Lemma 7.21 (2).*

(1) Let ${}^{\dagger}\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$ be a topological group which is isomorphic to $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$. Then there exists a group-theoretic algorithm for constructing

- *subquotients*

$$ {}^{\dagger}\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}, {}^{\dagger}\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}, {}^{\dagger}G_K, {}^{\dagger}(l\Delta_{\Theta}), {}^{\dagger}(\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^{\Theta}, {}^{\dagger}(\Pi^{\mathrm{temp}}_{\underline{\underline{X}}})^{\Theta}, {}^{\dagger}(\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}, {}^{\dagger}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta} $$

  of ${}^{\dagger}\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$, and

- *a collection of subgroups of ${}^{\dagger}\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$ for each element of $(\mathbb{Z}/l\mathbb{Z})/\{\pm 1\}$,*

such that any isomorphism ${}^{\dagger}\Pi^{\mathrm{temp}}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$ maps

- *the above subquotients to the subquotients*

$$ \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}, \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}, G_K, l\Delta_{\Theta}, (\Delta^{\mathrm{temp}}_{\underline{\underline{X}}})^{\Theta}, (\Pi^{\mathrm{temp}}_{\underline{\underline{X}}})^{\Theta}, (\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}, (\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta} $$

  of $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$ respectively, and

- *the above collection of subgroups to the collection of cuspidal decomposition groups of $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$ determined by the label in $(\mathbb{Z}/l\mathbb{Z})/\{\pm 1\}$,*

in a functorial manner with respect to isomorphisms of topological groups (and no need of any reference isomorphism to $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$).

(2) "$(\Pi \mapsto \mathbb{M})$":
There exists a group-theoretic algorithm for constructing a mod N mono-theta environment ${}^{\dagger}\mathbb{M} = ({}^{\dagger}\Pi, \mathcal{D}_{{}^{\dagger}\Pi}, s^{\Theta}_{{}^{\dagger}\Pi})$, where

$$ {}^{\dagger}\Pi := {}^{\dagger}\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}} \times_{{}^{\dagger}G_K} \left(({}^{\dagger}(l\Delta_{\Theta}) \otimes \mathbb{Z}/N\mathbb{Z}) \rtimes {}^{\dagger}G_K \right) $$

up to isomorphism in a functorial manner with respect to isomorphisms of topological groups (and no need of any reference isomorphism to $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$). (cf. also [EtTh, Corollary 2.18 (ii)] for a stronger form).

(3) "$(\mathbb{M} \mapsto \Pi)$":
Let ${}^{\dagger}\mathbb{M} = ({}^{\dagger}\Pi, \mathcal{D}_{{}^{\dagger}\Pi}, s^{\Theta}_{{}^{\dagger}\Pi})$ be a mod N mono-theta environment which is isomorphic

to $\mathbb{M}_N$. Then there exists a group-theoretic algorithm for constructing a quotient ${}^\dagger\Pi \twoheadrightarrow {}^\dagger\Pi^{\text{temp}}_{\underline{\underline{Y}}}$, such that any isomorphism ${}^\dagger\mathbb{M} \xrightarrow{\sim} \mathbb{M}_N$ maps this quotient to the quotient $\Pi^{\text{temp}}_{\underline{\underline{Y}}}[\mu_N] \twoheadrightarrow \Pi^{\text{temp}}_{\underline{\underline{Y}}}$ in a functorial manner with respect to isomorphisms of mono-theta environments (and no need of any reference isomorphism to $\mathbb{M}_N$). Furthermore, any isomorphism ${}^\dagger\mathbb{M} \xrightarrow{\sim} \mathbb{M}_N$ induces an isomorphism from

$${}^\dagger\Pi^{\text{temp}}_{\underline{\underline{X}}} := \mathrm{Aut}({}^\dagger\Pi^{\text{temp}}_{\underline{\underline{Y}}}) \times_{\mathrm{Out}({}^\dagger\Pi^{\text{temp}}_{\underline{\underline{Y}}})} \mathrm{Im}(\mathcal{D}_{{}^\dagger\Pi} \to \mathrm{Out}({}^\dagger\Pi^{\text{temp}}_{\underline{\underline{Y}}}))$$

to $\Pi^{\text{temp}}_{\underline{\underline{X}}}$, where we set the topology of ${}^\dagger\Pi^{\text{temp}}_{\underline{\underline{X}}}$ as the topology determined by taking

$${}^\dagger\Pi^{\text{temp}}_{\underline{\underline{Y}}} \xrightarrow{\sim} \mathrm{Aut}({}^\dagger\Pi^{\text{temp}}_{\underline{\underline{Y}}}) \times_{\mathrm{Out}({}^\dagger\Pi^{\text{temp}}_{\underline{\underline{Y}}})} \{1\} \subset {}^\dagger\Pi^{\text{temp}}_{\underline{\underline{X}}}$$

to be an open subgroup. Finally, if we apply the algorithm of (2) to ${}^\dagger\Pi^{\text{temp}}_{\underline{\underline{Y}}}$, then the resulting mono-theta environment is isomorphic to the original ${}^\dagger\mathbb{M}$, via an isomorphism which induces the identity on ${}^\dagger\Pi^{\text{temp}}_{\underline{\underline{Y}}}$.

(4) *Let ${}^\dagger\mathbb{M} = ({}^\dagger\Pi, \mathcal{D}_{{}^\dagger\Pi}, s^\Theta_{{}^\dagger\Pi})$, and ${}^\ddagger\mathbb{M} = ({}^\ddagger\Pi, \mathcal{D}_{{}^\ddagger\Pi}, s^\Theta_{{}^\ddagger\Pi})$ be mod N mono-theta environments. Let ${}^\dagger\Pi^{\text{temp}}_{\underline{\underline{X}}}$ and ${}^\ddagger\Pi^{\text{temp}}_{\underline{\underline{X}}}$ be the topological groups constructed in (3) from ${}^\dagger\mathbb{M}$ and ${}^\ddagger\mathbb{M}$ respectively. Then the functoriality of the algorithm in (3) gives us a natural map*

$$\mathrm{Isom}^{\mu_N\text{-conj}}({}^\dagger\mathbb{M}, {}^\ddagger\mathbb{M}) \to \mathrm{Isom}({}^\dagger\Pi^{\text{temp}}_{\underline{\underline{X}}}, {}^\ddagger\Pi^{\text{temp}}_{\underline{\underline{X}}}),$$

which is surjective with fibers of cardinality 1 (resp. 2) if N is odd (resp. even), where we write $\mathrm{Isom}^{\mu_N\text{-conj}}$ for the set of μ_N-conjugacy classes of isomorphisms. In particular, for any positive integer M with $M \mid N$, we have a natural homomorphism $\mathrm{Aut}^{\mu_N\text{-conj}}({}^\dagger\mathbb{M}) \to \mathrm{Aut}^{\mu_M\text{-conj}}({}^\dagger\mathbb{M}_M)$, where we write ${}^\dagger\mathbb{M}_M$ for the mod M mono-theta environment induced by ${}^\dagger\mathbb{M}$ such that the kernel and cokernel have the same cardinality (≤ 2) as the kernel and cokernel of the homomorphism $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/N\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/M\mathbb{Z})$ induced by the natural surjection $\mathbb{Z}/N\mathbb{Z} \twoheadrightarrow \mathbb{Z}/M\mathbb{Z}$, respectively.

Proof. (1): We can group-theoretically reconstruct a quotient ${}^\dagger\Pi^{\text{temp}}_{\underline{\underline{X}}} \twoheadrightarrow {}^\dagger G_K$ by Lemma 6.2, other subquotients by Lemma 7.8, Lemma 7.12 and the definitions, and the labels of cuspidal decomposition groups by Lemma 7.16.

(2) follows from the definitions (Note that we can reconstruct the set ${}^\dagger\underline{\underline{\ddot{\eta}}}^{\Theta, l\mathbb{Z}\times\mu_2}$ of theta classes by Remark 7.15.1, thus, the theta section $s^\Theta_{{}^\dagger\Pi}$ as well (cf. the construction of the theta section $s^\Theta_{\underline{\underline{\ddot{Y}}}}$ before Definition 7.20)).

(3): We can group-theoretically reconstruct a quotient ${}^\dagger\Pi \twoheadrightarrow {}^\dagger\Pi^{\text{temp}}_{\underline{\underline{Y}}}$ by Lemma 7.18 (2). The reconstruction of ${}^\dagger\Pi^{\text{temp}}_{\underline{\underline{X}}}$ comes from the definitions and the temp-slimness of

${}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$ (Lemma 6.4 (5)). The last claim of (3) follows from the definitions and the description of the algorithm in (2).

(4): The surjectivity of the map comes from the last claim of (3). The fiber of this map is a $\ker(\mathrm{Aut}^{\mu_N\text{-conj}}({}^\dagger\mathbb{M}) \to \mathrm{Aut}({}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}))$-torsor. By Theorem 7.23 (1) below (Note that there is no circular argument), the natural isomorphism ${}^\dagger(l\Delta_\Theta)\otimes\mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mu_N({}^\dagger(l\Delta_\Theta[\mu_N]))$ is preserved by automorphisms of ${}^\dagger\mathbb{M}$. Note that $\ker(\mathrm{Aut}^{\mu_N\text{-conj}}({}^\dagger\mathbb{M}) \to \mathrm{Aut}({}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}))$ consists of automorphisms acting as the identity on ${}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$, hence on $\ker({}^\dagger\Pi \to {}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})$ by the above natural isomorphism. Thus, we have

$$\ker(\mathrm{Aut}^{\mu_N\text{-conj}}({}^\dagger\mathbb{M}) \to \mathrm{Aut}({}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{X}}})) \cong \mathrm{Hom}({}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}/{}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}, \ker({}^\dagger\Pi \to {}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})),$$

where ${}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}/{}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}} \cong \mu_2$ and $\ker({}^\dagger\Pi \to {}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})) \cong \mu_N$. The cardinality of this group is 1 (resp. 2) is N is odd (resp. even). The last claim follows from this description. □

Theorem 7.23. (Three Rigidities of Mono-theta Environments, [EtTh, Corollary 2.19]) *Let $N \geq 1$ be an integer, l a prime number and $\underline{\underline{X}}$ a smooth log-curve of type $(1, (\mathbb{Z}/l\mathbb{Z})^\Theta)$ over a finite extension K of $\mathbb{Q}_p$. We assume that l and p are odd, and $K = \ddot{K}$. Let $\mathbb{M}_N$ be the resulting mod N model mono-theta environment (which is independent of the choice of a member of $\underline{\underline{\ddot{\eta}}}^{\Theta, l\underline{\mathbb{Z}}\times\mu_2}$, up to isomorphism over the identity of $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$ by Lemma 7.21 (2)).*

(1) **(Cyclotomic Rigidity)** *Let ${}^\dagger\mathbb{M} = ({}^\dagger\Pi, \mathcal{D}_{{}^\dagger\Pi}, s^\Theta_{{}^\dagger\Pi})$ be a mod N mono-theta environment which is isomorphic to $\mathbb{M}_N$. We write ${}^\dagger\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$ for the topological group obtained by applying Corollary 7.22 (3). Then there exists a group-theoretic algorithm for constructing subquotients*

$$^\dagger(l\Delta_\Theta[\mu_N]) \subset {}^\dagger((\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}})^\Theta[\mu_N]) \subset {}^\dagger((\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^\Theta[\mu_N])$$

of ${}^\dagger\Pi$ such that any isomorphism ${}^\dagger\mathbb{M} \xrightarrow{\sim} \mathbb{M}_N$ maps these subquotients to the subquotients

$$l\Delta_\Theta[\mu_N] \subset (\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}})^\Theta[\mu_N] \subset (\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^\Theta[\mu_N]$$

of $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]$, in a functorial manner with respect to isomorphisms of mono-theta environments (no need of any reference isomorphism to $\mathbb{M}_N$). Moreover, there exists a group-theoretic algorithm for constructing two splittings of the natural surjection

$$^\dagger(l\Delta_\Theta[\mu_N]) \twoheadrightarrow {}^\dagger(l\Delta_\Theta)$$

such that any isomorphism ${}^\dagger\mathbb{M} \xrightarrow{\sim} \mathbb{M}_N$ maps these two splittings to the two splittings of the surjection

$$l\Delta_\Theta[\mu_N] \twoheadrightarrow l\Delta_\Theta$$

determined by the mod N algebraic section $s^{\mathrm{alg}}_{\ddot{\underline{\underline{Y}}}}$ and the mod N theta section $s^{\Theta}_{\ddot{\underline{\underline{Y}}}}$. in a functorial manner with respect to isomorphisms of mono-theta environments (no need of any reference isomorphism to $\mathbb{M}$). Hence in particular, by taking the difference of these two splittings, there exists a group-theoretic algorithm for constructing an isomorphism of cyclotomes

$$\text{(Cyc. Rig. Mono-th.)} \qquad {}^{\dagger}(l\Delta_{\Theta}) \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mu_N({}^{\dagger}(l\Delta_{\Theta}[\mu_N]))$$

such that any isomorphism ${}^{\dagger}\mathbb{M} \xrightarrow{\sim} \mathbb{M}_N$ maps this isomorphism of the cyclotomes to the natural isomorphism of cyclotomes

$$l\Delta_{\Theta} \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mu_N(l\Delta_{\Theta}[\mu_N])$$

in a functorial manner with respect to isomorphisms of mono-theta environments (no need of any reference isomorphism to $\mathbb{M}_N$).

(2) **(Discrete Rigidity)** *Any projective system $({}^{\dagger}\mathbb{M}_N)_{N\geq 1}$ of mono-theta environments is isomorphic to the natural projective system of the model mono-theta environments $(\mathbb{M}_N)_{N\geq 1}$.*

(3) **(Constant Multiple Rigidity)** *Assume that $\ddot{\underline{\underline{\eta}}}^{\Theta, l\underline{\mathbb{Z}}\times\mu_2}$ is of standard type. Let $({}^{\dagger}\mathbb{M}_N)_{N\geq 1}$ be a projective system of mono-theta environments. Then there exists a group-theoretic algorithm for constructing a collection of classes of $H^1({}^{\dagger}\Pi^{\mathrm{temp}}_{\ddot{\underline{\underline{Y}}}}, {}^{\dagger}(l\Delta_{\Theta}))$ such that any isomorphism $({}^{\dagger}\mathbb{M}_N)_{N\geq 1} \xrightarrow{\sim} (\mathbb{M}_N)_{N\geq 1}$ to the projective systems of the model mono-theta environments maps the above collection of classes to the collection of classes of $H^1(\Pi^{\mathrm{temp}}_{\ddot{\underline{\underline{Y}}}}, l\Delta_{\Theta})$ given by some multiple of the collection of classes $\ddot{\underline{\underline{\eta}}}^{\Theta, l\underline{\mathbb{Z}}\times\mu_2}$ by an element of μ_l in a functorial manner with respect to isomorphisms of projective systems of mono-theta environments (no need of any reference isomorphism to $(\mathbb{M}_N)_{N\geq 1}$).*

We shall refer to ${}^{\dagger}(l\Delta_{\Theta}) \otimes \mathbb{Z}/N\mathbb{Z}$ as the (mod N) **internal cyclotome of the mono-theta environment** ${}^{\dagger}\mathbb{M}$, and $\mu_N({}^{\dagger}(l\Delta_{\Theta}[\mu_N]))$ the (mod N) **external cyclotome of the mono-theta environment** ${}^{\dagger}\mathbb{M}$. We shall refer to the above isomorphism (Cyc. Rig. Mono-th.) as the **cyclotomic rigidity of mono-theta environment**.

Proof. (1): Firstly, note that the restrictions of the algebraic section $s^{\mathrm{alg}}_{\ddot{\underline{\underline{Y}}}}$ and the theta section $s^{\Theta}_{\ddot{\underline{\underline{Y}}}}$ to $\ker\{\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}} \twoheadrightarrow (\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}\}$ coincide by Remark 7.2.1 (1). Hence we can reconstruct $\ker\{{}^{\dagger}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]) \twoheadrightarrow {}^{\dagger}((\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}[\mu_N])\}$ as the subset of (any μ_N-conjugacy class of) $s^{\Theta}_{{}^{\dagger}\Pi}$ whose elements project to $\ker\{{}^{\dagger}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}) \twoheadrightarrow {}^{\dagger}((\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta})\}$, via the projection ${}^{\dagger}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]) \twoheadrightarrow {}^{\dagger}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})$, where ${}^{\dagger}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N]) \twoheadrightarrow {}^{\dagger}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})$, ${}^{\dagger}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})$, and ${}^{\dagger}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}) \twoheadrightarrow {}^{\dagger}((\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}$ are reconstructed by Lemma 7.18 (2), Corollary 7.22 (3) and Corollary 7.22 (1) respectively. We can also reconstruct the subquotients ${}^{\dagger}(l\Delta_{\Theta}[\mu_N]) \subset {}^{\dagger}((\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}[\mu_N]) \subset {}^{\dagger}((\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}[\mu_N])$ as the inverse images of ${}^{\dagger}(l\Delta_{\Theta}) \subset {}^{\dagger}((\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}) \subset {}^{\dagger}((\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta})$, which are reconstructed by Corollary 7.22 (1) (3), via the quotient ${}^{\dagger}((\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta}[\mu_N]) \twoheadrightarrow {}^{\dagger}((\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^{\Theta})$. We can reconstruct the splitting of the natural surjection ${}^{\dagger}(l\Delta_{\Theta}[\mu_N]) \twoheadrightarrow {}^{\dagger}(l\Delta_{\Theta})$ given by the theta section directly as $s^{\Theta}_{{}^{\dagger}\Pi}$. On the other hand, we can reconstruct the splitting of the natural surjection ${}^{\dagger}(l\Delta_{\Theta}[\mu_N]) \twoheadrightarrow {}^{\dagger}(l\Delta_{\Theta})$ given by the algebraic section by the algorithm of Lemma 7.21 (1).

(2) follows from Corollary 7.22 (4), since $R^1 \varprojlim_N \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/N\mathbb{Z}) = 0$ and $R^1 \varprojlim_N \mu_N = 0$. cf. also Remark 7.23.1 (2).

(3) follows from Lemma 7.21 (3), Corollary 7.15, the cyclotomic rigidity (1), and the discrete rigidity (2). $\square$

Remark 7.23.1. In this remark, we compare rigidity properties of mono-theta environments and bi-theta environments (cf. Remark 7.20.1 for bi-theta environments).

(1) (Cyclotomic Rigidity) The proof of the cyclotimic rigidity for mono-theta environments comes from the reconstruction of the image of the algebraic section, and this reconstruction comes from the *quadratic* structure of theta group (=Heisenberg group) (cf. Remark 7.19.1). On the other hand, for a bi-theta environment, the image of the algebraic section is included as a datum of a bi-theta environment, hence the cyclotomic rigidity trivially holds for bi-theta environment.

(2) (Constant Multiple Rigidity) The proof of the constant multiple rigidity for mono-theta environments comes from the elliptic cuspidalisation (cf. Proposition 7.9). On the other hand, for a bi-theta environment, the image of the algebraic section is included as a datum of a bi-theta environment. This means that the ratio (i.e., the étale theta class) determined by the given data of theta section and algebraic section is independent of the simultaneous constant multiplications on theta section and algebraic section, hence the constant multiple rigidity trivially holds for bi-theta environment.

(3) (Discrete Rigidity) A mono-theta environment does not include a datum of algebraic section, it includes only a datum of theta section. By this reason, a mono-theta environment has "shifting automorphisms" $\ddot{\alpha}_{\delta}$ in Lemma 7.21 (2) (which comes from the "less-than-or-equal-to-quadratic" structure of theta group (=Heisenberg

group)). This means that there is no "basepoint" relative to the $l\underline{\mathbb{Z}}$ action on $\underline{\underline{Y}}$, i.e., no distinguished irreducible component of the special fiber. If we work with a projective system of mono-theta environments, then by the compatibility of mod N theta sections, where N runs through the positive integers, the mod N theta classes determine a single "discrete" $l\underline{\mathbb{Z}}$-torsor in the projective limit. The "shifting automorphisms" gives us a $l\underline{\mathbb{Z}}$-indeterminacy, which is *independent* of N (cf. Lemma 7.21 (3)), and to find a common basepoint for the $l\underline{\mathbb{Z}}/Nl\underline{\mathbb{Z}}$-torsor in the projective system is the same thing to trivialise a $\varprojlim_N l\underline{\mathbb{Z}}/l\underline{\mathbb{Z}}(=0)$-torsor, which remains discrete. This is the reason that the discrete rigidity holds for mono-theta environments. On the other hand, a bi-theta environment includes a datum of algebraic section as well. The basepoint indeterminacy is roughly $Nl\underline{\mathbb{Z}}$-indeterminacy (i.e., the surjectivity of Lemma 7.21 (3) does not hold for bi-theta environments. for the precise statement, see [EtTh, Proposition 2.14 (iii)]), which *depends* on N, and to find a common basepoint for the $l\underline{\mathbb{Z}}/Nl\underline{\mathbb{Z}}$-torsor in the projective system is the same thing to trivialise a $\varprojlim_N l\underline{\mathbb{Z}}/Nl\underline{\mathbb{Z}}(=l\widehat{\underline{\mathbb{Z}}})$-torsor, which does not remain discrete (it is profinite). Hence the discrete rigidity does not hold for bi-theta environments.

Note also that a short exact sequence of the projective systems

$$0 \to Nl\mathbb{Z} \to l\mathbb{Z} \to l\mathbb{Z}/Nl\mathbb{Z} \to 0 \quad (\text{resp. } 0 \to l\mathbb{Z} \to l\mathbb{Z} \to l\mathbb{Z}/l\mathbb{Z} \to 0 \ )$$

with respect to $N \geq 1$, which corresponds to bi-theta environments (resp. mono-theta environments), induces an exact sequence

$$0 \to \varprojlim_N Nl\mathbb{Z}\,(=0) \to l\mathbb{Z} \to l\widehat{\mathbb{Z}} \to R^1 \varprojlim_N Nl\mathbb{Z}(=l\widehat{\mathbb{Z}}/l\mathbb{Z}) \to 0$$

$$(\text{resp. } 0 \to l\mathbb{Z} \to l\mathbb{Z} \to 0 \to R^1 \varprojlim_N l\mathbb{Z}\,(=0) \ ),$$

and that $R^1 \varprojlim_N Nl\mathbb{Z} = l\widehat{\mathbb{Z}}/l\mathbb{Z}$ (resp. $R^1 \varprojlim_N l\mathbb{Z} = 0$) exactly corresponds to the non-discreteness (resp. discreteness) phenomenon of bi-theta environment (resp. mono-theta environment). cf. also [EtTh, Remark 2.16.1].

The following diagram is a summary of this remark (cf. also [EtTh, Introduction]):

	cycl. rig.	disc. rig.	const. mult. rig.
mono-theta env.	delicately OK (structure of theta group)	OK	delicately OK (elliptic cuspidalisation)
bi-theta env.	trivially OK	Fails	trivially OK

Remark 7.23.2. If we consider N-th power $\ddot{\Theta}^N$ $(N > 1)$ of the theta function $\ddot{\Theta}$ instead of the first power $\ddot{\Theta}^1 = \ddot{\Theta}$, then the cyclotomic rigidity of Theorem 7.23 (1) does not hold since it comes from the quadratic structure of the theta group (=Heisenberg group) (cf. Remark 7.19.1). The cyclotomic rigidity of the mono-theta environment is one of the most important tools in inter-universal Teichmüller theory, hence if we use $\ddot{\Theta}^N$ $(N > 1)$ instead of $\ddot{\Theta}$, then inter-universal Teichmüller theory does not work. If it worked, then it would give us a sharper Diophantine inequality, which would be a contradiction with the results in analytic number theory (cf. [Mass2]). cf. also Remark 11.10.1 (the principle of Galois evaluation) and Remark 13.13.3 (2) (N-th power does not work).

Remark 7.23.3. The cyclotomic rigidity rigidifies the $\widehat{\mathbb{Z}}^\times \cong \mathrm{Aut}(\widehat{\mathbb{Z}}(1))$-indeterminacy of an object which is isomorphic to "$\widehat{\mathbb{Z}}(1)$", hence rigidifies the induced $\widehat{\mathbb{Z}}^\times \cong \mathrm{Aut}(\widehat{\mathbb{Z}}(1))$-indeterminacy of $H^1(-, \text{“}\widehat{\mathbb{Z}}(1)\text{”})$. As for the cohomology class $\log(\ddot{\Theta})$ of the theta function $\ddot{\Theta}$, it ridigifies $\widehat{\mathbb{Z}}^\times \log(\ddot{\Theta})$. The constant multiple rigidity rigidifies $\log(\ddot{\Theta}) + \widehat{\mathbb{Z}}$. Hence the cyclotomic rigidity and the constant multiple rigidity rigidify the indeterminacy $\widehat{\mathbb{Z}}^\times \log(\ddot{\Theta}) + \widehat{\mathbb{Z}}$ of the affine transformation type. The discrete rigidity rigidifies $\widehat{\mathbb{Z}} \cong \mathrm{Hom}(\text{“}\widehat{\mathbb{Z}}(1)\text{”}, \text{“}\widehat{\mathbb{Z}}(1)\text{”})$. Here the second "$\widehat{\mathbb{Z}}(1)$" is a coefficient cyclotome, and it is subject to $\widehat{\mathbb{Z}}^\times \cong \mathrm{Aut}(\widehat{\mathbb{Z}}(1))$-indeterminacy which is rigidified by the cyclotomic rigidity. The first "$\widehat{\mathbb{Z}}(1)$" is a cyclotome which arises as a subquotient of a (tempered) fundamental group. Hence three rigidities of mono-theta environments in Theorem 7.23 correspond to the structure of the theta group (=Heisenberg group) $(\Delta_X^{\mathrm{temp}})^\Theta$:

$$\begin{pmatrix} \text{cyclotomic rigidity} & \text{constant multiple rigidity} \\ & \\ 0 & \text{discrete rigidity} \end{pmatrix}.$$

cf. also the filtration of Lemma 7.5 (1).

§ 7.5. Some Analogous Objects at Good Places.

In inter-unversal Teichüller theory, $\underline{\underline{X}}$ is the main actor for places in $\underline{\mathbb{V}}^{\mathrm{bad}}$. In this subsection, for the later use, we introduce a counterpart $\underset{\rightarrow}{X}$ of $\underline{\underline{X}}$ for places in $\underline{\mathbb{V}}^{\mathrm{good}}$ and related objects (However, the theory for the places in $\underline{\mathbb{V}}^{\mathrm{bad}}$ is more important than the one for the places in $\underline{\mathbb{V}}^{\mathrm{good}}$).

Let X be a hyperbolic curve of type $(1, 1)$ over a field K of characteristic 0, $\underline{C}$ a hyperbolic orbicurve of type $(1, l\text{-tors})_\pm$ (cf. Definition 7.10) whose K-core C is also the K-core of X. Then $\underline{C}$ determines a hyperbolic orbicurve $\underline{X} := \underline{C} \times_C X$ of type $(1, l\text{-tors})$. Let $\iota_{\underline{X}}$ be the nontrivial element in $\mathrm{Gal}(\underline{X}/\underline{C})(\cong \mathbb{Z}/2\mathbb{Z})$. We write G_K for the absolute Galois group of K for an algebraic closure $\overline{K}$. Let $l \geq 5$ be a prime number.

Assumption We assume that G_K acts trivially on $\Delta_X^{\mathrm{ab}} \otimes (\mathbb{Z}/l\mathbb{Z})$.

(In inter-universal Teichmüller theory, we will use for $K = F_{\mathrm{mod}}(E_{F_{\mathrm{mod}}}[l])$ later.) We write $\underline{\epsilon}^0$ for the unique zero-cusp of $\underline{X}$. We choose a non-zero cusp $\underline{\epsilon}$ and let $\underline{\epsilon}'$ and $\underline{\epsilon}''$ be the cusps of $\underline{X}$ over $\underline{\epsilon}$, and let $\Delta_{\underline{X}} \twoheadrightarrow \Delta_{\underline{X}}^{\mathrm{ab}} \otimes (\mathbb{Z}/l\mathbb{Z}) \twoheadrightarrow \Delta_{\underline{\epsilon}}$ be the quotient of $\Delta_{\underline{X}}^{\mathrm{ab}} \otimes (\mathbb{Z}/l\mathbb{Z})$ by the images of the inertia subgroups of all non-zero cusps except $\underline{\epsilon}'$ and $\underline{\epsilon}''$ of $\underline{X}$. Then we have the natural exact sequence

$$0 \to I_{\underline{\epsilon}'} \times I_{\underline{\epsilon}''} \to \Delta_{\underline{\epsilon}} \to \Delta_{\underline{E}} \otimes (\mathbb{Z}/l\mathbb{Z}) \to 0,$$

with the natural actions of G_K and $\mathrm{Gal}(\underline{X}/\underline{C})(\cong \mathbb{Z}/2\mathbb{Z})$, where $\underline{E}$ is the genus one compactification of $\underline{X}$, and $I_{\underline{\epsilon}'}$, $I_{\underline{\epsilon}''}$ are the images in $\Delta_{\underline{\epsilon}}$ of the inertia subgroups of the cusps $\underline{\epsilon}'$, $\underline{\epsilon}'$ respectively (we have non-canonically $I_{\underline{\epsilon}'} \cong I_{\underline{\epsilon}''} \cong \mathbb{Z}/l\mathbb{Z}$). Note that $\iota_{\underline{X}}$ induces an isomorphism $I_{\underline{\epsilon}'} \cong I_{\underline{\epsilon}''}$, and that $\iota_{\underline{X}}$ acts on $\Delta_{\underline{E}} \otimes (\mathbb{Z}/l\mathbb{Z})$ via the multiplication by -1. Since l is odd, the action of $\iota_{\underline{X}}$ on $\Delta_{\underline{\epsilon}}$ induces a decomposition

$$\Delta_{\underline{\epsilon}} \xrightarrow{\sim} \Delta_{\underline{\epsilon}}^{+} \times \Delta_{\underline{\epsilon}}^{-},$$

where $\iota_{\underline{X}}$ acts on $\Delta_{\underline{\epsilon}}^{+}$ and $\Delta_{\underline{\epsilon}}^{-}$ by $+1$ and -1 respectively. Note that the natural composites $I_{\underline{\epsilon}'} \hookrightarrow \Delta_{\underline{\epsilon}} \twoheadrightarrow \Delta_{\underline{\epsilon}}^{+}$ and $I_{\underline{\epsilon}''} \hookrightarrow \Delta_{\underline{\epsilon}} \twoheadrightarrow \Delta_{\underline{\epsilon}}^{+}$ are isomorphisms. We define $(\Pi_{\underline{X}} \twoheadrightarrow) J_{\underline{X}}$ by pushing the short exact sequences $1 \to \Delta_{\underline{X}} \to \Pi_{\underline{X}} \to G_K \to 1$ and by $\Delta_{\underline{X}} \twoheadrightarrow \Delta_{\underline{\epsilon}} \twoheadrightarrow \Delta_{\underline{\epsilon}}^{+}$:

$$\begin{tikzcd}
1 \arrow[r] & \Delta_{\underline{X}} \arrow[r] \arrow[d] & \Pi_{\underline{X}} \arrow[r] \arrow[d] & G_K \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \Delta_{\underline{\epsilon}}^{+} \arrow[r] & J_{\underline{X}} \arrow[r] & G_K \arrow[r] & 1.
\end{tikzcd}$$

Next, we consider the cusps "$2\underline{\epsilon}'$" and "$2\underline{\epsilon}''$" of $\underline{X}$ corresponding to the points of $\underline{E}$ obtained by multiplying $\underline{\epsilon}'$ and $\underline{\epsilon}''$ by 2 respectively, relative to the group law of the elliptic curve determined by the pair $(\underline{X}, \underline{\epsilon}^0)$. These cusps are not over the cusp $\underline{\epsilon}$ in $\underline{C}$, since $2 \not\equiv \pm 1 \pmod{l}$ by $l \geq 5$. Hence the decomposition groups of "$2\underline{\epsilon}'$" and "$2\underline{\epsilon}''$" give us sections $\sigma : G_K \to J_{\underline{X}}$ of the natural surjection $J_{\underline{X}} \twoheadrightarrow G_K$. The element $\iota_{\underline{X}} \in \mathrm{Gal}(\underline{X}/\underline{C})$, which interchange $I_{\underline{\epsilon}'}$ and $I_{\underline{\epsilon}''}$, acts trivially on $\Delta_{\underline{\epsilon}}^{+}$ (Note also $I_{\underline{\epsilon}'} \xrightarrow{\sim} \Delta_{\underline{\epsilon}} \xleftarrow{\sim} I_{\underline{\epsilon}''}$), hence these two sections to $J_{\underline{X}}$ coincides. This section is only determined by "$2\underline{\epsilon}'$" (or "$2\underline{\epsilon}''$") up to an inner automorphism of $J_{\underline{X}}$ given by an element $\Delta_{\underline{\epsilon}}^{+}$; however, since the natural outer action of G_K on $\Delta_{\underline{\epsilon}}^{+}$ is trivial by Assumption, it follows that the section completely determined by "$2\underline{\epsilon}'$" (or "$2\underline{\epsilon}''$") and the image of the section is normal in $J_{\underline{X}}$. By taking the quotient by this image, we obtain a surjection $(\Pi_{\underline{X}} \twoheadrightarrow) J_{\underline{X}} \twoheadrightarrow \Delta^{+}_{\underline{\epsilon}}$. Let

$$\underrightarrow{X} \to \underline{X}$$

be the corresponding covering with $\mathrm{Gal}(\underrightarrow{X}/\underline{X}) \cong \Delta^{+}_{\underline{\epsilon}} (\cong \mathbb{Z}/l\mathbb{Z})$.

Definition 7.24. ([IUTchI, Definition 1.1]) An orbicurve over K is called **of type $(1, l$-$\underrightarrow{\text{tors}})$** if it is isomorphic to $\underrightarrow{X}$ over K for some l and ϵ.

The arrow $\to$ in the notation $\underrightarrow{X}$ indicates a direction or an order on the $\{\pm 1\}$-orbits (i.e., the cusps of $\underline{C}$) of Q (in Assumption (1) before Definition 7.10) is determined by $\underline{\epsilon}$ (Remark [IUTchI, Remark 1.1.1]). We omit the construction of "$\underrightarrow{C}$" (cf. [IUTchI, §1]), since we do not use it. This $\underrightarrow{X}$ is the main actor for places in $\underline{\mathbb{V}}^{\mathrm{good}}$ in inter-universal Teichmüller theory :

	local $\underline{\mathbb{V}}^{\mathrm{bad}}$	local $\underline{\mathbb{V}}^{\mathrm{good}}$	global $\boxplus$	global $\boxtimes$
main actor	$\underline{\underline{X}}_{\underline{v}}$	$\underrightarrow{X}_{\underline{v}}$	$\underline{X}_K$	$\underline{C}_K$

Lemma 7.25. ([IUTchI, Corollary 1.2]) *We assume that K is an* NF *or an* MLF*. Then from $\Pi_{\underrightarrow{X}}$, there exists a group-theoretic algorithm to reconstruct $\Pi_{\underline{X}}$ and $\Pi_{\underline{C}}$ (as subgroups of* $\mathrm{Aut}(\underrightarrow{X})$*) together with the conjugacy classes of the decomposition group(s) determined by the set(s) of cusps $\{\underline{\epsilon}', \underline{\epsilon}''\}$ and $\{\epsilon\}$ respectively, in a functorial manner with respect to isomorphisms of topological groups.*

cf. also Lemma 7.8, Lemma 7.12 ([EtTh, Proposition 1.8, Proposition 2.4]).

Proof. First, since $\Pi_{\underrightarrow{X}}$, $\Pi_{\underline{X}}$ and $\Pi_{\underline{C}}$ are slim by Proposition 2.7 (2b), these are naturally embedded into $\mathrm{Aut}(\Pi_{\underrightarrow{X}})$ by conjugate actions. By the K-coricity of C, we can also group-theoretically reconstruct $(\Pi_{\underrightarrow{X}} \subset) \, \Pi_C \, (\subset \mathrm{Aut}(\Pi_{\underrightarrow{X}}))$. By Proposition 2.2 or Corollary 2.4, we can group-theoretically reconstruct the subgroups $\Delta_{\underrightarrow{C}} \subset \Pi_{\underrightarrow{C}}$ and $\Delta_{\underrightarrow{X}} \subset \Pi_{\underrightarrow{X}}$ (In particular, we can reconstruct l by the formula $[\Delta_C : \Delta_{\underrightarrow{X}}] = 2l^2$). We can reconstruct Δ_X as a unique torsion-free subgroup of Δ_C of index 2. Then we can reconstruct $\Pi_{\underline{X}} \, (\subset \Pi_C)$ as $\Pi_{\underline{X}} = H \cdot \Pi_{\underrightarrow{X}}$, where $H := \ker(\Delta_X \twoheadrightarrow \Delta_X^{\mathrm{ab}} \otimes (\mathbb{Z}/l\mathbb{Z}))$. The conjugacy classes of the decomposition groups of $\underline{\epsilon}^0$, $\underline{\epsilon}'$, and $\underline{\epsilon}''$ in $\Pi_{\underline{X}}$ can be reconstructed as the decomposition groups of cusps (Corollary 2.9 and Remark 2.9.2) whose image in $\Pi_{\underline{X}}/\Pi_{\underrightarrow{X}}$ is nontrivial. Then we can reconstruct the subgroup $\Pi_{\underline{C}} \subset \Pi_C$ by constructing a splitting of the natural surjection $\Pi_C/\Pi_{\underline{X}} \twoheadrightarrow \Pi_C/\Pi_X$ determined by $\Pi_{\underline{C}}/\Pi_{\underline{X}}$, where the splitting is characterised (since $l \nmid 3$) as the unique splitting (whose image $\subset \Pi_C/\Pi_{\underline{X}}$) stabilising (via the outer action on $\Pi_{\underline{X}}$) the collection of conjugacy classes of the decomposition groups in $\Pi_{\underline{X}}$ of $\underline{\epsilon}^0$, $\underline{\epsilon}'$, and $\underline{\epsilon}''$ (Note that if an involution of $\underline{X}$ fixed $\underline{\epsilon}'$ and interchanged $\underline{\epsilon}^0$ and $\underline{\epsilon}''$, then we would have $2 \equiv -1 \,(\mathrm{mod}\ l)$, i.e., $l \mid 3$). Finally, the decomposition groups of $\underline{\epsilon}'$ and $\underline{\epsilon}''$ in $\Pi_{\underline{X}}$ can be reconstructed as the decomposition group of cusps (Corollary 2.9 and Remark 2.9.2) whose image in $\Pi_{\underline{X}}/\Pi_{\underrightarrow{X}}$ is nontrivial, and is not fixed, up to conjugacy, by the outer action of $\Pi_C/\Pi_{\underline{X}}\,(\cong \mathbb{Z}/2\mathbb{Z})$ on $\Pi_{\underline{X}}$. □

Remark 7.25.1. ([IUTchI, Remark 1.2.1]) By Lemma 7.25, we have

$$\mathrm{Aut}_K(\underrightarrow{X}) = \mathrm{Gal}(\underrightarrow{X}/\underline{C})\,(\cong \mathbb{Z}/2l\mathbb{Z})$$

(cf. Remark 7.12.1).

§ 8. Frobenioids.

Roughly speaking, we have the following proportional formula:

Anabelioid (=Galois category) : Frobenioid = coverings : line bundles over coverings,

that is, the theory of Galois categories is a categorical formulation of coverings (i.e., it is formulated in terms of category, and geometric terms never appear), and the theory of Frobenioids is a categorical formulation of line bundles over coverings (i.e., it is formulated in terms of category, and geometric terms never appear). In [FrdI] and [FrdII], Mochizuki developed a general theory of Frobenioids; however, in this survey, we mainly forcus on model Frobenioids, which mainly used in inter-universal Teichmüller theory. The main theorems of the theory of Frobenioids are *category-theoretic reconstruction algorithms* of related objects (e.g., the base categories, the divisor monoids, and so on) under certain conditions; however, we avoid these theorems by including the objects, which we want to reconstruct, as input data, as suggested in [IUTchI, Remark 3.2.1 (ii)].

§ 8.1. Elementary Frobenioids and Model Frobenioids.

For a category $\mathcal{D}$, we shall refer to a contravariant functor $\Phi : \mathcal{D} \to \mathfrak{Mon}$ to the category of commutative monoids $\mathfrak{Mon}$ as a **monoid on $\mathcal{D}$** (In [FrdI, Definition 1.1], we write some conditions on Φ. However, this has no problem for our objects used in inter-universal Teichmüller theory.) If any element in $\Phi(A)$ is invertible for any $A \in \mathrm{Ob}(\mathcal{D})$, then we shall refer to Φ as **group-like**.

Definition 8.1. (Elementary Frobenioid, [FrdI, Definition 1.1 (iii)]) Let Φ be a monoid on a category $\mathcal{D}$. We consider the following category $\mathbb{F}_\Phi$:

(1) $\mathrm{Ob}(\mathbb{F}_\Phi) = \mathrm{Ob}(\mathcal{D})$.

(2) For $A, B \in \mathrm{Ob}(\mathcal{D})$, we put

$$\mathrm{Hom}_{\mathbb{F}_\Phi}(A, B) := \{\phi = (\mathrm{Base}(\phi), \mathrm{Div}(\phi), \deg_{\mathrm{Fr}}(\phi)) \in \mathrm{Hom}_{\mathcal{D}}(A, B) \times \Phi(A) \times \mathbb{N}_{\geq 1}\}.$$

We define the composition of $\phi = (\mathrm{Base}(\phi), \mathrm{Div}(\phi), \deg_{\mathrm{Fr}}(\phi)) : A \to B$ and $\psi = (\mathrm{Base}(\psi), \mathrm{Div}(\psi), \deg_{\mathrm{Fr}}(n)) : B \to C$ as

$$\psi \circ \phi := (\mathrm{Base}(\psi) \circ \mathrm{Base}(\phi), \Phi(\mathrm{Base}(\phi))(\mathrm{Div}(\psi)) + \deg_{\mathrm{Fr}}(\psi)\mathrm{Div}(\phi), \deg_{\mathrm{Fr}}(\psi)\deg_{\mathrm{Fr}}(\phi)) : A \to C.$$

We shall refer to $\mathbb{F}_\Phi$ as an **elementary Frobenioid associated to $\boldsymbol{\Phi}$**. Note that we have a natural functor $\mathbb{F}_\Phi \to \mathcal{D}$, which sends $A \in \mathrm{Ob}(\mathbb{F}_\Phi)$ to $A \in \mathrm{Ob}(\mathcal{D})$, and $\phi = (\mathrm{Base}(\phi), \mathrm{Div}(\phi), \deg_{\mathrm{Fr}}(\phi))$ to $\mathrm{Base}(\phi)$. We shall refer to $\mathcal{D}$ as the **base category of $\mathbb{F}_{\boldsymbol{\Phi}}$**.

For a category $\mathcal{C}$ and an elementary Frobenioid $\mathbb{F}_\Phi$, we shall refer to a covariant functor $\mathcal{C} \to \mathbb{F}_\Phi$ as a **pre-Frobenioid structure on $\mathcal{C}$** (In [FrdI, Definition 1.1 (iv)], we need conditions on Φ, $\mathcal{D}$, and $\mathcal{C}$ for the general theory of Frobenioids). We shall refer to a category $\mathcal{C}$ with a pre-Frobenioid structure as a **pre-Frobenioid**. For a pre-Frobenioid $\mathcal{C}$, we have a natural functor $\mathcal{C} \to \mathcal{D}$ by the composing with $\mathbb{F}_\Phi \to \mathcal{D}$. In a similar way, we obtain operations $\mathrm{Base}(-)$, $\mathrm{Div}(-)$, $\deg_{\mathrm{Fr}}(-)$ on $\mathcal{C}$ from the ones on $\mathbb{F}_\Phi$ by composing with $\mathbb{F}_\Phi \to \mathcal{D}$. We often use the same notation on $\mathcal{C}$ as well, by abuse of notation. We also shall refer to Φ and $\mathcal{D}$ as the **divisor monoid** and the **base category** of the pre-Frobenioid $\mathcal{C}$ respectively. We write

$$O^\times(A) := \{\phi \in \mathrm{Aut}_{\mathcal{C}}(A) \mid \mathrm{Base}(\phi) = \mathrm{id},\ \deg_{\mathrm{Fr}}(\phi) = 1\} \subset \mathrm{Aut}_{\mathcal{C}}(A),$$

and

$$O^\rhd(A) := \{\phi \in \mathrm{End}_{\mathcal{C}}(A) \mid \mathrm{Base}(\phi) = \mathrm{id},\ \deg_{\mathrm{Fr}}(\phi) = 1\} \subset \mathrm{End}_{\mathcal{C}}(A)$$

for $A \in \mathrm{Ob}(\mathcal{C})$. We also write $\mu_N(A) := \{a \in O^\times(A) \mid a^N = 1\}$ for $N \geq 1$.

Definition 8.2. ([IUTchI, Example 3.2 (v)]) When we are given a splitting $\mathrm{spl} : O^\rhd/O^\times \hookrightarrow O^\rhd$ (resp. a μ_N-orbit of a splitting $\mathrm{spl} : O^\rhd/O^\times \hookrightarrow O^\rhd$ for fixed N) of $O^\rhd \twoheadrightarrow O^\rhd/O^\times$, i.e., functorial splittings (resp. functorial μ_N-orbit of splittings) of $O^\rhd(A) \twoheadrightarrow O^\rhd(A)/O^\times(A)$ with respect to $A \in \mathrm{Ob}(\mathcal{C})$ and morphisms with $\deg_{\mathrm{Fr}} = 1$, then we shall refer to the pair $(\mathcal{C}, \mathrm{spl})$ as a **split pre-Frobenioid** (resp. a **$\boldsymbol{\mu_N}$-split pre-Frobenioid**).

If a pre-Frobenioid satisfies certain technical conditions, then we call it a **Frobenioid** (cf. [FrdI, Definition 1.3]). (Elementary Frobenioids are, in fact, Frobenioids ([FrdI, Proposition 1.5]).) In this survey, we do not recall the definition nor use the general theory of Frobenioids, and we mainly focus on model Frobenioids.

Definition 8.3. (Model Frobenioid, [FrdI, Theorem 5.2]) Let $\Phi : \mathcal{D} \to \mathfrak{Mon}$ be a monoid on a category $\mathcal{D}$. Let $\mathbb{B} : \mathcal{D} \to \mathfrak{Mon}$ be a group-like monoid on $\mathcal{D}$, and $\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \to \Phi^{\mathrm{gp}}$ a homomorphism. We write $\Phi^{\mathrm{birat}} := \mathrm{Im}(\mathrm{Div}_{\mathbb{B}}) \subset \Phi^{\mathrm{gp}}$. We consider the following category $\mathcal{C}$:

(1) The objects of $\mathcal{C}$ are pairs $A = (A_{\mathcal{D}}, \alpha)$, where $A_{\mathcal{D}} \in \mathrm{Ob}(\mathcal{D})$, and $\alpha \in \Phi(A_{\mathcal{D}})^{\mathrm{gp}}$. We write $\mathrm{Base}(A) := A_{\mathcal{D}}$, $\Phi(A) := \Phi(A_{\mathcal{D}})$, and $\mathbb{B}(A) := \mathbb{B}(A_{\mathcal{D}})$.

(2) For $A = (A_{\mathcal{D}}, \alpha), B = (B_{\mathcal{D}}, \beta) \in \mathrm{Ob}(\mathcal{C})$, we put

$$\mathrm{Hom}_{\mathcal{C}}(A,B) := \left\{ \begin{array}{l} \phi = (\mathrm{Base}(\phi), \mathrm{Div}(\phi), \deg_{\mathrm{Fr}}(\phi), u_\phi) \in \mathrm{Hom}_{\mathcal{D}}(A_{\mathcal{D}}, B_{\mathcal{D}}) \times \Phi(A) \times \mathbb{N}_{\geq 1} \times \mathbb{B}(A) \\ \text{such that } \ \deg_{\mathrm{Fr}}(\phi)\alpha + \mathrm{Div}(\phi) = \Phi(\mathrm{Base}(\phi))(\beta) + \mathrm{Div}_{\mathbb{B}}(u_\phi) \end{array} \right\}.$$

We define the composition of $\phi = (\mathrm{Base}(\phi), \mathrm{Div}(\phi), \deg_{\mathrm{Fr}}(\phi), u_\phi) : A \to B$ and $\psi = (\mathrm{Base}(\psi), \mathrm{Div}(\psi), \deg_{\mathrm{Fr}}(\psi), u_\psi) : B \to C$ as

$$\psi \circ \phi := \begin{pmatrix} \mathrm{Base}(\psi) \circ \mathrm{Base}(\phi), \Phi(\mathrm{Base}(\phi))(\mathrm{Div}(\psi)) + \deg_{\mathrm{Fr}}(\psi)\mathrm{Div}(\phi), \\ \deg_{\mathrm{Fr}}(\psi)\deg_{\mathrm{Fr}}(\phi), \mathbb{B}(\mathrm{Base}(\phi))(u_\psi) + \deg_{\mathrm{Fr}}(\psi)u_\phi \end{pmatrix}.$$

We equip $\mathcal{C}$ with a pre-Frobenioid structure $\mathcal{C} \to \mathbb{F}_\Phi$ by sending $(A_{\mathcal{D}}, \alpha) \in \mathrm{Ob}(\mathcal{C})$ to $A_{\mathcal{D}} \in \mathrm{Ob}(\mathbb{F}_\Phi)$ and $(\mathrm{Base}(\phi), \mathrm{Div}(\phi), \deg_{\mathrm{Fr}}(\phi), u_\phi)$ to $(\mathrm{Base}(\phi), \mathrm{Div}(\phi), \deg_{\mathrm{Fr}}(\phi))$. We shall refer to the category $\mathcal{C}$ as the **model Frobenioid** defined by the **divisor monoid Φ** and the **rational function monoid** $\mathbb{B}$ (Under some conditions, the model Frobenioid is in fact a Frobenioid).

The main theorems of the theory of Frobenioids are *category-theoretic reconstruction algorithms* of related objects (e.g., the base categories, the divisor monoids, and so on), under certain conditions. However, in this survey, we consider isomorphisms between pre-Frobenioids *not* to be just category equivalences, but to be category equivalences *including* pre-Frobenioid structures, i.e., for pre-Frobenioids $\mathcal{F}, \mathcal{F}'$ with pre-Frobenioid structures $\mathcal{F} \to \mathbb{F}_\Phi$, $\mathcal{F}' \to \mathbb{F}_{\Phi'}$, where $\mathbb{F}_\Phi, \mathbb{F}_{\Phi'}$ are defined by $\mathcal{D} \to \Phi$, $\mathcal{D}' \to \Phi'$ respectively, an **isomorphism of pre-Frobenioids** from $\mathcal{F}$ to $\mathcal{F}'$ consists of isomorphism classes (cf. also Definition 6.1 (5)) of equivalences $\mathcal{F}' \xrightarrow{\sim} \mathcal{F}$, $\mathcal{D}' \xrightarrow{\sim} \mathcal{D}$ of categories, and a natural transformation $\Phi' \to \Phi|_{\mathcal{D}'}$ (where $\Phi|_{\mathcal{D}'}$ is the restriction of Φ via $\mathcal{D}' \xrightarrow{\sim} \mathcal{D}$), such that it gives rise to an equivalence $\mathbb{F}_{\Phi'} \xrightarrow{\sim} \mathbb{F}_\Phi$ of categories, and the diagram

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{\sim} & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathbb{F}_{\Phi'} & \xrightarrow{\sim} & \mathbb{F}_{\Phi} \end{array}$$

is 1-commutative (i.e., one way of the composite of functors is isomorphic to the other way of the composite of functors) (cf. also [IUTchI, Remark 3.2.1 (ii)]).

Definition 8.4.

(1) (Trivial Line Bundle) For a model Frobenioid $\mathcal{F}$ with base category $\mathcal{D}$, we write $\mathcal{O}_A$ for the trivial line bundle over $A \in \mathrm{Ob}(\mathcal{D})$, i.e., the object determine by $(A, 0) \in \mathrm{Ob}(\mathcal{D}) \times \Phi(A)^{\mathrm{gp}}$ (These objects are called "Frobenius-trivial objects" in the terminology of [FrdI], which can category-theoretically be reconstructed only from $\mathcal{F}$ under some conditions).

(2) (Birationalisation, "$\mathbb{Z}_{\geq 0} \rightsquigarrow \mathbb{Z}$") Let $\mathcal{C}$ be a model Frebenioid. Let $\mathcal{C}^{\mathrm{birat}}$ be the category whose objects are the same as in $\mathcal{C}$, and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{birat}}}(A, B) := \varinjlim_{\phi: A' \to A,\ \mathrm{Base}(\phi): \mathrm{isom},\ \deg_{\mathrm{Fr}}(\phi)=1} \mathrm{Hom}_{\mathcal{C}}(A', B).$$

(For general Frobenioids, the definition of the birationalisation is a little more complicated. cf. [FrdI, Proposition 4.4]). We shall refer to $\mathcal{C}^{\mathrm{birat}}$ as the **birationalisation** of the model Frobenioid $\mathcal{C}$. We have a natural functor $\mathcal{C} \to \mathcal{C}^{\mathrm{birat}}$.

(3) (Realification, "$\mathbb{Z}_{\geq 0} \rightsquigarrow \mathbb{R}_{\geq 0}$") Let $\mathcal{C}$ be a model Frobenioid whose divisor monoid is Φ and whose rational function monoid is $\mathbb{B}$. Then let $\mathcal{C}^{\mathbb{R}}$ be the model Frobenioid obtained by replacing the divisor monoid Φ by $\Phi^{\mathbb{R}} := \Phi \otimes_{\mathbb{Z}_{\geq 0}} \mathbb{R}_{\geq 0}$, and the rational function monoid $\mathbb{B}$ by $\mathbb{B}^{\mathbb{R}} := \mathbb{R} \cdot \mathrm{Im}(\mathbb{B} \to \Phi^{\mathrm{gp}}) \subset (\Phi^{\mathbb{R}})^{\mathrm{gp}}$ (We need some conditions on $\mathcal{C}$, if we want to include more model Frobenioids which we do not treat in this survey. cf. [FrdI, Definition 2.4 (i), Proposition 5.2]). We shall refer to $\mathcal{C}^{\mathbb{R}}$ as the **realification** of the model Frobenioid $\mathcal{C}$. We have a natural functor $\mathcal{C} \to \mathcal{C}^{\mathbb{R}}$.

Definition 8.5. ($\times$-, $\times\mu$-Kummer structure on pre-Frobenioid, [IUTchII, Example 1.8 (iv), Definition 4.9 (i)])

(1) Let G be a toplogical group isomorphic to the absolute Galois group of an MLF. Then we can group-theoretically reconstruct an ind-topological monoid $G \curvearrowright O^{\rhd}(G)$ with G-action, by Proposition 5.2 (Step 1). Write $O^{\times}(G) := (O^{\rhd}(G))^{\times}$, $O^{\mu}(G) := (O^{\rhd}(G))_{\mathrm{tors}}$ and $O^{\times\mu}(G) := O^{\times}(G)/O^{\mu}(G)$ (We use the notation $O^{\times\mu}(-)$, not

$O^{\times}(-)/O^{\mu}(-)$, because we want to consider the object $O^{\times}(-)/O^{\mu}(-)$ as an abstract ind-topological module, i.e., without being equipped with the quotient structure $O^{\times}/O^{\mu}$). Write

$$\mathrm{Isomet}(G) = \Big\{G\text{-equivariant isomorphism } O^{\times\mu}(G) \xrightarrow{\sim} O^{\times\mu}(G) \text{ preserving the integral str. } \mathrm{Im}(O^{\times}(G)^{H} \to O^{\times\mu}(G)^{H}) \text{ for any open } H \subset G\Big\}.$$

We shall refer to the compact topological group $\mathrm{Isomet}(G)$ as the **group of G-isometries of $O^{\times\mu}(G)$**. If there is no confusion, we write just Isomet for $\mathrm{Isomet}(G)$.

(2) Let $\mathcal{C}$ be a pre-Frobenioid with base category $\mathcal{D}$. We assume that $\mathcal{D}$ is equivalent to the category of connected finite étale coverings of the spectrum of an MLF or a CAF. Let A_{∞} be a universal covering pro-object of $\mathcal{D}$. Write $G := \mathrm{Aut}(A_{\infty})$, hence G is isomorphic to the absolute Galois group of an MLF or a CAF. Then we have a natural action $G \curvearrowright O^{\rhd}(A_{\infty})$. For $N \geq 1$, we put

$$\mu_N(A_{\infty}) := \{a \in O^{\rhd}(A_{\infty}) \mid a^N = 1\} \ \subset \ O^{\mu}(A_{\infty}) := O^{\rhd}(A_{\infty})_{\mathrm{tors}} \ \subset \ O^{\rhd}(A_{\infty}),$$

and

$$O^{\times}(A_{\infty}) \twoheadrightarrow O^{\times\mu_N}(A_{\infty}) := O^{\times}(A_{\infty})/\mu_N(A_{\infty}) \twoheadrightarrow O^{\times\mu}(A_{\infty}) := O^{\times}(A_{\infty})/O^{\mu}(A_{\infty}).$$

These are equipped with natural G-actions. We assume that G is nontrivial (i.e., arising from an MLF). A **$\times$-Kummer structure** (resp. **$\times\mu$-Kummer structure**) on $\mathcal{C}$ is a $\widehat{\mathbb{Z}}^{\times}$-orbit (resp. an Isomet-orbit)

$$\kappa^{\times} : O^{\times}(G) \overset{\text{poly}}{\xrightarrow{\sim}} O^{\times}(A_{\infty}) \quad (\text{resp. } \kappa^{\times\mu} : O^{\times\mu}(G) \overset{\text{poly}}{\xrightarrow{\sim}} O^{\times\mu}(A_{\infty}) \ )$$

of isomorphisms of ind-topological G-modules. Note that the definition of a $\times$- (resp. $\times\mu$-) Kummer structure is independent of the choice of A_{∞}. Note also that any $\times$-Kummer structure on $\mathcal{C}$ is unique, since $\ker(\mathrm{Aut}(G \curvearrowright O^{\times}(G)) \twoheadrightarrow \mathrm{Aut}(G)) = \widehat{\mathbb{Z}}^{\times}(= \mathrm{Aut}(O^{\times}(G)))$ (cf. [IUTchII, Remark 1.11.1 (i) (b)]). We shall refer to a pre-Frobenioid equipped with a $\times$-Kummer structure (resp. $\times\mu$-Kummer structure) as a **$\times$-Kummer pre-Frobenioid** (resp. **$\times\mu$-Kummer pre-Frobenioid**). We shall refer to a split pre-Frobenioid equipped with a $\times$-Kummer structure (resp. $\times\mu$-Kummer structure) as a **split-$\times$-Kummer pre-Frobenioid** (resp. **split-$\times\mu$-Kummer pre-Frobenioid**).

Remark 8.5.1. ([IUTchII, Remark 1.8.1]) In the situation of Definition 8.5 (1), *no* automorphism of $O^{\times\mu}(G)$ induced by an element of $\mathrm{Aut}(G)$ is equal to an automorphism of $O^{\times\mu}(G)$ induced by an element of $\mathrm{Isomet}(G)$ which has nontrivial image in $\mathbb{Z}_p^{\times}$ (Here p is the residual characteristic of the MLF under consideration), since the composite with the p-adic logarithm of the cyclotomic character of G (which can be group-theoretically reconstructed by Proposition 2.1 (6)) determines a natural $\mathrm{Aut}(G) \times \mathrm{Isomet}(G)$-equivariant surjection $O^{\times\mu}(G) \twoheadrightarrow \mathbb{Q}_p$, where $\mathrm{Aut}(G)$ trivially acts on $\mathbb{Q}_p$ and $\mathrm{Isomet}(G)$ acts on $\mathbb{Q}_p$ via the natural surjection $\widehat{\mathbb{Z}}^\times \twoheadrightarrow \mathbb{Z}_p^\times$.

§ 8.2. Examples.

Example 8.6. (Geometric Frobenioid, [FrdI, Example 6.1]) Let V be a proper normal geometrically integral variety over a field k, $k(V)$ the function field of V, and $k(V)^\sim$ a (possibly inifinite) Galois extension. Write $G := \mathrm{Gal}(k(V)^\sim/k(V))$, and let $\mathbb{D}_{k(V)}$ be a set of $\mathbb{Q}$-Cartier prime divisors on V. The connected objects $\mathrm{Ob}(\mathcal{B}(G)^0)$ (cf. Section 0.2) of the Galois category (or connected anabelioid) $\mathcal{B}(G)$ can be thought of as schemes $\mathrm{Spec}\, L$, where $L \subset k(V)^\sim$ is a finite extension of $k(V)$. We write V_L for the normalisation of V in L, and we write $\mathbb{D}_L$ for the set of prime divisors of V_L which maps into (possibly subvarieties of codimension ≥ 1 of) prime divisors of $\mathbb{D}_{k(V)}$ We assume that any prime divisor of $\mathbb{D}_L$ is $\mathbb{Q}$-Cartier for any $\mathrm{Spec}\, L \in \mathrm{Ob}(\mathcal{B}(G)^0)$. We write $\Phi(L) \subset \mathbb{Z}_{\geq 0}[\mathbb{D}_L]$ for the monoid of effective Cartier divisors D on V_L such that every prime divisor in the support of D is in $\mathbb{D}_L$, and $\mathbb{B}(L) \subset L^\times$ for the group of rational functions f on V_L such that every prime divisor, at which f has a zero or a pole, is in $\mathbb{D}_L$. Note that we have a natural homomorphism $\mathbb{B}(L) \to \Phi(L)^{\mathrm{gp}}$ which sends f to $(f)_0 - (f)_\infty$ (Here, we write $(f)_0$ and $(f)_\infty$ for the zero-divisor and the pole-divisor of f respectively). This is functrial with respect to L. The data $(\mathcal{B}(G)^0, \Phi(-), \mathbb{B}(-), \mathbb{B} \to \Phi^{\mathrm{gp}})$ determines a model Frobenioid $\mathcal{C}_{V,k(V)^\sim,\mathbb{D}_K}$.

An object of $\mathcal{C}_{V,k(V)^\sim,\mathbb{D}_K}$, which is sent to $\mathrm{Spec}\, L \in \mathrm{Ob}(\mathcal{B}(G)^0)$, can be thought of as a line bundle $\mathcal{L}$ on V_L, which is representable by a Cartier divisor D with support in $\mathbb{D}_L$. For such line bundles $\mathcal{L}$ on $\mathrm{Spec}\, L$ and $\mathcal{M}$ on $\mathrm{Spec}\, M$ ($L, M \subset k(V)^\sim$ are finite extensions of $k(V)$), a morphism $\mathcal{L} \to \mathcal{M}$ in $\mathcal{C}_{V,k(V)^\sim,\mathbb{D}_K}$ can be thought of as consisting of a morphism $\mathrm{Spec}\, L \to \mathrm{Spec}\, M$ over $\mathrm{Spec}\, k(V)$, an element $d \in \mathbb{N}_{\geq 1}$, and a morphism of line bundles $\mathcal{L}^{\otimes d} \to \mathcal{M}|_{V_L}$ on V_L whose zero locus is a Cartier divisor supported in $\mathbb{D}_L$.

Example 8.7. (p-adic Frobenioid, [FrdII, Example 1.1], [IUTchI, Example 3.3]) Let $K_{\underline{v}}$ be a finite extension of $\mathbb{Q}_{p_{\underline{v}}}$ (In inter-universal Teichmüller theory, we use $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$). Write

$$\mathcal{D}_{\underline{v}} := \mathcal{B}(\underset{\rightarrow}{X}_{\underline{v}})^0, \quad \text{and} \quad \mathcal{D}^{\vdash}_{\underline{v}} := \mathcal{B}(K_{\underline{v}})^0,$$

where $\underset{\rightarrow}{X}_{\underline{v}}$ is a hyperbolic curve of type $(1, l\text{-}\underrightarrow{\mathrm{tors}})$ (cf. Definition 7.24). By pulling back finite étale coverings via the structure morphism $\underset{\rightarrow}{X}_{\underline{v}} \to \mathrm{Spec}\, K_{\underline{v}}$, we regard $\mathcal{D}^{\vdash}_{\underline{v}}$ as a full subcategory of $\mathcal{D}_{\underline{v}}$. We also have a left-adjoint $\mathcal{D}_{\underline{v}} \to \mathcal{D}^{\vdash}_{\underline{v}}$ to this functor, which is obtained by sending a $\Pi_{\underrightarrow{X}_{\underline{v}}}$-set E to the $G_{K_{\underline{v}}}$-set $E/\ker(\Pi_{\underrightarrow{X}_{\underline{v}}} \to G_{K_{\underline{v}}}) := \ker(\Pi_{\underrightarrow{X}_{\underline{v}}} \to G_{K_{\underline{v}}})$-orbits of E ([FrdII, Definition 1.3 (ii)]). Then

$$\Phi_{\mathcal{C}_{\underline{v}}} : \operatorname{Spec} L \mapsto \operatorname{ord}(O_L^{\triangleright})^{\mathrm{pf}} := (O_L/O_L^{\times})^{\mathrm{pf}}$$

(cf. Section 0.2 for the perfection $(-)^{\mathrm{pf}}$) gives us a monoid on $\mathcal{D}^{\vdash}_{\underline{v}}$. By composing the above $\mathcal{D}_{\underline{v}} \to \mathcal{D}^{\vdash}_{\underline{v}}$, it gives us a monoid $\Phi_{\mathcal{C}_{\underline{v}}}$ on $\mathcal{D}_{\underline{v}}$. Also,

$$\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} : \operatorname{Spec} L \mapsto \operatorname{ord}(\mathbb{Z}^{\triangleright}_{p_{\underline{v}}}) \, (\subset \operatorname{ord}(O_L^{\triangleright})^{\mathrm{pf}})$$

(cf. Section 0.2 for the perfection $(-)^{\mathrm{pf}}$) gives us a submonoid $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} \subset \Phi_{\mathcal{C}_{\underline{v}}}$ on $\mathcal{D}^{\vdash}_{\underline{v}}$. These monoids $\Phi_{\mathcal{C}_{\underline{v}}}$ on $\mathcal{D}_{\underline{v}}$ and $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ on $\mathcal{D}^{\vdash}_{\underline{v}}$ determine pre-Frobenioids (In fact, these are Frobenioid)

$$\mathcal{C}^{\vdash}_{\underline{v}} \subset \mathcal{C}_{\underline{v}}$$

whose base categories are $\mathcal{D}^{\vdash}_{\underline{v}}$ and $\mathcal{D}_{\underline{v}}$ respectively. These are called **$p_{\underline{v}}$-adic Frobenioids**. These pre-Frobenioid can be regarded as model Frobenioids whose rational function monoids $\mathbb{B}$ are given by $\operatorname{Ob}(\mathcal{D}^{\vdash}_{\underline{v}}) \ni \operatorname{Spec} L \mapsto L^{\times} \in \mathfrak{Mon}$, and $L^{\times} \ni f \mapsto (f)_0 - (f)_\infty :=$ image of $f \in \Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}(L) \subset \Phi_{\mathcal{C}_{\underline{v}}}(L)$ ([FrdII, Example 1.1]). Note that the element $p_{\underline{v}} \in \mathbb{Z}^{\triangleright}_{p_{\underline{v}}}$ gives us a splitting $\mathrm{spl}^{\vdash}_{\underline{v}} : O^{\triangleright}/O^{\times} \hookrightarrow O^{\triangleright}$, hence a split pre-Frobenioid

$$\mathcal{F}^{\vdash}_{\underline{v}} := (\mathcal{C}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}}).$$

We also write

$$\underline{\underline{\mathcal{F}}}_{\underline{v}} := \mathcal{C}_{\underline{v}}$$

for later use.

Example 8.8. (Tempered Frobenioid, [EtTh, Definition 3.3, Example 3.9, the beginning of §5], [IUTchI, Example 3.2]) Let $\underline{\underline{X}}_{\underline{v}} := \underline{\underline{X}}_{K_{\underline{v}}} \to \underline{X}_{\underline{v}} := \underline{X}_{K_{\underline{v}}}$ be a hyperbolic curve of type $(1, l\text{-tors}^{\Theta})$ and a hyperbolic curve of type $(1, \mathbb{Z}/l\mathbb{Z})$ respectively (Definition 7.13, Definition 7.11) over a finite extension $K_{\underline{v}}$ of $\mathbb{Q}_{p_{\underline{v}}}$ (As before, we always write the log-structure associated to the cusps, and consider the log-fundamental groups). Write

$$\mathcal{D}_{\underline{v}} := \mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})^0, \quad \mathcal{D}^{\vdash}_{\underline{v}} := \mathcal{B}(K_{\underline{v}})^0,$$

and $\mathcal{D}_0 := \mathcal{B}^{\mathrm{temp}}(\underline{X}_{\underline{v}})^0$ (cf. Section 0.2 for $(-)^0$. Note also that we have $\pi_1(\mathcal{D}_{\underline{v}}) \cong \Pi^{\mathrm{temp}}_{\underline{\underline{X}}_{\underline{v}}}$, and $\pi_1(\mathcal{D}^{\vdash}_{\underline{v}}) \cong G_{K_{\underline{v}}}$ (cf. Definition 6.1 (4))). We have a natural functor $\mathcal{D}_{\underline{v}} \to \mathcal{D}_0$, which sends $Y \to \underline{\underline{X}}_{\underline{v}}$ to the composite $Y \to \underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}}$.

For a tempered covering $Z \to \underline{X}_{\underline{v}}$ and its stable formal model $\mathfrak{Z}$ over O_L, where L is a finite extension of $K_{\underline{v}}$, let $\mathfrak{Z}_\infty \to \mathfrak{Z}$ be the universal combinatorial covering (i.e., the covering determined by the universal covering of the dual graph of the special fiber of $\mathfrak{Z}$), and Z_∞ the Raynaud generic fiber of $\mathfrak{Z}_\infty$.

Definition 8.9. ([EtTh, Definition 3.1], [IUTchI, Remark 3.2.4]) We write $\mathrm{Div}_+(\mathfrak{Z}_\infty)$ for the monoid of the effective Cartier divisors whose support lie in the union of the special fiber and the cusps of $\mathfrak{Z}_\infty$. We shall refer to such a divisor as an **effective Cartier log-divisor** on $\mathfrak{Z}_\infty$. Also, we write $\mathrm{Mero}(\mathfrak{Z}_\infty)$ for the group of meromorphic functions f on $\mathfrak{Z}_\infty$ such that, for any $N \geq 1, f$ admits an N-th root over some tempered covering of Z. We shall refer to such a function as a **log-meromorphic function** on $\mathfrak{Z}_\infty$.

Definition 8.10. ([EtTh, Definition 3.3, Example 3.9, the beginning of §5], [IUTchI, Example 3.2])

(1) Let Δ be a tempered group (Definition 6.1). We shall refer to a filtration $\{\Delta_i\}_{i\in I}$, (where I is countable) of Δ by characteristic open subgroups of finite index as a **tempred filter**, if the following conditions are satisfied:

  (a) We have $\bigcap_{i\in I} \Delta_i = \Delta$.

  (b) Every Δ_i admits an open characteristic subgroup Δ_i^∞ such that Δ_i/Δ_i^∞ is free, and, for any open normal subgroup $H \subset \Delta_i$ with free Δ_i/H, we have $\Delta_i^\infty \subset H$.

  (c) For each open subgroup $H \subset \Delta$, there exists unique $\Delta_{i_H}^\infty \subset H$, and, $\Delta_i^\infty \subset H$ implies $\Delta_i^\infty \subset \Delta_{i_H}^\infty$ for every $i \in I$.

(2) Let $\{\Delta_i\}_{i\in I}$ be a tempered filter of $\Delta_{\underline{X}_{\underline{v}}}^{\mathrm{temp}}$. Assume that, for any $i \in I$, the covering detemined by Δ_i has a stable model $\mathfrak{Z}_i$ over a ring of integers of a finite extension of $K_{\underline{v}}$, and all of the nodes and the irreducible components of the special fiber of $\mathfrak{Z}_i$ are rational (we say that $\mathfrak{Z}_i$ has **split** stable reduction). For any connected tempered covering $Y \to \underline{X}_{\underline{v}}$, which corresponds to an open subgroup $H \subset \Delta_{\underline{X}_{\underline{v}}}^{\mathrm{temp}}$, we put

$$\Phi_0(Y) := \varinjlim_{\Delta_i^\infty \subset H} \mathrm{Div}_+(\mathfrak{Z}_\infty)^{\mathrm{Gal}(Z_\infty/Y)}, \quad \mathbb{B}_0(Y) := \varinjlim_{\Delta_i^\infty \subset H} \mathrm{Mero}(\mathfrak{Z}_\infty)^{\mathrm{Gal}(Z_\infty/Y)}.$$

These determine functors $\Phi_0 : \mathcal{D}_0 \to \mathfrak{Mon}$, $\mathbb{B}_0 : \mathcal{D}_0 \to \mathfrak{Mon}$. We also have a natural functor $\mathbb{B}_0 \to \Phi_0^{\mathrm{gp}}$, by taking $f \mapsto (f)_0 - (f)_\infty$. We write $\mathbb{B}_0^{\mathrm{const}} \subset \mathbb{B}_0$ for the subfunctor defined by the constant log-meromorphic functions, and $\Phi_0^{\mathrm{const}} \subset \Phi_0^{\mathrm{gp}}$ for the image of $\mathbb{B}_0^{\mathrm{const}}$ in Φ_0^{gp}.

(3) We write $\mathcal{D}_0^{\mathrm{ell}} \subset \mathcal{D}_0$ for the full subcategory of tempered coverings which are unramified over the cusps of $\underline{X}_{\underline{v}}$ (i.e., tempered coverings of the underlying elliptic curve $\underline{E}_{\underline{v}}$ of $\underline{X}_{\underline{v}}$). We have a left adjoint $\mathcal{D}_0 \to \mathcal{D}_0^{\mathrm{ell}}$, which is obtained by sending a $\Pi_{\underline{X}_{\underline{v}}}$-set E to the $\Pi_{\underline{E}_{\underline{v}}}$-set $E/\ker(\Pi_{\underline{X}_{\underline{v}}} \to \Pi_{\underline{E}_{\underline{v}}}) := \ker(\Pi_{\underline{X}_{\underline{v}}} \to \Pi_{\underline{E}_{\underline{v}}})$-orbits of E

([FrdII, Definition 1.3 (ii)]). For $Y \in \mathrm{Ob}(\mathcal{D}_{\underline{v}})$, we write Y^{ell} for the image of Y by the composite $\mathcal{D}_{\underline{v}} \to \mathcal{D}_0 \to \mathcal{D}_0^{\mathrm{ell}}$. We put, for $Y \in \mathrm{Ob}(\mathcal{D}_{\underline{v}})$,

$$\Phi(Y) := \left(\varinjlim_{Z_\infty} \mathrm{Div}_+(\mathfrak{Z}_\infty)^{\mathrm{Gal}(Z_\infty/Y^{\mathrm{ell}})}\right)^{\mathrm{pf}} \subset \Phi_0(\text{the image of } Y \text{ in } \mathcal{D}_0)^{\mathrm{pf}},$$

where Z_∞ range over the connected tempered covering $Z_\infty \to Y^{\mathrm{ell}}$ in $\mathcal{D}_0^{\mathrm{ell}}$ such that the composite $Z_\infty \to Y^{\mathrm{ell}} \to \underline{X}_{\underline{v}}$ arises as the generic fiber of the universal combinatorial covering $\mathfrak{Z}_\infty$ of the stable model $\mathfrak{Z}$ of some finite étale covering $Z \to \underline{X}_{\underline{v}}$ in $\mathcal{D}_0^{\mathrm{ell}}$ with split stable reduction over the ring of integers of a finite extension of $K_{\underline{v}}$ (We use this Φ, not Φ_0, to consider only divisors related with the theta function). We write $(-)|_{\mathcal{D}_{\underline{v}}}$ for the restriction, via $\mathcal{D}_{\underline{v}} \to \mathcal{D}_0$, of a functor whose domain is $\mathcal{D}_0$. We also write $\Phi_0^{\mathbb{R}} := \Phi_0 \otimes_{\mathbb{Z}_{\geq 0}} \mathbb{R}_{\geq 0}$ and $\Phi^{\mathbb{R}} := \Phi \otimes_{\mathbb{Z}_{\geq 0}} \mathbb{R}_{\geq 0}$. Write

$$\mathbb{B} := \mathbb{B}_0|_{\mathcal{D}_{\underline{v}}} \times_{(\Phi^{\mathbb{R}})^{\mathrm{gp}}} \Phi^{\mathrm{gp}}, \qquad \Phi^{\mathrm{const}} := (\mathbb{R} \cdot \Phi_0^{\mathrm{const}})|_{\mathcal{D}_{\underline{v}}} \times_{(\Phi^{\mathbb{R}})^{\mathrm{gp}}} \Phi \subset \Phi^{\mathbb{R}},$$

and

$$\mathbb{B}^{\mathrm{const}} := \mathbb{B}_0^{\mathrm{const}}|_{\mathcal{D}_{\underline{v}}} \times_{(\Phi^{\mathbb{R}})^{\mathrm{gp}}} \Phi^{\mathrm{gp}} \to (\Phi^{\mathrm{const}})^{\mathrm{gp}} = (\mathbb{R} \cdot \Phi_0^{\mathrm{const}})|_{\mathcal{D}_{\underline{v}}} \times_{(\Phi^{\mathbb{R}})^{\mathrm{gp}}} \Phi^{\mathrm{gp}} \subset (\Phi^{\mathbb{R}})^{\mathrm{gp}}.$$

The data $(\mathcal{D}_{\underline{v}}, \Phi, \mathbb{B}, \mathbb{B} \to \Phi^{\mathrm{gp}})$ and $(\mathcal{D}_{\underline{v}}, \Phi^{\mathrm{const}}, \mathbb{B}^{\mathrm{const}}, \mathbb{B}^{\mathrm{const}} \to (\Phi^{\mathrm{const}})^{\mathrm{gp}})$ determine model Frobenioids

$$\underline{\underline{\mathcal{F}}}_{\underline{v}}, \quad \text{and} \quad \mathcal{C}_{\underline{v}}\,(= \underline{\underline{\mathcal{F}}}_{\underline{v}}^{\text{base-field}})$$

respectively (In fact, these are Frobenioids). We have a natural inclusion $\mathcal{C}_{\underline{v}} \subset \underline{\underline{\mathcal{F}}}_{\underline{v}}$. We shall refer to $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ a **tempered Frobenioid** and $\mathcal{C}_{\underline{v}}$ as its **base-field-theoretic hull**. Note that $\mathcal{C}_{\underline{v}}$ is also a $p_{\underline{v}}$-adic Frobenioid.

(4) We write $\underline{\underline{\Theta}}_{\underline{v}} \in O^\times(\mathcal{O}^{\mathrm{birat}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$ for the reciprocal (i.e., $1/(-)$) of the l-th root of the normalised theta function, which is well-defined up to μ_{2l} and the action of the group of automorphisms $l\underline{\mathbb{Z}} \subset \mathrm{Aut}(\mathcal{O}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$ (Note that we use the notation $\underline{\underline{\ddot{\Theta}}}$ in Section 8.3. This is not the reciprocal (i.e., not $1/(-)$) one). We also write $q_{\underline{v}}$ for the q-parameter of the elliptic curve $E_{\underline{v}}$ over $K_{\underline{v}}$. We consider $q_{\underline{v}}$ as an element $q_{\underline{v}} \in O^{\rhd}(\mathcal{O}_{\underline{\underline{X}}_{\underline{v}}})\,(\cong O^{\rhd}_{K_{\underline{v}}})$. We assume that any $2l$-torsion point of $E_{\underline{v}}$ is rational over $K_{\underline{v}}$. Then $q_{\underline{v}}$ admits a $2l$-root in $O^{\rhd}(\mathcal{O}_{\underline{\underline{X}}_{\underline{v}}})\,(\cong O^{\rhd}_{K_{\underline{v}}})$. Then we have

$$\underline{\underline{\Theta}}_{\underline{v}}(\sqrt{-q_{\underline{v}}}) = \underline{\underline{q}}_{\underline{v}} := q_{\underline{v}}^{1/2l} \in O^{\rhd}(\mathcal{O}_{\underline{\underline{X}}_{\underline{v}}}),$$

(which is well-defined up to μ_{2l}), since $\ddot{\Theta}(\sqrt{-q}) = -q^{-1/2}\sqrt{-1}^{-2}\ddot{\Theta}(\sqrt{-1}) = q^{-1/2}$ (in the notation of Lemma 7.4) by the formula $\ddot{\Theta}(q^{1/2}\ddot{U}) = -q^{-1/2}\ddot{U}^{-2}\ddot{\Theta}(\ddot{U})$ in

Lemma 7.4. The image of $\underline{\underline{q}}_{\underline{v}}$ determines a constant section, for which we write $\log_\Phi(\underline{\underline{q}}_{\underline{v}})$ of the monoid $\Phi_{\mathcal{C}_{\underline{v}}}$ of $\mathcal{C}_{\underline{v}}$. The submonoid

$$\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} := \mathbb{N}\log_\Phi(\underline{\underline{q}}_{\underline{v}})|_{\mathcal{D}^{\vdash}_{\underline{v}}} \subset \Phi_{\mathcal{C}_{\underline{v}}}|_{\mathcal{D}^{\vdash}_{\underline{v}}}$$

gives us a $p_{\underline{v}}$-adic Frobenioid

$$\mathcal{C}^{\vdash}_{\underline{v}}\ (\subset \mathcal{C}_{\underline{v}} = (\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\text{base-field}} \subset \underline{\underline{\mathcal{F}}}_{\underline{v}})$$

whose base category is $\mathcal{D}^{\vdash}_{\underline{v}}$. The element $\underline{\underline{q}}_{\underline{v}} \in K_{\underline{v}}$ determines a $\mu_{2l}(-)$-orbit $\mathrm{spl}^{\vdash}_{\underline{v}}$ of the splittings of $O^{\rhd} \twoheadrightarrow O^{\rhd}/O^{\times}$ on $\mathcal{C}^{\vdash}_{\underline{v}}$. Hence

$$\mathcal{F}^{\vdash}_{\underline{v}} := (\mathcal{C}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}})$$

is a μ_{2l}-split pre-Frobenioid.

Remark 8.10.1. We can category-theoretically reconstruct the base-field-theoretic hull $\mathcal{C}_{\underline{v}}$ from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ ([EtTh, Corollary 3.8]). However, in this survey, we include the base-field-theoretic hull in the deta of the tempered Frobenioid, i.e., we shall refer to a pair $\underline{\underline{\mathcal{F}}}_{\underline{v}} = (\underline{\underline{\mathcal{F}}}_{\underline{v}}, \mathcal{C}_{\underline{v}})$ as a tempered Frobenioid, by abuse of language/notation, in this survey.

Example 8.11. (Archimedean Frobenioid, [FrdII, Example 3.3], [IUTchI, Example 3.4]) This example is *not* a model Frobenioid (In fact, it is *not* of isotropic type, which any model Frobenioids should be). Let $K_{\underline{v}}$ be a complex Archimedean local field (In inter-universal Teichmüller theory, we use $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$). We define a category

$$\mathcal{C}_{\underline{v}}$$

as follows: The objects of $\mathcal{C}_{\underline{v}}$ are pairs $(V, \mathbb{A})$ of a one-dimensional $K_{\underline{v}}$-vector space V, and a subset $\mathbb{A} = B \times C \subset V \cong O^{\times}_{K_{\underline{v}}} \times \mathrm{ord}(K^{\times}_{\underline{v}})$ (Here we write $\mathrm{ord}(K^{\times}_{\underline{v}}) := K^{\times}_{\underline{v}}/O^{\times}_{K_{\underline{v}}}$. cf. Section 0.2 for $O_{K_{\underline{v}}}$), where $B \subset O^{\times}_{K_{\underline{v}}}(\cong \mathbb{S}^1)$ is a connected open subset, and $C \subset \mathrm{ord}(K^{\times}_{\underline{v}}) \cong \mathbb{R}_{>0}$ is an interval of the form $(0, \lambda]$ with $\lambda \in \mathbb{R}_{>0}$ (We shall refer to $\mathbb{A}$ as an **angular region**). The morphisms ϕ from $(V, \mathbb{A})$ to $(V', \mathbb{A}')$ in $\mathcal{C}_{\underline{v}}$ consist of an element $\deg_{\mathrm{Fr}}(\phi) \in \mathbb{N}_{\geq 1}$ and an isomorphism $V^{\otimes \deg_{\mathrm{Fr}}(\phi)} \xrightarrow{\sim} V'$ of $K_{\underline{v}}$-vector spaces which sends $\mathbb{A}^{\otimes \deg_{\mathrm{Fr}}(\phi)}$ into $\mathbb{A}'$. We write $\mathrm{Div}(\phi) := \log(\alpha) \in \mathbb{R}_{\geq 0}$ for the largest $\alpha \in \mathbb{R}_{>0}$ such that $\alpha \cdot \mathrm{Im}(\mathbb{A}^{\otimes \deg_{\mathrm{Fr}}(\phi)}) \subset \mathbb{A}'$. Let $\{\mathrm{Spec}\, K_{\underline{v}}\}$ be the category of connected finite étale coverings of $\mathrm{Spec}\, K_{\underline{v}}$ (Thus, there is only one object, and only one morphism), and $\Phi : \{\mathrm{Spec}\, K_{\underline{v}}\} \to \mathfrak{Mon}$ the functor defined by sending $\mathrm{Spec}\, K_{\underline{v}}$ (the unique object) to $\mathrm{ord}(O^{\rhd}_{K_{\underline{v}}}) \cong (0, 1] \overset{-\log}{\cong} \mathbb{R}_{\geq 0}$. Write also $\mathrm{Base}(V, \mathbb{A}) := \mathrm{Spec}\, K_{\underline{v}}$ for $(V, \mathbb{A}) \in \mathrm{Ob}(\mathcal{C}_{\underline{v}})$. Then the triple $(\mathrm{Base}(-), \Phi(-), \deg_{\mathrm{Fr}}(-))$ gives us a pre-Frobenioid structure $\mathcal{C}_{\underline{v}} \to \mathbb{F}_\Phi$ on $\mathcal{C}_{\underline{v}}$ (In fact, this is a Frobenioid). We shall refer to $\mathcal{C}_{\underline{v}}$ as an

Archimedean Frobenioid (cf. the Archimedean portion of arithmetic line bundles). Note also that we have a natural isomorphism $O^{\triangleright}(\mathcal{C}_{\underline{v}}) \cong O^{\triangleright}_{K_{\underline{v}}}$ of topological monoids (We can regarad $\mathcal{C}_{\underline{v}}$ as a Frobenioid-theoretic representation of the topological monoid $O^{\triangleright}_{K_{\underline{v}}}$).

Let $\underset{\rightarrow}{X}_{\underline{v}}$ be a hyperbolic curve of type $(1, l\text{-}\underrightarrow{\mathrm{tors}})$ (cf. Definition 7.24) over $K_{\underline{v}}$, and we write $\underset{\rightarrow}{\mathbb{X}}_{\underline{v}}$ for the Aut-holomorphic space (cf. Section 4) determined by $\underset{\rightarrow}{X}_{\underline{v}}$, and put

$$\mathcal{D}_{\underline{v}} := \underset{\rightarrow}{\mathbb{X}}_{\underline{v}}.$$

Note also that we have a natural isomorphism

$$K_{\underline{v}} \xrightarrow{\sim} \overline{\mathcal{A}^{\mathcal{D}_{\underline{v}}}}$$

of topological fields (cf. (Step 9) in Proposition 4.5), which determines an inclusion

$$\kappa_{\underline{v}} : O^{\triangleright}(\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}^{\mathcal{D}_{\underline{v}}}$$

of topological monoids. This gives us a Kummer structure (cf. Definition 4.6) on $\mathcal{D}_{\underline{v}}$. Write

$$\underline{\underline{\mathcal{F}}}_{\underline{v}} := (\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}}),$$

just as a triple. We define an isomorphism $\underline{\underline{\mathcal{F}}}_{\underline{v},1} \xrightarrow{\sim} \underline{\underline{\mathcal{F}}}_{\underline{v},2}$ of triples in an obvious manner.

Next, we consider the mono-analyticisation. Write

$$\mathcal{C}^{\vdash}_{\underline{v}} := \mathcal{C}_{\underline{v}}.$$

Note also that $\overline{\mathcal{A}^{\mathcal{D}_{\underline{v}}}}$ naturally determines a split monoid (cf. Definition 4.7) by transporting the natural splitting of $K_{\underline{v}}$ via the isomorphism $K_{\underline{v}} \xrightarrow{\sim} \overline{\mathcal{A}^{\mathcal{D}_{\underline{v}}}}$ of topological fields. This gives us a splitting $\mathrm{spl}^{\vdash}_{\underline{v}}$ on $\mathcal{C}^{\vdash}_{\underline{v}}$, hence a split-Frobenioid $(\mathcal{C}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}})$, as well as a split monoid

$$\mathcal{D}^{\vdash}_{\underline{v}} := (O^{\triangleright}(\mathcal{C}^{\vdash}_{\underline{v}}), \mathrm{spl}^{\vdash}_{\underline{v}}).$$

We put

$$\mathcal{F}^{\vdash}_{\underline{v}} := (\mathcal{C}^{\vdash}_{\underline{v}}, \mathcal{D}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}}),$$

just as a triple. We define an isomorphism $\mathcal{F}^{\vdash}_{\underline{v},1} \xrightarrow{\sim} \mathcal{F}^{\vdash}_{\underline{v},2}$ of triples in an obvious manner.

Example 8.12. (Global Realified Frobenioid, [FrdI, Example 6.3], [IUTchI, Example 3.5]) Let F_{mod} be a number field. Let $\{\mathrm{Spec}\, F_{\mathrm{mod}}\}$ be the category of the trivial connected finite étale covering of $\mathrm{Spec}\, F_{\mathrm{mod}}$. (Thus, there is only one object, and only one morphism.) Write

$$\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}}(F_{\mathrm{mod}}) := \bigoplus_{v \in \mathbb{V}(F_{\mathrm{mod}})^{\mathrm{non}}} \mathrm{ord}(O^{\triangleright}_v) \otimes_{\mathbb{Z}_{\geq 0}} \mathbb{R}_{\geq 0} \oplus \bigoplus_{v \in \mathbb{V}(F_{\mathrm{mod}})^{\mathrm{arc}}} \mathrm{ord}(O^{\triangleright}_v),$$

where $\mathrm{ord}(O_v^{\triangleright}) := O_v^{\triangleright}/O_v^{\times}$ (cf. Section 0.2 for O_v and $O_v^{\triangleright}$, $v \in \mathbb{V}(F_{\mathrm{mod}})^{\mathrm{arc}}$). We shall refer to an element of $\Phi(F_{\mathrm{mod}})$ (resp. $\Phi(F_{\mathrm{mod}})^{\mathrm{gp}}$) as an **effective arithmetic divisor** (resp. an **arithmetic divisor**). Note that $\mathrm{ord}(O_v^{\triangleright}) \cong \mathbb{Z}_{\geq 0}$ for $v \in \mathbb{V}(F_{\mathrm{mod}})^{\mathrm{non}}$, and $\mathrm{ord}(O_v^{\triangleright}) \cong \mathbb{R}_{\geq 0}$ for $v \in \mathbb{V}(F_{\mathrm{mod}})^{\mathrm{arc}}$. We have a natural homomorphism

$$\mathbb{B}(F_{\mathrm{mod}}) := F_{\mathrm{mod}}^{\times} \to \Phi(F_{\mathrm{mod}})^{\mathrm{gp}}.$$

Then the data $(\{\mathrm{Spec}\, F_{\mathrm{mod}}\}, \Phi_{\mathcal{C}_{\mathrm{mod}}^{\Vdash}}, \mathbb{B})$ determines a model Frobenioid

$$\mathcal{C}_{\mathrm{mod}}^{\Vdash}.$$

(In fact, it is a Frobenioid.) We shall refer to it as a **global realified Frobenioid**.

We have a natural bijection

$$\mathrm{Prime}(\mathcal{C}_{\mathrm{mod}}^{\Vdash}) \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$$

(by abuse of notation, we write $\mathrm{Prime}(\mathcal{C}_{\mathrm{mod}}^{\Vdash}) := \mathrm{Prime}(\Phi_{\mathcal{C}_{\mathrm{mod}}^{\Vdash}}(\mathrm{Spec}\, F_{\mathrm{mod}})))$, where $\mathrm{Prime}(-)$ is defined as follows:

Definition 8.13. Let M be a commutative monoid such that 0 is the only invertible element in M, the natural homomorphism $M \to M^{\mathrm{gp}}$ is injective, and any $a \in M^{\mathrm{gp}}$ with $na \in M$ for some $n \in \mathbb{N}_{\geq 1}$ is in the image of $M \hookrightarrow M^{\mathrm{gp}}$. We define the set $\mathrm{Prime}(M)$ of primes of M as follows ([FrdI, §0]):

(1) For $a, b \in M$, we write $a \leq b$, if there is $c \in M$ such that $a + c = b$.

(2) For $a, b \in M$, we write $a \preccurlyeq b$, if there is $n \in \mathbb{N}_{\geq 1}$ such that $a \leq nb$.

(3) For $0 \neq a \in M$, we say that a is **primary**, if $a \preccurlyeq b$ holds for any $M \ni b \preccurlyeq a$, $b \neq 0$.

(4) The relation $a \preccurlyeq b$ is an equivalence relation among the set of primary elements in M, and we shall refer to an equivalence class as a **prime** of M (this definition is different from a usual definition of primes of a monoid). We write $\mathrm{Prime}(M)$ for the set of primes of M.

Note that p_v determines an element

$$\log_{\mathrm{mod}}^{\vdash}(p_v) \in \Phi_{\mathcal{C}_{\mathrm{mod}}^{\Vdash}, v}$$

for $v \in \mathbb{V}_{\mathrm{mod}} \cong \mathrm{Prime}(\mathcal{C}_{\mathrm{mod}}^{\Vdash})$, where we write $\Phi_{\mathcal{C}_{\mathrm{mod}}^{\Vdash}, v} (\cong \mathbb{R}_{\geq 0})$ for the v-portion of $\Phi_{\mathcal{C}_{\mathrm{mod}}^{\Vdash}}$.

§ 8.3. From Tempered Frobenioids to Mono-theta Environments.

Let $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ be the tempered Frobenioid constructed in Example 8.8. Recall that it has a base category $\mathcal{D}_{\underline{v}}$ with $\pi_1(\mathcal{D}_{\underline{v}}) \cong \Pi_{\underline{\underline{X}}_{\underline{v}}}^{\mathrm{temp}} (=: \Pi_{\underline{v}})$. We write $\mathcal{O}_{\underline{\underline{\ddot{Y}}}}$ for the object in $\underline{\underline{\mathcal{F}}}_{\underline{v}}$

corresponding to the trivial line bundle on $\underline{\underline{\ddot{Y}}}$ (i.e., $\mathcal{O}_{\underline{\underline{\ddot{Y}}}} = (\underline{\underline{\ddot{Y}}}, 0) \in \mathrm{Ob}(\mathcal{D}_{\underline{v}}) \times \Phi(\underline{\underline{\ddot{Y}}})$. cf. Definition 8.4 (1)). Let $\ddot{Y}_{lN}$, $\mathfrak{Z}_{lN}$, $\ddot{\mathfrak{Z}}_{lN}$, $\mathfrak{L}_{lN}$, and $\ddot{\mathfrak{L}}_{lN}$ as in Section 7.1. We can interpret the pull-backs to $\ddot{\mathfrak{Z}}_{lN}$ of

(1) the algebraic section $s_{lN} \in \Gamma\left(\mathfrak{Z}_{lN}, \mathfrak{L}_{lN}|_{\mathfrak{Z}_{lN}}\right)$ of Lemma 7.1, and

(2) the theta trivialisation $\tau_{lN} \in \Gamma\left(\ddot{\mathfrak{Y}}_{lN}, \ddot{\mathfrak{L}}_{lN}\right)$ after Lemma 7.1.

as morphisms

$$s_N^{\sqcap}, s_N^{\sqcup} : \mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}} \to \ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}$$

in $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ respectively. For $A \in \mathrm{Ob}(\underline{\underline{\mathcal{F}}}_{\underline{v}})$, we write A^{birat} for the image of A in the birationalisation $\underline{\underline{\mathcal{F}}}_{\underline{v}} \to (\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\mathrm{birat}}$ (Definition 8.4 (2)). Then by definition, we have

$$s_N^{\sqcap} \circ (s_N^{\sqcup})^{-1} = \underline{\underline{\ddot{\Theta}}}^{1/N} \in O^{\times}\left(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}}^{\mathrm{birat}}\right)$$

for an N-th root of $\underline{\underline{\ddot{\Theta}}}$, where $\underline{\underline{\ddot{\Theta}}} := \ddot{\Theta}^{1/l}$ is a l-th root of the theta function $\ddot{\Theta}$ ([EtTh, Proposition 5.2 (i)]), as in Section 7.1 (cf. also the claim (7.2)). We write $H(\ddot{\mathfrak{Z}}_{lN})\,(\subset \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\ddot{\mathfrak{Z}}_{lN}))$ for the image of $\Pi_{\underline{\underline{\ddot{Y}}}}^{\mathrm{temp}}$ under the surjective outer homomorphism $\Pi_{\underline{\underline{X}}_{\underline{v}}}^{\mathrm{temp}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\ddot{\mathfrak{Z}}_{lN})$. We also write $H(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}})\,(\subset \mathrm{Aut}_{\underline{\underline{\mathcal{F}}}_{\underline{v}}}(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}})/O^{\times}(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}}))$ (resp. $H(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})\,(\subset \mathrm{Aut}_{\underline{\underline{\mathcal{F}}}_{\underline{v}}}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})/O^{\times}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}))$) for the inverse image of $H(\ddot{\mathfrak{Z}}_{lN})$ of the natural injection $\mathrm{Aut}_{\underline{\underline{\mathcal{F}}}_{\underline{v}}}(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}})/O^{\times}(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}}) \hookrightarrow \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\ddot{\mathfrak{Z}}_{lN})$ (resp. $\mathrm{Aut}_{\underline{\underline{\mathcal{F}}}_{\underline{v}}}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})/O^{\times}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}) \hookrightarrow \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\ddot{\mathfrak{Z}}_{lN})$):

$$\begin{tikzcd}
\Pi_{\underline{\underline{X}}_{\underline{v}}}^{\mathrm{temp}} \arrow[r, two heads] & \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\ddot{\mathfrak{Z}}_{lN}) & \mathrm{Aut}_{\underline{\underline{\mathcal{F}}}_{\underline{v}}}(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}})/O^{\times}(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}}) \text{ (resp. } \mathrm{Aut}_{\underline{\underline{\mathcal{F}}}_{\underline{v}}}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})/O^{\times}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})\text{)} \arrow[l, hook'] \\
\Pi_{\underline{\underline{\ddot{Y}}}}^{\mathrm{temp}} \arrow[u, hook] \arrow[r, two heads] & H(\ddot{\mathfrak{Z}}_{lN}) \arrow[u, hook] & H(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}}) \text{ (resp. } H(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})\text{).} \arrow[l, hook', "="'] \arrow[u, hook]
\end{tikzcd}$$

Note that we have natural isomorphisms $H(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}}) \cong H(\ddot{\mathfrak{Z}}_{lN}) \cong H(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})$. Choose a section of $\mathrm{Aut}_{\underline{\underline{\mathcal{F}}}_{\underline{v}}}(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}}) \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\ddot{\mathfrak{Z}}_{lN})$, which gives us a homomorphism

$$s_N^{\mathrm{triv}} : H(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}}) \to \mathrm{Aut}_{\underline{\underline{\mathcal{F}}}_{\underline{v}}}(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}}).$$

Then by taking the group actions of $H(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})$ on $s_N^{\sqcap}$, and $s_N^{\sqcup}$ (cf. the actions of $\Pi_{\ddot{Y}}^{\mathrm{temp}}$ on s_N and τ_N in Section 7.1), we have unique groups homomorphisms

$$s_N^{\sqcap\text{-gp}}, s_N^{\sqcup\text{-gp}} : H(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}) \to \mathrm{Aut}_{\underline{\underline{\mathcal{F}}}_{\underline{v}}}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}),$$

which make diagrams

$$\begin{array}{ccc} \mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}} & \xrightarrow{s_N^{\sqcap}} & \ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}} \\ {\scriptstyle (s_N^{\mathrm{triv}}|\ddot{\mathfrak{L}}_{lN})(h)}\downarrow & & \downarrow{\scriptstyle s_N^{\sqcap\text{-gp}}(h)} \\ \mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}} & \xrightarrow{s_N^{\sqcap}} & \ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}, \end{array} \qquad \begin{array}{ccc} \mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}} & \xrightarrow{s_N^{\sqcup}} & \ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}} \\ {\scriptstyle (s_N^{\mathrm{triv}}|\ddot{\mathfrak{L}}_{lN})(h)}\downarrow & & \downarrow{\scriptstyle s_N^{\sqcup\text{-gp}}(h)} \\ \mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}} & \xrightarrow{s_N^{\sqcup}} & \ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}, \end{array}$$

commutative for any $h \in H(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})$, where $s_N^{\mathrm{triv}}|\ddot{\mathfrak{L}}_{lN}$ is the composite of s_N^{triv} with the natural isomorphism $H(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}) \cong H(\mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}})$. Then the difference $s_N^{\sqcap\text{-gp}} \circ (s_N^{\sqcup\text{-gp}})^{-1}$ gives us a 1-cocycle $H(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}) \to \mu_N(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})$, whose cohomology class in

$$H^1(H(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}), \mu_N(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}))\,(\subset H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}, \mu_N(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})))$$

is, by construction, equal to the (mod N) Kummer class of an l-th root $\underline{\underline{\ddot{\Theta}}}$ of the theta function, and also equal to the $\underline{\underline{\ddot{\eta}}}^{\Theta}$ modulo N constructed before Definition 7.14 under the natural isomorphisms $l\Delta_\Theta \otimes (\mathbb{Z}/N\mathbb{Z}) \cong l\mu_{lN}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}) \cong \mu_N(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})$ ([EtTh, Proposition 5.2 (iii)]). (cf. also Remark 7.2.1.)

Note that the subquotients $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}} \twoheadrightarrow (\Pi^{\mathrm{temp}}_{X})^{\Theta}$, $l\Delta_\Theta \subset (\Pi^{\mathrm{temp}}_{X})^{\Theta}$ in Section 7.1 determine subquotients $\mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(S) \twoheadrightarrow \mathrm{Aut}^{\Theta}_{\mathcal{D}_{\underline{v}}}(S)$, $(l\Delta_\Theta)_S \subset \mathrm{Aut}^{\Theta}_{\mathcal{D}_{\underline{v}}}(S)$ for $S \in \mathrm{Ob}(\mathcal{D}_{\underline{v}})$. As in Remark 7.6.3, Remark 7.9.1, and Remark 7.15.1, by considering the zero-divisor and the pole-divisor (as seen in this subsection too) of the normalised theta function $\ddot{\Theta}(\sqrt{-1})^{-1}\ddot{\Theta}$, we can category-theoretically reconstruct the $l\underline{\mathbb{Z}} \times \mu_2$-orbit of the theta classes of standard type with $\mu_N(-)$-coefficient ([EtTh, Theorem 5.7]). As in the case of the cyclotomic rigidity on mono-theta environment (Theorem 7.23 (1)), by considering the difference of two splittings of the surjection $(l\Delta_\Theta)_S[\mu_N(S)] \twoheadrightarrow (l\Delta_\Theta)_S$, we can category-theoretically reconstruct the cyclotomic rigidity isomorphism

$$(l\Delta_\Theta)_S \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mu_N(S)\,(= l\mu_{lN}(S)) \qquad \text{(Cyc. Rig. Frd)}$$

for an object S of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ such that $\mu_{lN}(S) \cong \mathbb{Z}/lN\mathbb{Z}$, and $(l\Delta_\Theta)_S \otimes \mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}/N\mathbb{Z}$ as abstract groups ([EtTh, Theorem 5.6]). We shall refer to this isomorphism as the **cyclotomic rigidity in tempered Frobenioid**.

Write $(H(\ddot{\mathfrak{Z}}_{lN}) \subset)\,\mathrm{Im}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})\,(\subset \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\ddot{\mathfrak{Z}}_{lN}))$ to be the image of $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$ (Note that we used $\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}$ in the definition of $H(\ddot{\mathfrak{Z}}_{lN})$) under the natural surjective outer homomorphism $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}_{\underline{v}}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\ddot{\mathfrak{Z}}_{lN})$, and

$$\mathbb{E}_N := s_N^{\sqcap\text{-gp}}(\mathrm{Im}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})) \cdot \mu_N(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}) \subset \mathrm{Aut}_{\underline{\underline{\mathcal{F}}}_{\underline{v}}}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}).$$

Write also

$$\mathbb{E}^{\Pi}_N := \mathbb{E}_N \times_{\mathrm{Im}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})} \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}},$$

where the homomorphism $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}} \twoheadrightarrow \mathrm{Im}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})$ is well-defined up to $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}$-conjugate. Then the natural inclusions $\mu_N(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}) \hookrightarrow \mathbb{E}_N$ and $\mathrm{Im}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}) \hookrightarrow \mathbb{E}_N$ induce an isomorphism of topological groups

$$\mathbb{E}^{\Pi}_N \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N].$$

We write $(K^{\times}_{\underline{v}})^{1/N} \subset O^{\times}((\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})^{\mathrm{birat}})$ for the subgroup of elements whose N-th power is in the image of the natural inclusion $K^{\times}_{\underline{v}} \hookrightarrow O^{\times}((\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})^{\mathrm{birat}})$, and we write $(O^{\times}_{K_{\underline{v}}})^{1/N} := (K^{\times}_{\underline{v}})^{1/N} \cap O^{\times}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})$. Then the set of elements of $O^{\times}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})$ which normalise the subgroup $\mathbb{E}_N \subset \mathrm{Aut}_{\underline{\underline{\mathcal{F}}}_{\underline{v}}}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})$ is equal to the set of elements on which $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$ acts by multiplication by an element of $\mu_N(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})$, and it is equal to $(O^{\times}_{K_{\underline{v}}})^{1/N}$. Hence we have a natural outer action of $(O^{\times}_{K_{\underline{v}}})^{1/N}/\mu_N(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}) \xrightarrow{\sim} O^{\times}_{K_{\underline{v}}}$ on $\mathbb{E}_N$, and it extends to an outer action of $(K^{\times}_{\underline{v}})^{1/N}/\mu_N(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}) \xrightarrow{\sim} K^{\times}_{\underline{v}}$ on $\mathbb{E}_N$ ([EtTh, Lemma 5.8]). On the other hand, by composing the natural outer homomorphism $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}_{\underline{v}}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\ddot{\mathfrak{Z}}_{lN})$ with $s^{\sqcap\text{-gp}}_N$, we obtain a natural outer action $l\underline{\mathbb{Z}} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}/\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}} \to \mathrm{Out}(\mathbb{E}_N)$. We write $\mathcal{D}_{\mathcal{F}} := \langle \mathrm{Im}(K^{\times}_{\underline{v}}), l\underline{\mathbb{Z}}\rangle \subset \mathrm{Out}(\mathbb{E}^{\Pi}_N)$ for the subgroup generated by these outer actions of $K^{\times}_{\underline{v}}$ and $l\underline{\mathbb{Z}}$.

We also note that $s^{\sqcup\text{-gp}}_N : H(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}) \to \mathrm{Aut}_{\underline{\underline{\mathcal{F}}}_{\underline{v}}}(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})$ factors through $\mathbb{E}_N$, and we write $s^{\sqcup\text{-}\Pi}_N : \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}} \to \mathbb{E}^{\Pi}_N$ for the homomorphism induced by by taking $(-) \times_{\mathrm{Im}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})} \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}$ to the homomorphism $H(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}) \to \mathbb{E}_N$. We write $s^{\Theta}_{\mathcal{F}}$ for the $\mu_N(\ddot{\mathfrak{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}})$ -conjugacy classes of the subgroup given by the image of the homomorphism $s^{\sqcup\text{-}\Pi}_N$.

Then the triple

$$\mathbb{M}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) := (\mathbb{E}^{\Pi}_N, \mathcal{D}_{\mathcal{F}}, s^{\Theta}_{\mathcal{F}})$$

reconstructs a (mod N) mono-theta environment (We omitted the details here to verify that this is indeed a "category-theoretic" reconstruction algorithms. In fact, in inter-universal Teichmüller theory, for holomorphic Frobenioid-theoretic objects, we can use "copies" of the model object (category), instead of categories which are equivalent to the model object (category), and we can avoid "category-theoretic reconstruction algorithms" cf. also [IUTchI, Remark 3.2.1 (ii)]). Hence we obtain:

Theorem 8.14. ("$\mathcal{F} \mapsto \mathbb{M}$", [EtTh, Theorem 5.10], [IUTchII, Proposition 1.2 (ii)]) *We have a category-theoretic algorithm to reconstruct a (mod N) mono-theta environment $\mathbb{M}(\underline{\underline{\mathcal{F}}}_{\underline{v}})$ from a tempered Frobenioid $\underline{\underline{\mathcal{F}}}_{\underline{v}}$.*

Corollary 7.22 (2) reconstructs a mono-theta environment from a topological group ("$\Pi \mapsto \mathbb{M}$") and Theorem 8.14 reconstructs a mono-theta environment from a tempered Frobenioid ("$\mathcal{F} \mapsto \mathbb{M}$"). We relate group-theoretic constructions (étale-like objects) and Frobenioid-theoretic constructions (Frobenius-like objects) by transforming them into mono-theta environments (and by using Kummer theory, which is available by the cyclotomic rigidity of mono-theta environment), in inter-universal Teichmüller theory, especially, in the construction of Hodge-Arakelov-theoretic evaluation maps:

$$^{\dagger}\Pi_{\underline{v}} \longmapsto {}^{\dagger}\mathbb{M} \longleftarrow {}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}.$$

cf. Section 11.2.

§ 9. Preliminaries on the NF Counterpart of Theta Evaluation.

§ 9.1. Pseudo-Monoids of $\boldsymbol{\kappa}$-Coric Functions.

Definition 9.1. ([IUTchI, §0])

(1) A topological space P with a continuous map $P\times P \supset S \to P$ is called a **topological pseudo-monoid** if there exists a topological abelian group M (we write its group operation multiplicatively) and an embedding $\iota : P \hookrightarrow M$ of topological spaces such that $S = \{(a,b) \in P \times P \mid \iota(a)\cdot\iota(b) \in \iota(P) \subset M\}$ and the restriction of the group operation $M \times M \to M$ to S gives us the given map $S \to P$.

(2) If M is equipped with the discrete topology, we shall refer to P simply as a **pseudo-monoid**.

(3) A pseudo-monoid is called **divisible** if there exist M and ι as above such that, for any $n \geq 1$ and $a \in M$, there exists $b \in M$ with $b^n = a$, and if, for any $n \geq 1$ and $a \in M$, $a \in \iota(P)$ if and only if $a^n \in \iota(P)$.

(4) A pseudo-monoid is called **cyclotomic** if there exist M and ι as above such that, the subgroup $\mu_M \subset M$ of torsion elements of M is isomorphic to $\mathbb{Q}/\mathbb{Z}$, and if $\mu_M \subset \iota(P)$, $\mu_M \cdot \iota(P) \subset \iota(P)$ hold.

(5) For a cyclotomic pseudo-monoid P, write $\mu_{\widehat{\mathbb{Z}}}(P) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, P)$ and shall refer to it as the **cyclotome of a cycltomic pseudo-monoid** P.

Definition 9.2. ([IUTchI, Remark 3.1.7]) Let F_{mod} be a number field, and $C_{F_{\mathrm{mod}}} = (E_{F_{\mathrm{mod}}} \setminus \{O\})//\{\pm 1\}$ a semi-elliptic orbicurve (cf. Section 3.1) over F_{mod} which is an F_{mod}-core (Here, the model $E_{F_{\mathrm{mod}}}$ over F_{mod} is not unique in general). Let L be F_{mod} or $(F_{\mathrm{mod}})_v$ for some place v of F_{mod}, and write $C_L := C_{F_{\mathrm{mod}}} \times_{F_{\mathrm{mod}}} L$ and we write $|C_L|$ for the coarse scheme of the algebraic stack C_L (which is isomorphic to the affine line over L), and $\overline{|C_L|}$ the canonical smooth compactification of $|C_L|$. We write

L_C for the function field of C_L and take an algebraic closure $\overline{L_C}$ of L_C. Let $\overline{L}$ be the algebraic closure of L in $\overline{L_C}$. We put

$$L^{\bullet} := \begin{cases} F_{\text{mod}} & \text{if } L = F_{\text{mod}} \text{ or } L = (F_{\text{mod}})_v \text{ for } v : \text{non-Archimedean}, \\ (F_{\text{mod}})_v & \text{if } L = (F_{\text{mod}})_v \text{ for } v : \text{Archimedean}, \end{cases}$$

and

$$\mathcal{U}_{\overline{L}} := \begin{cases} \overline{L}^{\times} & \text{if } L = F_{\text{mod}}, \\ O_{\overline{L}}^{\times} & \text{if } L = (F_{\text{mod}})_v. \end{cases}$$

(1) A closed point of the proper smooth curve determined by some finite subextension of $L_C \subset \overline{L_C}$ is called a **critical point** if it maps to a closed point of $\overline{|C_L|}$ which arises from one of the 2-torsion points of $E_{F_{\text{mod}}}$.

(2) A critical point is called a **strictly critical point** if it does not map to the closed point of $\overline{|C_L|}$ which arises from the unique cusp of C_L.

(3) A rational function $f \in L_C$ on L_C is called κ**-coric** (κ stands for "Kummer") if the following conditions hold:

  (a) If $f \notin L$, then f has precisely one pole (of any order) and at least two distinct zeroes over $\overline{L}$.

  (b) The divisor $(f)_0$ of zeroes and the divisor $(f)_\infty$ of poles are defined over a finite extension of $L^{\bullet}$ and avoid the critical points.

  (c) The values of f at any strictly critical point of $\overline{|C_L|}$ are roots of unity.

(4) A rational function $f \in \overline{L_C}$ is called ${}_\infty\kappa$**-coric**, if there is a positive integer $n \geq 1$ such that f^n is κ-coric.

(5) A rational function $f \in \overline{L_C}$ is called ${}_\infty\kappa\times$**-coric**, if there is an element $c \in \mathcal{U}_{\overline{L}}$ such that $c \cdot f$ is ${}_\infty\kappa$-coric.

Remark 9.2.1.

(1) A rational function $f \in L_C$ is κ-coric if and only if f is ${}_\infty\kappa$-coric

(2) An ${}_\infty\kappa\times$-coric function $f \in \overline{L_C}$ is ${}_\infty\kappa$-coric if and only if the value at some strictly critical point of the proper smooth curve determined by some finite subextension of $L_C \subset \overline{L_C}$ containing f is a root of unity.

(3) The set of κ-coric functions ($\subset L_C$) forms a pseudo-monoid. The set of ${}_\infty\kappa$-coric functions ($\subset \overline{L_C}$) and the set of ${}_\infty\kappa\times$-coric functions ($\subset \overline{L_C}$) form divisible cyclotomic pseudo-monoids.

§ 9.2. Cyclotomic Rigidity via κ-Coric Functions.

Let F be a number field, $l \geq 5$ a prime number, $X_F = E_F \setminus \{O\}$ a once-punctured elliptic curve, and $F_{\mathrm{mod}}(\subset F)$ the field of moduli of X_F. Write $C_F := X_F//\{\pm 1\}$, and $K := F(E_F[l])$. Let $\underline{C}_K$ be a smooth log-orbicurve of type $(1, l\text{-tors})_\pm$ (cf. Definition 7.10) with K-core given by $C_K := C_F \times_F K$. Note that C_F admits a unique (up to unique isomorphism) model $C_{F_{\mathrm{mod}}}$ over F_{mod}, by the definition of F_{mod} and K-coricity of C_K. Note that $\underline{C}_K$ determines an orbicurve $\underline{X}_K$ of type $(1, l\text{-tors})$ (cf. Definition 7.10).

Let ${}^\dagger\mathcal{D}^{\circledcirc}$ be a category, which is equivalent to $\mathcal{D}^{\circledcirc} := \mathcal{B}(\underline{C}_K)^0$. We have an isomorphism ${}^\dagger\Pi^{\circledcirc} := \pi_1({}^\dagger\mathcal{D}^{\circledcirc}) \cong \Pi_{\underline{C}_K}$ (cf. Definition 6.1 (4) for $\pi_1((-)^0)$), well-defined up to inner automorphism.

Lemma 9.3. ([IUTchI, Remark 3.1.2] (i)) *From ${}^\dagger\mathcal{D}^{\circledcirc}$, we can group-theoretically reconstruct a profinite group ${}^\dagger\Pi^{\circledcirc\pm}(\subset {}^\dagger\Pi^{\circledcirc})$ corresponding to $\Pi_{\underline{X}_K}$.*

Proof. First, we can group-theoretically reconstruct an isomorph ${}^\dagger\Delta^{\circledcirc}$ of $\Delta_{\underline{C}_K}$ from ${}^\dagger\Pi^{\circledcirc}$, by Proposition 2.2 (1). Next, we can group-theoretically reconstruct an isomorph ${}^\dagger\Delta^{\circledcirc\pm}$ of $\Delta_{\underline{X}_K}$ from ${}^\dagger\Delta^{\circledcirc}$ as the unique torsion-free subgroup of ${}^\dagger\Delta^{\circledcirc}$ of index 2. Thirdly, we can group-theoretically reconstruct the decomposition subgroups of the non-zero cusps in ${}^\dagger\Delta^{\circledcirc\pm}$ by Remark 2.9.2 (Here, non-zero cusps can be group-theoretically grasped as the cusps whose inertia subgroups are contained in ${}^\dagger\Delta^{\circledcirc\pm}$). Finally, we can group-theoretically reconstruct an isomorph ${}^\dagger\Pi^{\circledcirc\pm}$ of $\Pi_{\underline{X}_K}$ as the subgroup of ${}^\dagger\Pi^{\circledcirc}$ generated by any of these decomposition groups and ${}^\dagger\Delta^{\circledcirc\pm}$. $\square$

Definition 9.4. ([IUTchI, Remark 3.1.2] (ii)) From ${}^\dagger\Pi^{\circledcirc}(= \pi_1({}^\dagger\mathcal{D}^{\circledcirc}))$, *instead of* reconstructing an isomorph of the function field of $\underline{C}_K$ directly from ${}^\dagger\Pi^{\circledcirc}$ by Theorem 3.17, we apply Theorem 3.17 to ${}^\dagger\Pi^{\circledcirc\pm}$ via Lemma 9.3 to reconstruct an isomorph of the function field of $\underline{X}_K$ with ${}^\dagger\Pi^{\circledcirc}/{}^\dagger\Pi^{\circledcirc\pm}$-action. We shall refer to this procedure as the **Θ-approach**. We also write $\mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^\dagger\Pi^{\circledcirc})$ to be the cyclotome defined in Definition 3.13 which we think of as being applied via Θ-approach.

Later, we may also use Θ-approach not only to $\Pi_{\underline{C}_K}$, but also $\Pi_{\underline{C}_{\underline{v}}}$, $\Pi_{\underline{\underline{X}}_{\underline{v}}}$, and $\Pi_{\underset{\rightarrow}{X}_{\underline{v}}}$ (cf. Section 10.1 for these objects). We will always apply Theorem 3.17 to these objects via Θ-approach (As for $\Pi_{\underline{\underline{X}}_{\underline{v}}}$ (resp. $\Pi_{\underset{\rightarrow}{X}_{\underline{v}}}$), see also Lemma 7.12 (resp. Lemma 7.25)).

Remark 9.4.1. ([IUTchI, Remark 3.1.2] (iii)) The extension

$$1 \to \Delta_\Theta \to \Delta_X^\Theta \to \Delta_X^{\mathrm{ell}} \to 1$$

in Section 7.1 gives us an extension class in

$$H^2(\Delta_X^{\mathrm{ell}}, \Delta_\Theta) \cong H^2(\Delta_X^{\mathrm{ell}}, \widehat{\mathbb{Z}}) \otimes \Delta_\Theta \cong \mathrm{Hom}(\mu_{\widehat{\mathbb{Z}}}(\Pi_X), \Delta_\Theta),$$

which determines an tautological isomorphism

$$\mu_{\widehat{\mathbb{Z}}}(\Pi_X) \xrightarrow{\sim} \Delta_\Theta.$$

This also gives us

$$\text{(Cyc. Rig. Ori. \&Theta)} \qquad \mu_{\widehat{\mathbb{Z}}}(\Pi_{\underline{X}}) \xrightarrow{\sim} l\Delta_\Theta.$$

As already seen in Section 7, the cyclotome $l\Delta_\Theta$ plays a central role in the theory of the étale theta function. In inter-universal Teichmüller theory, we need to use the above tautological isormophism in the construction of Hodge-Arakelov-theoretic evaluation map (cf. Section 11).

By applying Theorem 3.17 to ${}^\dagger\Pi^{\circledcirc}(=\pi_1({}^\dagger\mathcal{D}^{\circledcirc}))$, via the Θ-approach (Definition 9.4), we can group-theoretically reconstruct an isomorph

$$\overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})$$

of the *field* $\overline{F}$ with ${}^\dagger\Pi^{\circledcirc}$-action. We also write $\mathbb{M}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc}) := \overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})^\times$, which is an isomorph of $\overline{F}^\times$. We can also group-theoretically reconstruct a profinite group ${}^\dagger\Pi^{\circledast}(\supset {}^\dagger\Pi^{\circledcirc})$ corresponding to $\Pi_{C_{F_{\mathrm{mod}}}}$, by a similar way ("$\overline{\mathrm{Loc}}$") as in (Step 2) of the proof of Theorem 3.7 (We considered "Π's over G's" in (Step 2) of the proof of Theorem 3.7; however, in this case, we consider "Π's without surjections to G's"). Hence we obtain a morphism

$${}^\dagger\mathcal{D}^{\circledcirc} \to {}^\dagger\mathcal{D}^{\circledast} := \mathcal{B}({}^\dagger\Pi^{\circledast})^0,$$

which corresponding to $\underline{C}_K \to C_{F_{\mathrm{mod}}}$. Then the action of ${}^\dagger\Pi^{\circledcirc}$ on $\overline{\mathbb{M}}^{\circledast}({}^\dagger\Pi^{\circledcirc})$ naturally extends to an action of ${}^\dagger\Pi^{\circledast}$. In a similar way, by using Theorem 3.17 (especially *Belyi cuspidalisations*), we can group-theoretically reconstruct from ${}^\dagger\Pi^{\circledcirc}$ an isomorph

$$({}^\dagger\Pi^{\circledast})^{\mathrm{rat}} \quad (\twoheadrightarrow {}^\dagger\Pi^{\circledast})$$

of the absolute Galois group of the function field of $C_{F_{\mathrm{mod}}}$ in a functorial manner. By using *elliptic cuspidalisations* as well, we can also group-theoretically reconstruct from ${}^\dagger\Pi^{\circledcirc}$ isomorphs

$$\mathbb{M}^{\circledast}_{\kappa}({}^\dagger\mathcal{D}^{\circledcirc}), \quad \mathbb{M}^{\circledast}_{\infty\kappa}({}^\dagger\mathcal{D}^{\circledcirc}), \quad \mathbb{M}^{\circledast}_{\infty\kappa\times}({}^\dagger\mathcal{D}^{\circledcirc})$$

of the pseudo-monoids of κ-, ${}_\infty\kappa$-, and ${}_\infty\kappa\times$- coric rational functions associated with $C_{F_{\mathrm{mod}}}$ with natural $({}^\dagger\Pi^{\circledast})^{\mathrm{rat}}$-actions (Note that we can group-theoretically reconstruct evaluations at strictly critical points).

Example 9.5. (Global non-Realified Frobenioid, [IUTchI, Example 5.1 (i), (iii)]) By using the field structure on $\overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})$, we can group-theoretically reconstruct the set

$$\overline{\mathbb{V}}({}^\dagger\mathcal{D}^{\circledcirc})$$

of valuations on $\overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})$ with ${}^\dagger\Pi^{\circledast}$-action, which corresponds to $\mathbb{V}(\overline{F})$. Note also that the set

$${}^\dagger\mathbb{V}_{\mathrm{mod}} := \overline{\mathbb{V}}({}^\dagger\mathcal{D}^{\circledcirc})/{}^\dagger\Pi^{\circledast}, \quad (\text{resp. } \mathbb{V}({}^\dagger\mathcal{D}^{\circledcirc}) := \overline{\mathbb{V}}({}^\dagger\mathcal{D}^{\circledcirc})/{}^\dagger\Pi^{\circledcirc} \ )$$

of ${}^\dagger\Pi^{\circledast}$-orbits (resp. ${}^\dagger\Pi^{\circledcirc}$-orbits) of $\overline{\mathbb{V}}({}^\dagger\mathcal{D}^{\circledcirc})$ reconstructs $\mathbb{V}_{\mathrm{mod}}$ (resp. $\mathbb{V}(K)$), and that we have a natural bijection

$$\mathrm{Prime}({}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}) \xrightarrow{\sim} {}^\dagger\mathbb{V}_{\mathrm{mod}}$$

(cf. Definition 8.13 for $\mathrm{Prime}(-)$). Thus, we can also reconstruct the monoid

$$\Phi^{\circledast}({}^\dagger\mathcal{D}^{\circledast})(-)$$

on ${}^\dagger\mathcal{D}^{\circledast}$, which associates to $A \in \mathrm{Ob}({}^\dagger\mathcal{D}^{\circledast})$ the monoid $\Phi^{\circledast}({}^\dagger\mathcal{D}^{\circledast})(A)$ of stack-theoretic arithmetic divisors on $\overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})^A$ ($\subset \overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})$) (i.e., we are considering the coverings over the stack-theoretic quotient $(\mathrm{Spec}\, O_K)//\mathrm{Gal}(K/F_{\mathrm{mod}})(\cong \mathrm{Spec}\, O_{F_{\mathrm{mod}}})$) with the natural homomorphism $\overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})^A \to \Phi^{\circledast}({}^\dagger\mathcal{D}^{\circledast})(A)^{\mathrm{gp}}$ of monoids. Then these data $({}^\dagger\mathcal{D}^{\circledast}, \Phi^{\circledast}({}^\dagger\mathcal{D}^{\circledast}), \overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})^{(-)} \to \Phi^{\circledast}({}^\dagger\mathcal{D}^{\circledast})(-)^{\mathrm{gp}})$ determine a model Frobenioid

$$\mathcal{F}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})$$

whose base category is ${}^\dagger\mathcal{D}^{\circledast}$. We shall refer to this as a **global non-realified Frobenioid**.

Let ${}^\dagger\mathcal{F}^{\circledast}$ be a pre-Frobenioid, which is isomorphic to $\mathcal{F}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})$. Suppose that we are given a morphism ${}^\dagger\mathcal{D}^{\circledcirc} \to \mathrm{Base}({}^\dagger\mathcal{F}^{\circledast})$ which is abstractly equivalent (cf. Section 0.2) to the natural morphism ${}^\dagger\mathcal{D}^{\circledcirc} \to {}^\dagger\mathcal{D}^{\circledast}$. We identify $\mathrm{Base}({}^\dagger\mathcal{F}^{\circledast})$ with ${}^\dagger\mathcal{D}^{\circledast}$ (Note that this identification is uniquely determined by the F_{mod}-coricity of $C_{F_{\mathrm{mod}}}$ and Theorem 3.17). We write

$${}^\dagger\mathcal{F}^{\circledcirc} := {}^\dagger\mathcal{F}^{\circledast}|_{{}^\dagger\mathcal{D}^{\circledcirc}} \quad (\to {}^\dagger\mathcal{F}^{\circledast})$$

for the restriction of ${}^\dagger\mathcal{F}^{\circledast}$ to ${}^\dagger\mathcal{D}^{\circledcirc}$ via the natural ${}^\dagger\mathcal{D}^{\circledcirc} \to {}^\dagger\mathcal{D}^{\circledast}$. We also shall refer to this as a **global non-realified Frobenioid**. We also write

$${}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}} := {}^\dagger\mathcal{F}^{\circledast}|_{\text{terminal object in } {}^\dagger\mathcal{D}^{\circledast}} \quad (\subset {}^\dagger\mathcal{F}^{\circledast})$$

for the restriction of ${}^\dagger\mathcal{F}^{\circledast}$ to the full subcategory consisting of the terminal object in ${}^\dagger\mathcal{D}^{\circledast}$ (which corresponds to $C_{F_{\mathrm{mod}}}$). We also shall refer to this as a **global non-realified Frobenioid**. Note that the base category of ${}^\dagger\mathcal{F}_{\mathrm{mod}}$ has only one object and only one morphism. We can regard ${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}$ as the Frobenioid of (stack-theoretic) arithmetic line bundles over $(\mathrm{Spec}\, O_K)//\mathrm{Gal}(K/F_{\mathrm{mod}})\, (\cong \mathrm{Spec}\, F_{\mathrm{mod}})$. In inter-universal Teichmüller theory, we use the global non-realified Frobenioid for converting $\boxtimes$-line bundles into $\boxplus$-line bundles and vice versa (cf. Section 9.3 and Corollary 13.13).

Definition 9.6. (${}_{\infty}\kappa$-Coric and ${}_{\infty}\kappa\times$-Coric Structures, and Cyclotomic Rigidity via $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^{\times} = \{1\}$)

(1) (Global case, [IUTchI, Example 5.1 (ii), (iv), (v)]) We consider $O^{\times}(\mathcal{O}_A)$ (which is isomorphic to the multiplicative group of non-zero elements of a finite Galois extension of F_{mod}), varying Galois objects $A \in \mathrm{Ob}({}^{\dagger}\mathcal{D}^{\circledast})$ (Here $\mathcal{O}_A$ is a trivial line bundle on A. cf. Definition 8.4 (1)). Then we obtain a pair

$$ {}^{\dagger}\Pi^{\circledast} \curvearrowright {}^{\dagger}\widetilde{O}^{\circledast\times} $$

well-defined up to inner automorphisms of the pair arising from conjugation by ${}^{\dagger}\Pi^{\circledast}$. For each $\mathfrak{p} \in \mathrm{Prime}(\Phi_{{}^{\dagger}\mathcal{F}^{\circledast}}(\mathcal{O}_A))$, where we write $\Phi_{{}^{\dagger}\mathcal{F}^{\circledast}}$ for the divisor monoid of ${}^{\dagger}\mathcal{F}^{\circledast}$, we obtain a submonoid

$$ {}^{\dagger}O^{\rhd}_{\mathfrak{p}} \subset {}^{\dagger}O^{\times}(\mathcal{O}^{\mathrm{birat}}_A), $$

by taking the inverse image of $\mathfrak{p} \cup \{0\} \subset \Phi_{{}^{\dagger}\mathcal{F}^{\circledast}}(\mathcal{O}_A)$ via the natural homomorphism $O^{\times}(\mathcal{O}^{\mathrm{birat}}_A) \to \Phi_{{}^{\dagger}\mathcal{F}^{\circledast}}(\mathcal{O}_A)^{\mathrm{gp}}$ (i.e., the submonoid of integral elements of $O^{\times}(\mathcal{O}^{\mathrm{birat}}_A)$ with respect to $\mathfrak{p}$). Note that the natural action of $\mathrm{Aut}_{{}^{\dagger}\mathcal{F}^{\circledast}}(\mathcal{O}_A)$ on $O^{\times}(\mathcal{O}^{\mathrm{birat}}_A)$ permutes the $O^{\rhd}_{\mathfrak{p}}$'s. For each $\mathfrak{p}_0 \in \mathrm{Prime}(\Phi_{{}^{\dagger}\mathcal{F}^{\circledast}}(\mathcal{O}_{A_0}))$, where $A_0 \in \mathrm{Ob}({}^{\dagger}\mathcal{D}^{\circledast})$ is the terminal object, we obtain a closed subgroup

$$ {}^{\dagger}\Pi_{\mathfrak{p}_0} \subset {}^{\dagger}\Pi^{\circledast} $$

(well-defined up to conjugation) by varying Galois objects $A \in \mathrm{Ob}({}^{\dagger}\mathcal{D}^{\circledast})$, and by considering the elements of $\mathrm{Aut}_{{}^{\dagger}\mathcal{F}^{\circledast}}(\mathcal{O}_A)$ which fix the submonoid ${}^{\dagger}O^{\rhd}_{\mathfrak{p}}$ for system of $\mathfrak{p}$'s lying over $\mathfrak{p}_0$ (i.e., a decomposition group for some $v \in \mathbb{V}(F_{\mathrm{mod}})$). Note that $\mathfrak{p}_0$ is non-Archimedean if and only if the p-cohomological dimension of ${}^{\dagger}\Pi_{\mathfrak{p}_0}$ is equal to $2+1=3$ for inifinitely many prime numbers p (Here, 2 comes from the absolute Galois group of a local field, and 1 comes from "Δ-portion (or geometric portion)" of ${}^{\dagger}\Pi^{\circledast}$). By taking the completion of ${}^{\dagger}O^{\rhd}_{\mathfrak{p}}$ with respect to the corresponding valuation, varying Galois objects $A \in \mathrm{Ob}({}^{\dagger}\mathcal{D}^{\circledast})$, and considering a system of $\mathfrak{p}$'s lying over $\mathfrak{p}_0$, we also obtain a pair

$$ {}^{\dagger}\Pi_{\mathfrak{p}_0} \curvearrowright {}^{\dagger}\widetilde{O}^{\rhd}_{\widehat{\mathfrak{p}}_0} $$

of a toplogical group acting on an ind-topological monoid, which is well-defined up to the inner automorphisms of the pair arising from conjugation by ${}^{\dagger}\Pi_{\mathfrak{p}_0}$ (since ${}^{\dagger}\Pi_{\mathfrak{p}_0}$ is commensurably terminal in ${}^{\dagger}\Pi^{\circledast}$ (Proposition 2.7)).

We write

$$({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}} \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}$$

for the above pair $({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}} \curvearrowright {}^{\dagger}\widetilde{O}^{\circledast\times}$. Suppose that we are given isomorphs

$$({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}} \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}, \quad ({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}} \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times}$$

(Note that these are *Frobenius-like object*) of

$$({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}} \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) \quad ({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}} \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa\times}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

respectively (Note that these are *étale-like object*) as cyclotomic pseudo-monoids with a continuous action of $({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}$. We shall refer to such a pair as an $_{\infty}\boldsymbol{\kappa}$**-coric structure**, and an $_{\infty}\boldsymbol{\kappa}\times$**-coric structure** on ${}^{\dagger}\mathcal{F}^{\circledast}$ respectively.

We recall that the étale-like objects $\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})$, and $\mathbb{M}^{\circledast}_{\infty\kappa\times}({}^{\dagger}\mathcal{D}^{\circledcirc})$ are constructed as subsets of $_{\infty}H^1(({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}, \mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\dagger}\Pi^{\circledcirc})) := \varinjlim_{H\subset({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}:\,\mathrm{open}} H^1(H, \mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\dagger}\Pi^{\circledcirc}))$:

$$\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) \ \ (\text{resp. } \ \mathbb{M}^{\circledast}_{\infty\kappa\times}({}^{\dagger}\mathcal{D}^{\circledcirc}) \ \ ) \subset {}_{\infty}H^1(({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}, \mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\dagger}\Pi^{\circledcirc})).$$

On the other hand, by taking Kummer classes, we also have natural injections

$${}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa} \subset {}_{\infty}H^1(({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}, \mu_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa})), \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times} \subset {}_{\infty}H^1(({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}, \mu_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times})),$$

where $_{\infty}H^1(({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}, -) := \varinjlim_{H\subset({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}:\,\mathrm{open}} H^1(H, -)$. (The injectivity follows from the corresponding injectivity for $\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})$ and $\mathbb{M}^{\circledast}_{\infty\kappa\times}({}^{\dagger}\mathcal{D}^{\circledcirc})$ respectively.) Recall that the isomorphisms between two cyclotomes form a $\widehat{\mathbb{Z}}^{\times}$-torsor, and that κ-coric functions distinguish zeroes and poles (since it has precisely one pole (of any order) and at least two zeroes). Hence by $(\mathbb{Q}\otimes\widehat{\mathbb{Z}} \supset)\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^{\times} = \{1\}$, there exist unique isomorphisms

$$\text{(Cyc. Rig. NF1)} \qquad \mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\dagger}\Pi^{\circledcirc}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}), \quad \mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\dagger}\Pi^{\circledcirc}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times})$$

characterised as the ones which induce **Kummer isomorphisms**

$${}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}), \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathbb{M}^{\circledast}_{\infty\kappa\times}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

respectively. In a similar manner, for the isomorph ${}^{\dagger}\Pi^{\circledcirc} \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}$ of ${}^{\dagger}\Pi^{\circledcirc} \curvearrowright \widetilde{O}^{\circledast\times}$, there exists a unique isomorphism

$$\text{(Cyc. Rig. NF2)} \qquad \mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\dagger}\Pi^{\circledcirc}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}^{\circledast})$$

characterised as the one which induces a **Kummer isomorphism**

$${}^{\dagger}\mathbb{M}^{\circledast} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

between the direct limits of cohomology modules described in (Step 4) of Theorem 3.17, in a fashion which is *compatible with the integral submonoids "$O^{\triangleright}_{\mathfrak{p}}$"*. We shall refer to the isomorphism (Cyc. Rig. NF2) as the **cyclotmoic rigidity via $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^{\times} = \{1\}$** (cf. [IUTchI, Example 5.1 (v)]). By the above discussions, it follows that ${}^{\dagger}\mathcal{F}^{\circledast}$ always admits an ${}_{\infty}\kappa$-coric and an ${}_{\infty}\kappa\times$-coric structures, which are unique up to uniquely determined isomorphisms of pseudo-monoids with continuous actions of $({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}$ respectively. Thus, we regard ${}^{\dagger}\mathcal{F}^{\circledast}$ as being equipped with these uniquely determined ${}_{\infty}\kappa$-coric and ${}_{\infty}\kappa\times$-coric structures without notice. We also put

$$\mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) := (\mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}))^{({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}}, \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}} := ({}^{\dagger}\mathbb{M}^{\circledast})^{({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}},$$

$$\mathbb{M}^{\circledast}_{\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) := (\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}))^{({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}}, \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\kappa} := ({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa})^{({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}},$$

where we write $(-)^{({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}}$ for the $({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}$-invariant part.

(2) (Local non-Archimedean case, [IUTchI, Definition 5.2 (v), (vi)]) For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, let ${}^{\dagger}\mathcal{D}_{\underline{v}}$ be a category equivalent to $\mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})^0$ (resp. $\mathcal{B}(\underset{\rightarrow}{X}_{\underline{v}})^0$) over a finite extension $K_{\underline{v}}$ of $\mathbb{Q}_{p_{\underline{v}}}$, where $\underline{\underline{X}}_{\underline{v}}$ (resp. $\underset{\rightarrow}{X}_{\underline{v}}$) is a hyperbolic orbicurve of type $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$ (Definition 7.13) (resp. of type $(1, l\text{-}\underrightarrow{\mathrm{tors}})$ (Definition 7.24)) such that the field of moduli of the hyperbolic curve "X" of type $(1,1)$ in the start of the definition of hyperbolic orbicurve of type $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$ (resp. of type $(1, l\text{-}\underrightarrow{\mathrm{tors}})$) is a number field F_{mod}. By Corollary 3.19, we can group-theoretically reconstruct an isomorph

$${}^{\dagger}\Pi_{\underline{v}} \curvearrowright \mathbb{M}_{v}({}^{\dagger}\mathcal{D}_{\underline{v}})$$

of $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}_{\underline{v}}} \curvearrowright O^{\triangleright}_{\overline{K_{\underline{v}}}}$ (resp. $\Pi_{\underset{\rightarrow}{X}_{\underline{v}}} \curvearrowright O^{\triangleright}_{\overline{K_{\underline{v}}}}$) from ${}^{\dagger}\Pi_{\underline{v}} := \pi_1({}^{\dagger}\mathcal{D}_{\underline{v}})$.

Let $v \in \mathbb{V}_{\mathrm{mod}} = \mathbb{V}(F_{\mathrm{mod}})$ be the valuation lying under $\underline{v}$. From ${}^{\dagger}\Pi_{\underline{v}}$, we can group-theoretically reconstruct a profinite group ${}^{\dagger}\Pi_{v}$ corresponding to $C_{(F_{\mathrm{mod}})_v}$ by a similar way ("$\overline{\mathrm{Loc}}$") as in (Step 2) of the proof of Theorem 3.7. We write

$${}^{\dagger}\mathcal{D}_{v}$$

for $\mathcal{B}({}^{\dagger}\Pi_{v})^0$. We have a natural morphism ${}^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\mathcal{D}_{v}$ (This corresponds to $\underline{\underline{X}}_{\underline{v}} \to C_{(F_{\mathrm{mod}})_v}$ (resp. $\underset{\rightarrow}{X}_{\underline{v}} \to C_{(F_{\mathrm{mod}})_v}$)). In a similar way, by using Theorem 3.17 (especially *Belyi cuspidalisations*), we can group-theoretically reconstruct from ${}^{\dagger}\Pi_{\underline{v}}$ an isomorph

$$({}^{\dagger}\Pi_{v})^{\mathrm{rat}} \quad (\twoheadrightarrow {}^{\dagger}\Pi_{v})$$

of the absolute Galois group of the function field of $C_{(F_{\mathrm{mod}})_v}$ in a functorial manner. By using *elliptic cuspidalisations* as well, we can also group-theoretically reconstruct, from ${}^{\dagger}\Pi_{\underline{v}}$, isomorphs

$$\mathbb{M}_{\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}}), \quad \mathbb{M}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}}), \quad \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}})$$

of the pseudo-monoids of κ-, ${}_{\infty}\kappa$-, and ${}_{\infty}\kappa\times$- coric rational functions associated with $C_{(F_{\mathrm{mod}})_v}$ with natural $({}^{\dagger}\Pi_v)^{\mathrm{rat}}$-actions (Note that we can group-theoretically reconstruct evaluations at strictly critical points).

Let ${}^{\dagger}\mathcal{F}_{\underline{v}}$ be a pre-Frobenioid isomorphic to the $p_{\underline{v}}$-adic Frobenioid $\mathcal{C}_{\underline{v}} = (\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\text{base-field}}$ in Example 8.8 (resp. to the $p_{\underline{v}}$-adic Frobenioid $\mathcal{C}_{\underline{v}}$ in Example 8.7) whose base category is equal to ${}^{\dagger}\mathcal{D}_{\underline{v}}$. We write

$$({}^{\dagger}\Pi_v)^{\mathrm{rat}} \curvearrowright {}^{\dagger}\mathbb{M}_v$$

for an isomorph of $({}^{\dagger}\Pi_v)^{\mathrm{rat}} \curvearrowright \mathbb{M}_v({}^{\dagger}\mathcal{D}_{\underline{v}})$ determined by ${}^{\dagger}\mathcal{F}_{\underline{v}}$. Suppose that we are given isomorphs

$$({}^{\dagger}\Pi_v)^{\mathrm{rat}} \curvearrowright {}^{\dagger}\mathbb{M}_{\infty\kappa v}, \quad ({}^{\dagger}\Pi_v)^{\mathrm{rat}} \curvearrowright {}^{\dagger}\mathbb{M}_{\infty\kappa\times v}$$

(Note that these are *Frobenius-like object*) of

$$({}^{\dagger}\Pi_v)^{\mathrm{rat}} \curvearrowright \mathbb{M}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}}), \quad ({}^{\dagger}\Pi_v)^{\mathrm{rat}} \curvearrowright \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}})$$

(Note that these are *étale-like objects*) as cyclotomic pseudo-monoids with a continuous action of $({}^{\dagger}\Pi_v)^{\mathrm{rat}}$. We shall refer to such pairs as an **$_\infty\kappa$-coric structure**, and an **$_\infty\kappa\times$-coric structure** on ${}^{\dagger}\mathcal{F}_{\underline{v}}$ respectively.

We recall that the étale-like objects $\mathbb{M}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}})$, $\mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}})$ is constructed as subsets of ${}_{\infty}H^1(({}^{\dagger}\Pi_v)^{\mathrm{rat}}, \mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\dagger}\Pi_{\underline{v}})) := \varinjlim_{H\subset({}^{\dagger}\Pi_v)^{\mathrm{rat}}\,:\,\mathrm{open}} H^1(H, \mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\dagger}\Pi_{\underline{v}}))$:

$$\mathbb{M}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}}) \ \ (\text{resp. } \ \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}}) \ \ ) \subset {}_{\infty}H^1(({}^{\dagger}\Pi_v)^{\mathrm{rat}}, \mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\dagger}\Pi_{\underline{v}})).$$

On the other hand, by taking Kummer classes, we also have natural injections

$${}^{\dagger}\mathbb{M}_{\infty\kappa v} \subset {}_{\infty}H^1(({}^{\dagger}\Pi_v)^{\mathrm{rat}}, \mu_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}_{\infty\kappa v})), \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times} \subset {}_{\infty}H^1(({}^{\dagger}\Pi_v)^{\mathrm{rat}}, \mu_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}_{\infty\kappa\times v})).$$

(The injectivity follows from the corresponding injectivity for $\mathbb{M}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}})$ and $\mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}})$ respectively.) Recall that the isomorphisms between two cyclotomes form a $\widehat{\mathbb{Z}}^{\times}$-torsor, and that κ-coric functions distinguish zeroes and poles (since it has precisely one pole (of any order) and at least two zeroes). Hence by $(\mathbb{Q}\otimes\widehat{\mathbb{Z}} \supset)\mathbb{Q}_{>0}\cap\widehat{\mathbb{Z}}^{\times} = \{1\}$, there exist unique isomorphisms

$$\text{(Cyc. Rig. NF3)} \qquad \mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\dagger}\Pi_{\underline{v}}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}_{\infty\kappa v}), \quad \mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\dagger}\Pi_{\underline{v}}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}_{\infty\kappa\times v})$$

characterised as the ones which induce **Kummer isomorphisms**

$$ {}^\dagger\mathbb{M}_{\infty\kappa v} \overset{\text{Kum}}{\xrightarrow{\sim}} \mathbb{M}_{\infty\kappa v}({}^\dagger\mathcal{D}_{\underline{v}}), \quad {}^\dagger\mathbb{M}_{\infty\kappa\times v} \overset{\text{Kum}}{\xrightarrow{\sim}} \mathbb{M}_{\infty\kappa\times v}({}^\dagger\mathcal{D}_{\underline{v}}) $$

respectively. In a similar manner, for the isomorph ${}^\dagger\Pi_v \curvearrowright {}^\dagger\mathbb{M}_v$ of ${}^\dagger\Pi_v \curvearrowright \mathbb{M}_v({}^\dagger\mathcal{D}_{\underline{v}})$, there exists a unique isomorphism

$$ \text{(Cyc. Rig. NF4)} \qquad \mu^{\Theta}_{\widehat{\mathbb{Z}}}({}^\dagger\Pi_{\underline{v}}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}({}^\dagger\mathbb{M}_v) $$

characterised as the one which induces a **Kummer isomorphism**

$$ {}^\dagger\mathbb{M}_v \overset{\text{Kum}}{\xrightarrow{\sim}} \mathbb{M}_v({}^\dagger\mathcal{D}_{\underline{v}}) $$

between the direct limits of cohomology modules described in (Step 4) of Theorem 3.17. We also shall refer to the isomorphism (Cyc. Rig. NF4) as the **cyclotmoic rigidity via $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\}$** (cf. [IUTchI, Definition 5.2 (vi)]). By the above discussions, it follows that ${}^\dagger\mathcal{F}_{\underline{v}}$ always admits an ${}_\infty\kappa$-coric and ${}_\infty\kappa\times$-coric structures, which are unique up to uniquely determined isomorphisms of pseudo-monoids with continuous actions of $({}^\dagger\Pi_v)^{\text{rat}}$ respectively. Thus, we regard ${}^\dagger\mathcal{F}_{\underline{v}}$ as being equipped with these uniquely determined ${}_\infty\kappa$-coric and ${}_\infty\kappa\times$-coric structures without notice. We also put

$$ \mathbb{M}_{\kappa v}({}^\dagger\mathcal{D}_{\underline{v}}) := (\mathbb{M}_{\infty\kappa v}({}^\dagger\mathcal{D}_{\underline{v}}))^{({}^\dagger\Pi_v)^{\text{rat}}}, \quad {}^\dagger\mathbb{M}_{\kappa v} := ({}^\dagger\mathbb{M}_{\infty\kappa v})^{({}^\dagger\Pi_v)^{\text{rat}}}, $$

where we write $(-)^{({}^\dagger\Pi_v)^{\text{rat}}}$ for the $({}^\dagger\Pi_v)^{\text{rat}}$-invariant part.

(3) (Local Archimedean case, [IUTchI, Definition 5.2 (vii), (viii)]) For $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, let ${}^\dagger\mathcal{D}_{\underline{v}}$ be an Aut-holomorphic orbispace isomorphic to the Aut-holomorphic orbispace $\underset{\rightarrow}{\mathbb{X}}_{\underline{v}}$ associated to $\underset{\rightarrow}{X}_{\underline{v}}$, where $\underset{\rightarrow}{X}_{\underline{v}}$ is a hyperbolic orbicurve of type $(1, l\text{-}\underrightarrow{\text{tors}})$ (Definition 7.24) such that the field of moduli of the hyperbolic curve "X" of type $(1,1)$ in the start of the definition of hyperbolic orbicurve of type $(1, l\text{-}\underrightarrow{\text{tors}})$ is a number field F_{mod}.

Let $v \in \mathbb{V}_{\text{mod}} = \mathbb{V}(F_{\text{mod}})$ be the valuation lying under $\underline{v}$. By Proposition 4.5, we can algorithmically reconstruct an isomorph

$$ {}^\dagger\mathcal{D}_v $$

of the Aut-holomorphic orbispace $\mathbb{C}_v$ associated with $C_{(F_{\text{mod}})_v}$ from ${}^\dagger\mathcal{D}_{\underline{v}}$. We have a natural morphism ${}^\dagger\mathcal{D}_{\underline{v}} \to {}^\dagger\mathcal{D}_v$ (This corresponds to $\underset{\rightarrow}{X}_{\underline{v}} \to C_{(F_{\text{mod}})_v}$. Note that we have a natural isomorphism $\text{Aut}({}^\dagger\mathcal{D}_v) \xrightarrow{\sim} \text{Gal}(K_{\underline{v}}/(F_{\text{mod}})_v) \,(\subset \mathbb{Z}/2\mathbb{Z})$, since C_K is a K-core. Write

$$ {}^\dagger\mathcal{D}_v^{\text{rat}} := \varprojlim({}^\dagger\mathcal{D}_v \setminus \Sigma) \quad (\to {}^\dagger\mathcal{D}_v), $$

where we choose a projective system of $({}^\dagger\mathcal{D}_v \setminus \Sigma)$'s which arise as universal covering spaces of ${}^\dagger\mathcal{D}_v$ with $\Sigma \supset \{\text{strictly critical points}\}$, $\#\Sigma < \infty$ (cf. Definition 9.2 for strictly critical points). Note that ${}^\dagger\mathcal{D}_v^{\mathrm{rat}}$ is well-defined up to deck transformations over ${}^\dagger\mathcal{D}_v$. We write

$$\mathbb{M}_v({}^\dagger\mathcal{D}_{\underline{v}}) \subset \overline{\mathcal{A}^{{}^\dagger\mathcal{D}_{\underline{v}}}}$$

for the topological submonoid of non-zero elements with norm ≤ 1 (which is an isomorph of $O_{\mathbb{C}}^{\rhd}$) in the topological field $\overline{\mathcal{A}^{{}^\dagger\mathcal{D}_{\underline{v}}}}$ (cf. Proposition 4.5 for $\overline{\mathcal{A}^{{}^\dagger\mathcal{D}_{\underline{v}}}}$). By using *elliptic cuspidalisations*, we can also algorithmically reconstruct, from ${}^\dagger\mathcal{D}_{\underline{v}}$, isomorphs

$$\mathbb{M}_{\kappa v}({}^\dagger\mathcal{D}_{\underline{v}}),\quad \mathbb{M}_{\infty\kappa v}({}^\dagger\mathcal{D}_{\underline{v}}),\quad \mathbb{M}_{\infty\kappa\times v}({}^\dagger\mathcal{D}_{\underline{v}})\quad (\subset \mathrm{Hom}_{\text{co-hol}}({}^\dagger\mathcal{D}_{\underline{v}}^{\mathrm{rat}}, \mathbb{M}_v({}^\dagger\mathcal{D}_{\underline{v}})^{\mathrm{gp}})\ )$$

of the pseudo-monoids of κ-, ${}_\infty\kappa$-, and ${}_\infty\kappa\times$- coric rational functions associated with $C_{(F_{\mathrm{mod}})_v}$ as sets of morphisms of Aut-holomorphic orbispaces from ${}^\dagger\mathcal{D}_{\underline{v}}^{\mathrm{rat}}$ to $\mathbb{M}_v({}^\dagger\mathcal{D}_{\underline{v}})^{\mathrm{gp}}(= \mathcal{A}^{{}^\dagger\mathcal{D}_{\underline{v}}})$ which are compatible with the tautological co-holomorphicisation (Recall that $\overline{\mathcal{A}^{{}^\dagger\mathcal{D}_{\underline{v}}}}$ has a natural Aut-holomorphic structure and a tautological co-holomorphicisation (cf. Definition 4.1 (5) for co-holomorphicisation)).

Let ${}^\dagger\mathcal{F}_{\underline{v}} = ({}^\dagger\mathcal{C}_{\underline{v}}, {}^\dagger\mathcal{D}_{\underline{v}}, {}^\dagger\kappa_{\underline{v}} : O^{\rhd}({}^\dagger\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}^{{}^\dagger\mathcal{D}_{\underline{v}}})$ be a triple isomorphic to the triple $(\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}})$ in Example 8.11, where the second data is equal to the above ${}^\dagger\mathcal{D}_{\underline{v}}$. Write

$${}^\dagger\mathbb{M}_v := O^{\rhd}({}^\dagger\mathcal{C}_{\underline{v}}).$$

Then the Kummer structure ${}^\dagger\kappa_{\underline{v}}$ gives us an isomorphism

$${}^\dagger\kappa_{\underline{v}} : {}^\dagger\mathbb{M}_v \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathbb{M}_v({}^\dagger\mathcal{D}_{\underline{v}})$$

of topological monoids, to which we shall refer as a **Kummer isomorphism**. We can algorithmically reconstruct the pseudo-monoids

$${}^\dagger\mathbb{M}_{\infty\kappa v},\quad {}^\dagger\mathbb{M}_{\infty\kappa\times v}$$

of ${}_\infty\kappa$-coric and ${}_\infty\kappa\times$-coric rational functions associated to $C_{(F_{\mathrm{mod}})_v}$ as the sets of maps

$${}^\dagger\mathcal{D}_v^{\mathrm{rat}} \longrightarrow \mathbb{M}_v({}^\dagger\mathcal{D}_{\underline{v}})^{\mathrm{gp}} \coprod {}^\dagger\mathbb{M}_v^{\mathrm{gp}} \text{ (disjoint union)}$$

which send strictly critical points to ${}^\dagger\mathbb{M}_v^{\mathrm{gp}}$, otherwise to $\mathbb{M}_v({}^\dagger\mathcal{D}_{\underline{v}})^{\mathrm{gp}}$, such that the composite ${}^\dagger\mathcal{D}_v^{\mathrm{rat}} \to \mathbb{M}_v({}^\dagger\mathcal{D}_{\underline{v}})^{\mathrm{gp}} \coprod {}^\dagger\mathbb{M}_v^{\mathrm{gp}} \xrightarrow{\mathrm{id} \coprod (({}^\dagger\kappa_{\underline{v}})^{\mathrm{gp}})^{-1}} \mathbb{M}_v({}^\dagger\mathcal{D}_{\underline{v}})^{\mathrm{gp}}$ is an element of $\mathbb{M}_{\infty\kappa v}({}^\dagger\mathcal{D}_{\underline{v}})$, $\mathbb{M}_{\infty\kappa\times v}({}^\dagger\mathcal{D}_{\underline{v}})$ respectively. We shall refer to them as an **${}_\infty\boldsymbol{\kappa}$-coric structure**, and an **${}_\infty\boldsymbol{\kappa}\times$-coric structure** on ${}^\dagger\mathcal{F}_{\underline{v}}$ respectively. Note also that ${}^\dagger\mathbb{M}_{\kappa v}(\subset {}^\dagger\mathbb{M}_{\infty\kappa v})$ can be reconstructed as the subset of the maps which descend to some ${}^\dagger\mathcal{D}_{\underline{v}} \setminus \Sigma$ in the projective limit of ${}^\dagger\mathcal{D}^{\mathrm{rat}}_{\underline{v}}$, and are equivariant with the unique embedding $\mathrm{Aut}({}^\dagger\mathcal{D}_v) \hookrightarrow \mathrm{Aut}(\overline{\mathcal{A}^{{}^\dagger\mathcal{D}_{\underline{v}}}})$. Hence the Kummer structure ${}^\dagger\kappa_{\underline{v}}$ in ${}^\dagger\mathcal{F}_{\underline{v}}$ determines tautologically isomorphisms

$$ {}^\dagger\mathbb{M}_{\kappa v} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathbb{M}_{\kappa v}({}^\dagger\mathcal{D}_{\underline{v}}), \quad {}^\dagger\mathbb{M}_{\infty\kappa v} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathbb{M}_{\infty\kappa v}({}^\dagger\mathcal{D}_{\underline{v}}), \quad {}^\dagger\mathbb{M}_{\infty\kappa\times v} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathbb{M}_{\infty\kappa\times v}({}^\dagger\mathcal{D}_{\underline{v}}) $$

of pseudo-monoids, to which we also shall refer as **Kummer isomorphisms**.

Remark 9.6.1. (Mono-anabelian Transport) The technique of **mono-anabelian transport** is one of the main tools of *reconstructing an alien ring structure in a scheme theory from another* (after admitting mild indeterminacies). In this occasion, we explain it.

Let ${}^\dagger\Pi$, ${}^\ddagger\Pi$ be profinite groups isomorphic to Π_X, where X is a hyperbolic orbicurve of strictly Belyi type over non-Archimedean local field k (resp. isomorphic to $\Pi_{\underline{C}_K}$ as in this section). Then by Corollary 3.19 (resp. by Theorem 3.17 as mentioned in this subsection), we can group-theoretically construct isomorphs $O^{\rhd}({}^\dagger\Pi)$, $O^{\rhd}({}^\ddagger\Pi)$ (resp. $\mathbb{M}^{\circledast}({}^\dagger\Pi)$, $\mathbb{M}^{\circledast}({}^\ddagger\Pi)$) of $O^{\rhd}_{\overline{k}}$ (resp. $\overline{F}$) with ${}^\dagger\Pi$-, ${}^\ddagger\Pi$-action from the abstract topological groups ${}^\dagger\Pi$, ${}^\ddagger\Pi$ respectively (These are étale-like objects). Suppose that we are given isomorphs ${}^\dagger O^{\rhd}$, ${}^\ddagger O^{\rhd}$ (resp. ${}^\dagger\mathbb{M}^{\circledast}$, ${}^\ddagger\mathbb{M}^{\circledast}$) of $O^{\rhd}({}^\dagger\Pi)$, $O^{\rhd}({}^\ddagger\Pi)$ (resp. $\mathbb{M}^{\circledast}({}^\dagger\Pi)$, $\mathbb{M}^{\circledast}({}^\ddagger\Pi)$) respectively (This is a Frobenius-like object), and that an isomorphism ${}^\dagger\Pi \cong {}^\ddagger\Pi$ of topological groups. The topological monoids ${}^\dagger O^{\rhd}$ and ${}^\ddagger O^{\rhd}$ (resp. the multiplicative groups ${}^\dagger\mathbb{M}^{\circledast}$ and ${}^\ddagger\mathbb{M}^{\circledast}$ of fields) are *a priori* have no relation to each other, since an "isomorph" only means an isomorphic object, and an isomorphism is not specified. However, we can *canonically* relate them, by using the *Kummer theory* (cf. the Kummer isomorphism in Remark 3.19.2), which is available by relating two kinds of cyclotomes (i.e., cyclotomes arisen from Frobenius-like object and étale-like object) via the *cyclotomic rigidity via LCFT (resp. via $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\}$)*:

$$ \begin{array}{ccccccc} ({}^\dagger\Pi \curvearrowright {}^\dagger O^{\rhd}) & \overset{\mathrm{Kummer}}{\xrightarrow{\sim}} & ({}^\dagger\Pi \curvearrowright O^{\rhd}({}^\dagger\Pi)) & \underset{{}^\dagger\Pi\cong{}^\ddagger\Pi}{\overset{\text{induced by}}{\cong}} & ({}^\ddagger\Pi \curvearrowright O^{\rhd}({}^\ddagger\Pi)) & \overset{\mathrm{Kummer}}{\xleftarrow{\sim}} & ({}^\ddagger\Pi \curvearrowright {}^\dagger O^{\rhd}) \\ \text{Frobenius-like} & & \text{étale-like} & & \text{étale-like} & & \text{Frobenius-like} \end{array} $$

(resp.

$$ \begin{array}{ccccccc} ({}^\dagger\Pi \curvearrowright {}^\dagger\mathbb{M}^{\circledast}) & \overset{\mathrm{Kummer}}{\xrightarrow{\sim}} & ({}^\dagger\Pi \curvearrowright \mathbb{M}^{\circledast}({}^\dagger\Pi)) & \underset{{}^\dagger\Pi\cong{}^\ddagger\Pi}{\overset{\text{induced by}}{\cong}} & ({}^\ddagger\Pi \curvearrowright \mathbb{M}^{\circledast}({}^\ddagger\Pi)) & \overset{\mathrm{Kummer}}{\xleftarrow{\sim}} & ({}^\ddagger\Pi \curvearrowright {}^\dagger\mathbb{M}^{\circledast}) \\ \text{Frobenius-like} & & \text{étale-like} & & \text{étale-like} & & \text{Frobenius-like}). \end{array} $$

In short,

$$ {}^\dagger\Pi \cong {}^\ddagger\Pi, \quad ({}^\dagger\Pi \curvearrowright {}^\dagger\mathbb{M}^{\circledast}) \quad \underset{a\ priori}{\overset{\text{no relation}}{\longleftrightarrow}} \quad ({}^\ddagger\Pi \curvearrowright {}^\ddagger\mathbb{M}^{\circledast}) $$

$$\overset{\text{mono-anabelian}}{\underset{\text{transport}}{\Rightarrow}} \quad ({}^{\dagger}\Pi \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}) \overset{\text{canonically}}{\cong} ({}^{\ddagger}\Pi \curvearrowright {}^{\ddagger}\mathbb{M}^{\circledast}),$$

$$\text{cyclotomic rigidity} \overset{\text{makes available}}{\Rightarrow} \text{Kummer theory} \overset{\text{applied}}{\Rightarrow} \text{mono-anabelian transport}.$$

This technique is called the mono-anabelian transport.

Remark 9.6.2. (differences between three cyclotomic rigidities) We already met three kinds of cyclotomic rigidities: the *cyclotomic rigidity via LCFT* (Cyc. Rig. LCFT2) in Remark 3.19.2, *of mono-theta environment* (Cyc. Rig. Mono-th.) in Theorem 7.23 (1), and *via* $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^{\times} = \{1\}$ (Cyc. Rig. NF2) in Definition 9.6:

$$\mu_{\widehat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(M), \quad {}^{\dagger}(l\Delta_{\Theta}) \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mu_N({}^{\dagger}(l\Delta_{\Theta}[\mu_N])), \quad \mu_{\widehat{\mathbb{Z}}}^{\Theta}({}^{\dagger}\Pi^{\circledcirc}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}^{\circledast}).$$

In inter-universal Teichmüller theory, we use these three kinds of cyclotomic rigidities to *three kinds of Kummer theory* respectively, and they correspond to *three portions of* Θ*-links*, i.e.,

(U) we use the cyclotomic rigidity via LCFT (Cyc. Rig. LCFT2) for the *constant monoids* at local places in $\underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}$, which is related with the *local unit (modulo torsion) portion* of the Θ-links,

(V) we use the cyclotomic rigidity of mono-theta environment (Cyc. Rig. Mono-th.) for the *theta functions and their evaluations* at local places in $\underline{\mathbb{V}}^{\text{bad}}$, which is related with the *local value group portion* of the Θ-links, and

(G) we use the cyclotomic rigidity of via $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^{\times} = \{1\}$ (Cyc. Rig. NF2) for the *non-realified global Frobenioids*, which is related with the *global realified portion* of the Θ-links.

We explain more.

(1) In Remark 9.6.1, we used ${}^{\dagger}O^{\rhd}(\cong O^{\rhd}_{\overline{k}})$ and as examples to explain the technique of mono-anabelian transport. However, in inter-universal Teichmüller theory, the mono-anabelian transport using the cyclotomic rigidity via LCFT is useless in the important situation i.e., at local places in $\underline{\mathbb{V}}^{\text{bad}}$ (However, we use it in the less important situation i.e., at local places in $\underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}$), because *the cyclotomic rigidity via LCFT uses essentially the value group portion* in the construction, and, at places in $\underline{\mathbb{V}}^{\text{bad}}$ in inter-universal Teichmüller theory, we *deform the value group portion* in Θ-links! Since the value group portion is not shared under Θ-links, if

we use the cyclotomic rigidity via LCFT for the Kummer theory for theta functions/theta values at places in $\underline{\mathbb{V}}^{\mathrm{bad}}$ in a Hodge theatre, then the algorithm is only valid with in the same Hodge theatre, and we cannot see it from another Hodge theatre (i.e., the algorithm is **uniradial**. (cf. Remark 11.4.1, Proposition 11.15 (2), and Remark 11.17.2 (2)). Therefore, the cyclotomic rigidity via LCFT is *not* suitable at local places in $\underline{\mathbb{V}}^{\mathrm{bad}}$, which deforms the value group portion.

(2) Instead, we use the cyclotomic rigidity via LCFT at local places in $\underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$. In this case too, only the unit portion is shared in Θ-links, and the value group portion is not shared (even though the value group portion is not deformed in the case of $\underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$), thus, we ultimately admit $\widehat{\mathbb{Z}}^{\times}$-indeterminacy to make an algorithm **multiradial** (cf. Definition 11.1 (2), Example 11.2, and § A.4. cf. also Remark 11.4.1, and Proposition 11.5). Mono-analytic containers, or local log-volumes in algorithms have no effect by this $\widehat{\mathbb{Z}}^{\times}$-indeterminacy.

(3) In $\underline{\mathbb{V}}^{\mathrm{bad}}$, we use the cyclotomic rigidity of mono-theta environment for the Kummer theory of theta functions (cf. Proposition 11.14, and Theorem 12.7). The cyclotomic rigidity of mono-theta environment only uses μ_N-portion, and *does not use the value group portion*! Hence the Kummer theory using the cyclotomic rigidity of mono-theta environment in a Hodge theatre does not harm/affect the ones in other Hodge theatres. Therefore, these things make algorithms using the cyclotomic rigidity of mono-theta environment **multiradial** (cf. also Remark 11.4.1).

(4) In Remark 9.6.1, we used ${}^{\dagger}\mathbb{M}^{\circledast}(\cong \overline{F}^{\times})$ and as examples to explain the technique of mono-anabelian transport. However, in inter-universal Teichmüller theory, we cannot transport ${}^{\dagger}\mathbb{M}^{\circledast}(\cong \overline{F}^{\times})$ by the technique of the mono-anabelian transport by the following reason (cf. also [IUTchII, Remark 4.7.6]): In inter-universal Teichmüller theory, we consider Π_{C_F} as an abstract topological group. This means that the subgroups $\Pi_{\underline{C}_K}$, $\Pi_{\underline{X}_K}$ are *only well-defined up to* Π_{C_F}*-conjugacy*, i.e., the subgroups $\Pi_{\underline{C}_K}$, $\Pi_{\underline{X}_K}$ are only well-defined up to automorphisms arising from their normalisers in Π_{C_F}. Therefore, we need to consider these groups $\Pi_{\underline{C}_K}$, $\Pi_{\underline{X}_K}$ as being subject to indeterminacies of $\mathbb{F}_l^{\circledast}$-poly-actions (cf. Definition 10.16). However, $\mathbb{F}_l^{\circledast}$ nontrivially acts on ${}^{\dagger}\mathbb{M}^{\circledast}(\cong \overline{F}^{\times})$. Therefore, ${}^{\dagger}\mathbb{M}^{\circledast}(\cong \overline{F}^{\times})$ is inevitablyy subject to $\mathbb{F}_l^{\circledast}$-indeterminacies. Instead of ${}^{\dagger}\mathbb{M}^{\circledast}(\cong \overline{F}^{\times})$, we can transport the ${}^{\dagger}\Pi^{\circledast}$-invariant part ${}^{\dagger}\mathbb{M}_{\mathrm{mod}} := ({}^{\dagger}\mathbb{M}^{\circledast})^{{}^{\dagger}\Pi^{\circledast}}(\cong F_{\mathrm{mod}}^{\times})$, since $\mathbb{F}_l^{\circledast}$ trivially poly-acts on it, and there is no $\mathbb{F}_l^{\circledast}$-indeterminacies (cf. also Remark 11.22.1).

(5) Another important difference is as follows: The cyclotomic rigidity via LCFT and of mono-theta environment are compatible with the profinite topology, i.e., it is the projective limit of the "mod N" levels. On the other hand, the cyclotomic rigidity via $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^{\times} = \{1\}$ is not compatible with the profinite topology, i.e., it has no

such "mod N" levels. In the Kummer tower ($\widehat{k^\times} =) \varprojlim(k^\times \leftarrow k^\times \leftarrow \cdots)$, we have the field structures on each finite levels $k^\times(\cup\{0\})$; however, we have no field structure on the limit level $\widehat{k^\times}$. On the other hand, the logarithm "$\sum_n \frac{x^n}{n}$" needs field structure. Hence we need to work in "mod N" levels to construct $\mathfrak{log}$-links, and the Kummer theory using the cyclotomic rigidity via $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\}$ is *not* compatible with the $\mathfrak{log}$-links. Therefore, we cannot transport global non-realified Frobenioids under $\mathfrak{log}$-links. On the realified Frobenioids, we have the compatibility of the log-volumes with $\mathfrak{log}$-links (i.e., the formulae (5.1) and (5.2) in Proposition 5.2 and Proposition 5.4 respectively). (Note that N-th power maps are not compatible with addtions, hence we caanot work in a single scheme-theoretic basepoint over both the domain and the codomain of Kummer N-th power map. This means that we should work with different scheme-theoretic basepoints over both the domain and the codomain of Kummer N-th power map, hence the "isomorphism class compatibility" i.e., the compatibility with the convention that various objects of the tempered Frobenioids are known only up to isomorphism, is crucial here (cf. [IUTchII, Remark 3.6.4 (i)], [IUTchIII, Remark 2.1.1 (ii)]) (This is also related to Remark 13.13.3 (2b))).

Cyclotomic rigidity	via LCFT	of mono-theta env.	via $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\}$
Related Component of Θ-links	units modulo torsion	value group (theta values)	global realified component
Radiality	uniradial or multiradial up to $\widehat{\mathbb{Z}}^\times$-indet.	multiradial	multiradial
Compatibility with profinite top.	compatible	compatible	incompatible

§ 9.3. $\boxtimes$-Line Bundles and $\boxplus$-Line Bundles.

We continue to use the notation in the previous section. Moreover, we assume that we are given a subset $\underline{\mathbb{V}} \subset \mathbb{V}(K)$ such that the natural surjection $\mathbb{V}(K) \twoheadrightarrow \mathbb{V}(F_{\mathrm{mod}})$ induces a bijection $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}(F_{\mathrm{mod}})$ (Note that, as we will see in the following definitions, we are regarding $\underline{\mathbb{V}}$ as an "*analytic section*" of the morphism $\operatorname{Spec} O_K \twoheadrightarrow \operatorname{Spec} O_{F_{\mathrm{mod}}}$). Write $\underline{\mathbb{V}}^{\mathrm{non}} := \underline{\mathbb{V}} \cap \mathbb{V}(K)^{\mathrm{non}}$ and $\underline{\mathbb{V}}^{\mathrm{arc}} := \underline{\mathbb{V}} \cap \mathbb{V}(K)^{\mathrm{arc}}$.

Definition 9.7. ([IUTchIII, Example 3.6]) We write $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ (i.e., without "$\dagger$")

for the global non-realified Frobenioid which is constructed by the model $\mathcal{D}(\underline{C}_K)^0$ (i.e., without "†").

(1) ($\boxtimes$-line bundle) A **$\boxtimes$-line bundle** on $(\mathrm{Spec}\, O_K)//\mathrm{Gal}(K/F_{\mathrm{mod}})$ is a data $\mathcal{L}^{\boxtimes} = (T, \{t_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$, where

   (a) T is an $F^{\times}_{\mathrm{mod}}$-torsor, and

   (b) $t_{\underline{v}}$ is a trivialisation of the torsor $T_{\underline{v}} := T \otimes_{F^{\times}_{\mathrm{mod}}} (K^{\times}_{\underline{v}}/O^{\times}_{K_{\underline{v}}})$ for each $\underline{v} \in \underline{\mathbb{V}}$, where $F^{\times}_{\mathrm{mod}} \to K^{\times}_{\underline{v}}/O^{\times}_{K_{\underline{v}}}$ is the natural group homomorphism,

   satisfying the condition that there is an element $t \in T$ such that $t_{\underline{v}}$ is equal to the trivialisation determined by t for all but finitely many $\underline{v} \in \underline{\mathbb{V}}$. We can define a **tensor product** $(\mathcal{L}^{\boxtimes})^{\otimes n}$ of a $\boxtimes$-line bundle $\mathcal{L}^{\boxtimes}$ for $n \in \mathbb{Z}$ in an obvious manner.

(2) (morphism of $\boxtimes$-line bundles) Let $\mathcal{L}^{\boxtimes}_1 = (T_1, \{t_{1,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$, $\mathcal{L}^{\boxtimes}_2 = (T_2, \{t_{2,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ be $\boxtimes$-line bundles. An **elementary morphism $\mathcal{L}^{\boxtimes}_1 \to \mathcal{L}^{\boxtimes}_2$ of $\boxtimes$-line bundles** is an isomorphism $T_1 \xrightarrow{\sim} T_2$ of $F^{\times}_{\mathrm{mod}}$-torsors which sends the trivialisation $t_{1,\underline{v}}$ to an element of the $O^{\rhd}_{K_{\underline{v}}}$-orbit of $t_{2,\underline{v}}$ (i.e., the morphism is integral at $\underline{v}$) for each $\underline{v} \in \underline{\mathbb{V}}$. A **morphism of $\boxtimes$-line bundles** from $\mathcal{L}^{\boxtimes}_1$ to $\mathcal{L}^{\boxtimes}_2$ is a pair of a positive integer $n \in \mathbb{Z}_{>0}$ and an elementary morphism $(\mathcal{L}^{\boxtimes}_1)^{\otimes n} \to \mathcal{L}^{\boxtimes}_2$. We can define a composite of morphisms in an obvious manner. Then the $\boxtimes$-line bundles on $(\mathrm{Spec}\, O_K)//\mathrm{Gal}(K/F_{\mathrm{mod}})$ and the morphisms between them form a category (in fact, a Frobenioid)

$$\mathcal{F}^{\circledast}_{\mathrm{MOD}}.$$

   We have a natural isomorphism

$$\mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{MOD}}$$

   of (pre-)Frobenioids, which induces the identity morphism $F^{\times}_{\mathrm{mod}} \to F^{\times}_{\mathrm{mod}}$ on $\Phi((-)^{\mathrm{birat}})$. Note that the category $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ is defined by using *only the multiplicative ($\boxtimes$) structure.*

(3) ($\boxplus$-line bundle) A **$\boxplus$-line bundle** on $(\mathrm{Spec}\, O_K)//\mathrm{Gal}(K/F_{\mathrm{mod}})$ is a data $\mathcal{L}^{\boxplus} = \{J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, where $J_{\underline{v}} \subset K_{\underline{v}}$ is a fractional ideal for each $\underline{v} \in \underline{\mathbb{V}}$ (i.e., a finitely generated non-zero $O_{K_{\underline{v}}}$-submodule of $K_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, and a positive real multiple of $O_{K_{\underline{v}}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ (cf. Section 0.2 for $O_{K_{\underline{v}}}$)) such that $J_{\underline{v}} = O_{K_{\underline{v}}}$ for finitely many $\underline{v} \in \underline{\mathbb{V}}$. We can define a **tensor product** $(\mathcal{L}^{\boxplus})^{\otimes n}$ of a $\boxplus$-line bundle $\mathcal{L}^{\boxplus}$ for $n \in \mathbb{Z}$ in an obvious manner.

(4) (morphism of $\boxplus$-line bundles) Let $\mathcal{L}^{\boxplus}_1 = \{J_{1,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, $\mathcal{L}^{\boxplus}_2 = \{J_{2,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ be $\boxplus$-line bundles. An **elementary morphism $\mathcal{L}^{\boxplus}_1 \to \mathcal{L}^{\boxplus}_2$ of $\boxplus$-line bundles** is an element

$f \in F_{\mathrm{mod}}^{\times}$ such that $f \cdot J_{1,\underline{v}} \subset J_{2,\underline{v}}$ (i.e., f is integral at $\underline{v}$) for each $\underline{v} \in \underline{\mathbb{V}}$. A **morphism of $\boxplus$-line bundles** from $\mathcal{L}_1^{\boxplus}$ to $\mathcal{L}_2^{\boxplus}$ is a pair of a positive integer $n \in \mathbb{Z}_{>0}$ and an elementary morphism $(\mathcal{L}_1^{\boxplus})^{\otimes n} \to \mathcal{L}_2^{\boxplus}$. We can define a composite of morphisms in an obvious manner. Then the $\boxplus$-line bundles on $(\mathrm{Spec}\, O_K)//\mathrm{Gal}(K/F_{\mathrm{mod}})$ and the morphisms between them form a category (in fact, a Frobenioid)

$$\mathcal{F}_{\mathfrak{mod}}^{\circledast}.$$

We have a natural isomorphism

$$\mathcal{F}_{\mathrm{mod}}^{\circledast} \xrightarrow{\sim} \mathcal{F}_{\mathfrak{mod}}^{\circledast}$$

of (pre-)Frobenioids, which induces the identity morphism $F_{\mathrm{mod}}^{\times} \to F_{\mathrm{mod}}^{\times}$ on $\Phi((-)^{\mathrm{birat}})$. Note that the category $\mathcal{F}_{\mathfrak{mod}}^{\circledast}$ is defined by using *both the multiplicative ($\boxtimes$) and the additive ($\boxplus$) structures.*

Hence by combining the isomorphisms, we have a natural isomorphism

$$\mathcal{F}_{\mathfrak{mod}}^{\circledast} \xrightarrow{\sim} \mathcal{F}_{\mathrm{MOD}}^{\circledast} \qquad \text{(Convert)}$$

of (pre-)Frobenioids, which induces the identity morphism $F_{\mathrm{mod}}^{\times} \to F_{\mathrm{mod}}^{\times}$ on $\Phi((-)^{\mathrm{birat}})$.

§ 10. Hodge Theatres.

In this section, we construct Hodge theatres after fixing an *initial* Θ*-data* (Section 10.1). More precisely, we construct $\Theta^{\pm\mathrm{ell}}$NF-*Hodge theatres* (In this survey, we shall refer to them as $\boxtimes\boxplus$-*Hodge theatres*). We can consider $\mathbb{Z}/l\mathbb{Z}$ as a finite approximation of $\mathbb{Z}$ for $l >> 0$ (Note also that we take $l >> 0$ approximately of order of a value of height function. cf. Section). Then we can consider $\mathbb{F}_l^{\circledast}$ and $\mathbb{F}_l^{\rtimes\pm}$ as a "multiplicative finite approximation" and an "additive finite approximation" of $\mathbb{Z}$ respectively. Moreover, it is important that two operations (multiplication and addition) are separated in "these finite approximations" (cf. Remark 10.29.2). Like $\underline{\mathbb{Z}}/l\underline{\mathbb{Z}}$ is a finite approximation of $\underline{\mathbb{Z}}$ (Recall that $\underline{\mathbb{Z}} = \mathrm{Gal}(\mathfrak{Y}/\mathfrak{X})$), a Hodge theatre, which consists of various data involved by $\underline{\underline{X}}_{\underline{v}}$, $\underset{\rightarrow}{X}_{\underline{v}}$, $\underline{C}_K$ and so on, can be seen as a finite approximation of upper half plane.

Before preceeding to the detailed constructions, we briefly explain the structure of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theatre (or $\boxtimes\boxplus$-Hodge theatre). A $\Theta^{\pm\mathrm{ell}}$NF-Hodge theatre (or a $\boxtimes\boxplus$-Hodge theatre) will be obtained by "gluing" (Section 10.6)

- a ΘNF-*Hodge theatre*, which has a $\mathbb{F}_l^{\circledast}$-*symmetry*, is *related to a number field*, *of arithmetic nature*, and is used to *Kummer theory for* NF (In this survey, we shall refer to it as a $\boxtimes$-*Hodge theatre*, Section 10.4) and

- a $\Theta^{\pm\mathrm{ell}}$-*Hodge theatre*, which has a $\mathbb{F}_l^{\rtimes\pm}$-*symmetry*, is *related to an elliptic curve*, *of geometric nature*, and is used to *Kummer theory for* Θ (In this survey, we shall refer to it as a $\boxplus$-*Hodge theatre*, Section 10.5).

Separating the multiplicative ($\boxtimes$) symmetry and the additive ($\boxplus$) symmetry is also important (cf. [IUTchII, Remark 4.7.3, Remark 4.7.6]).

ΘNF-Hodge theatre	$\mathbb{F}_l^{\circledast}$-symmetry ($\boxtimes$)	arithmetic nature	Kummer theory for NF
$\Theta^{\pm\mathrm{ell}}$-Hodge theatre	$\mathbb{F}_l^{\rtimes\pm}$-symmetry ($\boxplus$)	geometric nature	Kummer theory for Θ

As for the analogy with upper half plane, the multiplicative symmetry (resp. the additive symmetry) corresponds to supersingular points of the reduction modulo p of modular curves (resp. the cusps of the modular curves). cf. the following tables ([IUTchI, Fig. 6.4]):

	$\boxtimes$-symmetry	Basepoint (cf. Remark 10.29.1)	Functions (cf. Corollary 11.23)
upper half plane	$z \mapsto \frac{z\cos(t)-\sin(t)}{z\sin(t)+\cos(t)}$, $z \mapsto \frac{\overline{z}\cos(t)+\sin(t)}{\overline{z}\sin(t)-\cos(t)}$	supersingular pts.	rat. fct. $w = \frac{z-i}{z+i}$
Hodge theatre	$\mathbb{F}_l^{\circledast}$-symm.	$\mathbb{F}_l^{\circledast} \curvearrowright \underline{\mathbb{V}}^{\mathrm{Bor}}$	elements of F_{mod}

	$\boxplus$-symmetry	Basepoint (cf. Remark 10.29.1)	Functions (cf. Corollary 11.21)
upper half plane	$z \mapsto z+a$, $z \mapsto -\overline{z}+a$	cusp	trans. fct. $q = e^{2\pi i}$
Hodge theatre	$\mathbb{F}_l^{\rtimes\pm}$-symm.	$\underline{\mathbb{V}}^{\pm}$	theta values $\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{1\leq j\leq l^{\circledast}}$

	Coric symmetry (cf. Proposition 10.34 (3))
upper half plane	$z \mapsto z, -\overline{z}$
Hodge theatre	$\{\pm 1\}$

These three kinds of Hodge theatres have base-Hodge theatres (like Frobenioids) respectively, i.e., a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theatre (or a ⊠⊞-Hodge theatre) has a *base-*$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theatre* (or $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-*Hodge theatre*, or $\mathcal{D}$-⊠⊞-*Hodge theatre*), which is obtained by "gluing"

- a *base-*ΘNF-*Hodge theatre* (or $\mathcal{D}$-ΘNF-*Hodge theatre*, or $\mathcal{D}$-⊠-*Hodge theatre*) and
- a *base-*$\Theta^{\pm\mathrm{ell}}$-*Hodge theatre* (or $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-*Hodge theatre*, or $\mathcal{D}$-⊞-*Hodge theatre*).

A $\mathcal{D}$-ΘNF-Hodge theatre (or $\mathcal{D}$-⊠-Hodge theatre) consists

- of three portions
  - (local object) a *holomorphic base-(or* $\mathcal{D}$*-)prime-strip* ${}^{\dagger}\mathfrak{D}_{>} = \{{}^{\dagger}\mathcal{D}_{>,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, where ${}^{\dagger}\mathcal{D}_{>,\underline{v}}$ is a category equivalent to $\mathcal{B}(\underset{\rightarrow}{X}_{\underline{v}})^0$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$, or a category equivalent to $\mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})^0$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, or an Aut-holomorphic orbispace isomorphic to $\underset{\rightarrow}{\mathbb{X}}_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ (Section 10.3),
  - (local object) a *capsule* ${}^{\dagger}\mathfrak{D}_J = \{{}^{\dagger}\mathfrak{D}_j\}_{j\in J}$ *of* $\mathcal{D}$*-prime-strips* indexed by $J\,(\cong \mathbb{F}_l^{\circledast})$ (cf. Section 0.2 for the term "capsule"), and
  - (global object) a category ${}^{\dagger}\mathcal{D}^{\circledcirc}$ equivalent to $\mathcal{B}(\underline{C}_K)^0$,
- and of two *base-bridges*
  - a *base-(or* $\mathcal{D}$*-)*Θ*-bridge* ${}^{\dagger}\phi^{\Theta}_{\circledast}$, which connects the capsule ${}^{\dagger}\mathfrak{D}_J$ of $\mathcal{D}$-prime-strips to the $\mathcal{D}$-prime-strip ${}^{\dagger}\mathfrak{D}_{>}$, and
  - a *base-(or* $\mathcal{D}$*-)*NF*-bridge* ${}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}$, which connects the capsule ${}^{\dagger}\mathfrak{D}_J$ of $\mathcal{D}$-prime-strips to the global object ${}^{\dagger}\mathcal{D}^{\circledcirc}$.

Here, for a holomorphic base-(or $\mathcal{D}$-)prime-strip ${}^{\dagger}\mathfrak{D} = \{{}^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, we can associate its *mono-analyticisation* (cf. Section 3.5) ${}^{\dagger}\mathfrak{D}^{\vdash} = \{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, which is a *mono-analytic base-(or* $\mathcal{D}^{\vdash}$*-)prime-strip.*

On the other hand, a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theatre (or $\mathcal{D}$-⊞-Hodge theatre) similarly consists

- of three portions
  - (local object) a $\mathcal{D}$-prime-strip ${}^\dagger\mathfrak{D}_\succ = \{{}^\dagger\mathcal{D}_{\succ,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$,
  - (local object) a capsule ${}^\dagger\mathfrak{D}_T = \{{}^\dagger\mathfrak{D}_t\}_{t\in T}$ of $\mathcal{D}$-prime-strips indexed by $T\,(\cong \mathbb{F}_l)$, and
  - (global object) a category ${}^\dagger\mathcal{D}^{\circledcirc\pm}$ equivalent to $\mathcal{B}(\underline{X}_K)^0$,
- and of two *base-bridges*
  - a *base-(or $\mathcal{D}$-)$\Theta^\pm$-bridge* ${}^\dagger\phi_\pm^{\Theta^\pm}$, which connects the capsule ${}^\dagger\mathfrak{D}_T$ of $\mathcal{D}$-prime-strips to the $\mathcal{D}$-prime-strip ${}^\dagger\mathfrak{D}_\succ$, and
  - a *base-(or $\mathcal{D}$-)Θ^{ell}-bridge* ${}^\dagger\phi_\pm^{\Theta^{\mathrm{ell}}}$, which connects the capsule ${}^\dagger\mathfrak{D}_T$ of $\mathcal{D}$-prime-strips to the global object ${}^\dagger\mathcal{D}^{\circledcirc\pm}$.

Hence the structure of a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theatre (or $\mathcal{D}$-$\boxtimes\boxplus$-Hodge theatre) is as follows (For the torsor structures, Aut, and gluing see Proposition 10.20, Proposition 10.34, Lemma 10.38, and Definition 10.39):

$$\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}\text{-}\mathcal{HT}$$

$$\begin{array}{ccccccc}
(\mathrm{Aut}=\{\pm1\})\ \mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-}\mathcal{HT} & & {}^\dagger\mathfrak{D}_\succ & \xrightarrow{\text{gluing } (>=\{0,\succ\})} & {}^\dagger\mathfrak{D}_> & & \mathcal{D}\text{-}\Theta\mathrm{NF}\text{-}\mathcal{HT}\ (\mathrm{Aut}=\{1\}) \\
& \underline{\mathcal{D}\text{-}\Theta^\pm\text{-bridge}} & {}^\dagger\phi_\pm^{\Theta^\pm}\Big\uparrow (\{\pm1\}\times\{\pm1\}^{\underline{\mathbb{V}}}\text{-torsor}) & & (\text{rigid})\Big\uparrow {}^\dagger\phi_*^{\Theta} & \underline{\mathcal{D}\text{-}\Theta\text{-bridge}} & \\
\boxed{\boxplus\text{-Symm.}} & (t\in T\,(\cong\mathbb{F}_l)) & {}^\dagger\mathfrak{D}_T & \xrightarrow{\text{gluing } (J=(T\setminus\{0\})/\{\pm1\})} & {}^\dagger\mathfrak{D}_J & (j\in J\,(\cong\mathbb{F}_l^*)) & \boxed{\boxtimes\text{-Symm.}} \\
& \underline{\mathcal{D}\text{-}\Theta^{\mathrm{ell}}\text{-bridge}} & {}^\dagger\phi_\pm^{\Theta^{\mathrm{ell}}}\Big\downarrow (\mathbb{F}_l^{\pm}\text{-torsor}) & & (\mathbb{F}_l^{*}\text{-torsor})\Big\downarrow {}^\dagger\phi_*^{\mathrm{NF}} & \underline{\mathcal{D}\text{-NF-bridge}} & \\
\text{Geometric} & (\underline{X}_K \rightsquigarrow) & {}^\dagger\mathcal{D}^{\circledcirc\pm} & & {}^\dagger\mathcal{D}^{\circledcirc} & (\leftsquigarrow \underline{C}_K) & \text{Arithmetic}
\end{array}$$

We can also draw a picture as follows (cf. [IUTchI, Fig. 6.5]):

$$\begin{array}{ccc}
\mathfrak{D}_\succ = /^\pm & \overset{>=\{0,\succ\}}{\Rightarrow} & \mathfrak{D}_> = /^* \\
\phi_\pm^{\Theta^\pm}\Big\uparrow & & \Big\uparrow\phi_*^{\Theta} \\
\{\pm1\} \curvearrowright \mathfrak{D}_T = /^\pm_{-l^*} \cdots /^\pm_{-1} /^\pm_0 /^\pm_1 \cdots /^\pm_{l^*} & \overset{J=(T\setminus\{0\})/\{\pm1\}}{\Rightarrow} & \mathfrak{D}_J = /^*_1 /^* \cdots /^*_{l^*} \\
\phi_\pm^{\Theta^{\mathrm{ell}}}\Big\downarrow & & \Big\downarrow\phi_*^{\mathrm{NF}} \\
\mathbb{F}_l^{\rtimes\pm} \curvearrowright \begin{smallmatrix}\pm\to\pm\\ \uparrow\quad\downarrow\\ \pm\leftarrow\pm\end{smallmatrix}\ \mathcal{D}^{\circledcirc\pm} = \mathcal{B}(\underline{X}_K)^0 & & \mathbb{F}_l^{*} \curvearrowright \begin{smallmatrix}*\to *\\ \uparrow\quad\downarrow\\ *\leftarrow *\end{smallmatrix}\ \mathcal{D}^{\circledcirc} = \mathcal{B}(\underline{C}_K)^0,
\end{array}$$

where /'s express prime-strips.

These are base Hodge theatres, and the structure of the total space of Hodge theatres is as follows: A ΘNF-Hodge theatre (or ⊠-Hodge theatre) consists

- of five portions
  - (local and global realified object) a Θ-*Hodge theatre* ${}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$, which consists of
    - (local object) a pre-Frobenioid ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ isomorphic to the $p_{\underline{v}}$-adic Frobenioid $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ (Example 8.7) for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$, or a pre-Frobenioid isomorphic to the tempered Frobenioid $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ (Example 8.8), or a triple ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}} = ({}^{\dagger}\mathcal{C}_{\underline{v}}, {}^{\dagger}\mathcal{D}_{\underline{v}}, {}^{\dagger}\kappa_{\underline{v}})$, isomorphic to the triple $\underline{\underline{\mathcal{F}}}_{\underline{v}} = (\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}})$ (Example 8.11) of the Archimedean Frobenioid $\mathcal{C}_{\underline{v}}$, the Aut-holomorphic orbispace $\mathcal{D}_{\underline{v}} = \underset{\rightarrow}{\mathbb{X}}_{\underline{v}}$ and its Kummer structure $\kappa_{\underline{v}} : O^{\rhd}(\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}^{\mathcal{D}_{\underline{v}}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, and
    - (global realified object with localisations) a quadruple ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} = ({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}, \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{{}^{\dagger}\rho^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ of a pre-Frobenioid isomorphic to the global realified Frobenioid $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ (Example 8.12), a bijection $\mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}}$, a mono-analytic Frobenioid-(or $\mathcal{F}^{\vdash}$-)prime-strip $\{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ (cf. below), and global-to-local homomorphisms $\{{}^{\dagger}\rho^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$.
  - (local object) a *holomorphic Frobenioid-(or* $\mathcal{F}$*-)prime-strip* ${}^{\dagger}\mathfrak{F}_{>} = \{{}^{\dagger}\mathcal{F}_{>,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, where ${}^{\dagger}\mathcal{F}_{>,\underline{v}}$ is equalto the ${}^{\dagger}\mathcal{F}_{\underline{v}}$'s in the above Θ-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\Theta}$.
  - (local object) a *capsule* ${}^{\dagger}\mathfrak{F}_J = \{{}^{\dagger}\mathfrak{F}_j\}_{j\in J}$ *of* $\mathcal{F}$*-prime-strips* indexed by J ($\cong \mathbb{F}_l^{\circledast}$) (cf. Section 0.2 for the term "capsule"),
  - (global object) a pre-Frobenioid ${}^{\dagger}\mathcal{F}^{\circledcirc}$ isomorphic to the global non-realified Frobenioid $\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$ (Example 9.5), and
  - (global object) a pre-Frobenioid ${}^{\dagger}\mathcal{F}^{\circledast}$ isomorphic to the global non-realified Frobenioid $\mathcal{F}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$ (Example 9.5).
- and of two *bridges*
  - a Θ-*bridge* ${}^{\dagger}\psi^{\Theta}_{\circledast}$, which connects the capsule ${}^{\dagger}\mathfrak{F}_J$ of prime-strips to the prime-strip ${}^{\dagger}\mathfrak{F}_{>}$, and to the Θ-Hodge theatre ${}^{\dagger}\mathfrak{F}_{>} \dashrightarrow {}^{\dagger}\mathcal{HT}^{\Theta}$, and
  - an NF-*bridge* ${}^{\dagger}\psi^{\mathrm{NF}}_{\circledast}$, which connects the capsule ${}^{\dagger}\mathfrak{F}_J$ of prime-strips to the global objects ${}^{\dagger}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{\dagger}\mathcal{F}^{\circledast}$.

and these objects are "lying over" the corresponding base objects.

Here, for a holomorphic Frobenioid-(or $\mathcal{F}$-)prime-strip ${}^{\dagger}\mathfrak{F} = \{{}^{\dagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, we can algorithmically associate its *mono-analyticisation* (cf. Section 3.5) ${}^{\dagger}\mathfrak{F}^{\vdash} = \{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, which is a *mono-analytic Frobenioid-(or* $\mathcal{F}^{\vdash}$*-)prime-strip.*

On the other hand, a $\Theta^{\pm\mathrm{ell}}$-Hodge theatre (or $\boxplus$-Hodge theatre) similarly consists

- of three portions
  - (local object) an *$\mathcal{F}$-prime-strip* ${}^{\dagger}\mathfrak{F}_{\succ} = \{{}^{\dagger}\mathcal{F}_{\succ,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$,
  - (local object) a *capsule* ${}^{\dagger}\mathfrak{F}_{T} = \{{}^{\dagger}\mathfrak{F}_{t}\}_{t\in T}$ *of $\mathcal{F}$-prime-strips* indexed by T $(\cong \mathbb{F}_l)$, and
  - (global object) the same global object ${}^{\dagger}\mathcal{D}^{\circledcirc\pm}$ as in the $\mathcal{D}$-$\boxplus$-Hodge theatre,
- and of two *bridges*
  - a $\Theta^{\pm}$*-bridge* ${}^{\dagger}\psi^{\Theta^{\pm}}_{\pm}$, which connects the capsule ${}^{\dagger}\mathfrak{F}_{T}$ of prime-strips to the prime-strip ${}^{\dagger}\mathfrak{F}_{\succ}$, and
  - a Θ^{ell}*-bridge* ${}^{\dagger}\psi^{\Theta^{\mathrm{ell}}}_{\pm}$ is equal to the $\mathcal{D}$-Θ^{ell}-bridge ${}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}$,

and these objects are "lying over" the corresponding base objects.

Hence the structure of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theatre (or $\boxtimes\boxplus$-Hodge theatre) is as follows (For the torsor structures, Aut, and gluing, see Lemma 10.25, Lemma 10.37, Lemma 10.38, and Definition 10.39):

$$\begin{tikzcd}[column sep=small]
 & \Theta^{\pm\mathrm{ell}}\text{NF-}\mathcal{HT} & {}^{\dagger}\mathcal{HT}^{\Theta} & \\
(\mathrm{Aut}=\{\pm1\})\ \Theta^{\pm\mathrm{ell}}\text{-}\mathcal{HT} & {}^{\dagger}\mathfrak{F}_{\succ} \arrow[r, dotted, "\text{gluing }(>=\{0,\succ\})"] & {}^{\dagger}\mathfrak{F}_{>} \arrow[u, dotted, "\mathcal{F}\text{-prime-strip}"'] & \Theta\text{NF-}\mathcal{HT}\ (\mathrm{Aut}=\{1\}) \\
\boxed{\boxplus\text{-Symm.}}\ (t\in T\,(\cong\mathbb{F}_l)) \quad \underline{\Theta^{\pm}\text{-bridge}} & {}^{\dagger}\mathfrak{F}_{T} \arrow[u, "{}^{\dagger}\psi^{\Theta^{\pm}}_{\pm}", "(\{\pm1\}\times\{\pm1\}^{\underline{\mathbb{V}}}\text{-torsor})"'] \arrow[r, dotted, "\text{gluing }(J=(T\setminus\{0\})/\{\pm1\})"] \arrow[d, "{}^{\dagger}\psi^{\Theta^{\mathrm{ell}}}_{\pm}"', "(\mathbb{F}_l^{\pm}\text{-torsor})"] & {}^{\dagger}\mathfrak{F}_{J} \arrow[u, "(\text{rigid})", "{}^{\dagger}\psi^{\Theta}_{\ast}"'] \arrow[d, "(\mathbb{F}_l^{\ast}\text{-torsor})"', "{}^{\dagger}\psi^{\mathrm{NF}}_{\ast}"] & \underline{\Theta\text{-bridge}}\quad (j\in J\,(\cong\mathbb{F}_l^{\ast}))\ \boxed{\boxtimes\text{-Symm.}} \\
({}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}:{}^{\dagger}\mathfrak{D}_{T}\to{}^{\dagger}\mathcal{D}^{\circledcirc\pm})\ \underline{\Theta^{\mathrm{ell}}\text{-bridge}} \quad \text{Geometric} & {}^{\dagger}\mathcal{D}^{\circledcirc\pm} & {}^{\dagger}\mathcal{F}^{\circledcirc} \arrow[d, dotted, "{}^{\dagger}\mathcal{D}^{\circledcirc}\to{}^{\dagger}\mathcal{D}^{\circledast}"'] & \underline{\text{NF-bridge}}\quad \text{Arithmetic} \\
\text{Kummer for }\Theta & & {}^{\dagger}\mathcal{F}^{\circledast} & \text{Kummer for NF}
\end{tikzcd}$$

§ 10.1. Initial Θ-Data.

Definition 10.1. We shall refer to a collection of data

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$$

as an **initial Θ-data**, if it satisfies the following conditions:

(1) F is a number field such that $\sqrt{-1} \in F$, and $\overline{F}$ is an algebraic closure of F. We write $G_F := \mathrm{Gal}(\overline{F}/F)$.

(2) X_F is a once-punctured elliptic curve over F, which admits stable reduction over all $v \in \mathbb{V}(F)^{\mathrm{non}}$. We write $E_F (\supset X_F)$ for the elliptic curve over F obtaine by the smooth compactification of X_F. We also write $C_F := X_F /\!/ \{\pm 1\}$, where we write "$/\!/$" for the stack-theoretic quotient, and -1 is the F-involution determined by the multiplication by -1 on E_F. Let F_{mod} be the field of moduli (i.e., the field generated by the j-invariant of E_F over $\mathbb{Q}$). We assume that F is Galois over F_{mod} of degree prime to l, and that $2 \cdot 3$-torsion points of E_F are rational over F.

(3) $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \subset \mathbb{V}_{\mathrm{mod}} := \mathbb{V}(F_{\mathrm{mod}})$ is a non-empty subset of $\mathbb{V}^{\mathrm{non}}_{\mathrm{mod}} \setminus \{v \in \mathbb{V}^{\mathrm{non}}_{\mathrm{mod}} \mid v \mid 2\}$ such that X_F has bad (multiplicative in this case by the condition above) reduction at the places of $\mathbb{V}(F)$ lying over $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$. Write $\mathbb{V}^{\mathrm{good}}_{\mathrm{mod}} := \mathbb{V}_{\mathrm{mod}} \setminus \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$ (Note that X_F may have bad reduction at some places $\mathbb{V}(F)$ lying over $\mathbb{V}^{\mathrm{good}}_{\mathrm{mod}}$), $\mathbb{V}(F)^{\mathrm{bad}} := \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \times_{\mathbb{V}_{\mathrm{mod}}} \mathbb{V}(F)$, and $\mathbb{V}(F)^{\mathrm{good}} := \mathbb{V}^{\mathrm{good}}_{\mathrm{mod}} \times_{\mathbb{V}_{\mathrm{mod}}} \mathbb{V}(F)$. We also write $\Pi_{X_F} := \pi_1(X_F) \subset \Pi_{C_F} := \pi_1(C_F)$, and $\Delta_{X_F} := \pi_1(X_F \times_F \overline{F}) \subset \Delta_{C_F} := \pi_1(C_F \times_F \overline{F})$.

(4) l is a prime number ≥ 5 such that the image of the outer homomorphism $G_F \to \mathrm{GL}_2(\mathbb{F}_l)$ determined by the l-torsion points of E_F contains the subgroup $\mathrm{SL}_2(\mathbb{F}_l) \subset \mathrm{GL}_2(\mathbb{F}_l)$. Write $K := F(E_F[l])$, which corresponds to the kernel of the above homomorphism (Thus, since 3-torsion points of E_F are rational, K is Galois over F_{mod} by Lemma 1.7 (4). We also assume that l is not divisible by any place in $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$, and that l does not divide the order (normalised as being 1 for a uniformiser) of the q-parameters of E_F at places in $\mathbb{V}(F)^{\mathrm{bad}}$.

(5) $\underline{C}_F$ is a hyperbolic orbicurve of type $(1, l\text{-tors})_{\pm}$ (cf. Definition 7.10) over K with K-core given by $C_K := C_F \times_F K$ (Thus, $\underline{C}_K$ is determined, up to K-isomorphism, by C_F by the above (4)). Let $\underline{X}_K$ be a hyperbolic curve of type $(1, l\text{-tors})$ (cf. Definition 7.10) over K determined, up to K-isomorphism, by $\underline{C}_K$. Recall that we have uniquely determined open subgroup $\Delta_{\underline{\underline{X}}} \subset \Delta_{\underline{C}}$ corresponding to the hyperbolic curve $\underline{\underline{X}}_{\overline{F}}$ of type $(1, l\text{-tors}^{\Theta})$ (cf. Definition 7.11), which is a finite étale covering of $\underline{C}_{\overline{F}} := \underline{C}_F \times_F \overline{F}$ (cf. the argument after Assumption (2) in Section 7.3, where the decomposition $\overline{\Delta}_{\underline{X}} \cong \overline{\Delta}^{\mathrm{ell}}_{\underline{X}} \times \overline{\Delta}_{\Theta}$ does not depend on the choice of $\epsilon_{\iota_{\underline{X}}}$).

(6) $\underline{\mathbb{V}} \subset \mathbb{V}(K)$ is a subset such that the composite $\underline{\mathbb{V}} \subset \mathbb{V}(K) \twoheadrightarrow \mathbb{V}_{\mathrm{mod}}$ is a bijection, i.e., $\underline{\mathbb{V}}$ is a section of the surjection $\mathbb{V}(K) \twoheadrightarrow \mathbb{V}_{\mathrm{mod}}$. Write $\underline{\mathbb{V}}^{\mathrm{non}} := \underline{\mathbb{V}} \cap \mathbb{V}(K)^{\mathrm{non}}$, $\underline{\mathbb{V}}^{\mathrm{arc}} := \underline{\mathbb{V}} \cap \mathbb{V}(K)^{\mathrm{arc}}$, $\underline{\mathbb{V}}^{\mathrm{good}} := \underline{\mathbb{V}} \cap \mathbb{V}(K)^{\mathrm{good}}$, and $\underline{\mathbb{V}}^{\mathrm{bad}} := \underline{\mathbb{V}} \cap \mathbb{V}(K)^{\mathrm{bad}}$. For a place $\underline{v} \in \underline{\mathbb{V}}$, write $(-)_{\underline{v}} := (-)_F \times_F K_{\underline{v}}$ or $(-)_{\underline{v}} := (-)_K \times_K K_{\underline{v}}$ for the base change of

a hyperbolic orbicurve over F and K respectively. For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, we assume that the hyperbolic orbicurve $\underline{C}_{\underline{v}}$ is of type $(1, \mathbb{Z}/l\mathbb{Z})_{\pm}$ (cf. Definition 7.13) (Note that we have "$K = \ddot{K}$" since 2-torsion points of E_F are rational). For a place $\underline{v} \in \underline{\mathbb{V}}$, it follows that $\underline{\underline{X}}_{\overline{F}} \times_{\overline{F}} \overline{F}_{\underline{v}}$ admits a natural model $\underline{\underline{X}}_{\underline{v}}$ over $K_{\underline{v}}$, which is hyperbolic curve of type $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$ (cf. Definition 7.13), where $\underline{v}$ is a place of $\overline{F}$ lying over $\underline{v}$ (Roughly speaking, $\underline{\underline{X}}_{\underline{v}}$ is defined by taking "l-root of the theta function"). For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, we write $\Pi_{\underline{v}} := \Pi^{\mathrm{temp}}_{\underline{\underline{X}}_{\underline{v}}}$.

(7) $\underline{\epsilon}$ is a non-zero cusp of the hyperbolic orbicurve $\underline{C}_K$. For $\underline{v} \in \underline{\mathbb{V}}$, we write $\underline{\epsilon}_{\underline{v}}$ for the cusp of $\underline{C}_{\underline{v}}$ determined by $\underline{\epsilon}$. If $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, we assume that $\underline{\epsilon}_{\underline{v}}$ is the cusp, which arises from the canonical generator (up to sign) of $\widehat{\mathbb{Z}}$ via the surjection $\Pi_X \twoheadrightarrow \widehat{\mathbb{Z}}$ determined by the natural surjection $\Pi^{\mathrm{temp}}_X \twoheadrightarrow \mathbb{Z}$ (cf. Section 7.1 and Definition 7.13). Thus, the data $(X_K := X_F \times_F K, \underline{C}_K, \underline{\epsilon})$ determines a hyperbolic curve $\underset{\rightarrow}{X}_K$ of type $(1, l\text{-}\underrightarrow{\mathrm{tors}})$ (cf. Definition 7.24). For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, we write $\Pi_{\underline{v}} := \Pi_{\underset{\rightarrow}{X}_{\underline{v}}}$.

Note that $\underline{C}_K$ and $\underline{\epsilon}$ can be regarded as "*a global multiplicative subspace and a canonical generator up to* $\{\pm 1\}$", which was one of main interests in Hodge-Arakelov theory (cf. Appendix A). At first glance, they do not seem to be a global multiplicative subspace and a canonical generator up to $\{\pm 1\}$; however, by going outside the scheme theory (Recall we cannot obtain (with finitely many exceptions) a global multiplicative subspace within a scheme theory), and using mono-anabelian reconstruction algorithms, they behave as though they are a global multiplicative subspace and a canonical generator up to $\{\pm 1\}$.

From now on, we take an initial Θ-data $(\overline{F}/F, X_F, l, \underline{C}_K, \underline{\mathbb{V}}, \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}, \underline{\epsilon})$, and fix it until the end of Section 13.

§ 10.2. Model Objects.

From now on, we often use the convention (cf. [IUTchI, §0]) that, for categories $\mathcal{C}, \mathcal{D}$, we call any isomorphism class of equivalences $\mathcal{C} \to \mathcal{D}$ of categories an **isomorphism** $\mathcal{C} \to \mathcal{D}$ (Note that this termniology differs from the standard terminology of category theory).

Definition 10.2. (Local Model Objects, [IUTchI, Example 3.2, Example 3.3, Example 3.4]) For the fixed initial Θ-data, we define model objects (i.e., without "†") as follows:

(1) ($\mathcal{D}_{\underline{v}}$: holomorphic, base) We write $\mathcal{D}_{\underline{v}}$ for the category $\mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})^0$ of connected objects of the connected temperoid $\mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, the category $\mathcal{B}(\underset{\rightarrow}{X}_{\underline{v}})^0$

of connected objects of the connected anabelioid $\mathcal{B}(\underset{\rightarrow}{X}_{\underline{v}})$ for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}$, and the Aut-holomorphic orbispace $\underset{\rightarrow}{\mathbb{X}}_{\underline{v}}$ associated with $\underset{\rightarrow}{X}_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$ (cf. Section 4).

(2) ($\mathcal{D}^{\vdash}_{\underline{v}}$: mono-analytic, base) We write $\mathcal{D}^{\vdash}_{\underline{v}}$ for the category $\mathcal{B}(K_{\underline{v}})^0$ of connected objects of the connected anabelioid $\mathcal{B}(K_{\underline{v}})$ for $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, and the split monoid $(O^{\rhd}(\mathcal{C}^{\vdash}_{\underline{v}}), \text{spl}^{\vdash}_{\underline{v}})$ in Example 8.11. We also write $G_{\underline{v}} := \pi_1(\mathcal{D}^{\vdash}_{\underline{v}})$ for $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$.

(3) ($\mathcal{C}_{\underline{v}}$: holomorphic, Frobenioid-theoretic) We write $\mathcal{C}_{\underline{v}}$ for the base-field-theoretic hull $(\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\text{base-field}}$ (with base category $\mathcal{D}_{\underline{v}}$) of the tempered Frobenioid $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ in Example 8.8 for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, the $p_{\underline{v}}$-adic Frobenioid $\mathcal{C}_{\underline{v}}$ (with base category $\mathcal{D}_{\underline{v}}$) in Example 8.7 for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}$, and the Archimedean Frobenioid $\mathcal{C}_{\underline{v}}$ (whose base category has only one object $\text{Spec}\, K_{\underline{v}}$ and only one morphism) in Example 8.11 for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$.

(4) ($\underline{\underline{\mathcal{F}}}_{\underline{v}}$: holomorphic, Frobenioid-theoretic) We write $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ for the tempered Frobenioid $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ (with base category $\mathcal{D}_{\underline{v}}$) in Example 8.8 for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, the $p_{\underline{v}}$-adic Frobenioid $\mathcal{C}_{\underline{v}}$ (with base category $\mathcal{D}_{\underline{v}}$) in Example 8.7 for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}$, and the triple $(\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}})$ of the Archimedean Frobenioid, the Aut-holomorphic orbispace, and the Kummer structure $\kappa_{\underline{v}} : O^{\rhd}(\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}^{\mathcal{D}_{\underline{v}}}$ in Example 8.11 for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$.

(5) ($\mathcal{C}^{\vdash}_{\underline{v}}$: mono-analytic, Frobenioid-theoretic) We write $\mathcal{C}^{\vdash}_{\underline{v}}$ for the $p_{\underline{v}}$-adic Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}}$ (with base category $\mathcal{D}^{\vdash}_{\underline{v}}$) in Example 8.8 for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, the $p_{\underline{v}}$-adic Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}}$ (with base category $\mathcal{D}^{\vdash}_{\underline{v}}$) in Example 8.7 for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}$, and the Archimedean Frobenioid $\mathcal{C}_{\underline{v}}$ (whose base category has only one object $\text{Spec}\, K_{\underline{v}}$ and only one morphism) in Example 8.11 for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$.

(6) ($\mathcal{F}^{\vdash}_{\underline{v}}$: mono-analytic, Frobenioid-theoretic) We write $\mathcal{F}^{\vdash}_{\underline{v}}$ for the μ_{2l}-split pre-Frobenioid $(\mathcal{C}^{\vdash}_{\underline{v}}, \text{spl}^{\vdash}_{\underline{v}})$ (with base category $\mathcal{D}^{\vdash}_{\underline{v}}$) in Example 8.8 for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, the split pre-Frobenioid $(\mathcal{C}^{\vdash}_{\underline{v}}, \text{spl}^{\vdash}_{\underline{v}})$ (with base category $\mathcal{D}^{\vdash}_{\underline{v}}$) in Example 8.7 for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}$, and the triple $(\mathcal{C}^{\vdash}_{\underline{v}}, \mathcal{D}^{\vdash}_{\underline{v}}, \text{spl}^{\vdash}_{\underline{v}})$, where $(\mathcal{C}^{\vdash}_{\underline{v}}, \text{spl}^{\vdash}_{\underline{v}})$ is the split Archimedean Frobenioid, and $\mathcal{D}^{\vdash}_{\underline{v}} = (O^{\rhd}(\mathcal{C}^{\vdash}_{\underline{v}}), \text{spl}^{\vdash}_{\underline{v}})$ is the split monoid (as above) in Example 8.11 for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$.

cf. the following table (We use $\mathcal{D}_{\underline{v}}$'s (resp. $\mathcal{D}^{\vdash}_{\underline{v}}$'s, resp. $\mathcal{F}^{\vdash}_{\underline{v}}$'s) with $\underline{v} \in \underline{\mathbb{V}}$ for $\mathcal{D}$-prime-strips (resp. $\mathcal{D}^{\vdash}$-prime-strips, $\mathcal{F}^{\vdash}$-prime-strips) later (cf. Definition 10.9 (1) (2)). However, we use $\mathcal{C}_{\underline{v}}$ (*not* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$) with $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$ and $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ with $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$ for $\mathcal{F}$-prime-strips (cf. Definition 10.9 (3)), and $\underline{\underline{\mathcal{F}}}_{\underline{v}}$'s with $\underline{v} \in \underline{\mathbb{V}}$ for Θ-Hodge theatres later (cf. Definition 10.7)):

	$\underline{\mathbb{V}}^{\mathrm{bad}}$ (Example 8.8)	$\underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$ (Example 8.7)	$\underline{\mathbb{V}}^{\mathrm{arc}}$ (Example 8.11)
$\mathcal{D}_{\underline{v}}$	$\mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})^0 \quad (\Pi_{\underline{v}})$	$\mathcal{B}(\underset{\to}{X}_{\underline{v}})^0 \quad (\Pi_{\underline{v}})$	$\underset{\to}{\mathbb{X}}_{\underline{v}}$
$\mathcal{D}^{\vdash}_{\underline{v}}$	$\mathcal{B}(K_{\underline{v}})^0 \quad (G_{\underline{v}})$	$\mathcal{B}(K_{\underline{v}})^0 \quad (G_{\underline{v}})$	$(O^{\rhd}(\mathcal{C}^{\vdash}_{\underline{v}}), \mathrm{spl}^{\vdash}_{\underline{v}})$
$\mathcal{C}_{\underline{v}}$	$(\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\text{base-field}} \quad (\Pi_{\underline{v}} \curvearrowright (O^{\rhd}_{\overline{F}_{\underline{v}}})^{\mathrm{pf}})$	$\Pi_{\underline{v}} \curvearrowright (O^{\rhd}_{\overline{F}_{\underline{v}}})^{\mathrm{pf}}$	Arch. Fr'd $\mathcal{C}_{\underline{v}}$ ($\leftsquigarrow$ang. region)
$\underline{\underline{\mathcal{F}}}_{\underline{v}}$	temp. Fr'd $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ ($\leftsquigarrow\Theta$-fct.)	equal to $\mathcal{C}_v$	$(\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}})$
$\mathcal{C}^{\vdash}_{\underline{v}}$	$G_{\underline{v}} \curvearrowright O^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$	$G_{\underline{v}} \curvearrowright O^{\times}_{\overline{F}_{\underline{v}}} \cdot p^{\mathbb{N}}_{\underline{v}}$	equal to $\mathcal{C}_{\underline{v}}$
$\mathcal{F}^{\vdash}_{\underline{v}}$	$(\mathcal{C}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}})$	$(\mathcal{C}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}})$	$(\mathcal{C}^{\vdash}_{\underline{v}}, \mathcal{D}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}})$

We continue to define model objects.

Definition 10.3. (Model Global Objects, [IUTchI, Definition 4.1 (v), Definition 6.1 (v)]) We put

$$\mathcal{D}^{\circledcirc} := \mathcal{B}(\underline{C}_K)^0, \quad \mathcal{D}^{\circledcirc\pm} := \mathcal{B}(\underline{X}_K)^0.$$

Isomorphs of the global objects will be used in Proposition 10.19 and Proposition 10.33 to write "labels" on each local objects in a consistent manner (cf. also Remark 6.11.1). We will use $\mathcal{D}^{\circledcirc}$ for ($\mathcal{D}$-)$\boxtimes$-Hodge theatre (Section 10.4), and $\mathcal{D}^{\circledcirc\pm}$ for ($\mathcal{D}$-)$\boxplus$-Hodge theatre (Section 10.5).

Definition 10.4. (Model Global Realified Frobenioid with Localisations, [IUTchI, Example 3.5]) Let $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ be the global realified Frobenioid in Example 8.12. Note that we have the natural bijection $\mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$, and an element $\log^{\vdash}_{\mathrm{mod}}(p_v) \in \Phi_{\mathcal{C}^{\Vdash}_{\underline{v}}, v}$ for each $v \in \mathbb{V}_{\mathrm{mod}}$. For $v \in \mathbb{V}_{\mathrm{mod}}$, we write $\underline{v} \in \underline{\mathbb{V}}$ for the corresponding element under the bijection $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$. For each $\underline{v} \in \underline{\mathbb{V}}$, we also have the (pre-)Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}}$ (cf. Definition 10.2 (5)). We write $\mathcal{C}^{\vdash\mathbb{R}}_{\underline{v}}$ for the realification of $\mathcal{C}^{\vdash}_{\underline{v}}$ (Definition 8.4 (3)) for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, and $\mathcal{C}_{\underline{v}}$ itself for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. We write $\log_{\Phi}(p_{\underline{v}}) \in \Phi^{\mathbb{R}}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ for the element determined by $p_{\underline{v}}$, where we write $\Phi^{\mathbb{R}}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ for the divisor monoid of $\mathcal{C}^{\vdash\mathbb{R}}_{\underline{v}}$. We have the natural restriction functor

$$\mathcal{C}^{\Vdash}_{\mathrm{mod}} \to \mathcal{C}^{\vdash\mathbb{R}}_{\underline{v}}$$

for each $\underline{v} \in \underline{\mathbb{V}}$. This is determined, up to isomorphism, by the isomorphism

$$\rho_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v} \overset{\text{gl. to loc.}}{\xrightarrow{\sim}} \Phi^{\mathbb{R}}_{\mathcal{C}^{\vdash}_{\underline{v}}} \qquad \log^{\vdash}_{\mathrm{mod}}(p_v) \mapsto \frac{1}{[K_{\underline{v}} : (F_{\mathrm{mod}})_v]} \log_{\Phi}(p_{\underline{v}})$$

of topological monoids (For the assignment, consider the volume interpretations of the arithmetic divisors, i.e., $\log_{p_v} \#(O_{(F_{\mathrm{mod}})_v}/p_v) = \frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]} \log_{p_{\underline{v}}} \#(O_{K_{\underline{v}}}/p_{\underline{v}})$). Recall also the point of view of regarding $\underline{\mathbb{V}}(\subset \mathbb{V}(K))$ as an "*analytic section*" of $\operatorname{Spec} O_K \twoheadrightarrow \operatorname{Spec} O_{F_{\mathrm{mod}}}$ (The left-hand side $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v}$ is an object on $(F_{\mathrm{mod}})_v$, and the right-hand side $\Phi^{\mathbb{R}}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ is an object on $K_{\underline{v}}$). We write $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ for the quadruple

$$\mathfrak{F}^{\Vdash}_{\mathrm{mod}} := (\mathcal{C}^{\Vdash}_{\mathrm{mod}}, \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \{\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

of the global realified Frobenioid, the bijection of primes, the model objects $\mathcal{F}^{\vdash}_{\underline{v}}$'s in Definition 10.2 (6), and the localisation homomorphisms. We define an isomorphism $\mathfrak{F}^{\Vdash}_{\mathrm{mod},1} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}_{\mathrm{mod},2}$ of quadruples in an obvious manner.

Isomorphs of the global realified Frobenioids are used to consider log-volume functions.

Definition 10.5. (Θ-version, [IUTchI, Example 3.2 (v), Example 3.3 (ii), Example 3.4 (iii), Example 3.5 (ii)])

(1) ($\underline{\mathbb{V}}^{\mathrm{bad}}$) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. We write $\mathcal{D}^{\Theta}_{\underline{v}}$ $(\subset \mathcal{D}_{\underline{v}})$ for the category whose objects are $A^{\Theta} := A \times \underline{\underline{\ddot{Y}}}_{\underline{v}}$ for $A \in \mathrm{Ob}(\mathcal{D}^{\vdash}_{\underline{v}})$, where $\times$ is the product in $\mathcal{D}_{\underline{v}}$, and morphisms are morphisms over $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ in $\mathcal{D}_{\underline{v}}$ (Note also that $\underline{\underline{\ddot{Y}}}_{\underline{v}} \in \mathrm{Ob}(\mathcal{D}_{\underline{v}})$ is defined over $K_{\underline{v}}$). Taking "$(-) \times \underline{\underline{\ddot{Y}}}_{\underline{v}}$" induces an equivalenc $\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{D}^{\Theta}_{\underline{v}}$ of categories. The assignment

$$\mathrm{Ob}(\mathcal{D}^{\Theta}_{\underline{v}}) \ni A^{\Theta} \mapsto O^{\times}(\mathcal{O}_{A^{\Theta}}) \cdot (\underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}|_{\mathcal{O}_{A^{\Theta}}}) \subset O^{\times}(\mathcal{O}^{\mathrm{birat}}_{A^{\Theta}})$$

determines a monoid $O^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ on $\mathcal{D}^{\Theta}_{\underline{v}}$ (cf. Example 8.8 for $\underline{\underline{\Theta}}_{\underline{v}} \in O^{\times}(\mathcal{O}^{\mathrm{birat}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$, and $\mathcal{O}_{(-)}$ for Definition 8.4 (1)). Under the above equivalence $\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{D}^{\Theta}_{\underline{v}}$ of categories, we have natural isomorphism $O^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \xrightarrow{\sim} O^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$. These are compatible with the assignment

$$\underline{\underline{q}}_{\underline{v}}|_{\mathcal{O}_A} \mapsto \underline{\underline{\Theta}}_{\underline{v}}|_{\mathcal{O}_{A^{\Theta}}}$$

and a natural isomorphism $O^{\times}(\mathcal{O}_A) \xrightarrow{\sim} O^{\times}(\mathcal{O}_{A^{\Theta}})$ induced by the projection $A^{\Theta} = A \times \underline{\underline{\ddot{Y}}}_{\underline{v}} \to A$ (cf. Example 8.8 for $\underline{\underline{q}}_{\underline{v}} \in O^{\rhd}(\mathcal{O}_{\underline{\underline{X}}_{\underline{v}}})$). Hence the monoid $O^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ determines a $p_{\underline{v}}$-adic Frobenioid

$$\mathcal{C}^{\Theta}_{\underline{v}} \ (\subset \underline{\underline{\mathcal{F}}}^{\mathrm{birat}}_{\underline{v}})$$

whose base category is $\mathcal{D}^{\Theta}_{\underline{v}}$. Note also $\underline{\underline{\Theta}}_{\underline{v}}$ determines a $\mu_{2l}(-)$-orbit of splittings $\mathrm{spl}^{\Theta}_{\underline{v}}$ of $\mathcal{C}^{\Theta}_{\underline{v}}$. We have a natural equivalence $\mathcal{C}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{C}^{\Theta}_{\underline{v}}$ of categories, which sends $\mathrm{spl}^{\vdash}_{\underline{v}}$ to $\mathrm{spl}^{\Theta}_{\underline{v}}$, hence we have an isomorphism

$$\mathcal{F}^{\vdash}_{\underline{v}}(=(\mathcal{C}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}})) \xrightarrow{\sim} \mathcal{F}^{\Theta}_{\underline{v}} := (\mathcal{C}^{\Theta}_{\underline{v}}, \mathrm{spl}^{\Theta}_{\underline{v}})$$

of μ_{2l}-split pre-Frobenioids.

(2) ($\underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$. Recall that the divisor monoid of $\mathcal{C}^{\vdash}_{\underline{v}}$ is of the form $O^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) = O^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \times \mathbb{N}\log(p_{\underline{v}})$, where we write $\log(p_{\underline{v}})$ for the element $p_{\underline{v}}$ considered additively. We put

$$O^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) := O^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) \times \mathbb{N}\log(p_{\underline{v}})\log(\underline{\underline{\Theta}}),$$

where $\log(p_{\underline{v}})\log(\underline{\underline{\Theta}})$ is just a formal symbol. We have a natural isomorphism $O^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \xrightarrow{\sim} O^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$. Then the monoid $O^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ determines a $p_{\underline{v}}$-adic Frobenioid

$$\mathcal{C}^{\Theta}_{\underline{v}}$$

whose base category is $\mathcal{D}^{\Theta}_{\underline{v}} := \mathcal{D}^{\vdash}_{\underline{v}}$. Note also that $\log(p_{\underline{v}})\log(\underline{\underline{\Theta}})$ determines a splitting $\mathrm{spl}^{\Theta}_{\underline{v}}$ of $\mathcal{C}^{\Theta}_{\underline{v}}$. We have a natural equivalence $\mathcal{C}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{C}^{\Theta}_{\underline{v}}$ of categories, which sends $\mathrm{spl}^{\vdash}_{\underline{v}}$ to $\mathrm{spl}^{\Theta}_{\underline{v}}$, hence we have an isomorphism

$$\mathcal{F}^{\vdash}_{\underline{v}}(=(\mathcal{C}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}})) \xrightarrow{\sim} \mathcal{F}^{\Theta}_{\underline{v}} := (\mathcal{C}^{\Theta}_{\underline{v}}, \mathrm{spl}^{\Theta}_{\underline{v}})$$

of split pre-Frobenioids.

(3) ($\underline{\mathbb{V}}^{\mathrm{arc}}$) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Recall that the image $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ of $\mathrm{spl}^{\vdash}_{\underline{v}}$ of the split monoid $(O^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}, \mathrm{spl}^{\vdash}_{\underline{v}})$ is isomorphic to $\mathbb{R}_{\geq 0}$. We write $\log(p_{\underline{v}}) \in \Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ for the element $p_{\underline{v}}$ considered additively (cf. Section 0.2 for $p_{\underline{v}}$ with Archimedean $\underline{v}$). We put

$$\Phi_{\mathcal{C}^{\Theta}_{\underline{v}}} := \mathbb{R}_{\geq 0}\log(p_{\underline{v}})\log(\underline{\underline{\Theta}}),$$

where $\log(p_{\underline{v}})\log(\underline{\underline{\Theta}})$ is just a formal symbol. We also write $O^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}} := (O^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}})^{\times}$, and $O^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}} := O^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$. Then we obtain a split pre-Frobenioid

$$(\mathcal{C}^{\Theta}_{\underline{v}}, \mathrm{spl}^{\Theta}_{\underline{v}}),$$

such that $O^{\rhd}(\mathcal{C}^{\Theta}_{\underline{v}}) = O^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}} \times \Phi_{\mathcal{C}^{\Theta}_{\underline{v}}}$. We have a natural equivalence $\mathcal{C}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{C}^{\Theta}_{\underline{v}}$ of categories, which sends $\mathrm{spl}^{\vdash}_{\underline{v}}$ to $\mathrm{spl}^{\Theta}_{\underline{v}}$, hence we have an isomorphism $(\mathcal{C}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}}) \xrightarrow{\sim} (\mathcal{C}^{\Theta}_{\underline{v}}, \mathrm{spl}^{\Theta}_{\underline{v}})$ of split pre-Frobenioids, and an isomorphism

$$\mathcal{F}^{\vdash}_{\underline{v}}(=(\mathcal{C}^{\vdash}_{\underline{v}}, \mathcal{D}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}})) \xrightarrow{\sim} \mathcal{F}^{\Theta}_{\underline{v}} := (\mathcal{C}^{\Theta}_{\underline{v}}, \mathcal{D}^{\Theta}_{\underline{v}}, \mathrm{spl}^{\Theta}_{\underline{v}})$$

of triples, where we write $\mathcal{D}^{\Theta}_{\underline{v}} := \mathcal{D}^{\vdash}_{\underline{v}}$.

(4) (Global Realified with Localisations) Let $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ be the global realified Frobenioid considered in Definition 10.4. For each $v \in \mathbb{V}_{\mathrm{mod}}$, we write $\underline{v}$ for the corresponding element under the bijection $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$. Write

$$\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{theta}}} := \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}} \cdot \log(\underline{\underline{\Theta}}),$$

where $\log(\underline{\underline{\Theta}})$ is just a formal symbol. This monoid $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{theta}}}$ determines a global realified Frobenioid

$$\mathcal{C}^{\Vdash}_{\mathrm{theta}}$$

with a natural equivalence $\mathcal{C}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathrm{theta}}$ of categories and a natural bijection $\mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{theta}}) \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$. For each $v \in \mathbb{V}_{\mathrm{mod}}$, the element $\log^{\vdash}_{\mathrm{mod}}(p_v) \in \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v} \subset \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}}$ determines an element $\log^{\vdash}_{\mathrm{mod}}(p_v)\log(\underline{\underline{\Theta}}) \in \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{theta}},v} \subset \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{theta}}}$. As in the case where $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$, We have the natural restriction functor

$$\mathcal{C}^{\Vdash}_{\mathrm{theta}} \to \mathcal{C}^{\Theta\mathbb{R}}_{\underline{v}}$$

for each $\underline{v} \in \underline{\mathbb{V}}$. This is determined, up to isomorphism, by the isomorphism

$$\rho^{\Theta}_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{theta}},v} \overset{\text{gl. to loc.}}{\xrightarrow{\sim}} \Phi^{\mathbb{R}}_{\mathcal{C}^{\Theta}_{\underline{v}}} \quad \log^{\vdash}_{\mathrm{mod}}(p_v)\log(\underline{\underline{\Theta}}) \mapsto \begin{cases} \frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\log_{\Phi}(p_{\underline{v}})\log(\underline{\underline{\Theta}}) & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}, \\ \frac{\log_{\Phi}(p_{\underline{v}})}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\frac{\log_{\Phi}(\underline{\underline{\Theta}}_{\underline{v}})}{\log_{\Phi}(\underline{\underline{q}}_{\underline{v}})} & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}} \end{cases}$$

of topological monoids, where we write $\log_{\Phi}(p_{\underline{v}})\log(\underline{\underline{\Theta}}) \in \Phi^{\mathbb{R}}_{\mathcal{C}^{\Theta}_{\underline{v}}}$ for the element determined by $\log_{\Phi}(p_{\underline{v}})$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, and $\log_{\Phi}(\underline{\underline{\Theta}}_{\underline{v}})$, $\log_{\Phi}(p_{\underline{v}})$, and we write $\log_{\Phi}(\underline{\underline{q}}_{\underline{v}})$ for the element determined by $\underline{\underline{\Theta}}_{\underline{v}}$, $p_{\underline{v}}$, and $\underline{\underline{q}}_{\underline{v}}$ respectively for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ (Note that $\log_{\Phi}(\underline{\underline{\Theta}}_{\underline{v}})$ is *not* a formal symbol). Note that for any $\underline{v} \in \underline{\mathbb{V}}$, the localisation homomorphisms $\rho_{\underline{v}}$ and $\rho^{\Theta}_{\underline{v}}$ are compatible with the natural equivalences $\mathcal{C}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathrm{theta}}$, and $\mathcal{C}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{C}^{\Theta}_{\underline{v}}$:

$$\begin{tikzcd}
\log^{\vdash}_{\mathrm{mod}}(p_v) \arrow[r, mapsto, "\text{``mod}\to\text{theta''}"] \arrow[d, mapsto, "\rho_{\underline{v}}"'] & \log^{\vdash}_{\mathrm{mod}}(p_v)\log(\underline{\underline{\Theta}}) \arrow[d, mapsto, "\rho^{\Theta}_{\underline{v}}"] \\
\frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\log_{\Phi}(p_{\underline{v}}) \arrow[r, mapsto, "\text{``}\vdash\to\Theta\text{''}"'] & \frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\log_{\Phi}(p_{\underline{v}})\log(\underline{\underline{\Theta}})
\end{tikzcd}$$

for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, and

$$\begin{tikzcd}
\log^{\vdash}_{\mathrm{mod}}(p_v) \arrow[r, mapsto, "\text{``mod}\to\text{theta''}"] \arrow[d, mapsto, "\rho_{\underline{v}}"'] & \log^{\vdash}_{\mathrm{mod}}(p_v)\log(\underline{\underline{\Theta}}) \arrow[d, mapsto, "\rho^{\Theta}_{\underline{v}}"] \\
\frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\log_{\Phi}(p_{\underline{v}}) \arrow[r, mapsto, "\text{``}\vdash\to\Theta\text{''}"'] & \frac{\log_{\Phi}(p_{\underline{v}})}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\frac{\log_{\Phi}(\underline{\underline{\Theta}}_{\underline{v}})}{\log_{\Phi}(\underline{\underline{q}}_{\underline{v}})}
\end{tikzcd}$$

for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. We write $\mathfrak{F}^{\Vdash}_{\mathrm{theta}}$ for the quadruple

$$\mathfrak{F}^{\Vdash}_{\mathrm{theta}} := (\mathcal{C}^{\Vdash}_{\mathrm{theta}}, \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{theta}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \{\mathcal{F}^{\Theta}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{\rho^{\Theta}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

of the global realified Frobenioid, the bijection of primes, the Θ-version of model objects $\mathcal{F}^{\Theta}_{\underline{v}}$'s in (1), (2), and (3), and the localisation homomorphisms.

Note that we have group-theoretic or category-theoretic reconstruction algorithms such as reconstructing $\mathcal{D}^{\vdash}_{\underline{v}}$ from $\mathcal{D}_{\underline{v}}$. We summarise these as follows ([IUTchI, Example 3.2 (vi), Example 3.3 (iii)]):

$$\begin{tikzcd}
\underline{\underline{\mathcal{F}}}_{\underline{v}} \arrow[r, mapsto] \arrow[dd, mapsto, bend right=60, "\substack{\text{up to } l\underline{\mathbb{Z}}\text{-indet.} \\ \text{on } \underline{\underline{\Theta}}_{\underline{v}} \text{ for } \underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}}"'] & \mathcal{C}_{\underline{v}} \arrow[r, mapsto, "\text{except}", "\underline{\mathbb{V}}^{\mathrm{arc}}"'] \arrow[dl, mapsto] & \mathcal{D}_{\underline{v}} \arrow[d, mapsto] \arrow[dd, mapsto, bend left=60] \\
\mathcal{F}^{\vdash}_{\underline{v}} \arrow[r, mapsto] & \mathcal{C}^{\vdash}_{\underline{v}} \arrow[r, mapsto] & \mathcal{D}^{\vdash}_{\underline{v}} \\
\mathcal{F}^{\Theta}_{\underline{v}} \arrow[r, mapsto] & \mathcal{C}^{\Theta}_{\underline{v}} \arrow[r, mapsto] & \mathcal{D}^{\Theta}_{\underline{v}}.
\end{tikzcd}$$

(Note also the remark given just before Theorem 8.14.)

Definition 10.6. ($\mathcal{D}$-version or "log-shell version", [IUTchI, Example 3.5 (ii), (iii)]) We write

$$\mathcal{D}^{\Vdash}_{\mathrm{mod}}$$

for a copy of $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$. Let $\Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}}}$, $\mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$, $\log^{\mathcal{D}}_{\mathrm{mod}}(p_v) \in \Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}},v} \subset \Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}}}$ be the corresponding objects under the tautological equivalence $\mathcal{C}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{D}^{\Vdash}_{\mathrm{mod}}$. For each $v \in \mathbb{V}_{\mathrm{mod}}$, we write $\underline{v}$ for the corresponding element under the bijection $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$.

For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, we can group-theoretically reconstruct from $\mathcal{D}^{\vdash}_{\underline{v}}$

$$(\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}} := \mathbb{R}_{\mathrm{non}}(G_{\underline{v}})\,(\cong \mathbb{R}_{\geq 0})$$

and Frobenius element $\mathbb{F}(G_{\underline{v}}) \in (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$ by (Step 3) in Proposition 5.2 (Recall that $G_{\underline{v}} = \pi_1(\mathcal{D}^{\vdash}_{\underline{v}})$). Write also

$$\log^{\mathcal{D}}_{\Phi}(p_{\underline{v}}) := e_{\underline{v}}\mathbb{F}(G_{\underline{v}}) \in (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}},$$

where we write $e_{\underline{v}}$ for the absolute ramification index of $K_{\underline{v}}$.

For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, we can also group-theoretically reconstruct from the split monoid $\mathcal{D}^{\vdash}_{\underline{v}} = (O^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}, \mathrm{spl}^{\vdash}_{\underline{v}})$

$$(\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}} := \mathbb{R}_{\mathrm{arc}}(\mathcal{D}^{\vdash}_{\underline{v}})\,(\cong \mathbb{R}_{\geq 0})$$

and Frobenius element $\mathbb{F}(\mathcal{D}^{\vdash}_{\underline{v}}) \in (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$ by (Step 4) in Proposition 5.4. Write also

$$\log^{\mathcal{D}}_{\Phi}(p_{\underline{v}}) := \frac{\mathbb{F}(\mathcal{D}^{\vdash}_{\underline{v}})}{2\pi} \in (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}},$$

where $2\pi \in \mathbb{R}^{\times}$ is the length of the perimeter of the unit circle (Note that $(\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$ has a natural $\mathbb{R}^{\times}$-module structure).

Hence for any $\underline{v} \in \underline{\mathbb{V}}$, we obtain a uniquely determined isomorphism

$$\rho^{\mathcal{D}}_{\underline{v}} : \Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}}, v} \overset{\text{gl. to loc.}}{\overset{\sim}{\longrightarrow}} (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}} \quad \log^{\mathcal{D}}_{\mathrm{mod}}(p_v) \mapsto \frac{1}{[K_{\underline{v}} : (F_{\mathrm{mod}})_v]} \log^{\mathcal{D}}_{\Phi}(p_{\underline{v}})$$

of topological monoids.

We write $\mathfrak{F}^{\Vdash}_{\mathcal{D}}$ for the quadruple

$$\mathfrak{F}^{\Vdash}_{\mathcal{D}} := (\mathcal{D}^{\Vdash}_{\mathrm{mod}}, \mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{mod}}) \overset{\sim}{\to} \underline{\mathbb{V}}, \{\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}, \{\rho^{\mathcal{D}}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}})$$

of the global realified Frobenioid, the bijection of primes, the $\mathcal{D}^{\vdash}$-version of model objects $\mathcal{D}^{\vdash}_{\underline{v}}$'s, and the localisation homomorphisms.

§ 10.3. Θ-Hodge Theatres and Prime-strips.

Definition 10.7. (Θ-Hodge theatre, [IUTchI, Definition 3.6]) A **Θ-Hodge theatre** is a collection

$$^{\dagger}\mathcal{HT}^{\Theta} = (\{^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}),$$

where

(1) (local object) $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is a pre-Frobenioid (resp. a triple $(^{\dagger}\mathcal{C}_{\underline{v}}, {}^{\dagger}\mathcal{D}_{\underline{v}}, {}^{\dagger}\kappa_{\underline{v}})$) isomorphic to the model $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ (resp. isomorphic to the model triple $\underline{\underline{\mathcal{F}}}_{\underline{v}} = (\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}})$) in Definition 10.2 (4) for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ (resp. for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$). We write $^{\dagger}\mathcal{D}_{\underline{v}}$, $^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$, $^{\dagger}\mathcal{D}^{\Theta}_{\underline{v}}$, $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$, $^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}$ (resp. $^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$, $^{\dagger}\mathcal{D}^{\Theta}_{\underline{v}}$, $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$,$^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}$) for the objects algorithmically reconstructed from $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ corresponding to the model objects (i.e., the objects without †).

(2) (global realified object with localisations) $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ is a quadruple

$$(^{\dagger}\mathcal{C}^{\vdash}_{\mathrm{mod}}, \mathrm{Prime}(^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \overset{\sim}{\to} \underline{\mathbb{V}}, \{^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}, \{^{\dagger}\rho_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}),$$

where $^{\dagger}\mathcal{C}^{\vdash}_{\mathrm{mod}}$ is a category equivalent to the model $\mathcal{C}^{\vdash}_{\mathrm{mod}}$ in Definition 10.4, $\mathrm{Prime}(^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \overset{\sim}{\to} \underline{\mathbb{V}}$ is a bijection of sets, $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$ is the reconstructed object from the above local data $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$, and $^{\dagger}\rho_v : \Phi_{^{\dagger}\mathcal{C}^{\Vdash}_{\underline{v}}, v} \overset{\text{gl. to loc.}}{\overset{\sim}{\longrightarrow}} \Phi^{\mathbb{R}}_{^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}}$ is an isomorphism of topological monoids (Here $^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}$ is the reconstructed object from the above local data $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$), such that there exists an isomorphism of quadruples $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \overset{\sim}{\to} \mathfrak{F}^{\Vdash}_{\mathrm{mod}}$. We write $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{theta}}$, $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathcal{D}}$ for the algorithmically reconstructed object from $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ corresponding to the model objects (i.e., the objects without †).

Definition 10.8. (Θ-link, [IUTchI, Corollary 3.7 (i)]) Let ${}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$, ${}^{\ddagger}\mathcal{HT}^{\Theta} = (\{{}^{\ddagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$ be Θ-Hodge theatres (with respect to the fixed initial Θ-data). We shall refer to the full poly-isomorphism (cf. Section 0.2)

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{theta}} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} $$

as the **Θ-link** from ${}^{\dagger}\mathcal{HT}$ to ${}^{\ddagger}\mathcal{HT}$ (Note that the full poly-isomorphism is non-empty), and we write it as

$$ {}^{\dagger}\mathcal{HT}^{\Theta} \quad \xrightarrow{\Theta} \quad {}^{\ddagger}\mathcal{HT}^{\Theta}, $$

and we shall refer to this diagram as the **Frobenius-picture of Θ-Hodge theatres** ([IUTchI, Corollary 3.8]). Note that the essential meaning of the above link is

$$ \text{“ } \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}} \xrightarrow{\sim} \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}} \text{ ”} $$

for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$.

Remark 10.8.1. ([IUTchI, Corollary 3.7 (ii), (iii)])

(1) (Preservation of $\mathcal{D}^{\vdash}$) For each $\underline{v} \in \underline{\mathbb{V}}$, we have a natural composite full poly-isomorphism

$$ {}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} {}^{\dagger}\mathcal{D}^{\Theta}_{\underline{v}} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}, $$

where the first isomorphism is the natural one (Recall that it is tautological for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, and that it is induced by $(-)\times\underline{\underline{\ddot{Y}}}_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$), and the second full poly-isomorphism is the full poly-isomorphism of the Θ-link. Hence *the mono-analytic base “$\mathcal{D}^{\vdash}_{\underline{v}}$” is preserved (or “shared”) under the* Θ*-link* (i.e., $\mathcal{D}^{\vdash}_{\underline{v}}$ *is horizontally coric*). Note that the holomorphic base “$\mathcal{D}_{\underline{v}}$” is *not* shared under the Θ-link (i.e., Θ-link shares the underlying mono-analytic base structures, but *not* the arithmetically holomorphic base structures).

(2) (Preservation of $O^{\times}$) For each $\underline{v} \in \underline{\mathbb{V}}$, we have a natural composite full poly-isomorphism

$$ O^{\times}_{{}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} O^{\times}_{{}^{\dagger}\mathcal{C}^{\Theta}_{\underline{v}}} \overset{\text{full poly}}{\xrightarrow{\sim}} O^{\times}_{{}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}}, $$

where the first isomorphism is the natural one (Recall that it is tautological for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, and that it is induced by $(-) \times \underline{\underline{\ddot{Y}}}_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$), and the second full poly-isomorphism is induced by the full poly-isomorphism of the Θ-link. Hence *“$O^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$” is preserved (or “shared”) under the* Θ*-link* (i.e., $O^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ *is horizontally coric*). Note also that the value group portion is *not* shared under the Θ-link.

We can visualise the "shared" and "non-shared" relation as follows:

$$\boxed{{}^{\dagger}\mathcal{D}_{\underline{v}}} - - > \boxed{\left({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \curvearrowright O^{\times}_{{}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}}\right) \cong \left({}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}} \curvearrowright O^{\times}_{{}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}}\right)} < - - \boxed{{}^{\ddagger}\mathcal{D}_{\underline{v}}}$$

We shall refer to this diagram as the **étale-picture of Θ-Hodge theatres** ([IUTchI, Corollary 3.9]). Note that, *there is the notion of the order in the Frobenius-picture* (i.e., ${}^{\dagger}(-)$ is on the left, and ${}^{\ddagger}(-)$ is on the right), on the other hand, there is no such an order and *it has a permutation symmetry in the étale-picture* (cf. also the last table in Section 4.3).

This Θ-link is the primitive one. We will update the Θ-link to $\Theta^{\times\mu}$-link, $\Theta^{\times\mu}_{\mathrm{gau}}$-link (cf. Corollary 11.24), and $\Theta^{\times\mu}_{\mathrm{LGP}}$-link (resp. $\Theta^{\times\mu}_{\mathfrak{lgp}}$-link) (cf. Definition 13.9 (2)) in inter-universal Teichmüller theory :

$$\Theta\text{-link} \underset{\substack{\text{"theta fct.}\mapsto\text{theta values"} \\ \text{and } O^{\times}\mapsto O^{\times}/\mu}}{\overset{\text{"Hodge-Arakelov-theoretic eval."}}{\rightsquigarrow}} \Theta^{\times\mu}_{\mathrm{gau}}\text{-link} \overset{\text{" log -link"}}{\rightsquigarrow} \Theta^{\times\mu}_{\mathrm{LGP}}\text{-link (resp. } \Theta^{\times\mu}_{\mathfrak{lgp}}\text{-link).}$$

Definition 10.9. ([IUTchI, Definition 4.1 (i), (iii), (iv) Definition 5.2 (i), (ii), (iii), (iv)])

(1) ($\mathcal{D}$: holomorphic, base) A **holomorphic base-prime-strip**, or **$\mathcal{D}$-prime-strip** is a collection

$${}^{\dagger}\mathfrak{D} = \{{}^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

of data such that ${}^{\dagger}\mathcal{D}_{\underline{v}}$ is a category equivalent to the model $\mathcal{D}_{\underline{v}}$ in Definition 10.2 (1) for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, and ${}^{\dagger}\mathcal{D}_{\underline{v}}$ is an Aut-holomorphic orbispace isomorphic to the model $\mathcal{D}_{\underline{v}}$ in Definition 10.2 (1). A **morphism of $\mathcal{D}$-prime-strips** is a collection of morphisms indexed by $\underline{\mathbb{V}}$ between each component.

(2) ($\mathcal{D}^{\vdash}$: mono-analytic, base) A **mono-analytic base-prime-strip**, or **$\mathcal{D}^{\vdash}$-prime-strip** is a collection

$${}^{\dagger}\mathfrak{D}^{\vdash} = \{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

of data such that ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$ is a category equivalent to the model $\mathcal{D}^{\vdash}_{\underline{v}}$ in Definition 10.2 (2) for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, and ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$ is a split monoid isomorphic to the model $\mathcal{D}^{\vdash}_{\underline{v}}$ in Definition 10.2 (2). A **morphism of $\mathcal{D}^{\vdash}$-prime-strips** is a collection of morphisms indexed by $\underline{\mathbb{V}}$ between each component.

(3) ($\mathcal{F}$: holomorphic, Frobenioid-theoretic) A **holomorphic Frobenioid-prime-strip**, or **$\mathcal{F}$-prime-strip** is a collection

$$^{\dagger}\mathfrak{F} = \{^{\dagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

of data such that $^{\dagger}\mathcal{F}_{\underline{v}}$ is a pre-Frobenioid isomorphic to the model $\mathcal{C}_{\underline{v}}$ (*not* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$) in Definition 10.2 (3) for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, and $^{\dagger}\mathcal{F}_{\underline{v}} = (^{\dagger}\mathcal{C}_{\underline{v}}, {}^{\dagger}\mathcal{D}_{\underline{v}}, {}^{\dagger}\kappa_{\underline{v}})$ is a triple of a category, an Aut-holomorphic orbispace, and a Kummer structure, which is isomorphic to the model $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ in Definition 10.2 (3). An **isomorphism of $\mathcal{F}$-prime-strips** is a collection of isomorphisms indexed by $\underline{\mathbb{V}}$ between each component.

(4) ($\mathcal{F}^{\vdash}$: mono-analytic, Frobenioid-theoretic) A **mono-analytic Frobenioid-prime-strip**, or **$\mathcal{F}^{\vdash}$-prime-strip** is a collection

$$^{\dagger}\mathfrak{F}^{\vdash} = \{^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

of data such that $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$ is a μ_{2l}-split pre-Frobenioid (resp. split pre-Frobenioid) isomorphic to the model $\mathcal{F}^{\vdash}_{\underline{v}}$ in Definition 10.2 (6) for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$), and $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}} = (^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}, {}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}, {}^{\dagger}\mathrm{spl}^{\vdash}_{\underline{v}})$ is a triple of a category, a split monoid, and a splitting of $^{\dagger}\mathcal{C}_{\underline{v}}$, which is isomorphic to the model $\mathcal{F}^{\vdash}_{\underline{v}}$ in Definition 10.2 (6). An **isomorphism of $\mathcal{F}^{\vdash}$-prime-strips** is a collection of isomorphisms indexed by $\underline{\mathbb{V}}$ between each component.

(5) ($\mathcal{F}^{\Vdash}$: global realified with localisations) A **global realified mono-analytic Frobenioid-prime-strip**, or **$\mathcal{F}^{\Vdash}$-prime-strip** is a quadruple

$$^{\dagger}\mathfrak{F}^{\Vdash} = (^{\dagger}\mathcal{C}^{\Vdash}, \mathrm{Prime}(^{\dagger}\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^{\dagger}\mathfrak{F}^{\vdash}, \{^{\dagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}),$$

where $^{\dagger}\mathcal{C}^{\Vdash}$ is a pre-Frobenioid isomorphic to the model $\mathcal{C}^{\vdash}_{\mathrm{mod}}$ in Definition 10.4, $\mathrm{Prime}(^{\dagger}\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}$ is a bijection of sets, $^{\dagger}\mathfrak{F}^{\vdash}$ is an $\mathcal{F}^{\vdash}$-prime-strip, and $^{\dagger}\rho_{\underline{v}} : \Phi_{^{\dagger}\mathcal{C}^{\Vdash},v} \xrightarrow[\sim]{\text{gl. to loc.}} \Phi^{\mathbb{R}}_{^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}}$ is an isomorphism of topological monoids (Here, $^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}$ is the object reconstructed from $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$), such that the quadruple $^{\dagger}\mathfrak{F}^{\Vdash}$ is isomorphic to the model $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ in Definition 10.4. An **isomorphism of $\mathcal{F}^{\Vdash}$-prime-strips** is an isomorphism of quadruples.

(6) Let $\mathrm{Aut}_{\mathcal{D}}(-)$, $\mathrm{Isom}_{\mathcal{D}}(-,-)$ (resp. $\mathrm{Aut}_{\mathcal{D}^{\vdash}}(-)$, $\mathrm{Isom}_{\mathcal{D}^{\vdash}}(-,-)$ resp. $\mathrm{Aut}_{\mathcal{F}}(-)$, $\mathrm{Isom}_{\mathcal{F}}(-,-)$ resp. $\mathrm{Aut}_{\mathcal{F}^{\vdash}}(-)$, $\mathrm{Isom}_{\mathcal{F}^{\vdash}}(-,-)$ resp. $\mathrm{Aut}_{\mathcal{F}^{\Vdash}}(-)$, $\mathrm{Isom}_{\mathcal{F}^{\Vdash}}(-,-)$) be the group of automorphisms of a $\mathcal{D}$-(resp. $\mathcal{D}^{\vdash}$-, resp. $\mathcal{F}$-, resp. $\mathcal{F}^{\vdash}$-, resp. $\mathcal{F}^{\Vdash}$-)prime-strip, and the set of isomorphisms between $\mathcal{D}$-(resp. $\mathcal{D}^{\vdash}$-, resp. $\mathcal{F}$-, resp. $\mathcal{F}^{\vdash}$-, resp. $\mathcal{F}^{\Vdash}$-)prime-strips.

Remark 10.9.1. We use global realified prime-strips with localisations for calculating (group-theoretically reconstructed) local log-volumes (cf. Section 5) *with the global product formula*. Another necessity of global realified prime-strips with localisations is as follows: If we were working only with the various local Frobenioids for $\underline{v} \in \underline{\mathbb{V}}$ (which are directly related to computations of the log-volumes), then we *could not distinguish*, for example, $p_{\underline{v}}^m O_{K_{\underline{v}}}$ from $O_{K_{\underline{v}}}$ with $m \in \mathbb{Z}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, since the isomorphism of these Frobenioids arising from (the updated version of) Θ-link *preserves only the isomorphism classes of objects* of these Frobenioids. By using global realified prime-strips with localisations, we can distinguish them (cf. [IUTchIII, (xii) of the proof of Corollary 3.12]).

Note that we can algorithmically associate $\mathcal{D}^{\vdash}$-prime-strip ${}^{\dagger}\mathfrak{D}^{\vdash}$ to any $\mathcal{D}$-prime-strip ${}^{\dagger}\mathfrak{D}$ and so on. We summarise this as follows (cf. also [IUTchI, Remark 5.2.1 (i), (ii)]):

$$\begin{tikzcd}
{}^{\dagger}\mathcal{HT}^{\Theta} \arrow[r, mapsto] \arrow[d, mapsto] & {}^{\dagger}\mathfrak{F} \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dl, mapsto] & {}^{\dagger}\mathfrak{D} \arrow[d, mapsto] \\
{}^{\dagger}\mathfrak{F}^{\Vdash} \arrow[r, mapsto] & {}^{\dagger}\mathfrak{F}^{\vdash} \arrow[r, mapsto] & {}^{\dagger}\mathfrak{D}^{\vdash}.
\end{tikzcd}$$

Lemma 10.10. ([IUTchI, Corollary 5.3, Corollary 5.6 (i)])

(1) Let ${}^{1}\mathcal{F}^{\circledast}$, ${}^{2}\mathcal{F}^{\circledast}$ (resp. ${}^{1}\mathcal{F}^{\circledcirc}$, ${}^{2}\mathcal{F}^{\circledcirc}$) be pre-Frobenioids isomorphic to the global non-realifed Frobenioid ${}^{\dagger}\mathcal{F}^{\circledast}$ (resp. ${}^{\dagger}\mathcal{F}^{\circledcirc}$) in Example 9.5 , then the natural map

$$\mathrm{Isom}({}^{1}\mathcal{F}^{\circledast}, {}^{2}\mathcal{F}^{\circledast}) \to \mathrm{Isom}(\mathrm{Base}({}^{1}\mathcal{F}^{\circledast}), \mathrm{Base}({}^{2}\mathcal{F}^{\circledast}))$$

$$(\textit{resp.}\ \ \mathrm{Isom}({}^{1}\mathcal{F}^{\circledcirc}, {}^{2}\mathcal{F}^{\circledcirc}) \to \mathrm{Isom}(\mathrm{Base}({}^{1}\mathcal{F}^{\circledcirc}), \mathrm{Base}({}^{2}\mathcal{F}^{\circledcirc}))\ \ )$$

is bijective.

(2) For $\mathcal{F}$-prime-strips ${}^{1}\mathfrak{F}$, ${}^{2}\mathfrak{F}$, whose associated $\mathcal{D}$-prime-strips are ${}^{1}\mathfrak{D}$, ${}^{2}\mathfrak{D}$ respectively, the natural map

$$\mathrm{Isom}_{\mathcal{F}}({}^{1}\mathfrak{F}, {}^{2}\mathfrak{F}) \to \mathrm{Isom}_{\mathcal{D}}({}^{1}\mathfrak{D}, {}^{2}\mathfrak{D})$$

is bijective.

(3) For $\mathcal{F}^{\vdash}$-prime-strips ${}^{1}\mathfrak{F}^{\vdash}$, ${}^{2}\mathfrak{F}^{\vdash}$, whose associated $\mathcal{D}^{\vdash}$-prime-strips are ${}^{1}\mathfrak{D}^{\vdash}$, ${}^{2}\mathfrak{D}^{\vdash}$ respectively, the natural map

$$\mathrm{Isom}_{\mathcal{F}^{\vdash}}({}^{1}\mathfrak{F}^{\vdash}, {}^{2}\mathfrak{F}^{\vdash}) \to \mathrm{Isom}_{\mathcal{D}^{\vdash}}({}^{1}\mathfrak{D}^{\vdash}, {}^{2}\mathfrak{D}^{\vdash})$$

is bijective.

(4) For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, let $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ be the tempered Frobenioid in Example 8.8, whose base category is $\mathcal{D}_{\underline{v}}$ then the natural map

$$\mathrm{Aut}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \to \mathrm{Aut}(\mathcal{D}_{\underline{v}})$$

is bijective.

(5) For Th-Hodge theatres ${}^{1}\mathcal{HT}^{\Theta}$, ${}^{2}\mathcal{HT}^{\Theta}$, whose associated $\mathcal{D}$-prime-strips are ${}^{1}\mathfrak{D}_{>}$, ${}^{2}\mathfrak{D}_{>}$ respectively, the natural map

$$\mathrm{Isom}({}^{1}\mathcal{HT}^{\Theta}, {}^{2}\mathcal{HT}^{\Theta}) \to \mathrm{Isom}_{\mathcal{D}}({}^{1}\mathfrak{D}_{>}, {}^{2}\mathfrak{D}_{>})$$

is bijective.

Proof. (1) follows from the category-theoretic construction of the isomorphism $\mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} {}^{\dagger}\mathbb{M}^{\circledast}$ in Example 9.5. (2) follows from the mono-anabelian reconstruction algorithms via Belyi cuspidalisation (Corollary 3.19), and the Kummer isomorphism in Remak 3.19.2) for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, and the definition of the Kummer structure for Aut-holomorphis orbispaces (Definition 4.6) for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. (3) follows from Proposition 5.2 and Proposition 5.4. We show (4). By Theorem 3.17, automorphisms of $\mathcal{D}_{\underline{v}}$ arises from automorphisms of $\underline{\underline{X}}_{\underline{v}}$, thus, the surjectivity of (4) holds. To show the injectivity of (4), let α be in the kernel. Then it suffices to show that α induces the identity on the rational functions and divisor monoids of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. By the category-theoretic reconstruction of cyclotomic rigidity (cf. isomorphism (Cyc. Rig. Frd)) and the naturality of Kummer map, (which is injective), it follows that α induces the identity on the rational functions of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. Since α preserves the base-field-theoretic hull, α also preserves the non-cuspidal portion of the divisor of the Frobenioid-theoretic theta function and its conjugate (these are preserved by α since we already show that α preserves the rational function monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$), hence α induces the identity on the non-cuspidal elements of the divisor monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. Similary, since any divisor of degree 0 on an elliptic curve supported on the torsion points admits a positive multiple which is principal, it follows that α induces the identityo on the cuspidal elements of the divisor monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ as well. by considering the cuspidal portions of divisor of a suitable rational functions (these are preserved by α since we already show that α preserves the rational function monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$). (Note that we can simplify the proof by suitably adding $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ more data, and considering the isomorphisms preserving these data. cf. also the remark given just before Theorem 8.14 and [IUTchI, Remark 3.2.1 (ii)]). (5) follows from (4). $\square$

Remark 10.10.1. ([IUTchI, Remark 5.3.1]) Let ${}^{1}\mathfrak{F}$, ${}^{2}\mathfrak{F}$ be $\mathcal{F}$-prime-strips, whose associated $\mathcal{D}$-prime-strips are ${}^{1}\mathfrak{D}$, ${}^{2}\mathfrak{D}$ respectively. Let

$$\phi : {}^{1}\mathfrak{D} \to {}^{2}\mathfrak{D}$$

be a morphism of $\mathcal{D}$-prime-strips, which is not necessarily an isomorphism, such that all of the $\underline{v}(\in \underline{\mathbb{V}}^{\mathrm{good}})$-components are isomorphisms, and the induced morphism $\phi^{\vdash} : {}^1\mathfrak{D}^{\vdash} \to {}^2\mathfrak{D}^{\vdash}$ on the associated $\mathcal{D}^{\vdash}$-prime-strips is also an isomorphism. Then ϕ uniquely lifts to an "**arrow**"

$$\psi : {}^1\mathfrak{F} \to {}^2\mathfrak{F},$$

which we say that $\boldsymbol{\psi}$ **is lying over** $\boldsymbol{\phi}$, as follows: By pulling-back (or making categorical fiber products) of the (pre-)Frobenioids in ${}^2\mathfrak{F}$ via the various $\underline{v}(\in \underline{\mathbb{V}})$-components of ϕ, we obtain the pulled-back $\mathcal{F}$-prime-strip $\phi^*({}^2\mathfrak{F})$ whose associated $\mathcal{D}$-prime-strip is tautologically equal to ${}^1\mathfrak{D}$. Then this tautological equality uniquely lifts to an isomorphism ${}^1\mathfrak{F} \xrightarrow{\sim} \phi^*({}^2\mathfrak{F})$ by Lemma 10.10 (2):

$$\begin{tikzcd}
{}^1\mathfrak{F} \arrow[r, dotted, "\sim"] \arrow[dr, mapsto] & \phi^*({}^2\mathfrak{F}) \arrow[r, "\text{pull back}"] \arrow[d, mapsto] & {}^2\mathfrak{F} \arrow[d, mapsto] \\
& {}^1\mathfrak{D} \arrow[r, "\phi"] & {}^2\mathfrak{D}.
\end{tikzcd}$$

Definition 10.11. ([IUTchI, Definition 4.1 (v), (vi), Definition 6.1 (vii)]) Let ${}^\dagger\mathcal{D}^{\circledcirc}$ (resp. ${}^\dagger\mathcal{D}^{\circledcirc\pm}$) is a category equivalent to the model global object $\mathcal{D}^{\circledcirc}$ (resp. $\mathcal{D}^{\circledcirc\pm}$) in Definition 10.3.

(1) Recall that, from ${}^\dagger\mathcal{D}^{\circledcirc}$ (resp. ${}^\dagger\mathcal{D}^{\circledcirc\pm}$), we can group-theoretically reconstruct a set $\mathbb{V}({}^\dagger\mathcal{D}^{\circledcirc})$ (resp. $\mathbb{V}({}^\dagger\mathcal{D}^{\circledcirc\pm})$) of valuations corresponding to $\mathbb{V}(K)$ by Example 9.5 (resp. in a slimilar way as in Example 9.5, i.e., firstly group-theoretically reconstructing an isomorph of the field $\overline{F}$ from $\pi_1({}^\dagger\mathcal{D}^{\circledcirc\pm})$ by Theorem 3.17 via the Θ-approach (Definition 9.4), secondly group-theoretically reconstructing an isomorph $\overline{\mathbb{V}}({}^\dagger\mathcal{D}^{\circledcirc\pm})$ of $\mathbb{V}(\overline{F})$ with $\pi_1({}^\dagger\mathcal{D}^{\circledcirc\pm})$-action, by the valuations on the field, and finally consider the set of $\pi_1({}^\dagger\mathcal{D}^{\circledcirc\pm})$-orbits of $\overline{\mathbb{V}}({}^\dagger\mathcal{D}^{\circledcirc\pm})$).

For $\underline{w} \in \mathbb{V}({}^\dagger\mathcal{D}^{\circledcirc})^{\mathrm{arc}}$ (resp. $\underline{w} \in \mathbb{V}({}^\dagger\mathcal{D}^{\circledcirc\pm})^{\mathrm{arc}}$), by Proposition 4.8 and Lemma 4.9, we can group-theoretically reconstruct, from ${}^\dagger\mathcal{D}^{\circledcirc}$ (resp. ${}^\dagger\mathcal{D}^{\circledcirc\pm}$), an Aut-holomorphic orbispace

$$\underline{\mathbb{C}}({}^\dagger\mathcal{D}^{\circledcirc}, \underline{w}) \quad (\text{resp. } \underline{\mathbb{X}}({}^\dagger\mathcal{D}^{\circledcirc\pm}, \underline{w}) \ \ )$$

corresponding to $\underline{C}_{\underline{w}}$ (resp. $\underline{X}_{\underline{w}}$). For an Aut-holomorphic orbispace $\mathbb{U}$, a **morphism**

$$\mathbb{U} \to {}^\dagger\mathcal{D}^{\circledcirc} \quad (\text{resp. } \mathbb{U} \to {}^\dagger\mathcal{D}^{\circledcirc\pm} \ \ )$$

is a morphism of Aut-holomorphic orbispaces $\mathbb{U} \to \underline{\mathbb{C}}({}^\dagger\mathcal{D}^{\circledcirc}, \underline{w})$ (resp. $\mathbb{U} \to \underline{\mathbb{X}}({}^\dagger\mathcal{D}^{\circledcirc\pm}, \underline{w})$) for some $\underline{w} \in \mathbb{V}({}^\dagger\mathcal{D}^{\circledcirc})^{\mathrm{arc}}$ (resp. $\underline{w} \in \mathbb{V}({}^\dagger\mathcal{D}^{\circledcirc\pm})^{\mathrm{arc}}$).

(2) For a $\mathcal{D}$-prime-strip ${}^\dagger\mathfrak{D} = \{{}^\dagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, a **poly-morphism**

$${}^\dagger\mathfrak{D} \overset{\text{poly}}{\longrightarrow} {}^\dagger\mathcal{D}^{\circledcirc} \quad (\text{resp. } {}^\dagger\mathfrak{D} \overset{\text{poly}}{\longrightarrow} {}^\dagger\mathcal{D}^{\circledcirc\pm} \ \ )$$

is a collection of poly-morphisms $\{{}^{\dagger}\mathcal{D}_{\underline{v}} \stackrel{\text{poly}}{\longrightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc}\}_{\underline{v}\in\underline{\mathbb{V}}}$ (resp. $\{{}^{\dagger}\mathcal{D}_{\underline{v}} \stackrel{\text{poly}}{\longrightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}\}_{\underline{v}\in\underline{\mathbb{V}}}$) indexed by $\underline{v} \in \underline{\mathbb{V}}$ (cf. Definition 6.1 (5) for $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, and the above definition in (1) for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$).

(3) For a capsule ${}^{E}\mathfrak{D} = \{{}^{e}\mathfrak{D}\}_{e\in E}$ of $\mathcal{D}$-prime-strips and a $\mathcal{D}$-prime-strip ${}^{\dagger}\mathfrak{D}$, a **poly-morphism**

$$ {}^{E}\mathfrak{D} \stackrel{\text{poly}}{\longrightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc} \quad (\text{resp. } {}^{E}\mathfrak{D} \stackrel{\text{poly}}{\longrightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}, \quad \text{resp. } {}^{E}\mathfrak{D} \stackrel{\text{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{D}\ \ ) $$

is a collection of poly-morphisms $\{{}^{e}\mathfrak{D} \stackrel{\text{poly}}{\longrightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc}\}_{e\in E}$ (resp. $\{{}^{e}\mathfrak{D} \stackrel{\text{poly}}{\longrightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}\}_{e\in E}$, resp. $\{{}^{e}\mathfrak{D} \stackrel{\text{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{D}\}_{e\in E}$).

Definition 10.12. ([IUTchII, Definition 4.9 (ii), (iii), (iv), (v), (vi), (vii), (viii)]) Let ${}^{\ddagger}\mathfrak{F}^{\vdash} = \{{}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ be an $\mathcal{F}^{\vdash}$-prime-strip with associated $\mathcal{D}^{\vdash}$-prime-strip ${}^{\ddagger}\mathfrak{D}^{\vdash} = \{{}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$.

(1) Recall that ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}$ is a μ_{2l}-split pre-Frobenioid (resp. a split pre-Frobenioid, resp. a triple $({}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}, {}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}, {}^{\ddagger}\mathrm{spl}^{\vdash}_{\underline{v}})$) for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}$, resp. $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$). Let ${}^{\ddagger}A_{\infty}$ be a universal covering pro-object of ${}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}$, and write ${}^{\ddagger}G := \mathrm{Aut}({}^{\ddagger}A_{\infty})$ (hence ${}^{\ddagger}G$ is a profinite group isomorphic to $G_{\underline{v}}$). For $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}$), we write

$$ O^{\perp}({}^{\ddagger}A_{\infty})\ (\subset O^{\rhd}({}^{\ddagger}A_{\infty})) $$

for the submonoid generated by $\mu_{2l}({}^{\ddagger}A_{\infty})$ and the image of the splittings on ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}$ (resp. the submonoid determined by the image of the splittings on ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}$), and put

$$ O^{\blacktriangleright}({}^{\ddagger}A_{\infty}) := O^{\perp}({}^{\ddagger}A_{\infty})/\mu_{2l}({}^{\ddagger}A_{\infty}) \ \ (\text{resp. } O^{\blacktriangleright}({}^{\ddagger}A_{\infty}) := O^{\perp}({}^{\ddagger}A_{\infty})\ \ ), $$

and

$$ O^{\blacktriangleright\times\mu}({}^{\ddagger}A_{\infty}) := O^{\blacktriangleright}({}^{\ddagger}A_{\infty})\times O^{\times\mu}({}^{\ddagger}A_{\infty}) \ \ (\text{resp. } O^{\blacktriangleright\times\mu}({}^{\ddagger}A_{\infty}) := O^{\blacktriangleright}({}^{\ddagger}A_{\infty})\times O^{\times\mu}({}^{\ddagger}A_{\infty})\ \ ). $$

These are equipped with natural ${}^{\ddagger}G$-actions.

Next, for $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, we can group-theoretically reconstruct, from ${}^{\ddagger}G$, ind-topological modules ${}^{\ddagger}G \curvearrowright O^{\times}({}^{\ddagger}G)$, ${}^{\ddagger}G \curvearrowright O^{\times\mu}({}^{\ddagger}G)$ with G-action, by Proposition 5.2 (Step 1) (cf. Definition 8.5 (1)). Then by Definition 8.5 (2), there exists a unique $\widehat{\mathbb{Z}}^{\times}$-orbit of isomorphisms

$$ {}^{\ddagger}\kappa^{\vdash\times}_{\underline{v}} : O^{\times}({}^{\ddagger}G) \stackrel{\text{poly}}{\stackrel{\sim}{\to}} O^{\times}({}^{\ddagger}A_{\infty}) $$

of ind-topological modules with ${}^{\ddagger}G$-actions. Moreover, ${}^{\ddagger}\kappa^{\vdash\times}_{\underline{v}}$ induces an Isomet-orbit

$$ {}^{\ddagger}\kappa^{\vdash\times\mu}_{\underline{v}} : O^{\times\mu}({}^{\ddagger}G) \stackrel{\text{poly}}{\stackrel{\sim}{\to}} O^{\times\mu}({}^{\ddagger}A_{\infty}) $$

of isomorphisms.

For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the rational function monoid determined by $O^{\blacktriangleright \times \mu}({}^{\ddagger}A_{\infty})^{\mathrm{gp}}$ with ${}^{\ddagger}G$-action and the divisor monoid of ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}$ determine a model Frobenioid with a splitting. The Isomet-orbit of isomorphisms ${}^{\ddagger}\kappa^{\vdash \times \mu}_{\underline{v}}$ determines a $\times\mu$-Kummer structure (Definition 8.5 (2)) on this model Frobenioid. For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$), we write

$$^{\ddagger}\mathcal{F}^{\vdash \blacktriangleright \times \mu}_{\underline{v}}$$

for the resulting split-$\times\mu$-Kummer pre-Frobenioid (resp. the collection of data obtained by replacing the split pre-Frobenioid ${}^{\ddagger}\mathcal{C}_{\underline{v}}$ in ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}} = ({}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}, {}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}, {}^{\ddagger}\mathrm{spl}^{\vdash}_{\underline{v}})$ by the inductive system, indexed by the multiplicative monoid $\mathbb{N}_{\geq 1}$, of split pre-Frobenioids obtained from ${}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}$ by taking the quotients by the N-torsions for $N \in \mathbb{N}_{\geq 1}$. Thus, the units of the split pre-Frobenioids of this inductive system give rise to an inductive system $\cdots \twoheadrightarrow O^{\times \mu_N}(A_{\infty}) \twoheadrightarrow \cdots \twoheadrightarrow O^{\times \mu_{NM}}(A_{\infty}) \twoheadrightarrow \cdots$, and a system of compatible surjections $\{({}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}})^{\times} \twoheadrightarrow O^{\times \mu_N}(A_{\infty})\}_{N \in \mathbb{N}_{\geq 1}}$ (which can be regard as a kind of Kummer structure on ${}^{\ddagger}\mathcal{F}^{\vdash \blacktriangleright \times \mu}_{\underline{v}}$) for the split monoid ${}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}$), and, by abuse of notation,

$$^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}$$

for the split-$\times$-Kummer pre-Frobenioid determined by the split pre-Frobenioid ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}$ with the $\times$-Kummer structure determined by ${}^{\ddagger}\kappa^{\vdash \times}_{\underline{v}}$.

(2) Write

$$^{\ddagger}\mathfrak{F}^{\vdash \blacktriangleright \times \mu} := \{{}^{\ddagger}\mathcal{F}^{\vdash \blacktriangleright \times \mu}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}.$$

We also write

$$^{\ddagger}\mathfrak{F}^{\vdash \times} = \{{}^{\ddagger}\mathcal{F}^{\vdash \times}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}} \quad (\text{resp. } {}^{\ddagger}\mathfrak{F}^{\vdash \times \mu} := \{{}^{\ddagger}\mathcal{F}^{\vdash \times \mu}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}\ )$$

for the collection of data obtained by replacing the various split pre-Frobenioids of ${}^{\ddagger}\mathfrak{F}^{\vdash}$ (resp. ${}^{\ddagger}\mathfrak{F}^{\vdash \blacktriangleright \times \mu}$) by the split Frobenioid with trivial splittings obtained by considering the subcategories determined by morphisms ϕ with $\mathrm{Div}(\phi) = 0$ (i.e., the "units" for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$) in the pre-Frobenioid structure. Note that ${}^{\ddagger}\mathcal{F}^{\vdash \times}_{\underline{v}}$ (resp. ${}^{\ddagger}\mathcal{F}^{\vdash \times \mu}_{\underline{v}}$) is a split-$\times$-Kummer pre-Frobenioid (resp. a split-$\times\mu$-Kummer pre-Frobenioid).

(3) An **$\mathcal{F}^{\vdash \times}$-prime-strip** (resp. an **$\mathcal{F}^{\vdash \times \mu}$-prime-strip**, resp. an **$\mathcal{F}^{\vdash \blacktriangleright \times \mu}$-prime-strip**) is a collection

$$^{*}\mathfrak{F}^{\vdash \times} = \{{}^{*}\mathcal{F}^{\vdash \times}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}} \ (\text{resp. } {}^{*}\mathfrak{F}^{\vdash \times \mu} = \{{}^{*}\mathcal{F}^{\vdash \times \mu}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}, \ \text{resp. } {}^{*}\mathfrak{F}^{\vdash \blacktriangleright \times \mu} = \{{}^{*}\mathcal{F}^{\vdash \blacktriangleright \times \mu}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}\ )$$

of data such that ${}^{*}\mathcal{F}^{\vdash \times}_{\underline{v}}$ (resp. ${}^{*}\mathcal{F}^{\vdash \times \mu}_{\underline{v}}$, resp. ${}^{*}\mathcal{F}^{\vdash \blacktriangleright \times \mu}_{\underline{v}}$) is isomorphic to ${}^{\ddagger}\mathcal{F}^{\vdash \times}_{\underline{v}}$ (resp. ${}^{\ddagger}\mathcal{F}^{\vdash \times \mu}_{\underline{v}}$, resp. ${}^{\ddagger}\mathcal{F}^{\vdash \blacktriangleright \times \mu}_{\underline{v}}$) for each $\underline{v} \in \underline{\mathbb{V}}$. An **isomorphism of $\mathcal{F}^{\vdash \times}$-prime-strips** (resp. **$\mathcal{F}^{\vdash \times}$-prime-strips**, resp. **$\mathcal{F}^{\vdash \times}$-prime-strips**) is a collection of isomorphisms indexed by $\underline{\mathbb{V}}$ between each component.

(4) An $\boldsymbol{\mathcal{F}^{\Vdash\blacktriangleright\times\mu}}$**-prime-strip** is a quadruple

$$^*\mathfrak{F}^{\Vdash\blacktriangleright\times\mu} = ({}^*\mathcal{C}^{\Vdash}, \mathrm{Prime}({}^*\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^*\mathfrak{F}^{\vdash\blacktriangleright\times\mu}, \{{}^*\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

where ${}^*\mathcal{C}^{\Vdash}$ is a pre-Frobenioid isomorphic to the model $\mathcal{C}^{\vdash}_{\mathrm{mod}}$ in Definition 10.4, $\mathrm{Prime}({}^*\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}$ is a bijection of sets, ${}^*\mathfrak{F}^{\vdash\blacktriangleright\times\mu}$ is an $\mathcal{F}^{\vdash\blacktriangleright\times\mu}$-prime-strip, and ${}^*\rho_{\underline{v}}$: $\Phi_{{}^*\mathcal{C}^{\Vdash},v} \overset{\text{gl. to loc.}}{\xrightarrow{\sim}} \Phi^{\mathbb{R}}_{{}^*\mathcal{C}^{\vdash}_{\underline{v}}}$ is an isomorphism of topological monoids (Here, ${}^*\mathcal{C}^{\vdash}_{\underline{v}}$ is the object reconstructed from ${}^*\mathcal{F}^{\vdash\blacktriangleright\times\mu}_{\underline{v}}$), such that the quadruple ${}^*\mathfrak{F}^{\Vdash}$ is isomorphic to the model $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ in Definition 10.4. An **isomorphism of** $\boldsymbol{\mathcal{F}^{\Vdash\blacktriangleright\times\mu}}$**-prime-strips** is a collection of isomorphisms indexed by $\underline{\mathbb{V}}$ between each component.

(5) Let $\mathrm{Aut}_{\mathcal{F}^{\vdash\times}}(-)$, $\mathrm{Isom}_{\mathcal{F}^{\vdash\times}}(-,-)$ (resp. $\mathrm{Aut}_{\mathcal{F}^{\vdash\times\mu}}(-)$, $\mathrm{Isom}_{\mathcal{F}^{\vdash\times\mu}}(-,-)$ resp. $\mathrm{Aut}_{\mathcal{F}^{\vdash\blacktriangleright\times\mu}}(-)$, $\mathrm{Isom}_{\mathcal{F}^{\vdash\blacktriangleright\times\mu}}(-,-)$ resp. $\mathrm{Aut}_{\mathcal{F}^{\Vdash\blacktriangleright\times\mu}}(-)$, $\mathrm{Isom}_{\mathcal{F}^{\Vdash\blacktriangleright\times\mu}}(-,-)$) be the group of automorphisms of an $\mathcal{F}^{\vdash\times}$-(resp. $\mathcal{F}^{\vdash\times\mu}$-, resp. $\mathcal{F}^{\vdash\blacktriangleright\times\mu}$-, resp. $\mathcal{F}^{\Vdash\blacktriangleright\times\mu}$-)prime-strip, and the set of isomorphisms between $\mathcal{F}^{\vdash\times}$-(resp. $\mathcal{F}^{\vdash\times\mu}$-, resp. $\mathcal{F}^{\vdash\blacktriangleright\times\mu}$-, resp. $\mathcal{F}^{\Vdash\blacktriangleright\times\mu}$-)prime-strips.

Remark 10.12.1. In the definition of ${}^\ddagger\mathcal{F}^{\vdash\blacktriangleright\times\mu}_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ in Definition 10.12, we consider an inductive system. We use this as follows: For the crucial non-interference property for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, we use the fact that the $p_{\underline{v}}$-adic logarithm kills the torsion $\mu(-) \subset O^{\times}(-)$. However, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, the Archimedean logarithm *does not* kill the torsion. Instead, in the notation of Section 5.2, we replace a part of $\mathfrak{log}$-link by $k^{\sim} \twoheadrightarrow (O^{\rhd}_k)^{\mathrm{gp}} \twoheadrightarrow (O^{\rhd}_k)^{\mathrm{gp}}/\mu_N(k)$ and consider $k^{\sim}$ as being reconstructed from $(O^{\rhd}_k)^{\mathrm{gp}}/\mu_N(k)$, not from $(O^{\rhd}_k)^{\mathrm{gp}}$, and write weight N on the corrsponding log-volume. Then there is no problem. cf. also Definition 12.1 (2), (4), Proposition 12.2 (2) (cf. [IUTchIII, Remark 1.2.1]), Proposition 13.7, and Proposition 13.11.

Definition 10.13. ([IUTchIII, Definition 2.4])

(1) Let

$$^\ddagger\mathfrak{F}^{\vdash} = \{{}^\ddagger\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

be an $\mathcal{F}^{\vdash}$-prime-strip. Then by Definition 10.12 (1), for each $\underline{w} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, the splittings of the μ_{2l}-split-Frobenioid ${}^\ddagger\mathcal{F}^{\vdash}_{\underline{w}}$ determine submonoids $O^{\perp}(-) \subset O^{\rhd}(-)$ and quotient monoids $O^{\perp}(-) \twoheadrightarrow O^{\blacktriangleright}(-) = O^{\perp}(-)/O^{\mu}(-)$. Similarly, for each $\underline{w} \in \underline{\mathbb{V}}^{\mathrm{good}}$, the splitting of the split Frobenioid ${}^\ddagger\mathcal{F}^{\vdash}_{\underline{w}}$ determines a submonoid $O^{\perp}(-) \subset O^{\rhd}(-)$. In this case, we write $O^{\blacktriangleright}(-) := O^{\perp}(-)$. We write

$$^\ddagger\mathfrak{F}^{\vdash\perp} = \{{}^\ddagger\mathcal{F}^{\vdash\perp}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \quad {}^\ddagger\mathfrak{F}^{\vdash\blacktriangleright} = \{{}^\ddagger\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

for the collection of data obtained by replacing the μ_{2l}-split/split Frobenioid portion of each ${}^\ddagger\mathcal{F}^{\vdash}_{\underline{v}}$ by the pre-Frobenioids determined by the subquotient monoids $O^{\perp}(-) \subset O^{\rhd}(-)$ and $O^{\blacktriangleright}(-)$, respectively.

(2) An $\mathcal{F}^{\vdash\perp}$**-prime-strip** (resp. an $\mathcal{F}^{\vdash\blacktriangleright}$**-prime-strip**) is a collection

$$^*\mathfrak{F}^{\vdash\perp} = \{^*\mathcal{F}^{\vdash\perp}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \quad (\text{resp. } ^*\mathfrak{F}^{\vdash\blacktriangleright} = \{^*\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \quad)$$

of data such that $^*\mathcal{F}^{\vdash\perp}_{\underline{v}}$ (resp. $^*\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$) is isomorphic to $^\ddagger\mathcal{F}^{\vdash\perp}_{\underline{v}}$ (resp. $^\ddagger\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$) for each $\underline{v} \in \underline{\mathbb{V}}$. An **isomorphism of** $\mathcal{F}^{\vdash\perp}$**-prime-strips** (resp. $\mathcal{F}^{\vdash\blacktriangleright}$**-prime-strips**) is a collection of isomorphisms indexed by $\underline{\mathbb{V}}$ between each component.

(3) An $\mathcal{F}^{\Vdash\perp}$**-prime-strip** (resp. $\mathcal{F}^{\Vdash\blacktriangleright}$**-prime-strip**) is a quadruple

$$^*\mathfrak{F}^{\Vdash\perp} = (^*\mathcal{C}^{\Vdash}, \text{Prime}(^*\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^*\mathfrak{F}^{\vdash\perp}, \{^*\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

$$(\text{resp. } ^*\mathfrak{F}^{\Vdash\blacktriangleright} = (^*\mathcal{C}^{\Vdash}, \text{Prime}(^*\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^*\mathfrak{F}^{\vdash\blacktriangleright}, \{^*\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) \quad)$$

where $^*\mathcal{C}^{\Vdash}$ is a pre-Frobenioid isomorphic to the model $\mathcal{C}^{\vdash}_{\text{mod}}$ in Definition 10.4, $\text{Prime}(^*\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}$ is a bijection of sets, $^*\mathfrak{F}^{\vdash\perp}$ (resp. $^*\mathfrak{F}^{\vdash\blacktriangleright}$) is an $\mathcal{F}^{\vdash\perp}$-prime-strip (resp. $\mathcal{F}^{\vdash\blacktriangleright}$-prime-strip), and $^*\rho_{\underline{v}} : \Phi_{^*\mathcal{C}^{\Vdash},v} \overset{\text{gl. to loc.}}{\xrightarrow{\sim}} \Phi^{\mathbb{R}}_{^*\mathcal{C}^{\vdash}_{\underline{v}}}$ is an isomorphism of topological monoids (Here, $^*\mathcal{C}^{\vdash}_{\underline{v}}$ is the object reconstructed from $^*\mathcal{F}^{\vdash\perp}_{\underline{v}}$ (resp. $^*\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$)), such that the quadruple $^*\mathfrak{F}^{\Vdash\perp}$ (resp. $^*\mathfrak{F}^{\Vdash\blacktriangleright}$) is isomorphic to the model $\mathfrak{F}^{\Vdash}_{\text{mod}}$ in Definition 10.4. An **isomorphism of** $\mathcal{F}^{\Vdash\perp}$**-prime-strips** (resp. $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips) is a collection of isomorphisms indexed by $\underline{\mathbb{V}}$ between each component.

§ 10.4. The Multiplicative Symmetry $\boxtimes$: ΘNF-Hodge Theatres and NF-, Θ-Bridges.

We begin constructing the multiplicative portion of full Hodge theatres.

Definition 10.14. ([IUTchI, Definition 4.1 (i), (ii), (v)]) Let $^\dagger\mathfrak{D} = \{^\dagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ be a $\mathcal{D}$-prime-strip.

(1) For $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}$), we can group-theoretically reconstruct in a functorial manner, from $\pi_1(^\dagger\mathcal{D}_{\underline{v}})$, a tempered group (resp. a profinite group) $(\supset \pi_1(^\dagger\mathcal{D}_{\underline{v}}))$ corresponding to $\underline{C}_{\underline{v}}$ by Lemma 7.12 (resp. by Lemma 7.25). We write

$$^\dagger\underline{\mathcal{D}}_{\underline{v}}$$

for its $\mathcal{B}(-)^0$. We have a natural morphism $^\dagger\mathcal{D}_{\underline{v}} \to {}^\dagger\underline{\mathcal{D}}_{\underline{v}}$ (This corresponds to $\underline{\underline{X}}_{\underline{v}} \to \underline{C}_{\underline{v}}$ (resp. $\underset{\rightarrow}{X}_{\underline{v}} \to \underline{C}_{\underline{v}}$)). Similarly, for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, we can algorithmically reconstruct, in a functorial manner, from $^\dagger\mathcal{D}_{\underline{v}}$, an Aut-holomorphic orbispace $^\dagger\underline{\mathcal{D}}_{\underline{v}}$ corresponding to $\underline{C}_{\underline{v}}$ by translating Lemma 7.25 into the theory of Aut-holomorphic spaces (since $\underset{\rightarrow}{X}_{\underline{v}}$ admits a $K_{\underline{v}}$-core) with a natural morphism $^\dagger\mathcal{D}_{\underline{v}} \to {}^\dagger\underline{\mathcal{D}}_{\underline{v}}$. Write

$$^\dagger\underline{\mathfrak{D}} := \{^\dagger\underline{\mathcal{D}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}.$$

(2) Recall that we can algorithmically reconstruct the set of conjugacy classes of cuspidal decomposition groups of $\pi_1({}^\dagger\mathcal{D}_{\underline{v}})$ or $\pi_1({}^\dagger\underline{\mathcal{D}}_{\underline{v}})$ by Corollary 6.12 for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, by Corollary 2.9 for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$, and by considering $\pi_0(-)$ of a cofinal collection of the complements of compact subsets of the underlying topological space of ${}^\dagger\mathcal{D}_{\underline{v}}$ or ${}^\dagger\underline{\mathcal{D}}_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. We say them the **set of cusps of $^\dagger\mathcal{D}_{\underline{v}}$ or $^\dagger\underline{\mathcal{D}}_{\underline{v}}$**.

For $\underline{v} \in \underline{\mathbb{V}}$, a **label class of cusps of $^\dagger\mathcal{D}_{\underline{v}}$** is the set of cusps of ${}^\dagger\mathcal{D}_{\underline{v}}$ lying over a single non-zero cusp of ${}^\dagger\underline{\mathcal{D}}_{\underline{v}}$ (Note that each label class of cusps consists of two cusps). We write

$$\mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}})$$

for the set of label classes of cusps of ${}^\dagger\mathcal{D}_{\underline{v}}$. Note that $\mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}})$ has a natural $\mathbb{F}_l^{\circledast}$-torsor structure (which comes from the action of $\mathbb{F}_l^\times$ on Q in the definition of $\underline{X}$ in Section 7.1). Note also that, for any $\underline{v} \in \underline{\mathbb{V}}$, we can algorithmically reconstruct a canonical element

$${}^\dagger\eta_{\underline{v}} \in \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}})$$

corresponding to $\underline{\epsilon}_{\underline{v}}$ in the initial Θ-data, by Lemma 7.16 for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, Lemma 7.25 for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$, and a translation of Lemma 7.25 into the theory of Aut-holomorphic spaces for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$.

(Note that, if we used ${}^\dagger\underline{\mathcal{D}}_{\underline{v}}$ (i.e., "$\underline{C}_{\underline{v}}$") instead of ${}^\dagger\mathcal{D}_{\underline{v}}$ (i.e., "$\underline{X}_{\to\underline{v}}$") for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$, then we could not reconstruct ${}^\dagger\eta_{\underline{v}}$. In fact, we could make the action of the automorphism group of ${}^\dagger\underline{\mathcal{D}}_{\underline{v}}$ on LabCusp *transitive* for some $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$, by using Chebotarev density theorem (i.e., by making a decomposition group in $\mathrm{Gal}(K/F) \hookrightarrow \mathrm{GL}_2(\mathbb{F}_l)$ to be the subgroup of diagonal matrices with determinant 1). cf. [IUTchI, Remark 4.2.1].)

(3) Let ${}^\dagger\mathcal{D}^{\circledcirc}$ is a category equivalent to the model global object $\mathcal{D}^{\circledcirc}$ in Definition 10.3. Then by Remark 2.9.2, similarly we can define the **set of cusps of $^\dagger\mathcal{D}^{\circledcirc}$** and the **set of label classes of cusps**

$$\mathrm{LabCusp}({}^\dagger\mathcal{D}^{\circledcirc}),$$

which has a natural $\mathbb{F}_l^{\circledast}$-torsor structure.

From the definitions, we immediately obtain the following proposition:

Proposition 10.15. ([IUTchI, Proposition 4.2]) *Let ${}^\dagger\mathfrak{D} = \{{}^\dagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ be a $\mathcal{D}$-prime-strip. Then for any $\underline{v}, \underline{w} \in \underline{\mathbb{V}}$, there exist unique bijections*

$$\mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{w}})$$

which are compatible with the $\mathbb{F}_l^{\circledast}$*-torsor structures and send the canonical element* ${}^{\dagger}\underline{\eta}_{\underline{v}}$ *to the canonical element* ${}^{\dagger}\underline{\eta}_{\underline{w}}$*. By these identifications, we can write*

$$\mathrm{LabCusp}({}^{\dagger}\mathfrak{D})$$

for them. Note that it has a canonical element which comes from ${}^{\dagger}\underline{\eta}_{\underline{v}}$*'s. The* $\mathbb{F}_l^{\circledast}$*-torsor structure and the canonical element give us a natural bijection*

$$\mathrm{LabCusp}({}^{\dagger}\mathfrak{D}) \xrightarrow{\sim} \mathbb{F}_l^{\circledast}.$$

Definition 10.16. (Model $\mathcal{D}$-NF-Bridge, [IUTchI, Example 4.3]) We write

$$\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \quad \subset \mathrm{Aut}(\underline{C}_K) \cong \mathrm{Out}(\Pi_{\underline{C}_K}) \cong \mathrm{Aut}(\mathcal{D}^{\circledcirc})$$

for the subgroup of elements which fix the cusp $\underline{\epsilon}$ (The firs isomorphisms follows from Theorem 3.17). By Theorem 3.7, we can group-theoretically reconstruct Δ_X from $\Pi_{\underline{C}_K}$. We obtain a natural homomorphism

$$\mathrm{Out}(\Pi_{\underline{C}_K}) \to \mathrm{Aut}(\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l)/\{\pm 1\},$$

since inner automorphisms of $\Pi_{\underline{C}_K}$ act by multiplication by ± 1 on $E_{\overline{F}}[l]$. By choosing a suitable basis of $\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l$, which induces an isomorphism $\mathrm{Aut}(\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l)/\{\pm 1\} \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{F}_l)/\{\pm 1\}$, the images of $\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$ and $\mathrm{Aut}(\underline{C}_K)$ are identified with the following subgroups

$$\left\{\begin{pmatrix} * & * \\ 0 & \pm 1 \end{pmatrix}\right\} \subset \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \subset \mathrm{Im}(G_{F_{\mathrm{mod}}}) \quad (\supset \mathrm{SL}_2(\mathbb{F}_l)/\{\pm 1\})$$

of $\mathrm{GL}_2(\mathbb{F}_l)/\{\pm 1\}$, where $\mathrm{Im}(G_{F_{\mathrm{mod}}}) \subset \mathrm{GL}_2(\mathbb{F}_l)/\{\pm 1\}$ is the image of the natural action of $G_{F_{\mathrm{mod}}} := \mathrm{Gal}(\overline{F}/F_{\mathrm{mod}})$ on $E_{\overline{F}}[l]$. Write also

$$\underline{\mathbb{V}}^{\pm \mathrm{un}} := \mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \cdot \underline{\mathbb{V}} \quad \subset \underline{\mathbb{V}}^{\mathrm{Bor}} := \mathrm{Aut}(\underline{C}_K) \cdot \underline{\mathbb{V}} \quad \subset \mathbb{V}(K).$$

Hence we have a natural isomorphism

$$\mathrm{Aut}(\underline{C}_K)/\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \xrightarrow{\sim} \mathbb{F}_l^{\circledast},$$

thus, $\underline{\mathbb{V}}^{\mathrm{Bor}}$ is the $\mathbb{F}_l^{\circledast}$-orbit of $\underline{\mathbb{V}}^{\pm \mathrm{un}}$. By the above discussions, from $\pi_1(\mathcal{D}^{\circledcirc})$, we can group-theoretically reconstruct

$$\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \subset \mathrm{Aut}(\mathcal{D}^{\circledcirc})$$

corresponding to $\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \subset \mathrm{Aut}(\underline{C}_K)$ (cf. also Definition 10.11 (1), (2)).

For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$, resp. $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$), We write

$$\phi^{\mathrm{NF}}_{\bullet,\underline{v}} : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc}$$

for the natural morphism correponding to $\underline{\underline{X}}_{\underline{v}} \to \underline{C}_{\underline{v}} \to \underline{C}_K$ (resp. $\underrightarrow{X}_{\underline{v}} \to \underline{C}_{\underline{v}} \to \underline{C}_K$, resp. a tautological morphism $\mathcal{D}_{\underline{v}} = \underrightarrow{\mathbb{X}}_{\underline{v}} \to \underline{\mathbb{C}}_{\underline{v}} \xrightarrow{\sim} \underline{\mathbb{C}}(\mathcal{D}^{\circledcirc}, \underline{v})$) (cf. Definition 10.11 (1)). Write

$$\phi^{\mathrm{NF}}_{\underline{v}} := \mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \circ \phi^{\mathrm{NF}}_{\bullet,\underline{v}} \circ \mathrm{Aut}(\mathcal{D}_{\underline{v}}) : \mathcal{D}_{\underline{v}} \xrightarrow{\mathrm{poly}} \mathcal{D}^{\circledcirc}.$$

Let $\mathfrak{D}_j = \{\mathcal{D}_{\underline{v}_j}\}_{\underline{v}\in\underline{\mathbb{V}}}$ be a copy of the tautological $\mathcal{D}$-prime-strip $\{\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ for each $j \in \mathbb{F}_l^{\circledast}$ (Here, we write $\underline{v}_j$ for the pair $(j, \underline{v})$). Write

$$\phi^{\mathrm{NF}}_1 := \{\phi^{\mathrm{NF}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} : \mathfrak{D}_1 \xrightarrow{\mathrm{poly}} \mathcal{D}^{\circledcirc}$$

(cf. Definition 10.11 (2)). Since ϕ^{NF}_1 is stable under the action of $\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc})$, we obtain a poly-morphism

$$\phi^{\mathrm{NF}}_j := (\text{action of } j) \circ \phi^{\mathrm{NF}}_1 : \mathfrak{D}_j \xrightarrow{\mathrm{poly}} \mathcal{D}^{\circledcirc},$$

by post-composing a lift of $j \in \mathbb{F}_l^{\circledast} \cong \mathrm{Aut}(\mathcal{D}^{\circledcirc})/\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc})$ to $\mathrm{Aut}(\mathcal{D}^{\circledcirc})$. Hence we obtain a poly-morphism

$$\phi^{\mathrm{NF}}_{\circledast} := \{\phi^{\mathrm{NF}}_j\}_{j\in\mathbb{F}_l^{\circledast}} : \mathfrak{D}_{\circledast} := \{\mathfrak{D}_j\}_{j\in\mathbb{F}_l^{\circledast}} \xrightarrow{\mathrm{poly}} \mathcal{D}^{\circledcirc}$$

from a capsule of $\mathcal{D}$-prime-strip to the global object $\mathcal{D}^{\circledcirc}$ (cf. Definition 10.11 (3)). This is called the **model base-(or $\mathcal{D}$-)NF-bridge**. Note that $\phi^{\mathrm{NF}}_{\circledast}$ is equivariant with the natural poly-action (cf. Section 0.2) of $\mathbb{F}_l^{\circledast}$ on $\mathcal{D}^{\circledcirc}$ and the natural permutation poly-action of $\mathbb{F}_l^{\circledast}$ (via capsule-full poly-automorphisms (cf. Section 0.2)) on the components of the cupsule $\mathfrak{D}_{\circledast}$. In particular, we obtain a poly-action of $\mathbb{F}_l^{\circledast}$ on $(\mathfrak{D}_{\circledast}, \mathcal{D}^{\circledcirc}, \phi^{\mathrm{NF}}_{\circledast})$.

Definition 10.17. (Model $\mathcal{D}$-Θ-Bridge, [IUTchI, Example 4.4]) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Recall that we have a natural bijection between the set of cusps of $\underline{C}_{\underline{v}}$ and $|\mathbb{F}_l|$ by Lemma 7.16. Thus, we can write labels ($\in |\mathbb{F}_l|$) on the collections of cusps of $\underline{X}_{\underline{v}}$, $\underline{\underline{X}}_{\underline{v}}$ by considering fibers over $\underline{C}_{\underline{v}}$. We write

$$\mu_- \in \underline{X}_{\underline{v}}(K_{\underline{v}})$$

for the unique torsion point of order 2 such that the closures of the cusp labelled $0 \in |\mathbb{F}_l|$ and μ_- in the stable model of $\underline{X}_{\underline{v}}$ over $O_{K_{\underline{v}}}$ intersect the same irreducible component of the special fiber (i.e., "-1" in $\mathbb{G}_m^{\mathrm{rig}}/q^{\mathbb{Z}}_{\underline{X}_{\underline{v}}}$). We shall refer to the points obtained by translating the cusps labelled by $j \in |\mathbb{F}_l|$ by μ_- with respect to the group scheme structure of $\underline{E}_{\underline{v}} (\supset \underline{X}_{\underline{v}})$ (Recall that the origin of $\underline{E}_{\underline{v}}$ is the cusp labelled by $0 \in |\mathbb{F}_l|$) as the **evaluation points of $\underline{\boldsymbol{X}}_{\underline{\boldsymbol{v}}}$ labelled by $\boldsymbol{j}$**. Note that the value of $\underline{\underline{\Theta}}_{\underline{v}}$ in Example 8.8 at a point of $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ lying over an evaluation point labelled by $j \in |\mathbb{F}_l|$ is in the μ_{2l}-orbit of

$$\left\{ \quad \underline{\underline{q}}_{\underline{v}}^{\underline{\underline{j}}^2} \quad \right\}_{\underline{\underline{j}} \in \mathbb{Z} \text{ such that } \underline{\underline{j}} \equiv j \text{ in } |\mathbb{F}_l|}$$

by calculation $\ddot{\Theta}\left(\sqrt{-q_{\underline{v}}^{\underline{\underline{j}}}}\right) = (-1)^{\underline{\underline{j}}} q_{\underline{v}}^{-\underline{\underline{j}}^2/2} \sqrt{-1}^{-2\underline{\underline{j}}} \ddot{\Theta}(\sqrt{-1}) = q_{\underline{v}}^{-\underline{\underline{j}}^2/2}$ in the notation of Lemma 7.4 (cf. the formula $\ddot{\Theta}(q^{\underline{\underline{j}}/2}\ddot{U}) = (-1)^{\underline{\underline{j}}} q^{-1/2} \ddot{U}^{-2} \ddot{\Theta}(\ddot{U})$ in Lemma 7.4). In particular, the points of $\underline{\underline{X}}_{\underline{v}}$ lying over evaluation points of $\underline{X}_{\underline{v}}$ are all defined over $K_{\underline{v}}$, by the definition of $\underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}}$ (Note that the image of a point in the domain of $\underline{\underline{\ddot{Y}}} \overset{(\text{covering map}, \underline{\underline{\ddot{\Theta}}})}{\hookrightarrow} \ddot{Y} \times \mathbb{A}^1$ is rational over $K_{\underline{v}}$, then the point is rational over $K_{\underline{v}}$. cf. also Assumption (5) of Definition 7.13). We shall refer to the points in $\underline{\underline{X}}(K_{\underline{v}})$ lying over the evaluation points of $\underline{X}_{\underline{v}}$ (labelled by $j \in |\mathbb{F}_l|$) as the **evaluation points of $\underline{\underline{\boldsymbol{X}}}_{\underline{\boldsymbol{v}}}$** (labelled by $j \in |\mathbb{F}_l|$). We also shall refer to the sections $G_{\underline{v}} \hookrightarrow \Pi_{\underline{v}} (= \Pi_{\underline{\underline{X}}_{\underline{v}}})$ given by the evaluation points (labelled by $j \in |\mathbb{F}_l|$) as the **evaluation section** of $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$ (labelled by $j \in |\mathbb{F}_l|$). Note that, by using Theorem 3.7 (elliptic cuspidalisation) and Remark 6.12.1 (together with Lemma 7.16, Lemma 7.12), we can group-theoretically reconstruct the evaluation sections from (an isomorph of) $\Pi_{\underline{v}}$.

Let $\mathfrak{D}_{>} = \{\mathcal{D}_{>,\underline{w}}\}_{\underline{w} \in \underline{\mathbb{V}}}$ be a copy of the tautological $\mathcal{D}$-prime-strip $\{\mathcal{D}_{\underline{w}}\}_{\underline{w} \in \underline{\mathbb{V}}}$. Write

$$\begin{aligned} \phi^{\Theta}_{\underline{v}_j} :=& \operatorname{Aut}(\mathcal{D}_{>,\underline{v}}) \circ (\mathcal{B}^{\text{temp}}(\Pi_{\underline{v}})^0 \overset{\text{natural}}{\longrightarrow} \mathcal{B}(K_{\underline{v}})^0 \underset{\text{labelled by } j}{\overset{\text{eval. section}}{\longrightarrow}} \mathcal{B}^{\text{temp}}(\Pi_{\underline{v}})^0) \circ \operatorname{Aut}(\mathcal{D}_{\underline{v}_j}) \\ &: \mathcal{D}_{\underline{v}_j} \overset{\text{poly}}{\longrightarrow} \mathcal{D}_{>,\underline{v}}. \end{aligned}$$

Note that the homomorphism $\pi_1(\mathcal{D}_{\underline{v}_j}) \to \pi_1(\mathcal{D}_{>,\underline{v}})$ induced by any constituent of the poly-morphism $\phi^{\Theta}_{\underline{v}_j}$ (which is well-defined up to inner automorphisms) is compatible with the respective outer actions on $\pi_1^{\text{geo}}(\mathcal{D}_{\underline{v}_j})$ and $\pi_1^{\text{geo}}(\mathcal{D}_{>,\underline{v}})$ (Here we write π_1^{geo} for the geometric portion of π_1, which can be group-theoretically reconstructed by Lemma 6.2) for some outer isomorphism $\pi_1^{\text{geo}}(\mathcal{D}_{\underline{v}_j}) \overset{\sim}{\to} \pi_1^{\text{geo}}(\mathcal{D}_{>,\underline{v}})$ (which is determined up to finite ambiguity by Remark 6.10.1). We say this fact, in short, as $\phi^{\Theta}_{\underline{v}_j}$ is compatible with the outer actions on the respective geometric tempered fundamental groups.

Let $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$. Write

$$\phi^{\Theta}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} \overset{\text{full poly}}{\overset{\sim}{\to}} \mathcal{D}_{>,\underline{v}}$$

to be the full poly-isomorphism for each $j \in \mathbb{F}_l^{*}$,

$$\phi^{\Theta}_j := \{\phi^{\Theta}_{\underline{v}_j}\}_{\underline{v} \in \underline{\mathbb{V}}} : \mathfrak{D}_j \overset{\text{poly}}{\longrightarrow} \mathfrak{D}_{>},$$

and

$$\phi^{\Theta}_{*} := \{\phi^{\Theta}_j\}_{j \in \mathbb{F}_l^{*}} : \mathfrak{D}_{*} \overset{\text{poly}}{\longrightarrow} \mathfrak{D}_{>}.$$

This is called the **model base-(or $\mathcal{D}$-)Θ-bridge** (Note that this is *not* a poly-*iso*morphism). Note that $\mathfrak{D}_{\divideontimes}$ has a natural permutation poly-action by $\mathbb{F}_l^{\divideontimes}$, and that, on the other hand, the labels $\in |\mathbb{F}_l|$ (or $\in \mathrm{LabCusp}(\mathfrak{D}_{>})$) determined by the evaluation sections corresponding to a given $j \in \mathbb{F}_l^{\divideontimes}$ are fixed by any automorphisms of $\mathfrak{D}_{>}$.

Definition 10.18. ($\mathcal{D}$-NF-Bridge, $\mathcal{D}$-Θ-Bridge, and $\mathcal{D}$-⊠-Hodge Theatre, [IUTchI, Definition 4.6])

(1) A **base-(or $\mathcal{D}$-)NF-bridge** is a poly-morphism

$$^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}} : {}^{\dagger}\mathfrak{D}_J \xrightarrow{\text{poly}} {}^{\dagger}\mathcal{D}^{\circledcirc},$$

where $^{\dagger}\mathcal{D}^{\circledcirc}$ is a category equivalent to the model global object $\mathcal{D}^{\circledcirc}$, and $^{\dagger}\mathfrak{D}_J$ is a capsule of $\mathcal{D}$-prime-strips indexed by a finite set J, such that there exist isomorphisms $\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^{\dagger}\mathcal{D}^{\circledcirc}$, $\mathfrak{D}_{\divideontimes} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_J$, conjugation by which sends $\phi_{\divideontimes}^{\mathrm{NF}} \mapsto {}^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}$. An **isomorphism of $\mathcal{D}$-NF-bridges** $\left({}^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}} : {}^{\dagger}\mathfrak{D}_J \xrightarrow{\text{poly}} {}^{\dagger}\mathcal{D}^{\circledcirc}\right) \xrightarrow{\sim} \left({}^{\ddagger}\phi_{\divideontimes}^{\mathrm{NF}} : {}^{\ddagger}\mathfrak{D}_{J'} \xrightarrow{\text{poly}} {}^{\ddagger}\mathcal{D}^{\circledcirc}\right)$ is a pair of a capsule-full poly-isomorphism ${}^{\dagger}\mathfrak{D}_J \overset{\text{capsule-full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{D}_{J'}$ and an $\mathrm{Aut}_{\underline{\epsilon}}({}^{\dagger}\mathcal{D}^{\circledcirc})$-orbit (or, equivalently, an $\mathrm{Aut}_{\underline{\epsilon}}({}^{\ddagger}\mathcal{D}^{\circledcirc})$-orbit) ${}^{\dagger}\mathcal{D}^{\circledcirc} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathcal{D}^{\circledcirc}$ of isomorphisms, which are compatible with ${}^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}$, ${}^{\ddagger}\phi_{\divideontimes}^{\mathrm{NF}}$. We define compositions of them in an obvious manner.

(2) A **base-(or $\mathcal{D}$-)Θ-bridge** is a poly-morphism

$$^{\dagger}\phi_{\divideontimes}^{\Theta} : {}^{\dagger}\mathfrak{D}_J \xrightarrow{\text{poly}} {}^{\dagger}\mathfrak{D}_{>},$$

where ${}^{\dagger}\mathfrak{D}_{>}$ is a $\mathcal{D}$-prime-strip, and ${}^{\dagger}\mathfrak{D}_J$ is a cupsule of $\mathcal{D}$-prime-strips indexed by a finite set J, such that there exist isomorphisms $\mathfrak{D}_{>} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_{>}$, $\mathfrak{D}_{\divideontimes} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_J$, conjugation by which sends $\phi_{\divideontimes}^{\Theta} \mapsto {}^{\dagger}\phi_{\divideontimes}^{\Theta}$. An **isomorphism of $\mathcal{D}$-Θ-bridges** $\left({}^{\dagger}\phi_{\divideontimes}^{\Theta} : {}^{\dagger}\mathfrak{D}_J \xrightarrow{\text{poly}} {}^{\dagger}\mathfrak{D}_{>}\right) \xrightarrow{\sim} \left({}^{\ddagger}\phi_{\divideontimes}^{\Theta} : {}^{\ddagger}\mathfrak{D}_{J'} \xrightarrow{\text{poly}} {}^{\ddagger}\mathfrak{D}_{>}\right)$ is a pair of a capsule-full poly-isomorphism ${}^{\dagger}\mathfrak{D}_J \overset{\text{capsule-full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{D}_{J'}$ and the full-poly isomorphism ${}^{\dagger}\mathfrak{D}_{>} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{D}_{>}$, which are compatible with ${}^{\dagger}\phi_{\divideontimes}^{\Theta}$, ${}^{\ddagger}\phi_{\divideontimes}^{\Theta}$. We define compositions of them in an obvious manner.

(3) A **base-(or $\mathcal{D}$-)ΘNF-Hodge theatre** (or a **$\mathcal{D}$-⊠-Hodge theatre**) is a collection

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes} = \left({}^{\dagger}\mathcal{D}^{\circledcirc} \xleftarrow{{}^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}} {}^{\dagger}\mathfrak{D}_J \xrightarrow{{}^{\dagger}\phi_{\divideontimes}^{\Theta}} {}^{\dagger}\mathfrak{D}_{>}\right),$$

where ${}^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}$ is a $\mathcal{D}$-NF-bridge, and ${}^{\dagger}\phi_{\divideontimes}^{\Theta}$ is a $\mathcal{D}$-Θ-bridge, such that there exist isomorphisms $\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^{\dagger}\mathcal{D}^{\circledcirc}$, $\mathfrak{D}_{\divideontimes} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_J$, $\mathfrak{D}_{>} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_{>}$, conjugation by which sends

$\phi^{\mathrm{NF}}_{\divideontimes} \mapsto {}^\dagger\phi^{\mathrm{NF}}_{\divideontimes}$, $\phi^{\Theta}_{\divideontimes} \mapsto {}^\dagger\phi^{\Theta}_{\divideontimes}$. An **isomorphism of $\mathcal{D}$-$\boxtimes$-Hodge theatres** is a pair of isomorphisms of $\mathcal{D}$-NF-bridges and $\mathcal{D}$-Θ-bridges such that they induce the same bijection between the index sets of the respective capsules of $\mathcal{D}$-prime-strips. We define compositions of them in an obvious manner.

Proposition 10.19. (Transport of Label Classes of Cusps via Base-Bridges, [IUTchI, Proposition 4.7]) *Let* ${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes} = ({}^\dagger\mathcal{D}^{\circledcirc} \overset{{}^\dagger\phi^{\mathrm{NF}}_{\divideontimes}}{\longleftarrow} {}^\dagger\mathfrak{D}_J \overset{{}^\dagger\phi^{\Theta}_{\divideontimes}}{\longrightarrow} {}^\dagger\mathfrak{D}_{>})$ *be a* $\mathcal{D}$-$\boxtimes$-*Hodge theatre.*

(1) The structure of $\mathcal{D}$-Θ-*bridge* ${}^\dagger\phi^{\Theta}_{\divideontimes}$ *at* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *involving the evaluation sections determines a bijection*

$$ {}^\dagger\chi : J \xrightarrow{\sim} \mathbb{F}_l^{\divideontimes}. $$

(2) For $j \in J$, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ *(resp.* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*), we consider the various outer homomorphisms* $\pi_1({}^\dagger\mathcal{D}_{\underline{v}_j}) \to \pi_1({}^\dagger\mathcal{D}^{\circledcirc})$ *induced by the* $(\underline{v}, j)$*-portion* ${}^\dagger\phi^{\mathrm{NF}}_{\underline{v}_j} : {}^\dagger\mathcal{D}_{\underline{v}_j} \to {}^\dagger\mathcal{D}^{\circledcirc}$ *of the* $\mathcal{D}$-NF-*bridge* ${}^\dagger\phi^{\mathrm{NF}}_{\divideontimes}$*. By considering cuspidal inertia subgroups of* $\pi_1({}^\dagger\mathcal{D}^{\circledcirc})$ *whose unique subgroup of index* l *is contained in the image of this homomorphism (resp. the closures in* $\pi_1({}^\dagger\mathcal{D}^{\circledcirc})$ *of the images of cuspidal inertia subgroups of* $\pi_1({}^\dagger\mathcal{D}_{\underline{v}_j})$ *(cf. Definition 10.14 (2) for the group-theoretic reconstruction of cuspidal inertia subgroups for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*), these homomorphisms induce a natural isomorphism*

$$ \mathrm{LabCusp}({}^\dagger\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}_j}) $$

of $\mathbb{F}_l^{\divideontimes}$*-torsors. These isomorphisms are compatible with the isomorphism* $\mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}_j}) \xrightarrow{\sim} \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{w}_j})$ *of* $\mathbb{F}_l^{\divideontimes}$*-torsors in Proposition 10.15 when we vary* $\underline{v} \in \underline{\mathbb{V}}$*. Hence we obtaine a natural isomorphism*

$$ \mathrm{LabCusp}({}^\dagger\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}({}^\dagger\mathfrak{D}_j) $$

of $\mathbb{F}_l^{\divideontimes}$*-torsors.*

Next, for each $j \in J$*, the various* $\underline{v}(\in \underline{\mathbb{V}})$*-portions of the* j*-portion* ${}^\dagger\phi^{\Theta}_j : {}^\dagger\mathfrak{D}_j \to {}^\dagger\mathfrak{D}_{>}$ *of the* $\mathcal{D}$-Θ-*bridge* ${}^\dagger\phi^{\Theta}_{\divideontimes}$ *determine an isomorphism*

$$ \mathrm{LabCusp}({}^\dagger\mathfrak{D}_j) \xrightarrow{\sim} \mathrm{LabCusp}({}^\dagger\mathfrak{D}_{>}) $$

of $\mathbb{F}_l^{\divideontimes}$*-torsors. Therefore, for each* $j \in J$*, by composing isomorphisms of* $\mathbb{F}_l^{\divideontimes}$*-torsors obtained via* ${}^\dagger\phi^{\mathrm{NF}}_j$, ${}^\dagger\phi^{\Theta}_j$*, we get an isomorphism*

$$ {}^\dagger\phi^{\mathrm{LC}}_j : \mathrm{LabCusp}({}^\dagger\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}({}^\dagger\mathfrak{D}_{>}) $$

of $\mathbb{F}_l^{\divideontimes}$*-torsors, such that* ${}^\dagger\phi^{\mathrm{LC}}_j$ *is obtained from* ${}^\dagger\phi^{\mathrm{LC}}_1$ *by the action by* ${}^\dagger\chi(j) \in \mathbb{F}_l^{\divideontimes}$*.*

(3) *By considering the canonical elements* ${}^\dagger\underline{\eta}_{\underline{v}} \in \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}})$ *for* $\underline{v}$*'s, we obtain a unique element*

$$[{}^\dagger\underline{\epsilon}] \in \mathrm{LabCusp}({}^\dagger\mathcal{D}^{\circledcirc})$$

such that, for each $j \in J$*, the natural bijection* $\mathrm{LabCusp}({}^\dagger\mathfrak{D}_{>}) \xrightarrow{\sim} \mathbb{F}_l^{\circledast}$ *in Proposition 10.15 sends* ${}^\dagger\phi_j^{\mathrm{LC}}([{}^\dagger\underline{\epsilon}]) = {}^\dagger\phi_1^{\mathrm{LC}}({}^\dagger\chi(j) \cdot [{}^\dagger\underline{\epsilon}]) \mapsto {}^\dagger\chi(j)$*. In particular, the element* $[{}^\dagger\underline{\epsilon}]$ *determines an isomorphism*

$${}^\dagger\zeta_{\circledast} : \mathrm{LabCusp}({}^\dagger\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} J \ (\xrightarrow{\sim} \mathbb{F}_l^{\circledast})$$

of $\mathbb{F}_l^{\circledast}$*-torsors.*

Remark 10.19.1. (cf. [IUTchI, Remark 4.5.1]) We consider the group-theoretic algorithm in Proposition 10.19 (2) for $\underline{v} \in \underline{\mathbb{V}}$. Here, the morphism $\pi_1({}^\dagger\mathcal{D}_{\underline{v}_j}) \to \pi_1({}^\dagger\mathcal{D}^{\circledcirc})$ is only known up to $\pi_1({}^\dagger\mathcal{D}^{\circledcirc})$-conjugacy, and a cuspidal inertia subgroup labelled by an element $\in \mathrm{LabCusp}({}^\dagger\mathcal{D}^{\circledcirc})$ is also well-defined up to $\pi_1({}^\dagger\mathcal{D}^{\circledcirc})$-conjugacy. *We have no natural way to synchronise these indeterminacies.* Let J be the unique open subgroup of index l of a cuspidal inertia subgroup. A nontrivial fact is that, if we use Theorem 6.11, then we can factorise $J \hookrightarrow \pi_1({}^\dagger\mathcal{D}^{\circledcirc})$ up to $\pi_1({}^\dagger\mathcal{D}^{\circledcirc})$-conjugacy into $J \hookrightarrow \pi_1({}^\dagger\mathcal{D}_{\underline{v}_j})$ up to $\pi_1({}^\dagger\mathcal{D}_{\underline{v}_j})$-conjugacy and $\pi_1({}^\dagger\mathcal{D}_{\underline{v}_j}) \hookrightarrow \pi_1({}^\dagger\mathcal{D}^{\circledcirc})$ up to $\pi_1({}^\dagger\mathcal{D}^{\circledcirc})$-conjugacy (i.e., factorise $J \overset{\mathrm{out}}{\hookrightarrow} \pi_1({}^\dagger\mathcal{D}^{\circledcirc})$ as $J \overset{\mathrm{out}}{\hookrightarrow} \pi_1({}^\dagger\mathcal{D}_{\underline{v}_j}) \overset{\mathrm{out}}{\hookrightarrow} \pi_1({}^\dagger\mathcal{D}^{\circledcirc})$). This can be regarded as a *partial synchronisation of the indeterminacies.*

Proof. The proposition immediately follows from the described algorithms. □

The following proposition follows from the definitions:

Proposition 10.20. (Properties of $\mathcal{D}$-NF-Brideges, $\mathcal{D}$-Θ-Bridges, $\mathcal{D}$-$\boxtimes$-Hodge theatres, [IUTchI, Proposition 4.8])

(1) *For* $\mathcal{D}$-NF*-bridges* ${}^\dagger\phi_{\circledast}^{\mathrm{NF}}$, ${}^\ddagger\phi_{\circledast}^{\mathrm{NF}}$*, the set* $\mathrm{Isom}({}^\dagger\phi_{\circledast}^{\mathrm{NF}}, {}^\ddagger\phi_{\circledast}^{\mathrm{NF}})$ *is an* $\mathbb{F}_l^{\circledast}$*-torsor.*

(2) *For* $\mathcal{D}$-Θ*-bridges* ${}^\dagger\phi_{\circledast}^{\Theta}$, ${}^\ddagger\phi_{\circledast}^{\Theta}$*, we have* $\#\mathrm{Isom}({}^\dagger\phi_{\circledast}^{\mathrm{NF}}, {}^\ddagger\phi_{\circledast}^{\mathrm{NF}}) = 1$.

(3) *For* $\mathcal{D}$-$\boxtimes$*-Hodge theatres* ${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$, ${}^\ddagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$*, we have* $\#\mathrm{Isom}({}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}, {}^\ddagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}) = 1$.

(4) *For a* $\mathcal{D}$-NF*-bridge* ${}^\dagger\phi_{\circledast}^{\mathrm{NF}}$ *and a* $\mathcal{D}$-Θ*-bridge* ${}^\dagger\phi_{\circledast}^{\Theta}$*, the set*

$$\left\{ \textit{capsule-full poly-isom. } {}^\dagger\mathfrak{D}_J \overset{\text{capsule-full poly}}{\xrightarrow{\sim}} {}^\dagger\mathfrak{D}_{J'} \textit{ by which } {}^\dagger\phi_{\circledast}^{\mathrm{NF}}, {}^\dagger\phi_{\circledast}^{\Theta} \textit{ form a } \mathcal{D}\text{-}\boxtimes\text{-}\textit{Hodge theatre} \right\}$$

is an $\mathbb{F}_l^{\circledast}$*-torsor.*

(5) For a $\mathcal{D}$-NF*-bridge* ${}^{\dagger}\phi_{\circledast}^{\mathrm{NF}}$*, we have a functorial algorithm to construct, up to* $\mathbb{F}_l^{\circledast}$*-indeterminacy, a* $\mathcal{D}$-$\boxtimes$*-Hodge theatre whose* $\mathcal{D}$-NF*-bridge is* ${}^{\dagger}\phi_{\circledast}^{\mathrm{NF}}$*.*

Definition 10.21. ([IUTchI, Corollary 4.12]) Let ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$, ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$ be $\mathcal{D}$-$\boxtimes$-Hodge theatres. the **base-(or $\mathcal{D}$-)ΘNF-link** (or $\mathcal{D}$-$\boxtimes$-link)

$$ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes} \xrightarrow{\mathcal{D}} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes} $$

is the full poly-isomorphism

$$ {}^{\dagger}\mathfrak{D}^{\vdash}_{>} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{D}^{\vdash}_{>} $$

between the mono-analyticisations of the codomains of the $\mathcal{D}$-Θ-bridges.

Remark 10.21.1. In $\mathcal{D}$-$\boxtimes$-link, the $\mathcal{D}^{\vdash}$-prime-strips are shared, but not the arithmetically holomorphic structures. We can visualise the "shared" and "non-shared" relation as follows:

$$ \boxed{{}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}} \dashrightarrow \boxed{{}^{\dagger}\mathfrak{D}^{\vdash}_{>} \cong {}^{\ddagger}\mathfrak{D}^{\vdash}_{>}} \dashleftarrow \boxed{{}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}} $$

We shall refer to this diagram as the **étale-picture of $\mathcal{D}$-$\boxtimes$-Hodge theatres**. Note that *we have a permutation symmetry in the étale-picture.*

We constructed $\mathcal{D}$-$\boxtimes$-Hodge theatres. These are base objects. Now, we begin constructing the total spaces, i.e., $\boxtimes$-Hodge theatres, by putting Frobenioids on them.

We start with the following situation: Let ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes} = ({}^{\dagger}\mathcal{D}^{\circledcirc} \overset{{}^{\dagger}\phi_{\circledast}^{\mathrm{NF}}}{\longleftarrow} {}^{\dagger}\mathfrak{D}_J \overset{{}^{\dagger}\phi_{\circledast}^{\Theta}}{\longrightarrow} {}^{\dagger}\mathfrak{D}_{>})$ be a $\mathcal{D}$-$\boxtimes$-Hodge theatre (with respect to the fixed initial Θ-data). Let ${}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$ be a Θ-Hodge theatre, whose associseted $\mathcal{D}$-prime strip is equal to ${}^{\dagger}\mathfrak{D}_{>}$ in the given $\mathcal{D}$-$\boxtimes$-Hodge theatre. We write ${}^{\dagger}\mathfrak{F}_{>}$ for the $\mathcal{F}$-prime-strip tautologically associated to (the $\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$-portion of) the Θ-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\Theta}$. Note that ${}^{\dagger}\mathfrak{D}_{>}$ can ben identified with the $\mathcal{D}$-prime-strip associated to ${}^{\dagger}\mathfrak{F}_{>}$:

$$ \begin{array}{ccc} {}^{\dagger}\mathcal{HT}^{\Theta} & \longmapsto & {}^{\dagger}\mathfrak{F}_{>} \\ & & \downarrow \\ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes} & \longmapsto & {}^{\dagger}\mathfrak{D}_{>}. \end{array} $$

Definition 10.22. ([IUTchI, Example 5.4 (iii), (iv)]) Let ${}^{\dagger}\mathcal{F}^{\circledast}$ be a pre-Frobenioid isomorphic to $\mathcal{F}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$ as in Example 9.5, where ${}^{\dagger}\mathcal{D}^{\circledcirc}$ is the data in the given $\mathcal{D}$-$\boxtimes$-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$. We write ${}^{\dagger}\mathcal{F}^{\circledcirc} := {}^{\dagger}\mathcal{F}^{\circledast}|_{{}^{\dagger}\mathcal{D}^{\circledcirc}}$, and ${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}} := {}^{\dagger}\mathcal{F}^{\circledast}|_{\text{terminal object in } {}^{\dagger}\mathcal{D}^{\circledast}}$, as in Example 9.5.

(1) For $\delta \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$, a **$\boldsymbol{\delta}$-valuation** $\in \mathbb{V}({}^{\dagger}\mathcal{D}^{\circledcirc})$ is a valuation which lies in the "image" (in the obvious sense) via ${}^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}$ of the unique $\mathcal{D}$-prime-strip ${}^{\dagger}\mathfrak{D}_j$ of the capsule ${}^{\dagger}\mathfrak{D}_J$ such that the bijection $\mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}({}^{\dagger}\mathfrak{D}_j)$ induced by ${}^{\dagger}\phi_j^{\mathrm{NF}}$ sends δ to the element of $\mathrm{LabCusp}({}^{\dagger}\mathfrak{D}_j) \xrightarrow{\sim} \mathbb{F}_l^{\divideontimes}$ (cf. Proposition 10.15) labelled by $1 \in \mathbb{F}_l^{\divideontimes}$ (Note that, if we allow ourselves to use the model object $\mathcal{D}^{\circledcirc}$, then a δ-valuation $\in \mathbb{V}({}^{\dagger}\mathcal{D}^{\circledcirc})$ is an element, which is sent to an elemento of $\underline{\mathbb{V}}^{\pm\mathrm{un}} \subset \mathbb{V}(K)$ under the bijection $\mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(\mathcal{D}^{\circledcirc})$ induced by a unique $\mathrm{Aut}_{\underline{\epsilon}}({}^{\dagger}\mathcal{D}^{\circledcirc})$-orbit of isomorphisms ${}^{\dagger}\mathcal{D}^{\circledcirc} \xrightarrow{\sim} \mathcal{D}^{\circledcirc}$ sending $\delta \mapsto [\underline{\epsilon}] \in \mathrm{LabCusp}(\mathcal{D}^{\circledcirc})$).

(2) For $\delta \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$, by localising at each of the δ-valuations $\in \mathbb{V}({}^{\dagger}\mathcal{D}^{\circledcirc})$, from ${}^{\dagger}\mathcal{F}^{\circledcirc}$ (or, from $(({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}} \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}) = (\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc}) \curvearrowright \widetilde{O}^{\circledast\times})$ in Definition 9.6), we can construct an $\mathcal{F}$-prime-strip

$$ {}^{\dagger}\mathcal{F}^{\circledcirc}|_{\delta} $$

which is *well-defined up to isomorphism* (Note that the natural projection $\underline{\mathbb{V}}^{\pm\mathrm{un}} \twoheadrightarrow \mathbb{V}_{\mathrm{mod}}$ is *not* injective, hence it is necessary to think that ${}^{\dagger}\mathcal{F}|_{\delta}$ is *well-defined only up to isomorphism*, since there is *no* canonical choice of an element of a fiber of the natural projection $\underline{\mathbb{V}}^{\pm\mathrm{un}} \twoheadrightarrow \mathbb{V}_{\mathrm{mod}}$) as follows: For a non-Archimedean δ-valuation $\underline{v}$, it is the $p_{\underline{v}}$-adic Frobenioid associated to the restrictions to "the open subgroup" of ${}^{\dagger}\Pi_{\mathfrak{p}_0} \cap \pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$ determined by $\delta \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$ (i.e., corresponding to "$\underline{\underline{X}}$" or "$\underrightarrow{X}$") (cf. Definition 9.6 for ${}^{\dagger}\Pi_{\mathfrak{p}_0}$). Here, if $\underline{v}$ lies over an element of $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$, then we have to replace the above "open subgroup" by its tempered analogue, which can be done by reconstructing, from the open subgroup of ${}^{\dagger}\Pi_{\mathfrak{p}_0} \cap \pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$, the semi-graph of anabelioids by Remark 6.12.1 (cf. also [SemiAnbd, Theorem 6.6]). For an Archimedean δ-valuation $\underline{v}$, this follows from Proposition 4.8, Lemma 4.9, and the isomorphism $\mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} {}^{\dagger}\mathbb{M}^{\circledast}$ in Example 9.5.

(3) For an $\mathcal{F}$-prime-strip ${}^{\ddagger}\mathfrak{F}$ whose associated $\mathcal{D}$-prime-strip is ${}^{\ddagger}\mathfrak{D}$, a **poly-morphism**

$$ {}^{\ddagger}\mathfrak{F} \xrightarrow{\mathrm{poly}} {}^{\dagger}\mathcal{F}^{\circledcirc} $$

is a full poly-isomorphism ${}^{\ddagger}\mathfrak{F} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\dagger}\mathcal{F}^{\circledcirc}|_{\delta}$ for some $\delta \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$ (Note that the fact that ${}^{\dagger}\mathcal{F}^{\circledcirc}|_{\delta}$ is well-defined only up to isomorphism is harmless here). We regard such a poly-morphism ${}^{\ddagger}\mathfrak{F} \xrightarrow{\mathrm{poly}} {}^{\dagger}\mathcal{F}^{\circledcirc}$ is lying over an induced poly-morphism ${}^{\ddagger}\mathfrak{D} \xrightarrow{\mathrm{poly}} {}^{\dagger}\mathcal{D}^{\circledcirc}$. Note also that such a poly-morphism ${}^{\ddagger}\mathfrak{F} \xrightarrow{\mathrm{poly}} {}^{\dagger}\mathcal{F}^{\circledcirc}$ is compatible with the local and global ${}_{\infty}\kappa$-coric structures (cf. Definition 9.6) in the following sense: The restriction of associated Kummer classes determines a collection of poly-morphisms of pseudo-monoids

$$ \left\{ ({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}} \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{{}_{\infty}\kappa} \xrightarrow{\mathrm{poly}} {}^{\ddagger}\mathbb{M}_{{}_{\infty}\kappa v} \subset {}^{\ddagger}\mathbb{M}_{{}_{\infty}\kappa\times v} \right\}_{\underline{v}\in\underline{\mathbb{V}}} $$

indexed by $\underline{\mathbb{V}}$, where the left-hand side $({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}} \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}$ is well-defined up to automorphisms induced by the inner automorphisms of $({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}$, and the right-hand side ${}^{\ddagger}\mathbb{M}_{\infty\kappa v} \subset {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}$ is well-defined up to automorphisms induced by the automorphisms of the $\mathcal{F}$-prime strip ${}^{\ddagger}\mathfrak{F}$. For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the above poly-morphism is equivariant with respect to the homomorphisms $({}^{\ddagger}\Pi_v)^{\mathrm{rat}} \to ({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}$ (cf. Definition 9.6 (2) for $({}^{\ddagger}\Pi_v)^{\mathrm{rat}}$) induced by the given poly-morphism ${}^{\ddagger}\mathfrak{F} \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathcal{F}^{\circledcirc}$.

(4) For a capsule ${}^{E}\mathfrak{F} = \{{}^{e}\mathfrak{F}\}$ of $\mathcal{F}$-prime-strips, whose associated capsule of $\mathcal{D}$-prime-strips is ${}^{E}\mathfrak{D}$, and an $\mathcal{F}$-prime-strip ${}^{\dagger}\mathfrak{F}$ whose associated $\mathcal{D}$-prime-strip is ${}^{\dagger}\mathcal{D}$, a **poly-morphism**

$$ {}^{E}\mathfrak{F} \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathcal{F}^{\circledcirc} \quad (\text{resp. } {}^{E}\mathfrak{F} \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{F}\ ) $$

is a collection of poly-morphisms $\{{}^{e}\mathfrak{F} \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathcal{F}^{\circledcirc}\}_{e\in E}$ (resp. $\{{}^{e}\mathfrak{F} \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{F}\}_{e\in E}$). We consider a poly-morphism ${}^{E}\mathfrak{F} \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathcal{F}^{\circledcirc}$ (resp. ${}^{E}\mathfrak{F} \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{F}$) as lying over the induced poly-morphism ${}^{E}\mathfrak{D} \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc}$ (resp. ${}^{E}\mathfrak{D} \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{D}$).

We return to the situation of

$$\begin{tikzcd} {}^{\dagger}\mathcal{HT}^{\Theta} \arrow[r, mapsto] & {}^{\dagger}\mathfrak{F}_{>} \arrow[d, mapsto] \\ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes} \arrow[r, mapsto] & {}^{\dagger}\mathfrak{D}_{>}. \end{tikzcd}$$

Definition 10.23. (Model Θ-Bridge, Model NF-Bridge, Diagonal $\mathcal{F}$-Objects, Localisation Functors, [IUTchI, Example 5.4 (ii), (v), (i), (vi), Example 5.1 (vii)]) For $j \in J$, let ${}^{\dagger}\mathfrak{F}_j = \{{}^{\dagger}\mathcal{F}_{\underline{v}_j}\}_{j\in J}$ be an $\mathcal{F}$-prime-strip whose associated $\mathcal{D}$-prime-strip is equal to ${}^{\dagger}\mathfrak{D}_j$. We also write ${}^{\dagger}\mathfrak{F}_J := \{{}^{\dagger}\mathfrak{F}_j\}_{j\in J}$ (i.e., a capsule indexed by $j \in J$).

Let ${}^{\dagger}\mathcal{F}^{\circledast}$ be a pre-Frobenioid isomorphic to $\mathcal{F}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$ as in Example 9.5, where ${}^{\dagger}\mathcal{D}^{\circledcirc}$ is the data in the given $\mathcal{D}$-$\boxtimes$-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$. We write ${}^{\dagger}\mathcal{F}^{\circledcirc} := {}^{\dagger}\mathcal{F}^{\circledast}|_{{}^{\dagger}\mathcal{D}^{\circledcirc}}$, and ${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}} := {}^{\dagger}\mathcal{F}^{\circledast}|_{\text{terminal object in } {}^{\dagger}\mathcal{D}^{\circledast}}$, as in Example 9.5.

(1) For $j \in J$, we write

$$ {}^{\dagger}\psi^{\Theta}_j : {}^{\dagger}\mathfrak{F}_j \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{F}_{>} $$

for the poly-morphism (cf. Definition 10.22 (4)) uniquely determined by ${}^{\dagger}\phi_j$ by Remark 10.10.1. Write

$$ {}^{\dagger}\psi^{\Theta}_{\divideontimes} := \{{}^{\dagger}\psi^{\Theta}_j\}_{j\in\mathbb{F}^{\divideontimes}_l} : {}^{\dagger}\mathfrak{F}_J \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{F}_{>}. $$

We regard ${}^{\dagger}\psi^{\Theta}_{\divideontimes}$ as lying over ${}^{\dagger}\phi^{\Theta}_{\divideontimes}$. We shall refer to ${}^{\dagger}\psi^{\Theta}_{\divideontimes}$ as the **model Θ-bridge**.

cf. also the following diagram:

$$\begin{tikzcd}
{}^{\dagger}\mathfrak{F}_j,\ {}^{\dagger}\mathfrak{F}_J \arrow[d, mapsto] \arrow[rr, dotted, bend left, "{}^{\dagger}\psi_j^{\Theta},\ {}^{\dagger}\psi_{\divideontimes}^{\Theta}"] & {}^{\dagger}\mathcal{HT}^{\Theta} \arrow[r, mapsto] & {}^{\dagger}\mathfrak{F}_{>} \arrow[d, mapsto] \\
{}^{\dagger}\mathfrak{D}_j,\ {}^{\dagger}\mathfrak{D}_J \arrow[rr, bend right, "{}^{\dagger}\phi_j^{\Theta},\ {}^{\dagger}\phi_{\divideontimes}^{\Theta}"'] & {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes} \arrow[l, mapsto] \arrow[r, mapsto] & {}^{\dagger}\mathfrak{D}_{>}.
\end{tikzcd}$$

(2) For $j \in J$, we write

$$ {}^{\dagger}\psi_j^{\mathrm{NF}} : {}^{\dagger}\mathfrak{F}_j \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathcal{F}^{\circledcirc} $$

for the poly-morphism (cf. Definition 10.22 (3)) uniquely determined by ${}^{\dagger}\phi_j$ by Lemma 10.10 (2). Write

$$ {}^{\dagger}\psi_{\divideontimes}^{\mathrm{NF}} := \{{}^{\dagger}\psi_j^{\mathrm{NF}}\}_{j \in \mathbb{F}_l^{\divideontimes}} : {}^{\dagger}\mathfrak{F}_J \overset{\mathrm{poly}}{\longrightarrow} {}^{\dagger}\mathcal{F}^{\circledcirc}. $$

We regard ${}^{\dagger}\psi_{\divideontimes}^{\mathrm{NF}}$ as lying over ${}^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}$. We shall refer to ${}^{\dagger}\psi_{\divideontimes}^{\mathrm{NF}}$ as the **model** NF-**bridge**. cf. also the following diagram:

$$\begin{tikzcd}
{}^{\dagger}\mathfrak{F}_j,\ {}^{\dagger}\mathfrak{F}_J \arrow[d, mapsto] \arrow[rr, dotted, bend left, "{}^{\dagger}\psi_j^{\mathrm{NF}},\ {}^{\dagger}\psi_{\divideontimes}^{\mathrm{NF}}"] & & {}^{\dagger}\mathcal{F}^{\circledcirc} \arrow[d, mapsto] \\
{}^{\dagger}\mathfrak{D}_j,\ {}^{\dagger}\mathfrak{D}_J \arrow[rr, bend right, "{}^{\dagger}\phi_j^{\mathrm{NF}},\ {}^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}"'] & {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes} \arrow[l, mapsto] \arrow[r, mapsto] & {}^{\dagger}\mathcal{D}^{\circledcirc}.
\end{tikzcd}$$

(3) Let also ${}^{\dagger}\mathfrak{F}_{\langle J\rangle} = \{{}^{\dagger}\mathcal{F}_{\underline{v}_{\langle J\rangle}}\}_{\underline{v}_{\langle J\rangle} \in \underline{\mathbb{V}}_{\langle J\rangle}}$ be an $\mathcal{F}$-prime-strip. We write ${}^{\dagger}\mathfrak{D}_{\langle J\rangle}$ for the associated $\mathcal{D}$-prime-strip to ${}^{\dagger}\mathfrak{F}_{\langle J\rangle}$. We write $\underline{\mathbb{V}}_j := \{\underline{v}_j\}_{\underline{v} \in \underline{\mathbb{V}}}$. We have a natural bijection $\underline{\mathbb{V}}_j \overset{\sim}{\to} \underline{\mathbb{V}} : \underline{v}_j \mapsto \underline{v}$. These bijections determine the diagonal subset

$$ \underline{\mathbb{V}}_{\langle J\rangle} \subset \underline{\mathbb{V}}_J := \prod_{j \in J} \underline{\mathbb{V}}_j, $$

which admits a natural bijection $\underline{\mathbb{V}}_{\langle J\rangle} \overset{\sim}{\to} \underline{\mathbb{V}}$. Hence we obtain a natural bijection $\underline{\mathbb{V}}_{\langle J\rangle} \overset{\sim}{\to} \underline{\mathbb{V}}_j$ for $j \in J$.

We have the full poly-isomorphism

$$^\dagger\mathfrak{F}_{\langle J\rangle} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^\dagger\mathfrak{F}_{>}$$

and the "diagonal arrow"

$$^\dagger\mathfrak{F}_{\langle J\rangle} \longrightarrow {}^\dagger\mathfrak{F}_{J},$$

which is the collection of the full poly-isomorphisms $^\dagger\mathfrak{F}_{\langle J\rangle} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^\dagger\mathfrak{F}_{j}$ indexed by $j \in J$. We regard $^\dagger\mathfrak{F}_{j}$ (resp. $^\dagger\mathfrak{F}_{\langle J\rangle}$) as a copy of $^\dagger\mathfrak{F}_{>}$ "situated on" the constituent labelled by $j \in J$ (resp. "situated in a diagonal fashion on" all the consitutents) of the capsule $^\dagger\mathfrak{D}_J$.

We have natural bijections

$$\underline{\mathbb{V}}_{\langle J\rangle} \xrightarrow{\sim} \underline{\mathbb{V}}_{j} \xrightarrow{\sim} \mathrm{Prime}({}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}) \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$$

for $j \in J$. Write

$$^\dagger\mathcal{F}^{\circledast}_{\langle J\rangle} := \{{}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}, \underline{\mathbb{V}}_{\langle J\rangle} \xrightarrow{\sim} \mathrm{Prime}({}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}})\},$$

$$^\dagger\mathcal{F}^{\circledast}_{j} := \{{}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}, \underline{\mathbb{V}}_{j} \xrightarrow{\sim} \mathrm{Prime}({}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}})\}$$

for $j \in J$. We regard $^\dagger\mathcal{F}^{\circledast}_{j}$ (resp. $^\dagger\mathcal{F}^{\circledast}_{\langle J\rangle}$) as a copy of $^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}$ "situated on" the constituent labelled by $j \in J$ (resp. "situated in a diagonal fashion on" all the consitutents) of the capsule $^\dagger\mathfrak{D}_J$. When we write $^\dagger\mathcal{F}^{\circledast}_{\langle J\rangle}$ for the underlying category (i.e., $^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}$) of $^\dagger\mathcal{F}^{\circledast}_{\langle J\rangle}$ by abuse of notation, we have a natural embedding of categories

$$^\dagger\mathcal{F}^{\circledast}_{\langle J\rangle} \hookrightarrow {}^\dagger\mathcal{F}^{\circledast}_{J} := \prod_{j\in J} {}^\dagger\mathcal{F}^{\circledast}_{j}.$$

Note that we do not regard the category $^\dagger\mathcal{F}^{\circledast}_{J}$ as being a (pre-)Frobenioid. We write $^\dagger\mathcal{F}^{\circledast\mathbb{R}}_{j}$, $^\dagger\mathcal{F}^{\circledast\mathbb{R}}_{\langle J\rangle}$ for the realifications (Definition 8.4) of $^\dagger\mathcal{F}^{\circledast}_{\langle J\rangle}$, $^\dagger\mathcal{F}^{\circledast}_{\langle J\rangle}$ respectively, and write $^\dagger\mathcal{F}^{\circledast\mathbb{R}}_{J} := \prod_{j\in J} {}^\dagger\mathcal{F}^{\circledast\mathbb{R}}_{j}$.

Since $^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}$ is defined by the restriction to the terminal object of $^\dagger\mathcal{D}^{\circledast}$, *any* poly-morphism $^\dagger\mathfrak{F}_{\langle J\rangle} \xrightarrow{\text{poly}} {}^\dagger\mathcal{F}^{\circledcirc}$ (resp. $^\dagger\mathfrak{F}_{j} \xrightarrow{\text{poly}} {}^\dagger\mathcal{F}^{\circledcirc}$) (cf. Definition 10.22 (3)) induces, via restriction (in the obvious sense), the *same* isomorphism class

$$({}^\dagger\mathcal{F}^{\circledcirc} \to {}^\dagger\mathcal{F}^{\circledast} \supset){}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} {}^\dagger\mathcal{F}^{\circledast}_{\langle J\rangle} \overset{\text{gl. to loc.}}{\longrightarrow} {}^\dagger\mathcal{F}_{\underline{v}_{\langle J\rangle}}$$

$$(\text{resp.}\quad ({}^\dagger\mathcal{F}^{\circledcirc} \to {}^\dagger\mathcal{F}^{\circledast} \supset){}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} {}^\dagger\mathcal{F}^{\circledast}_{j} \overset{\text{gl. to loc.}}{\longrightarrow} {}^\dagger\mathcal{F}_{\underline{v}_{j}}\quad)$$

of restriction functors, for each $\underline{v}_{\langle J\rangle} \in \underline{\mathbb{V}}_{\langle J\rangle}$ (resp. $\underline{v}_j \in \underline{\mathbb{V}}_j$) (Here, for $\underline{v}_{\langle J\rangle} \in \underline{\mathbb{V}}^{\mathrm{arc}}_{\langle J\rangle}$ (resp. $\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{arc}}_j$), we write ${}^{\dagger}\mathcal{F}_{\underline{v}_{\langle J\rangle}}$ (resp. ${}^{\dagger}\mathcal{F}_{\underline{v}_j}$) for the category component of the triple, by abuse of notation), i.e., it is *independent* of the choice (among its $\mathbb{F}^{\circledast}_l$-conjugates) of the poly-morphism ${}^{\dagger}\mathcal{F}_{\langle J\rangle} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$ (resp. ${}^{\dagger}\mathcal{F}_j \to {}^{\dagger}\mathcal{F}^{\circledcirc}$). cf. also Remark 11.22.1 and Remark 9.6.2 (4) (in the second numeration). We write

$$({}^{\dagger}\mathcal{F}^{\circledcirc} \to {}^{\dagger}\mathcal{F}^{\circledast} \supset){}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledast}_{\langle J\rangle} \overset{\text{gl. to loc.}}{\longrightarrow} {}^{\dagger}\mathfrak{F}_{\langle J\rangle}$$

$$(\text{resp.}\quad ({}^{\dagger}\mathcal{F}^{\circledcirc} \to {}^{\dagger}\mathcal{F}^{\circledast} \supset){}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledast}_{j} \overset{\text{gl. to loc.}}{\longrightarrow} {}^{\dagger}\mathfrak{F}_{j}\quad)$$

for the collection of the above *isomorphism classes* of restriction functors, as $\underline{v}_{\langle J\rangle}$ (resp. $\underline{v}_j$) ranges over the elements of $\underline{\mathbb{V}}_{\langle J\rangle}$ (resp. $\underline{\mathbb{V}}_j$). By combining $j \in J$, we also obtain a natural *isomorphism classes*

$${}^{\dagger}\mathcal{F}^{\circledast}_{J} \overset{\text{gl. to loc.}}{\longrightarrow} {}^{\dagger}\mathfrak{F}_{J}$$

of restriction functors. We also obtain their natural realifications

$${}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\langle J\rangle} \overset{\text{gl. to loc.}}{\longrightarrow} {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{\langle J\rangle},\quad {}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{J} \overset{\text{gl. to loc.}}{\longrightarrow} {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{J},\quad {}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{j} \overset{\text{gl. to loc.}}{\longrightarrow} {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{j}.$$

Definition 10.24. (NF-Bridge, Θ-Bridge, $\boxtimes$-Hodge Theatre, [IUTchI, Definition 5.5])

(1) an **NF-bridge** is a collection

$$\left({}^{\ddagger}\mathfrak{F}_J \overset{{}^{\ddagger}\psi^{\mathrm{NF}}_{\circledast}}{\longrightarrow} {}^{\ddagger}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{\ddagger}\mathcal{F}^{\circledast}\right)$$

as follows:

(a) ${}^{\ddagger}\mathfrak{F}_J = \{{}^{\ddagger}\mathfrak{F}_j\}_{j\in J}$ is a capsule of $\mathcal{F}$-prime-strip indexed by J. We write ${}^{\ddagger}\mathfrak{D}_J = \{{}^{\ddagger}\mathfrak{D}_j\}_{j\in J}$ for the associated capsule of $\mathcal{D}$-prime-strips.

(b) ${}^{\ddagger}\mathcal{F}^{\circledcirc}$, ${}^{\ddagger}\mathcal{F}^{\circledast}$ are pre-Frobenioids isomorphic toy ${}^{\ddagger}\mathcal{F}^{\circledcirc}$, ${}^{\ddagger}\mathcal{F}^{\circledast}$ in the definition of the model NF-bridge (Definition 10.23), respectively. We write ${}^{\ddagger}\mathcal{D}^{\circledcirc}$, ${}^{\ddagger}\mathcal{D}^{\circledast}$ for the base categories of ${}^{\ddagger}\mathcal{F}^{\circledcirc}$, ${}^{\ddagger}\mathcal{F}^{\circledast}$ respectively.

(c) The arrow $\dashrightarrow$ consists of a morphism ${}^{\ddagger}\mathcal{D}^{\circledcirc} \to {}^{\ddagger}\mathcal{D}^{\circledast}$, which is abstractly equivalent (cf. Section 0.2) to the morphism ${}^{\dagger}\mathcal{D}^{\circledcirc} \to {}^{\dagger}\mathcal{D}^{\circledast}$ definition of the model NF-bridge (Definition 10.23), and an isomorphism ${}^{\ddagger}\mathcal{F}^{\circledcirc} \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}^{\circledast}|_{{}^{\ddagger}\mathcal{D}^{\circledcirc}}$.

(d) ${}^{\ddagger}\psi^{\mathrm{NF}}_{\circledast}$ is a poly-morphism which is a unique lift of a poly-morphism ${}^{\ddagger}\phi^{\mathrm{NF}}_{\circledast}$: ${}^{\ddagger}\mathfrak{D}_J \overset{\text{poly}}{\longrightarrow} {}^{\ddagger}\mathcal{D}^{\circledcirc}$ such that ${}^{\ddagger}\phi^{\mathrm{NF}}_{\circledast}$ forms a $\mathcal{D}$-NF-bridge.

Note that we can associate an $\mathcal{D}$-NF-bridge ${}^{\ddagger}\phi_{\circledast}^{\mathrm{NF}}$ to any NF-bridge ${}^{\ddagger}\psi_{\circledast}^{\mathrm{NF}}$. An **isomorphism of NF-bridges**

$$\left({}^{1}\mathfrak{F}_{J_1} \xrightarrow{{}^{1}\psi_{\circledast}^{\mathrm{NF}}} {}^{1}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{1}\mathcal{F}^{\circledast}\right) \xrightarrow{\sim} \left({}^{2}\mathfrak{F}_{J_2} \xrightarrow{{}^{2}\psi_{\circledast}^{\mathrm{NF}}} {}^{2}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{2}\mathcal{F}^{\circledast}\right)$$

is a triple

$${}^{1}\mathfrak{F}_{J_1} \overset{\text{capsule-full poly}}{\xrightarrow{\sim}} {}^{2}\mathfrak{F}_{J_2},\ {}^{1}\mathcal{F}^{\circledcirc} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{2}\mathcal{F}^{\circledcirc},\ {}^{1}\mathcal{F}^{\circledast} \xrightarrow{\sim} {}^{2}\mathcal{F}^{\circledast}$$

of a capsule-full poly-isomorphism ${}^{1}\mathfrak{F}_{J_1} \overset{\text{capsule-full poly}}{\xrightarrow{\sim}} {}^{2}\mathfrak{F}_{J_2}$ (We write ${}^{1}\mathfrak{D}_{J_1} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{2}\mathfrak{D}_{J_2}$ for the induced poly-isomorphism), a poly-isomorphism ${}^{1}\mathcal{F}^{\circledcirc} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{2}\mathcal{F}^{\circledcirc}$ (We write ${}^{1}\mathcal{D}^{\circledcirc} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{2}\mathcal{D}^{\circledcirc}$ for the induced poly-isomorphism) such that the pair ${}^{1}\mathfrak{D}_{J_1} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{2}\mathfrak{D}_{J_2}$ and ${}^{1}\mathcal{D}^{\circledcirc} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{2}\mathcal{D}^{\circledcirc}$ forms a morphism of the associated $\mathcal{D}$-NF-bridges, and an isomoprhism ${}^{1}\mathcal{F}^{\circledast} \xrightarrow{\sim} {}^{2}\mathcal{F}^{\circledast}$, such that this triple is compatible (in the obvious sense) with ${}^{1}\psi_{\circledast}^{\mathrm{NF}}$, ${}^{2}\psi_{\circledast}^{\mathrm{NF}}$, and the respective $\dashrightarrow$'s. Note that we can associate an isomorphism of $\mathcal{D}$-NF-bridges to any isomorphism of NF-bridges.

(2) A **Θ-bridge** is a collection

$$\left({}^{\ddagger}\mathfrak{F}_{J} \xrightarrow{{}^{\ddagger}\psi_{\circledast}^{\Theta}} {}^{\ddagger}\mathfrak{F}_{>} \dashrightarrow {}^{\ddagger}\mathcal{HT}^{\Theta}\right)$$

as follows:

(a) ${}^{\ddagger}\mathfrak{F}_{J} = \{{}^{\ddagger}\mathfrak{F}_{j}\}_{j\in J}$ is a capsule of $\mathcal{F}$-prime-strips indexed by J We write ${}^{\ddagger}\mathfrak{D}_{J} = \{{}^{\ddagger}\mathfrak{D}_{j}\}_{j\in J}$ for the associated capsule of $\mathcal{D}$-prime-strips.

(b) ${}^{\ddagger}\mathcal{HT}^{\Theta}$ is a Θ-Hodge theatre.

(c) ${}^{\ddagger}\mathfrak{F}_{>}$ is the $\mathcal{F}$-prime-strip tautologically associated to ${}^{\ddagger}\mathcal{HT}^{\Theta}$. We use the notation $\dashrightarrow$ to write this relationship between ${}^{\ddagger}\mathfrak{F}_{>}$ and ${}^{\ddagger}\mathcal{HT}^{\Theta}$. We write ${}^{\ddagger}\mathfrak{D}_{>}$ for the $\mathcal{D}$-prime-strip associated to ${}^{\ddagger}\mathfrak{F}_{>}$.

(d) ${}^{\ddagger}\psi_{\circledast}^{\Theta} = \{{}^{\ddagger}\psi_{j}^{\Theta}\}_{j\in\mathbb{F}_l^{\circledast}}$ is the collection of poly-morphisms ${}^{\ddagger}\psi_{j}^{\Theta} : {}^{\ddagger}\mathfrak{F}_{j} \overset{\text{poly}}{\longrightarrow} {}^{\ddagger}\mathfrak{F}_{>}$ determined by a $\mathcal{D}$-Θ-bridge ${}^{\ddagger}\phi_{\circledast}^{\Theta} = \{{}^{\ddagger}\phi_{j}^{\Theta}\}_{j\in\mathbb{F}_l^{\circledast}}$ by Remark 10.10.1.

Note that we can associate an $\mathcal{D}$-Θ-bridge ${}^{\ddagger}\phi_{\circledast}^{\Theta}$ to any Θ-bridge ${}^{\ddagger}\psi_{\circledast}^{\Theta}$. An **isomorphism of Θ-bridges**

$$\left({}^{1}\mathfrak{F}_{J_1} \xrightarrow{{}^{1}\psi_{\circledast}^{\Theta}} {}^{1}\mathfrak{F}_{>} \dashrightarrow {}^{\ddagger}\mathcal{HT}^{\Theta}\right) \xrightarrow{\sim} \left({}^{2}\mathfrak{F}_{J_2} \xrightarrow{{}^{2}\psi_{\circledast}^{\Theta}} {}^{2}\mathfrak{F}_{>} \dashrightarrow {}^{2}\mathcal{HT}^{\Theta}\right)$$

is a triple

$$^1\mathfrak{F}_{J_1} \overset{\text{capsule-full poly}}{\xrightarrow{\sim}} {}^2\mathfrak{F}_{J_2},\ {}^1\mathfrak{F}_{>} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^2\mathfrak{F}_{>},\ {}^1\mathcal{HT}^{\Theta} \xrightarrow{\sim} {}^2\mathcal{HT}^{\Theta}$$

of a capsule-full poly-isomorphism ${}^1\mathfrak{F}_{J_1} \overset{\text{capsule-full poly}}{\xrightarrow{\sim}} {}^2\mathfrak{F}_{J_2}$ the full poly-isomorphism ${}^1\mathcal{F}^{\circledcirc} \overset{\text{poly}}{\xrightarrow{\sim}} {}^2\mathcal{F}^{\circledcirc}$ and an isomoprhism ${}^1\mathcal{F}^{\circledast} \xrightarrow{\sim} {}^2\mathcal{F}^{\circledast}$ of $\mathcal{HT}$-Hodge theatres, such that this triple is compatible (in the obvious sense) with ${}^1\psi^{\Theta}_{\divideontimes}$, ${}^2\psi^{\Theta}_{\divideontimes}$, and the respective $\dashrightarrow$'s. Note that we can associate an isomorphism of $\mathcal{D}$-Θ-bridges to any isomorphism of Θ-bridges.

(3) A **ΘNF-Hodge theatre** (or **⊠-Hodge theatre**) is a collection

$${}^{\ddagger}\mathcal{HT}^{\boxtimes} = \left({}^{\ddagger}\mathcal{F}^{\circledast} \dashleftarrow {}^{\ddagger}\mathcal{F}^{\circledcirc} \xleftarrow{{}^{\ddagger}\psi^{\mathrm{NF}}_{\divideontimes}} {}^{\ddagger}\mathfrak{F}_J \xrightarrow{{}^{\ddagger}\psi^{\Theta}_{\divideontimes}} {}^{\ddagger}\mathfrak{F}_{>} \dashrightarrow {}^{\ddagger}\mathcal{HT}^{\Theta} \right),$$

where $\left({}^{\ddagger}\mathcal{F}^{\circledast} \dashleftarrow {}^{\ddagger}\mathcal{F}^{\circledcirc} \xleftarrow{{}^{\ddagger}\psi^{\mathrm{NF}}_{\divideontimes}} {}^{\ddagger}\mathfrak{F}_J \right)$ forms an NF-bridge, and $\left({}^{\ddagger}\mathfrak{F}_J \xrightarrow{{}^{\ddagger}\psi^{\Theta}_{\divideontimes}} {}^{\ddagger}\mathfrak{F}_{>} \dashrightarrow {}^{\ddagger}\mathcal{HT}^{\Theta} \right)$ forms a Θ-bridge, such that the associated $\mathcal{D}$-NF-bridge ${}^{\ddagger}\phi^{\mathrm{NF}}_{\divideontimes}$ and the associated $\mathcal{D}$-Θ-bridge ${}^{\ddagger}\phi^{\Theta}_{\divideontimes}$ form a $\mathcal{D}$-⊠-Hodge theatre. An **isomorphism of ⊠-Hodge theatres** is a pair of a morphism of NF-bridge and a morphism of Θ-bridge, which induce the same bijection between the index sets of the respective capsules of $\mathcal{F}$-prime-strips. We define compositions of them in an obvious manner.

Lemma 10.25. (Properties of NF-Brideges, Θ-Bridges, ⊠-Hodge theatres, [IUTchI, Corollary 5.6])

(1) For NF*-bridges* ${}^1\psi^{\mathrm{NF}}_{\divideontimes}$, ${}^2\psi^{\mathrm{NF}}_{\divideontimes}$ *(resp.* Θ*-bridges* ${}^1\psi^{\Theta}_{\divideontimes}$, ${}^2\psi^{\Theta}_{\divideontimes}$*, resp.* ⊠*-Hodge theatres* ${}^1\mathcal{HT}^{\boxtimes}$, ${}^2\mathcal{HT}^{\boxtimes}$*) whose associated* $\mathcal{D}$-NF*-bridges (resp.* $\mathcal{D}$-Θ*-bridges, resp.* $\mathcal{D}$-⊠*-Hodge theatres) are* ${}^1\phi^{\mathrm{NF}}_{\divideontimes}$, ${}^2\phi^{\mathrm{NF}}_{\divideontimes}$ *(resp.* ${}^1\phi^{\Theta}_{\divideontimes}$, ${}^2\phi^{\Theta}_{\divideontimes}$*, resp.* ${}^1\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$, ${}^2\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$*) respectively, the natural map*

$$\begin{aligned} &\mathrm{Isom}({}^1\psi^{\mathrm{NF}}_{\divideontimes}, {}^2\psi^{\mathrm{NF}}_{\divideontimes}) \to \mathrm{Isom}({}^1\phi^{\mathrm{NF}}_{\divideontimes}, {}^2\phi^{\mathrm{NF}}_{\divideontimes}) \\ (\textit{resp.}\ \ &\mathrm{Isom}({}^1\psi^{\Theta}_{\divideontimes}, {}^2\psi^{\Theta}_{\divideontimes}) \to \mathrm{Isom}({}^1\phi^{\Theta}_{\divideontimes}, {}^2\phi^{\Theta}_{\divideontimes}), \\ \textit{resp.}\ \ &\mathrm{Isom}({}^1\mathcal{HT}^{\boxtimes}, {}^2\mathcal{HT}^{\boxtimes}) \to \mathrm{Isom}({}^1\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}, {}^2\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}) \quad) \end{aligned}$$

is bijective.

(2) For an NF*-bridge* ${}^{\ddagger}\psi^{\mathrm{NF}}_{\divideontimes}$ *and a* Θ*-bridge* ${}^{\ddagger}\psi^{\Theta}_{\divideontimes}$*, the set*

$$\left\{ \textit{capsule-full poly-isom.}\ {}^{\ddagger}\mathfrak{F}_J \overset{\text{capsule-full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{F}_{J'}\ \ \textit{by which}\ {}^{\ddagger}\psi^{\mathrm{NF}}_{\divideontimes}, {}^{\ddagger}\psi^{\Theta}_{\divideontimes}\ \textit{form a}\ \boxtimes\textit{-Hodge theatre} \right\}$$

is an $\mathbb{F}^{\divideontimes}_l$*-torsor.*

Proof. By using Lemma 10.10 (5), the claim (1) (resp. (2)) follows from Lemma 10.10 (1) (resp. (2)). □

§ 10.5. The Additive Symmetry $\boxplus$: $\Theta^{\pm\mathrm{ell}}$-Hodge Theatres and Θ^{ell}-, $\Theta^{\pm}$-Bridges.

We begin constructing the additive portion of full Hodge theatres.

Definition 10.26. ([IUTchI, Definition 6.1 (i)]) We shall refer to an element of $\mathbb{F}_l^{\rtimes\pm}$ as **positive** (resp. **negative**) if it is sent to $+1$ (resp. -1) by the natural surjction $\mathbb{F}_l^{\rtimes\pm} \twoheadrightarrow \{\pm 1\}$.

(1) An $\boldsymbol{\mathbb{F}_l^{\pm}}$**-group** is a set E with a $\{\pm 1\}$-orbit of bijections $E \xrightarrow{\sim} \mathbb{F}_l$. Hence any $\mathbb{F}_l^{\pm}$-group has a natural $\mathbb{F}_l$-module structure.

(2) An $\boldsymbol{\mathbb{F}_l^{\pm}}$**-torsor** is a set T with an $\mathbb{F}_l^{\rtimes\pm}$-orbit of bijections $T \xrightarrow{\sim} \mathbb{F}_l$ (Here, $\mathbb{F}_l^{\pm} \ni (\lambda, \pm 1)$ is actingg on $z \in \mathbb{F}_l$ via $z \mapsto \pm z + \lambda$). For an $\mathbb{F}_l^{\pm}$-torsor T, take an bijection $f : T \xrightarrow{\sim} \mathbb{F}_l$ in the given $\mathbb{F}_l^{\rtimes\pm}$-orbit, then we obtain a subgroup

$$\mathrm{Aut}_{+}(T) \quad (\text{resp. } \ \mathrm{Aut}_{\pm}(T) \quad)$$

of $\mathrm{Aut}_{(\mathrm{Sets})}(T)$ by transporting the subgroup $\mathbb{F}_l \cong \{z \mapsto z + \lambda \text{ for } \lambda \in \mathbb{F}_l\} \subset \mathrm{Aut}_{(\mathrm{Sets})}(\mathbb{F}_l)$ (resp. $\mathbb{F}_l^{\rtimes\pm} \cong \{z \mapsto \pm z + \lambda \text{ for } \lambda \in \mathbb{F}_l\} \subset \mathrm{Aut}_{(\mathrm{Sets})}(\mathbb{F}_l)$) via f. Note that this subgroup is independent of the choice of f in its $\mathbb{F}_l^{\rtimes\pm}$-orbit. Moreover, any element of $\mathrm{Aut}_{+}(T)$ is independent of the choice of f in its $\mathbb{F}_l$-orbit, hence if we consider f up to $\mathbb{F}_l^{\rtimes\pm}$-orbit, then it gives us a $\{\pm 1\}$-orbit of bijections $\mathrm{Aut}_{+}(T) \xrightarrow{\sim} \mathbb{F}_l$, i.e., $\mathrm{Aut}_{+}(T)$ has a natural $\mathbb{F}_l^{\pm}$-group structure. We shall refer to $\mathrm{Aut}_{+}(T)$ as the $\mathbb{F}_l^{\pm}$-group of **positive automorphisms of** $\boldsymbol{T}$. Note that we have $[\mathrm{Aut}_{\pm}(T); \mathrm{Aut}_{+}(T)] = 2$.

The following is an additive counterpart of Definition 10.14

Definition 10.27. ([IUTchI, Definition 6.1 (ii), (iii), (vi)]) Let ${}^{\dagger}\mathfrak{D} = \{{}^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ be a $\mathcal{D}$-prime-strip.

(1) For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$), we can group-theoretically reconstruct in a functorial manner, from $\pi_1({}^{\dagger}\mathcal{D}_{\underline{v}})$, a tempered group (resp. a profinite group) ($\supset \pi_1({}^{\dagger}\mathcal{D}_{\underline{v}})$) corresponding to $\underline{X}_{\underline{v}}$ by Lemma 7.12 (resp. by Lemma 7.25). We write

$${}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$$

for its $\mathcal{B}(-)^0$. We have a natural morphism ${}^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$ (This corresponds to $\underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}}$ (resp. $\underset{\rightarrow}{X}_{\underline{v}} \to \underline{X}_{\underline{v}}$)). Similarly, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, we can algorithmically reconstruct, in a functorial manner, from ${}^{\dagger}\mathcal{D}_{\underline{v}}$, an Aut-holomorphic orbispace ${}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$ corresponding to $\underline{X}_{\underline{v}}$ by translating Lemma 7.25 into the theory of Aut-holomorphic spaces (since $\underset{\rightarrow}{X}_{\underline{v}}$ admits a $K_{\underline{v}}$-core) with a natural morphism ${}^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$. Write

$$ {}^{\dagger}\underline{\mathfrak{D}}^{\pm} := \{{}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}. $$

(2) Recall that we can algorithmically reconstruct the set of conjugacy classes of cuspidal decomposition groups of $\pi_1({}^{\dagger}\mathcal{D}_{\underline{v}})$ or $\pi_1({}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}})$ by Corollary 6.12 for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, by Corollary 2.9 for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$, and by considering $\pi_0(-)$ of a cofinal collection of the complements of compact subsets of the underlying topological space of ${}^{\dagger}\mathcal{D}_{\underline{v}}$ or ${}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. We say them the **set of cusps of** $\boldsymbol{{}^{\dagger}\mathcal{D}_{\underline{v}}}$ **or** $\boldsymbol{{}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}}$.

For $\underline{v} \in \underline{\mathbb{V}}$, a **$\pm$-label class of cusps of** $\boldsymbol{{}^{\dagger}\mathcal{D}_{\underline{v}}}$ is the set of cusps of ${}^{\dagger}\mathcal{D}_{\underline{v}}$ lying over a single (not necessarily non-zero) cusp of ${}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$. We write

$$ \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}}) $$

for the set of $\pm$-label classes of cusps of ${}^{\dagger}\mathcal{D}_{\underline{v}}$. Note that $\mathrm{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}})$ has a natural $\mathbb{F}_l^{\times}$-action. Note also that, for any $\underline{v} \in \underline{\mathbb{V}}$, we can algorithmically reconstruct a zero element

$$ {}^{\dagger}\underline{\eta}^{0}_{\underline{v}} \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}}), $$

and a canonical element

$$ {}^{\dagger}\underline{\eta}^{\pm}_{\underline{v}} \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}}) $$

which is well-defined up to multiplication by ± 1, such that we have ${}^{\dagger}\underline{\eta}^{\pm}_{\underline{v}} \mapsto {}^{\dagger}\underline{\eta}_{\underline{v}}$ under the natural bijection

$$ \left\{\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}}) \setminus \{{}^{\dagger}\underline{\eta}^{0}_{\underline{v}}\}\right\} / \{\pm 1\} \xrightarrow{\sim} \mathrm{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}}). $$

Hence we have a natural bijection

$$ \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} \mathbb{F}_l, $$

which is *well-defined up to multiplication by* ± 1, and compatible with the bijection $\mathrm{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} \mathbb{F}_l^{*}$ in Proposition 10.15, i.e., $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}})$ *has a natural* $\mathbb{F}_l^{\pm}$*-group structure.* This structure $\mathbb{F}_l^{\pm}$-group gives us a natural surjection

$$ \mathrm{Aut}({}^{\dagger}\mathcal{D}_{\underline{v}}) \twoheadrightarrow \{\pm 1\} $$

by considering the induced automorphism of $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}})$. We write

$$ \mathrm{Aut}_{+}({}^{\dagger}\mathcal{D}_{\underline{v}}) \subset \mathrm{Aut}({}^{\dagger}\mathcal{D}_{\underline{v}}) $$

for the kernel of the above surjection, and we shall refer to it as the subgroup of **positive automorphisms** Write $\mathrm{Aut}_-({}^\dagger\mathcal{D}_{\underline{v}}) := \mathrm{Aut}({}^\dagger\mathcal{D}_{\underline{v}}) \setminus \mathrm{Aut}_+({}^\dagger\mathcal{D}_{\underline{v}})$,and we shall refer to it as the set of **negative automorphisms**. Similarly, for $\alpha \in \{\pm 1\}^{\underline{\mathbb{V}}}$, we write

$$\mathrm{Aut}_+({}^\dagger\mathfrak{D}) \subset \mathrm{Aut}_+({}^\dagger\mathfrak{D}) \ \ (\text{resp. } \ \mathrm{Aut}_\alpha({}^\dagger\mathfrak{D}) \subset \mathrm{Aut}_+({}^\dagger\mathfrak{D}) \ \ )$$

for the subgroup of automorphisms such that any $\underline{v}(\in \underline{\mathbb{V}})$-component is positive (resp. $\underline{v}(\in \underline{\mathbb{V}})$-component is positive if $\alpha(\underline{v}) = +1$ and negetive if $\alpha(\underline{v}) = -1$), and we shall refer to it as the subgroup of **positive automorphisms** (resp. the subgroup of ***α*-signed automorphisms**).

(3) Let ${}^\dagger\mathcal{D}^{\circledcirc\pm}$ is a category equivalent to the model global object $\mathcal{D}^{\circledcirc\pm}$ in Definition 10.3. Then by Remark 2.9.2, similarly we can define the **set of cusps of** $\boldsymbol{{}^\dagger\mathcal{D}^{\circledcirc\pm}}$ and the **set of ±-label classes of cusps**

$$\mathrm{LabCusp}^\pm({}^\dagger\mathcal{D}^{\circledcirc\pm}),$$

which can be identified with the set of cusps of ${}^\dagger\mathcal{D}^{\circledcirc\pm}$.

Definition 10.28. ([IUTchI, Definition 6.1 (iv)]) Let ${}^\dagger\mathfrak{D} = \{{}^\dagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, ${}^\ddagger\mathfrak{D} = \{{}^\ddagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ be $\mathcal{D}$-prime-strips. For any $\underline{v} \in \underline{\mathbb{V}}$, a **+-full poly-isomorphism** ${}^\dagger\mathcal{D}_{\underline{v}} \overset{+\text{-full poly}}{\xrightarrow{\sim}} {}^\ddagger\mathcal{D}_{\underline{v}}$ (resp. ${}^\dagger\mathfrak{D} \overset{+\text{-full poly}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{D}$) is a poly-isomorphism obtained as the $\mathrm{Aut}_+({}^\dagger\mathcal{D}_{\underline{v}})$-orbit (resp. $\mathrm{Aut}_+({}^\dagger\mathfrak{D})$-orbit) (or equivalently, $\mathrm{Aut}_+({}^\ddagger\mathcal{D}_{\underline{v}})$-orbit (resp. $\mathrm{Aut}_+({}^\ddagger\mathfrak{D})$-orbit)) of an isomorphism ${}^\dagger\mathcal{D}_{\underline{v}} \xrightarrow{\sim} {}^\ddagger\mathcal{D}_{\underline{v}}$ (resp. ${}^\dagger\mathfrak{D} \xrightarrow{\sim} {}^\ddagger\mathfrak{D}$). If ${}^\dagger\mathfrak{D} = {}^\ddagger\mathfrak{D}$, then there are precisely two +-full poly-isomorphisms ${}^\dagger\mathcal{D}_{\underline{v}} \overset{+\text{-full poly}}{\xrightarrow{\sim}} {}^\dagger\mathcal{D}_{\underline{v}}$ (resp. the set of +-full poly-isomorphisms ${}^\dagger\mathcal{D}_{\underline{v}} \xrightarrow{\sim} {}^\dagger\mathcal{D}_{\underline{v}}$ has a natural bijection with $\{\pm 1\}^{\underline{\mathbb{V}}}$). We shall refer to the +-full poly-isomorphism determined by the identity automorphism as **positive**, and the other one **negative** (resp. the +-full poly-isomorphism corresponding to $\alpha \in \{\pm 1\}^{\underline{\mathbb{V}}}$ an ***α*-signed +-full poly-automorphism**). A **capsule-+-full poly-morphism** between capsules of $\mathcal{D}$-prime-strips

$$\{{}^\dagger\mathfrak{D}_t\}_{t\in T} \overset{\text{capsule-}+\text{-full poly}}{\xrightarrow{\sim}} \{{}^\ddagger\mathfrak{D}_{t'}\}_{t'\in T'}$$

is a collection of +-full poly-isomorphisms ${}^\dagger\mathfrak{D}_t \overset{+\text{-full poly}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{D}_{\iota(t)}$, relative to some injection $\iota : T \hookrightarrow T'$.

Definition 10.29. ([IUTchI, Definition 6.1 (v)]) As in Definition 10.16, we can group-theoretically construct, from the model global object $\mathcal{D}^{\circledcirc\pm}$ in Definition 10.3, the outer homomorphism

$$(\mathrm{Aut}(\underline{X}_K) \cong)\, \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \to \mathrm{GL}_2(\mathbb{F}_l)/\{\pm 1\}$$

determined by $E_{\overline{F}}[l]$, by considering the Galois action on $\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l$ (The first isomorphism follows from Theorem 3.17). Note that the image of the above outer homomorphism contains the Borel subgroup $\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ of $\mathrm{SL}_2(\mathbb{F}_l)/\{\pm 1\}$ since the covering $\underline{X}_K \twoheadrightarrow X_K$ corresponds to the rank one quotient $\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l \twoheadrightarrow Q$. This rank one quotient determines a natural surjective homomorphism

$$\mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \twoheadrightarrow \mathbb{F}_l^{\divideontimes},$$

which can be reconstructed group-theoretically from $\mathcal{D}^{\circledcirc\pm}$. We write $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \subset \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathrm{Aut}(\underline{X}_K)$ for the kernel of the above homomorphism. Note that the subgroup $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \subset \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathrm{Aut}(\underline{X}_K)$ contains $\mathrm{Aut}_K(\underline{X}_K)$, and acts transitively on the cusps of $\underline{X}_K$. Next, we write $\mathrm{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm}) \subset \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm})$ for the subgroup of automorphisms which fix the cusps of $\underline{X}_K$ (Note that we can group-theoretically reconstruct this subgroup by Remark 2.9.2). Then we obtain natural outer isomorphisms

$$\mathrm{Aut}_K(\underline{X}_K) \xrightarrow{\sim} \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm},$$

where the second isomorphism depends on the choice of the cusp $\underline{\epsilon}$ of $\underline{C}_K$. cf. also the following diagram:

$$\begin{tikzcd}
\mathrm{Aut}(\underline{X}_K) \arrow[r, "\sim"] & \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \arrow[r, two heads] & \mathbb{F}_l^{\divideontimes} \\
\mathrm{Aut}_K(\underline{X}_K) \arrow[u, hook] \arrow[r, hook] \arrow[rr, bend right=60, "\sim"'] & \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \arrow[u, hook, "\mathbb{F}_l^{\divideontimes} \leftsquigarrow \begin{pmatrix} * & * \\ 0 & \circledast \end{pmatrix} \subset \mathrm{SL}_2(\mathbb{F}_l)/\{\pm 1\}"'] \arrow[r, two heads] & \mathbb{F}_l^{\rtimes\pm} \\
 & \mathrm{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm}). \arrow[u, hook, "\mathbb{F}_l^{\rtimes\pm} \leftsquigarrow \begin{pmatrix} 1 & \circledast \\ 0 & \pm \end{pmatrix}"'] &
\end{tikzcd}$$

If we write $\mathrm{Aut}_{+}(\mathcal{D}^{\circledcirc\pm}) \subset \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})$ for the unique subgroup of index 2 containing $\mathrm{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm})$, then the cusp $\underline{\epsilon}$ determines a natural $\mathbb{F}_l^{\pm}$-group structure on the subgroup

$$\mathrm{Aut}_{+}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm}) \ \subset \ \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm})$$

(corresponding to $\mathrm{Gal}(\underline{X}_K/X_K) \subset \mathrm{Aut}_K(\underline{X}_K)$), and a natural $\mathbb{F}_l^{\pm}$-torsor structure on $\mathrm{LabCusp}^{\pm}(\mathcal{D}^{\circledcirc\pm})$. Write also

$$\underline{\mathbb{V}}^{\pm} := \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \cdot \underline{\mathbb{V}} = \mathrm{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm}) \cdot \underline{\mathbb{V}} \ \subset \ \mathbb{V}(K).$$

Note also that the subgoup $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \subset \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \cong \mathrm{Aut}(\underline{X}_K)$ can be identified with the subgroup of $\mathrm{Aut}(\underline{X}_K)$ which stabilises $\underline{\mathbb{V}}^{\pm}$, and also that we can easily show that $\underline{\mathbb{V}}^{\pm} = \underline{\mathbb{V}}^{\pm\mathrm{un}}$ (Definition 10.16) (cf. [IUTchI, Remark 6.1.1]).

Remark 10.29.1. Note that $\mathbb{F}_l^{\rtimes\pm}$-symmetry permutes the cusps of $\underline{X}_K$ *without* permuting $\underline{\mathbb{V}}^{\pm}$ $(\subset \mathbb{V}(K))$, and is *of geometric nature*, which is suited to construct Hodge-Arakelov-theoretic evaluation map (Section 11).

On the other hand, $\mathbb{F}_l^{*}$ is a subquotient of $\mathrm{Gal}(K/F)$ and $\mathbb{F}_l^{*}$-symmetry *permutes* various $\mathbb{F}_l^{*}$-translates of $\underline{\mathbb{V}}^{\pm} = \underline{\mathbb{V}}^{\pm\mathrm{un}} \subset \underline{\mathbb{V}}^{\mathrm{Bor}}$ $(\subset \mathbb{V}(K))$, and is *of arithmetic nature* (cf. [IUTchI, Remark 6.12.6 (i)]), which is suite to the situation where we have to consider descend from K to F_{mod}. Such a situation induces global Galois permutations of various copies of $G_{\underline{v}}$ ($\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$) associated to distinct labels $\in \mathbb{F}_l^{*}$ which are only well-defined up to conjugacy indeterminacies, hence $\mathbb{F}_l^{*}$-symmetry is ill-suited to construct Hodge-Arakelov-theoretic evaluation map.

Remark 10.29.2. (cf. [IUTchII, Remark 4.7.6]) One of the important differences of $\mathbb{F}_l^{*}$-symmetry and $\mathbb{F}_l^{\rtimes\pm}$-symmetry is that $\mathbb{F}_l^{*}$-symmetry *does not permute the label* 0 *with the other labels*, on the other hand, $\mathbb{F}_l^{\rtimes\pm}$-symmetry *does*.

We need to *permute* the label 0 with the other labels in $\mathbb{F}_l^{\rtimes\pm}$-symmetry to perform the conjugate synchronisation (cf. Corollary 11.16 (1)), which is used to construct "diagonal objects" or "horizontally coric objects" (cf. Corollary 11.16, Corollary 11.17, and Corollary 11.24) or "*mono-analytic cores*" (In this sense, *label* 0 *is closely related to the units and additive symmetry.* cf. [IUTchII, Remark 4.7.3]),

On the other hand, we need to *separate* the label 0 from the other labels in $\mathbb{F}_l^{*}$-symmetry, since *the simultaneous excutions of the final multiradial algorithms on objects in each non-zero labels are compatible with each other by separating from mono-analytic cores (objects in the label* 0*)*, i.e., the algorithm is **multiradial** (cf. Section 11.1, and § A.4), and we perform Kummer theory for NF (Corollary 11.23) with $\mathbb{F}_l^{*}$-symmetry (since $\mathbb{F}_l^{*}$-symmetry is of arithmetic nature, and suited to the situation involved Galois group $\mathrm{Gal}(K/F_{\mathrm{mod}})$) in the NF portion of the final multiradial algorithm. Note also that the value group portion of the final multiradial algorithm, which involves theta values arising from non-zero labels, need to be separated from 0-labelled objects (i.e., mono-analytic cores, or units). In this sense, *the non-zero labels are closely related to the value groups and multiplicative symmetry.*

Definition 10.30. (Model $\mathcal{D}$-$\Theta^{\pm}$-Bridge, [IUTchI, Example 6.2]) In this definition, we regard $\mathbb{F}_l$ as an $\mathbb{F}_l^{\pm}$-group. Let $\mathfrak{D}_{\succ} = \{\mathcal{D}_{\succ,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, $\mathfrak{D}_t = \{\mathcal{D}_{\underline{v}_t}\}_{\underline{v}\in\underline{\mathbb{V}}}$ be copies of the tautological $\mathcal{D}$-prime-strip $\{\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ for each $t \in \mathbb{F}_l$ (Here, we write $\underline{v}_t$ for the pair $(t, \underline{v})$). For each $t \in \mathbb{F}_l$, let

$$\phi^{\Theta^{\pm}}_{\underline{v}_t} : \mathcal{D}_{\underline{v}_t} \overset{+\text{-full poly}}{\xrightarrow{\sim}} \mathcal{D}_{\succ,\underline{v}}, \quad \phi^{\Theta^{\pm}}_{t} : \mathcal{D}_{\underline{v}_t} \overset{+\text{-full poly}}{\xrightarrow{\sim}} \mathcal{D}_{\succ,\underline{v}}$$

be the positive +-full poly-isomorphisms respectively, with respect to the identifications with the tautological $\mathcal{D}$-prime-strip $\{\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$. Then we put

$$\phi_{\pm}^{\Theta^{\pm}} := \{\phi_t^{\Theta^{\pm}}\}_{t\in\mathbb{F}_l} \,:\, \mathfrak{D}_{\pm} := \{\mathfrak{D}_t\}_{t\in\mathbb{F}_l} \overset{\text{poly}}{\longrightarrow} \mathfrak{D}_{\succ}.$$

We shall refer to $\phi_{\pm}^{\Theta^{\pm}}$ as **model base-(or $\mathcal{D}$-)$\Theta^{\pm}$-bridge**.

We have a natural poly-automorphism $-1_{\mathbb{F}_l}$ of order 2 on the triple $(\mathfrak{D}_{\pm}, \mathfrak{D}_{\succ}, \phi_{\pm}^{\Theta^{\pm}})$ as follows: The poly-automorphism $-1_{\mathbb{F}_l}$ acts on $\mathbb{F}_l$ as multiplication by -1, and induces the poly-morphisms $\mathfrak{D}_t \overset{\text{poly}}{\xrightarrow{\sim}} \mathfrak{D}_{-t}$ $(t \in \mathbb{F}_l)$ and $\mathfrak{D}_{\succ} \overset{+\text{-full poly}}{\xrightarrow{\sim}} \mathfrak{D}_{\succ}$ determined by the $+$-full poly-automorphism whose sign at every $\underline{v} \in \underline{\mathbb{V}}$ is negative, with respect to the identifications with the tautological $\mathcal{D}$-prime-strip $\{\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$. This $-1_{\mathbb{F}_l}$ is compatible with $\phi_{\pm}^{\Theta^{\pm}}$ in the obvious sense. Similarly, each $\alpha \in \{\pm1\}^{\underline{\mathbb{V}}}$ determines a natural poly-automorphism $\alpha^{\Theta^{\pm}}$ of order 1 or 2 as follows: The poly-automorphism $\alpha^{\Theta^{\pm}}$ acts on $\mathbb{F}_l$ as the identity and the α-signed $+$-full poly-automorphism on $\mathfrak{D}_t$ $(t \in \mathbb{F}_l)$ and $\mathfrak{D}_{\succ}$. This $\alpha^{\Theta^{\pm}}$ is compatible with $\phi_{\pm}^{\Theta^{\pm}}$ in the obvious sense.

Definition 10.31. (Model $\mathcal{D}$-Θ^{ell}-Bridge, [IUTchI, Example 6.3]) In this definition, we regard $\mathbb{F}_l$ as an $\mathbb{F}_l^{\pm}$-torsor. Let $\mathfrak{D}_t = \{\mathcal{D}_{\underline{v}_t}\}_{\underline{v}\in\underline{\mathbb{V}}}$ be a copy of the tautological $\mathcal{D}$-prime-strip $\{\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ for each $t \in \mathbb{F}_l$, and write $\mathfrak{D}_{\pm} := \{\mathfrak{D}_t\}_{t\in\mathbb{F}_l}$ as in Definition 10.30. Let $\mathcal{D}^{\circledcirc\pm}$ be the model global object in Definition 10.3. In the following, fix an isomorphism $\mathrm{LabCusp}^{\pm}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_l$ of $\mathbb{F}_l^{\pm}$-torsor (cf. Definition 10.29). This identification induces an isomorphism $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}/\mathrm{Aut}_{\text{cusp}}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm}$ of groups For $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}$, resp. $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$), we write

$$\phi_{\bullet,\underline{v}}^{\Theta^{\text{ell}}} \,:\, \mathcal{D}_{\underline{v}} \longrightarrow \mathcal{D}^{\circledcirc\pm}$$

for the natural morphism correpsonding to $\underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}} \to \underline{X}_K$ (resp. $\underline{X}_{\to\underline{v}} \to \underline{X}_{\underline{v}} \to \underline{X}_K$, resp. a tautological morphism $\mathcal{D}_{\underline{v}} = \underline{\mathbb{X}}_{\to\underline{v}} \to \underline{\mathbb{X}}_{\underline{v}} \xrightarrow{\sim} \underline{\mathbb{X}}(\mathcal{D}^{\circledcirc\pm}, \underline{v})$ (cf. also Definition 10.11 (1), (2)).

Write

$$\phi_{\underline{v}_0}^{\Theta^{\text{ell}}} := \mathrm{Aut}_{\text{cusp}}(\mathcal{D}^{\circledcirc\pm}) \circ \phi_{\bullet,\underline{v}}^{\Theta^{\text{ell}}} \circ \mathrm{Aut}_{+}(\mathcal{D}_{\underline{v}_0}) \,:\, \mathcal{D}_{\underline{v}_0} \overset{\text{poly}}{\longrightarrow} \mathcal{D}^{\circledcirc\pm},$$

and

$$\phi_0^{\Theta^{\text{ell}}} := \{\phi_{\underline{v}_0}^{\Theta^{\text{ell}}}\}_{\underline{v}\in\underline{\mathbb{V}}} \,:\, \mathfrak{D}_0 \overset{\text{poly}}{\longrightarrow} \mathcal{D}^{\circledcirc\pm}.$$

Since $\phi_0^{\Theta^{\text{ell}}}$ is stable under the action of $\mathrm{Aut}_{\text{cusp}}(\mathcal{D}^{\circledcirc\pm})$, we obtain a poly-morphism

$$\phi_t^{\Theta^{\text{ell}}} := (\text{action of } t) \circ \phi_0^{\Theta^{\text{ell}}} \,:\, \mathfrak{D}_t \overset{\text{poly}}{\longrightarrow} \mathcal{D}^{\circledcirc\pm},$$

by post-composing a lift of $t \in \mathbb{F}_l \cong \mathrm{Aut}_{+}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\text{cusp}}(\mathcal{D}^{\circledcirc\pm})\,(\subset \mathbb{F}_l^{\rtimes\pm} \cong \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})$ $/\mathrm{Aut}_{\text{cusp}}(\mathcal{D}^{\circledcirc\pm}))$ to $\mathrm{Aut}_{+}(\mathcal{D}^{\circledcirc\pm})$. Hence we obtain a poly-morphism

$$\phi_{\pm}^{\Theta^{\text{ell}}} := \{\phi_t^{\Theta^{\text{ell}}}\}_{t\in\mathbb{F}_l} \,:\, \mathfrak{D}_{\pm} \overset{\text{poly}}{\longrightarrow} \mathcal{D}^{\circledcirc\pm}$$

from a capsule of $\mathcal{D}$-prime-strip to the global object $\mathcal{D}^{\circledcirc\pm}$ (cf. Definition 10.11 (3)). This is called the **model base-(or $\mathcal{D}$-)Θ^{ell}-bridge**.

Note that each $\gamma \in \mathbb{F}_l^{\rtimes\pm}$ gives us a natural poly-automorphism $\gamma_\pm$ of $\mathfrak{D}_\pm$ as follows: The automorphism $\gamma_\pm$ acts on $\mathbb{F}_l$ via the usual action of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathbb{F}_l$, and induces the $+$-full poly-isomorphism $\mathfrak{D}_t \overset{+\text{-full poly}}{\xrightarrow{\sim}} \mathfrak{D}_{\gamma(t)}$ whose sign at every $\underline{v} \in \underline{\mathbb{V}}$ is equal to the sign of γ. In this way, we obtain a natural poly-action of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathfrak{D}_\pm$. On the other hand, the isomorphism $\mathrm{Aut}_\pm(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm}$ determines a natural poly-action of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathcal{D}^{\circledcirc\pm}$. Note that $\phi_\pm^{\Theta^{\mathrm{ell}}}$ is equivariant with respect to these natural poly-actions of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathfrak{D}_\pm$ and $\mathcal{D}^{\circledcirc\pm}$. Hence we obtain a natural poly-action of $\mathbb{F}_l^{\rtimes\pm}$ on $(\mathfrak{D}_\pm, \mathcal{D}^{\circledcirc\pm}, \phi_\pm^{\Theta^{\mathrm{ell}}})$.

Definition 10.32. ($\mathcal{D}$-$\Theta^\pm$-Bridge, $\mathcal{D}$-Θ^{ell}-Bridge, $\mathcal{D}$-$\boxplus$-Hodge Theatre, [IUTchI, Definition 6.4])

(1) A **base-(or $\mathcal{D}$-)$\Theta^\pm$-bridge** is a poly-morphism

$$^\dagger\phi_\pm^{\Theta^\pm} : {}^\dagger\mathfrak{D}_T \overset{\text{poly}}{\longrightarrow} {}^\dagger\mathfrak{D}_\succ,$$

where $^\dagger\mathfrak{D}_\succ$ is a $\mathcal{D}$-prime-strip, and $^\dagger\mathfrak{D}_T$ is a cupsule of $\mathcal{D}$-prime-strips indexed by an $\mathbb{F}_l^\pm$-group T, such that there exist isomorphisms $\mathfrak{D}_\succ \xrightarrow{\sim} {}^\dagger\mathfrak{D}_\succ$, $\mathfrak{D}_\pm \xrightarrow{\sim} {}^\dagger\mathfrak{D}_T$, whose induced morphism $\mathbb{F}_l \xrightarrow{\sim} T$ on the index sets is an isomorphism of $\mathbb{F}_l^\pm$-groups, and conjugation by which sends $\phi_\pm^{\Theta^\pm} \mapsto {}^\dagger\phi_\pm^{\Theta^\pm}$. An **isomorphism of $\mathcal{D}$-$\Theta^\pm$-bridges** $\left({}^\dagger\phi_\pm^{\Theta^\pm} : {}^\dagger\mathfrak{D}_T \overset{\text{poly}}{\longrightarrow} {}^\dagger\mathfrak{D}_\succ\right) \xrightarrow{\sim} \left({}^\ddagger\phi_\pm^{\Theta^\pm} : {}^\ddagger\mathfrak{D}_{T'} \overset{\text{poly}}{\longrightarrow} {}^\ddagger\mathfrak{D}_\succ\right)$ is a pair of a capsule-$+$-full poly-isomorphism ${}^\dagger\mathfrak{D}_T \overset{\text{capsule-}+\text{full poly}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{D}_{T'}$ whose induced morphism $T \xrightarrow{\sim} T'$ on the index sets is an isomorphism of $\mathbb{F}_l^\pm$-groups, and a $+$-full-poly isomorphism ${}^\dagger\mathfrak{D}_\succ \overset{+\text{-full poly}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{D}_\succ$, which are compatible with ${}^\dagger\phi_\pm^{\Theta^\pm}$, ${}^\ddagger\phi_\pm^{\Theta^\pm}$. We define compositions of them in an obvious manner.

(2) A **base-(or $\mathcal{D}$-)Θ^{ell}-bridge** is a poly-morphism

$$^\dagger\phi_\pm^{\Theta^{\mathrm{ell}}} : {}^\dagger\mathfrak{D}_T \overset{\text{poly}}{\longrightarrow} {}^\dagger\mathcal{D}^{\circledcirc\pm},$$

where ${}^\dagger\mathcal{D}^{\circledcirc\pm}$ is a category equivalent to the model global object $\mathcal{D}^{\circledcirc\pm}$, and ${}^\dagger\mathfrak{D}_T$ is a cupsule of $\mathcal{D}$-prime-strips indexed by an $\mathbb{F}_l^\pm$-torsor T, such that there exist isomorphisms $\mathcal{D}^{\circledcirc\pm} \xrightarrow{\sim} {}^\dagger\mathcal{D}^{\circledcirc\pm}$, $\mathfrak{D}_\pm \xrightarrow{\sim} {}^\dagger\mathfrak{D}_T$, whose induced morphism $\mathbb{F}_l \xrightarrow{\sim} T$ on the index sets is an isomorphism of $\mathbb{F}_l^\pm$-torsors, and conjugation by which sends $\phi_\pm^{\Theta^{\mathrm{ell}}} \mapsto {}^\dagger\phi_\pm^{\Theta^{\mathrm{ell}}}$. An **isomorphism of $\mathcal{D}$-Θ^{ell}-bridges** $\left({}^\dagger\phi_\pm^{\Theta^{\mathrm{ell}}} : {}^\dagger\mathfrak{D}_T \overset{\text{poly}}{\longrightarrow} {}^\dagger\mathcal{D}^{\circledcirc\pm}\right) \xrightarrow{\sim} \left({}^\ddagger\phi_\pm^{\Theta^{\mathrm{ell}}} : {}^\ddagger\mathfrak{D}_{T'} \overset{\text{poly}}{\longrightarrow} {}^\ddagger\mathcal{D}^{\circledcirc\pm}\right)$ is a pair of a capsule-$+$-full poly-isomorphism ${}^\dagger\mathfrak{D}_T \overset{\text{capsule-}+\text{-full poly}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{D}_{T'}$

whose induced morphism $T \xrightarrow{\sim} T'$ on the index sets is an isomorphism of $\mathbb{F}_l^{\pm}$-torsors, and an $\mathrm{Aut}_{\mathrm{cusp}}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})$-orbit (or, equivalently, an $\mathrm{Aut}_{\mathrm{cusp}}({}^{\ddagger}\mathcal{D}^{\circledcirc\pm})$-orbit) ${}^{\dagger}\mathcal{D}^{\circledcirc\pm} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathcal{D}^{\circledcirc\pm}$ of isomorphisms, which are compatible with ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$, ${}^{\ddagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$. We define compositions of them in an obvious manner.

(3) A **base-(or $\mathcal{D}$-)$\Theta^{\pm\mathrm{ell}}$-Hodge theatre** (or a **$\mathcal{D}$-$\boxplus$-Hodge theatre**) is a collection

$$ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus} = \left({}^{\dagger}\mathfrak{D}_{\succ} \overset{{}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}}{\longleftarrow} {}^{\dagger}\mathfrak{D}_{T} \overset{{}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}}{\longrightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc\pm} \right), $$

where T is an $\mathbb{F}_l^{\pm}$-group, ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$ is a $\mathcal{D}$-Θ^{ell}-bridge, and ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$ is a $\mathcal{D}$-$\Theta^{\pm}$-bridge, such that there exist isomorphisms $\mathcal{D}^{\circledcirc\pm} \xrightarrow{\sim} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$, $\mathfrak{D}_{\pm} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_{T}$, $\mathfrak{D}_{\succ} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_{\succ}$, conjugation by which sends $\phi_{\pm}^{\Theta^{\mathrm{ell}}} \mapsto {}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$, $\phi_{\pm}^{\Theta^{\pm}} \mapsto {}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$. An **isomorphism of $\mathcal{D}$-$\boxplus$-Hodge theatres** is a pair of isomorphisms of $\mathcal{D}$-Θ^{ell}-bridges and $\mathcal{D}$-$\Theta^{\pm}$-bridges such that they induce the same poly-isomorphism of the respective capsules of $\mathcal{D}$-prime-strips. We define compositions of them in an obvious manner.

The following proposition is an additive analogue of Proposition 10.33, and follows by the same manner as Proposition 10.33:

Proposition 10.33. (Transport of $\pm$-Label Classes of Cusps via Base-Bridges, [IUTchI, Proposition 6.5]) *Let* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus} = ({}^{\dagger}\mathfrak{D}_{\succ} \overset{{}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}}{\longleftarrow} {}^{\dagger}\mathfrak{D}_{T} \overset{{}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}}{\longrightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc\pm})$ *be a* $\mathcal{D}$-$\boxplus$-*Hodge theatre.*

(1) The $\mathcal{D}$-Θ^{ell}-*bridge* ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$ *induces an isomorphism*

$$ {}^{\dagger}\zeta_{\underline{v}_t}^{\Theta^{\mathrm{ell}}} : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}_t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm}) $$

of $\mathbb{F}_l^{\pm}$-*torsors of* $\pm$-*label classes of cusps for each* $\underline{v} \in \underline{\mathbb{V}}$, $t \in T$. *Moreover, the composite*

$$ {}^{\dagger}\xi_{\underline{v}_t,\underline{w}_t}^{\Theta^{\mathrm{ell}}} := ({}^{\dagger}\zeta_{\underline{w}_t}^{\Theta^{\mathrm{ell}}})^{-1} \circ ({}^{\dagger}\zeta_{\underline{v}_t}^{\Theta^{\mathrm{ell}}}) : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}_t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{w}_t}) $$

is an isomorphism of $\mathbb{F}_l^{\pm}$-*groups for* $\underline{w} \in \underline{\mathbb{V}}$. *By these identifications* ${}^{\dagger}\xi_{\underline{v}_t,\underline{w}_t}^{\Theta^{\mathrm{ell}}}$ *of* $\mathbb{F}_l^{\pm}$-*groups* $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}_t})$ *when we vary* $\underline{v} \in \underline{\mathbb{V}}$, *we can write*

$$ \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{t}) $$

for them, and we can write the above isomorphism as an isomorphism

$$ {}^{\dagger}\zeta_{t}^{\Theta^{\mathrm{ell}}} : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm}) $$

of $\mathbb{F}_l^{\pm}$-*torsors.*

(2) The $\mathcal{D}$-$\Theta^{\pm}$-bridge ${}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}$ induces an isomorphism

$$ {}^{\dagger}\zeta^{\Theta^{\pm}}_{\underline{v}_t} \,:\, \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}_t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ,\underline{v}}) $$

of $\mathbb{F}_l^{\pm}$-groups of $\pm$-label classes of cusps for each $\underline{v} \in \underline{\mathbb{V}}$, $t \in T$. Moreover, the composites

$$ {}^{\dagger}\xi^{\Theta^{\pm}}_{\succ,\underline{v},\underline{w}} := ({}^{\dagger}\zeta^{\Theta^{\pm}}_{\underline{w}_0}) \circ {}^{\dagger}\xi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0,\underline{w}_0} \circ ({}^{\dagger}\zeta^{\Theta^{\pm}}_{\underline{v}_0})^{-1} \,:\, \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\succ,\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\succ,\underline{w}}), $$

$$ {}^{\dagger}\xi^{\Theta^{\pm}}_{\succ,\underline{v}_t,\underline{w}_t} := ({}^{\dagger}\zeta^{\Theta^{\pm}}_{\underline{w}_t})^{-1} \circ {}^{\dagger}\xi^{\Theta^{\pm}}_{\succ,\underline{v},\underline{w}} \circ ({}^{\dagger}\zeta^{\Theta^{\pm}}_{\underline{v}_t}) \,:\, \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}_t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{w}_t}) $$

(Here we write 0 for the zero element of the $\mathbb{F}_l^{\pm}$-group T) are isomorphisms of $\mathbb{F}_l^{\pm}$-groups for $\underline{w} \in \underline{\mathbb{V}}$, and we also have ${}^{\dagger}\xi^{\Theta^{\pm}}_{\underline{v}_t,\underline{w}_t} = {}^{\dagger}\xi^{\Theta^{\mathrm{ell}}}_{\underline{v}_t,\underline{w}_t}$. By these identifications ${}^{\dagger}\xi^{\Theta^{\pm}}_{\succ,\underline{v},\underline{w}}$ of $\mathbb{F}_l^{\pm}$-groups $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\succ,\underline{v}})$ when we vary $\underline{v} \in \underline{\mathbb{V}}$, we can write

$$ \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ}) $$

for them, and the various ${}^{\dagger}\zeta^{\Theta^{\pm}}_{\underline{v}_t}$'s, and ${}^{\dagger}\zeta^{\Theta^{\mathrm{ell}}}_{\underline{v}_t}$'s determine a single (well-defined) isomorphism

$$ {}^{\dagger}\zeta^{\Theta^{\mathrm{ell}}}_{t} \,:\, \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ}) $$

of $\mathbb{F}_l^{\pm}$-groups.

(3) We have a natural isomorphism

$$ {}^{\dagger}\zeta_{\pm} \,:\, \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} T $$

of $\mathbb{F}_l^{\pm}$-torsors, by considering the inverse of the map $T \ni t \mapsto {}^{\dagger}\zeta^{\Theta^{\mathrm{ell}}}_{t}(0) \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})$, where we write 0 for the zero element of the $\mathbb{F}_l^{\pm}$-group $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{t})$. Moreover, the composite

$$ ({}^{\dagger}\zeta^{\Theta^{\mathrm{ell}}}_{0})^{-1} \circ ({}^{\dagger}\zeta^{\Theta^{\mathrm{ell}}}_{t}) \circ ({}^{\dagger}\zeta^{\Theta^{\pm}}_{t})^{-1} \circ ({}^{\dagger}\zeta^{\Theta^{\pm}}_{0}) \,:\, \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{0}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{0}) $$

is equal to the action of $({}^{\dagger}\zeta^{\Theta^{\mathrm{ell}}}_{0})^{-1}(({}^{\dagger}\zeta_{\pm})^{-1}(t))$.

(4) For $\alpha \in \mathrm{Aut}_{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{cusp}}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})$, if we replece ${}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}$ by $\alpha \circ {}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}$, then the resulting "${}^{\dagger}\zeta^{\Theta^{\mathrm{ell}}}_{t}$" is related to the original ${}^{\dagger}\zeta^{\Theta^{\mathrm{ell}}}_{t}$ by post-composing with the image of α via the natural bijection

$$ \mathrm{Aut}_{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{cusp}}({}^{\dagger}\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathrm{Aut}_{\pm}(\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm}))(\cong \mathbb{F}_l^{\rtimes\pm}) $$

(cf. also Definition 10.29).

The following is an additive analogue of Proposition 10.20, and it follows from the definitions:

Proposition 10.34. (Properties of $\mathcal{D}$-$\Theta^{\pm}$-Brideges, $\mathcal{D}$-Θ^{ell}-Bridges, $\mathcal{D}$-$\boxplus$-Hodge theatres, [IUTchI, Proposition 6.6])

(1) For $\mathcal{D}$-$\Theta^{\pm}$-bridges ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$, ${}^{\ddagger}\phi_{\pm}^{\Theta^{\pm}}$, the set $\mathrm{Isom}({}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}, {}^{\ddagger}\phi_{\pm}^{\Theta^{\pm}})$ *is a $\{\pm 1\} \times \{\pm 1\}^{\underline{\mathbb{V}}}$-torsor, where the first factor $\{\pm 1\}$ (resp. the second factor $\{\pm 1\}^{\underline{\mathbb{V}}}$) corresponds to the poly-automorphism $-1_{\mathbb{F}_l}$ (resp. $\alpha^{\Theta^{\pm}}$) in Definition 10.30.*

(2) For $\mathcal{D}$-Θ^{ell}-bridges ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$, ${}^{\ddagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$, the set $\mathrm{Isom}({}^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}, {}^{\ddagger}\phi_{\divideontimes}^{\mathrm{NF}})$ *is an $\mathbb{F}_l^{\rtimes\pm}$-torsor, and we have a natural isomorphism* $\mathrm{Isom}({}^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}, {}^{\ddagger}\phi_{\divideontimes}^{\mathrm{NF}}) \cong \mathrm{Isom}_{\mathbb{F}_l^{\pm}\text{-torsors}}(T, T')$ *of $\mathbb{F}_l^{\rtimes\pm}$-torsors.*

(3) For $\mathcal{D}$-$\boxplus$-Hodge theatres ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$, ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$, the set $\mathrm{Isom}({}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}, {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus})$ *is an $\{\pm 1\}$-torsor, and we have a natural isomorphism* $\mathrm{Isom}({}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}, {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}) \cong \mathrm{Isom}_{\mathbb{F}_l^{\pm}\text{-groups}}(T, T')$ *of $\{\pm 1\}$-torsors.*

(4) For a $\mathcal{D}$-$\Theta^{\pm}$-bridge ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$ and a $\mathcal{D}$-Θ^{ell}-bridge ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$, the set

$$\left\{\textit{capsule-+-full poly-isom.}\ {}^{\dagger}\mathfrak{D}_T \overset{\text{capsule-+-full poly}}{\xrightarrow{\sim}} {}^{\dagger}\mathfrak{D}_{T'} \quad \textit{by which } {}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}, {}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}} \textit{ form a } \mathcal{D}\text{-}\boxplus\textit{-Hodge theatre}\right\}$$

is an $\mathbb{F}_l^{\rtimes\pm} \times \{\pm 1\}^{\underline{\mathbb{V}}}$-torsor, where the first factor $\mathbb{F}_l^{\rtimes\pm}$ (resp. the subgroup $\{\pm 1\} \times \{\pm 1\}^{\underline{\mathbb{V}}}$) corresponds to the $\mathbb{F}_l^{\rtimes\pm}$ in (2) (resp. to the $\{\pm 1\} \times \{\pm 1\}^{\underline{\mathbb{V}}}$ in (1)). Moreover, the first factor can be regarded as corresponding to the structure group of the $\mathbb{F}_l^{\rtimes\pm}$-torsor $\mathrm{Isom}_{\mathbb{F}_l^{\pm}\text{-torsors}}(T, T')$.

(5) For a $\mathcal{D}$-Θ^{ell}-bridge ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$, we have a functorial algorithm to construct, up to $\mathbb{F}_l^{\rtimes\pm}$-indeterminacy, a $\mathcal{D}$-$\boxplus$-Hodge theatre whose $\mathcal{D}$-Θ^{ell}-bridge is ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$.

Definition 10.35. ([IUTchI, Corollary 6.10]) Let ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$, ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$ be $\mathcal{D}$-$\boxplus$-Hodge theatres. the **base-(or $\mathcal{D}$-)$\Theta^{\pm\mathrm{ell}}$-link** (or **$\mathcal{D}$**-$\boxplus$-link)

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus} \quad \xrightarrow{\mathcal{D}} \quad {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$$

is the full poly-isomorphism

$${}^{\dagger}\mathfrak{D}_{>}^{\vdash} \quad \overset{\text{full poly}}{\xrightarrow{\sim}} \quad {}^{\ddagger}\mathfrak{D}_{>}^{\vdash}$$

between the mono-analyticisations of the $\mathcal{D}$-prime-strips constructed in Lemma 10.38 in the next subsection.

Remark 10.35.1. In $\mathcal{D}$-$\boxplus$-link, the $\mathcal{D}^{\vdash}$-prime-strips are shared, but not the arithmetically holomorphic structures. We can visualise the "shared" and "non-shared" relation as follows:

$$\boxed{{}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}} \dashrightarrow \boxed{{}^{\dagger}\mathcal{D}_{>}^{\vdash} \cong {}^{\ddagger}\mathcal{D}_{>}^{\vdash}} \dashleftarrow \boxed{{}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}}$$

We shall refer to this diagram as the **étale-picture of $\mathcal{D}$-$\boxplus$-Hodge theatres**. Note that *we have a permutation symmetry in the étale-picture.*

Definition 10.36. ($\Theta\pm$-Bridge, Θ^{ell}-Bridge, $\boxplus$-Hodge Theatre, [IUTchI, Deifinition 6.11])

(1) A $\boldsymbol{\Theta^{\pm}}$**-bridge** is a poly-morphism

$$^{\dagger}\psi_{\pm}^{\Theta^{\pm}} : {}^{\dagger}\mathfrak{F}_T \xrightarrow{\text{poly}} {}^{\ddagger}\mathfrak{F}_{\succ},$$

where ${}^{\dagger}\mathfrak{F}_{\succ}$ is an $\mathcal{F}$-prime-strip, and ${}^{\dagger}\mathfrak{F}_T$ is a cupsule of $\mathcal{F}$-prime-strips indexed by an $\mathbb{F}_l^{\pm}$-group T, which lifts (cf. Lemma 10.10 (2)) a $\mathcal{D}$-$\Theta^{\pm}$-bridge ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}} : {}^{\dagger}\mathfrak{D}_T \xrightarrow{\text{poly}} {}^{\dagger}\mathfrak{D}_{\succ}$. An **isomorphism of** $\boldsymbol{\Theta^{\pm}}$**-bridges** $\left({}^{\dagger}\psi_{\pm}^{\Theta^{\pm}} : {}^{\dagger}\mathfrak{F}_T \xrightarrow{\text{poly}} {}^{\dagger}\mathfrak{F}_{\succ}\right) \xrightarrow{\sim} \left({}^{\ddagger}\psi_{\pm}^{\Theta^{\pm}} : {}^{\ddagger}\mathfrak{F}_{T'} \xrightarrow{\text{poly}} {}^{\ddagger}\mathfrak{F}_{\succ}\right)$ is a pair of poly-isomorphisms ${}^{\dagger}\mathfrak{F}_T \overset{\text{poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{F}_{T'}$ and ${}^{\dagger}\mathfrak{F}_{\succ} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{F}_{\succ}$, which lifts a morphism between the associated $\mathcal{D}$-$\Theta^{\pm}$-bridges ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$, ${}^{\ddagger}\phi_{\pm}^{\Theta^{\pm}}$. We define compositions of them in an obvious manner.

(2) A $\boldsymbol{\Theta^{\text{ell}}}$**-bridge**

$$^{\dagger}\psi_{\pm}^{\Theta^{\text{ell}}} : {}^{\dagger}\mathfrak{F}_T \xrightarrow{\text{poly}} {}^{\dagger}\mathcal{D}^{\circledcirc\pm},$$

where ${}^{\dagger}\mathcal{D}^{\circledcirc\pm}$ is a category equivalent to the model global object $\mathcal{D}^{\circledcirc\pm}$ in Definition 10.3, and ${}^{\dagger}\mathfrak{F}_T$ is a capsule of $\mathcal{F}$-prime-strips indexed by an $\mathbb{F}_l^{\pm}$-torsor T, is a $\mathcal{D}$-Θ^{ell}-bridge ${}^{\dagger}\phi_{\pm}^{\Theta^{\text{ell}}} : {}^{\dagger}\mathfrak{D}_T \xrightarrow{\text{poly}} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$, where ${}^{\dagger}\mathfrak{D}_T$ is the associated capsule of $\mathcal{D}$-prime-strips to ${}^{\dagger}\mathfrak{F}_T$. An **isomorphism of** $\boldsymbol{\Theta^{\text{ell}}}$**-bridges** $\left({}^{\dagger}\psi_{\pm}^{\Theta^{\text{ell}}} : {}^{\dagger}\mathfrak{F}_T \xrightarrow{\text{poly}} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}\right) \xrightarrow{\sim} \left({}^{\ddagger}\psi_{\pm}^{\Theta^{\text{ell}}} : {}^{\ddagger}\mathfrak{F}_{T'} \xrightarrow{\text{poly}} {}^{\ddagger}\mathcal{D}^{\circledcirc\pm}\right)$ is a pair of poly-isomorphisms ${}^{\dagger}\mathfrak{F}_T \overset{\text{poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{F}_{T'}$ and ${}^{\dagger}\mathcal{D}^{\circledcirc\pm} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathcal{D}^{\circledcirc\pm}$, which determines a morphism between the associated $\mathcal{D}$-Θ^{ell}-bridges ${}^{\dagger}\phi_{\pm}^{\Theta^{\text{ell}}}$, ${}^{\ddagger}\phi_{\pm}^{\Theta^{\text{ell}}}$. We define compositions of them in an obvious manner.

(3) A $\boldsymbol{\Theta^{\pm\text{ell}}}$**-Hodge theatre** (or a $\boldsymbol{\boxplus}$**-Hodge theatre**) is a collection

$$^{\dagger}\mathcal{HT}^{\boxplus} = \left({}^{\dagger}\mathfrak{F}_{\succ} \xleftarrow{{}^{\dagger}\psi_{\pm}^{\Theta^{\pm}}} {}^{\dagger}\mathfrak{F}_T \xrightarrow{{}^{\dagger}\psi_{\pm}^{\Theta^{\text{ell}}}} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}\right),$$

where ${}^{\dagger}\psi_{\pm}^{\Theta^{\pm}}$ is a $\Theta^{\pm}$-bridge, and ${}^{\dagger}\psi_{\pm}^{\Theta^{\text{ell}}}$ is a Θ^{ell}-bridge, such that the associated $\mathcal{D}$-$\Theta^{\pm}$-bridge ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$ and the associated $\mathcal{D}$-Θ^{ell}-bridge ${}^{\dagger}\phi_{\pm}^{\Theta^{\text{ell}}}$ form a $\mathcal{D}$-$\boxplus$-Hodge theatre. An **isomorphism of** $\boldsymbol{\boxplus}$**-Hodge theatres** is a pair of a morphism of $\Theta^{\pm}$-bridge and a morphism of Θ^{ell}-bridge, which induce the same bijection between the respective capsules of $\mathcal{F}$-prime-strips. We define compositions of them in an obvious manner.

The following lemma follows from the definitions:

Lemma 10.37. (Properties of $\Theta^{\pm}$-Brideges, Θ^{ell}-Bridges, $\boxplus$-Hodge theatres, [IUTchI, Corollary 6.12])

(1) For $\Theta^{\pm}$-bridges ${}^1\psi_{\pm}^{\Theta^{\pm}}$, ${}^2\psi_{\pm}^{\Theta^{\pm}}$ (resp. Θ^{ell}-bridges ${}^1\psi_{\pm}^{\Theta^{\text{ell}}}$, ${}^2\psi_{\pm}^{\Theta^{\text{ell}}}$, resp. $\boxplus$-Hodge theatres ${}^1\mathcal{HT}^{\boxplus}$, ${}^2\mathcal{HT}^{\boxplus}$) whose associated $\mathcal{D}$-$\Theta^{\pm}$-bridges (resp. $\mathcal{D}$-Θ^{ell}-bridges, resp. $\mathcal{D}$-$\boxplus$-Hodge theatres) are ${}^1\phi_{\pm}^{\Theta^{\pm}}$, ${}^2\phi_{\pm}^{\Theta^{\pm}}$ (resp. ${}^1\phi_{\pm}^{\Theta^{\text{ell}}}$, ${}^2\phi_{\pm}^{\Theta^{\text{ell}}}$, resp. ${}^1\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$, ${}^2\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$) respectively, the natural map

$$\begin{aligned} &\mathrm{Isom}({}^1\psi_{\pm}^{\Theta^{\pm}}, {}^2\psi_{\pm}^{\Theta^{\pm}}) \to \mathrm{Isom}({}^1\phi_{\pm}^{\Theta^{\pm}}, {}^2\phi_{\pm}^{\Theta^{\pm}}) \\ (\textit{resp.}\quad &\mathrm{Isom}({}^1\psi_{\pm}^{\Theta^{\text{ell}}}, {}^2\psi_{\pm}^{\Theta^{\text{ell}}}) \to \mathrm{Isom}({}^1\phi_{\pm}^{\Theta^{\text{ell}}}, {}^2\phi_{\pm}^{\Theta^{\text{ell}}}), \\ \textit{resp.}\quad &\mathrm{Isom}({}^1\mathcal{HT}^{\boxplus}, {}^2\mathcal{HT}^{\boxplus}) \to \mathrm{Isom}({}^1\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}, {}^2\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}) \quad) \end{aligned}$$

is bijective.

(2) For a $\Theta^{\pm}$-bridge ${}^{\ddagger}\psi_{\pm}^{\Theta^{\pm}}$ and a Θ^{ell}-bridge ${}^{\ddagger}\psi_{\pm}^{\Theta^{\text{ell}}}$, the set

$$\left\{ \textit{capsule-+-full poly-isom.} \quad {}^{\ddagger}\mathfrak{F}_T \overset{\text{capsule-+-full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{F}_{T'} \quad \textit{by which } {}^{\ddagger}\psi_{\pm}^{\Theta^{\pm}}, {}^{\ddagger}\psi_{\pm}^{\Theta^{\text{ell}}} \textit{ form a } \boxplus\textit{-Hodge theatre} \right\}$$

is an $\mathbb{F}_l^{\rtimes\pm} \times \{\pm 1\}^{\underline{\mathbb{V}}}$-torsor. Moreover, the first factor can be regarded as corresponding to the structure group of the $\mathbb{F}_l^{\rtimes\pm}$-torsor $\mathrm{Isom}_{\mathbb{F}_l^{\pm}\text{-torsors}}(T, T')$.

§ 10.6. $\Theta^{\pm\text{ell}}$NF-Hodge Theatres — An Arithmetic Analogue of the Upper Half Plane.

In this subsection, we combine the multiplicative portion of Hodge theatre and the additive portion of Hodge theature to obtain full Hodge theatre.

Lemma 10.38. (From ($\mathcal{D}$-)$\Theta^{\pm}$-Bridge To ($\mathcal{D}$-)Θ-Bridge, [IUTchI, Definition 6.4 (i), Proposition 6.7, Definition 6.11 (i), Remark 6.12 (i)]) *Let ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}} : {}^{\dagger}\mathfrak{D}_T \overset{\text{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{D}_{\succ}$ (resp. ${}^{\dagger}\psi_{\pm}^{\Theta^{\pm}} : {}^{\dagger}\mathfrak{F}_T \overset{\text{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{F}_{\succ}$) be a $\mathcal{D}$-$\Theta^{\pm}$-bridge (resp. $\Theta^{\pm}$-bridge). We write*

$${}^{\dagger}\mathfrak{D}_{|T|} \quad (\textit{resp.} \quad {}^{\dagger}\mathfrak{F}_{|T|} \quad)$$

for the $l^{\pm}$-capsule (cf. Section 0.2 for $l^{\pm}$) of $\mathcal{D}$-prime-strips (resp. $\mathcal{F}$-prime-strips) obtained from l-capsule ${}^{\dagger}\mathfrak{D}_T$ (resp. ${}^{\dagger}\mathfrak{F}_T$) of $\mathcal{D}$-prime-strips (resp. $\mathcal{F}$-prime-strips) by forming the quotient $|T|$ of the index set T by $\{\pm 1\}$, and identifying the components of the cupsule ${}^{\dagger}\mathfrak{D}_T$ (resp. ${}^{\dagger}\mathfrak{F}_T$) in the same fibers of $T \twoheadrightarrow |T|$ via the components of the poly-morphism ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}} = \{{}^{\dagger}\phi_t^{\Theta^{\pm}}\}_{t\in T}$ (resp. ${}^{\dagger}\psi_{\pm}^{\Theta^{\pm}} = \{{}^{\dagger}\psi_t^{\Theta^{\pm}}\}_{t\in T}$) (Hence each component *of* ${}^\dagger\mathfrak{D}_{|T|}$ *(resp.* ${}^\dagger\mathfrak{F}_{|T|}$*) is only well-defined up to a positive automorphism). We also write*

$$ {}^\dagger\mathfrak{D}_{T^*} \quad (\textit{resp.}\ \ {}^\dagger\mathfrak{F}_{T^*}\ \ ) $$

for the l^**-capsule determined by the subset* $T^* := |T| \setminus \{0\}$ *of non-zero elements of* $|T|$*.*

We identify ${}^\dagger\mathfrak{D}_0$ *(resp.* ${}^\dagger\mathfrak{F}_0$*) with* ${}^\dagger\mathfrak{D}_\succ$ *(resp.* ${}^\dagger\mathfrak{F}_\succ$*) via* ${}^\dagger\phi_0^{\Theta^\pm}$ *(resp.* ${}^\dagger\psi_0^{\Theta^\pm}$*), and we write* ${}^\dagger\mathfrak{D}_>$ *(resp.* ${}^\dagger\mathfrak{F}_>$*) for the resulting* $\mathcal{D}$*-prime-strip (resp.* $\mathcal{F}$*-prime-strip) (i.e.,* $> = \{0, \succ\}$*). For* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$*, we replace the* $+$*-full poly-morphism at* $\underline{v}$*-component of* ${}^\dagger\phi_\pm^{\Theta^\pm}$ *(resp.* ${}^\dagger\psi_\pm^{\Theta^\pm}$*) by the full poly-morphism. For* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*, we replace the* $+$*-full poly-morphism at* $\underline{v}$*-component of* ${}^\dagger\phi_\pm^{\Theta^\pm}$ *(resp.* ${}^\dagger\psi_\pm^{\Theta^\pm}$*) by the poly-morphism determined by (group-theoretically reconstructed) evaluation section as in Definition 10.17 (resp. by the poly-morphism lying over (cf. Definition 10.23 (1), (2), and Remark 10.10.1) the poly-morphism determined by (group-theoretically reconstructed) evaluation section as in Definition 10.17). Then we algorithmically obtain a* $\mathcal{D}$*-*Θ*-bridge (resp. a potion of* Θ*-bridge)*

$$ {}^\dagger\phi_*^{\Theta} : {}^\dagger\mathfrak{D}_{T^*} \stackrel{\mathrm{poly}}{\longrightarrow} {}^\dagger\mathfrak{D}_> \quad (\textit{resp.}\ \ {}^\dagger\psi_*^{\Theta} : {}^\dagger\mathfrak{F}_{T^*} \stackrel{\mathrm{poly}}{\longrightarrow} {}^\dagger\mathfrak{F}_>\ \ ) $$

in a functorial manner. cf. also the following:

$$
\begin{array}{rcl rcl}
{}^\dagger\mathfrak{D}_0,\ {}^\dagger\mathfrak{D}_\succ & \mapsto & {}^\dagger\mathfrak{D}_>, & {}^\dagger\mathfrak{F}_0,\ {}^\dagger\mathfrak{F}_\succ & \mapsto & {}^\dagger\mathfrak{F}_>, \\
{}^\dagger\mathfrak{D}_t,\ {}^\dagger\mathfrak{D}_{-t}\ (t \neq 0) & \mapsto & {}^\dagger\mathfrak{D}_{|t|}, & {}^\dagger\mathfrak{F}_t,\ {}^\dagger\mathfrak{F}_{-t}\ (t \neq 0) & \mapsto & {}^\dagger\mathfrak{F}_{|t|} \\
{}^\dagger\mathfrak{D}_T|_{T\setminus\{0\}} & \mapsto & {}^\dagger\mathfrak{D}_{T^*}, & {}^\dagger\mathfrak{F}_T|_{T\setminus\{0\}} & \mapsto & {}^\dagger\mathfrak{F}_{T^*},
\end{array}
$$

where we write $|t|$ *for the image of* $t \in T$ *under the surjection* $T \twoheadrightarrow |T|$*.*

Definition 10.39. ([IUTchI, Remark 6.12.2]) Let ${}^\dagger\mathfrak{F}_T \stackrel{\mathrm{poly}}{\longrightarrow} {}^\dagger\mathfrak{F}_\succ$ be a $\Theta^\pm$-bridge, whose associated $\mathcal{D}$-$\Theta^\pm$-bridge is ${}^\dagger\mathfrak{D}_T \stackrel{\mathrm{poly}}{\longrightarrow} {}^\dagger\mathfrak{D}_\succ$. Then we have a group-theoretically functorial algorithm for constructing a $\mathcal{D}$-Θ-bridge ${}^\dagger\mathfrak{D}_{T^*} \stackrel{\mathrm{poly}}{\longrightarrow} {}^\dagger\mathfrak{D}_>$ from the $\mathcal{D}$-$\Theta^\pm$-bridge ${}^\dagger\mathfrak{D}_T \stackrel{\mathrm{poly}}{\longrightarrow} {}^\dagger\mathfrak{D}_\succ$ by Lemma 10.38. Suppose that this $\mathcal{D}$-Θ-bridge ${}^\dagger\mathfrak{D}_{T^*} \stackrel{\mathrm{poly}}{\longrightarrow} {}^\dagger\mathfrak{D}_>$ arises as the $\mathcal{D}$-Θ-bridge associated to a Θ-bridge ${}^\ddagger\mathfrak{F}_J \stackrel{\mathrm{poly}}{\longrightarrow} {}^\ddagger\mathfrak{D}_> \dashrightarrow {}^\ddagger\mathcal{HT}^\Theta$, where $J = T^*$:

$$
\begin{array}{ccc}
{}^\dagger\mathfrak{F}_T \stackrel{\mathrm{poly}}{\longrightarrow} {}^\dagger\mathfrak{F}_\succ & & {}^\ddagger\mathfrak{F}_J \stackrel{\mathrm{poly}}{\longrightarrow} {}^\ddagger\mathfrak{D}_> \dashrightarrow {}^\ddagger\mathcal{HT}^\Theta \\
\downarrow & & \downarrow \\
{}^\dagger\mathfrak{D}_T \stackrel{\mathrm{poly}}{\longrightarrow} {}^\dagger\mathfrak{D}_\succ & \longmapsto & {}^\dagger\mathfrak{D}_{T^*} \stackrel{\mathrm{poly}}{\longrightarrow} {}^\dagger\mathfrak{D}_>.
\end{array}
$$

Then the poly-morphism ${}^\ddagger\mathfrak{F}_J \stackrel{\mathrm{poly}}{\longrightarrow} {}^\ddagger\mathfrak{F}_>$ lying over ${}^\dagger\mathfrak{D}_{T^*} \stackrel{\mathrm{poly}}{\longrightarrow} {}^\dagger\mathfrak{D}_>$ is completely determined (cf. Definition 10.23 (1), (2), and Remark 10.10.1). Hence we can regard this portion ${}^\ddagger\mathfrak{F}_J \stackrel{\mathrm{poly}}{\longrightarrow} {}^\ddagger\mathfrak{F}_>$ of the Θ-bridge as having been constructed via the functorial algorithm of Lemma 10.38. Moreover, by Lemma 10.25 (1), the isomorphisms between Θ-bridges have a natural bijection with the the isomorphisms between the "${}^{\ddagger}\mathfrak{F}_J \overset{\text{poly}}{\longrightarrow} {}^{\ddagger}\mathfrak{F}_{>}$"-portion of Θ-bridges.

In this situation, we say that the Θ-bridge ${}^{\ddagger}\mathfrak{F}_J \overset{\text{poly}}{\longrightarrow} {}^{\ddagger}\mathfrak{D}_{>} \dashrightarrow {}^{\ddagger}\mathcal{HT}^{\Theta}$ (resp. $\mathcal{D}$-Θ-bridge ${}^{\dagger}\mathfrak{D}_{T^*} \overset{\text{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{D}_{>}$) **is glued** to the $\Theta^{\pm}$-bridge ${}^{\dagger}\mathfrak{F}_T \overset{\text{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{F}_{\succ}$ (resp. $\mathcal{D}$-$\Theta^{\pm}$-bridge ${}^{\dagger}\mathfrak{D}_T \overset{\text{poly}}{\longrightarrow} {}^{\dagger}\mathfrak{D}_{\succ}$) via the functorial algorithm in Lemma 10.38. Note that, by Proposition 10.20 (2) and Lemma 10.25 (1), the gluing isomorphism is *unique*.

Definition 10.40. ($\mathcal{D}$-$\boxtimes\boxplus$-Hodge Theatre, $\boxtimes\boxplus$-Hodge Theatre, [IUTchI, Definition 6.13])

(1) A **base-(or $\boldsymbol{\mathcal{D}}$-)$\boldsymbol{\Theta^{\pm\mathrm{ell}}}$NF-Hodge theatre** ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$ is a tripe of a $\mathcal{D}$-$\boxtimes$-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$, a $\mathcal{D}$-$\boxplus$-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$, and the (necessarily unique) gluing isomorphism between ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$ and ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$. We define an **isomorphism of $\boldsymbol{\mathcal{D}}$-$\boxtimes\boxplus$-Hodge theatres** in an obvious manner.

(2) A **$\boldsymbol{\Theta^{\pm\mathrm{ell}}}$NF-Hodge theatre** ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$ is a tripe of a $\boxtimes$-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\boxtimes}$, a $\boxplus$-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\boxplus}$, and the (necessarily unique) gluing isomorphism between ${}^{\dagger}\mathcal{HT}^{\boxtimes}$ and ${}^{\dagger}\mathcal{HT}^{\boxplus}$. We define an **isomorphism of $\boxtimes\boxplus$-Hodge theatres** in an obvious manner.

§ 11. Hodge-Arakelov-theoretic Evaluation Maps.

§ 11.1. Radial Environments.

In inter-universal Teichmüller theory, not only the existence of functorial group-theoretic algorithms, but also the *contents* of algorithms are important. In this subsection, we introduce important notions of coricity, uniradiality, and multiradiality for the *contents* of algorithms.

Definition 11.1. (Radial Environment, [IUTchII, Example 1.7, Example 1.9])

(1) A **radial environment** is a triple $(\mathcal{R}, \mathcal{C}, \Phi)$, where $\mathcal{R}$, $\mathcal{C}$ are groupoids (i.e., categories in which all morphisms are isomorphisms) such that all objects are isomorphic, and $\Phi : \mathcal{R} \to \mathcal{C}$ is an essentially surjective functor (In fact, in our mind, we expect that $\mathcal{R}$ and $\mathcal{C}$ are collections of certain "type of mathematical data" (i.e., **species**), and Φ is "algorithmically defined" functor (i.e., **mutations**). In this survey, we avoid the rigorous formulation of the language of species and mutations (cf. [IUTchIV, §3]), and we just assume that $\mathcal{R}, \mathcal{C}$ to be as above, and Φ to be a functor. cf. also Remark 3.4.4 (2)). We shall refer to $\mathcal{C}$ as a **coric category** an object of $\mathcal{C}$ a **coric data**, $\mathcal{R}$ a **radial category** an object of $\mathcal{R}$ a **radial data**, and Φ a **radial algorithm**.

(2) We shall refer to Φ as **multiradial**, if Φ is full. We shall refer to Φ as **uniradial**, if Φ is not full. We shall refer to $(\mathcal{R}, \mathcal{C}, \Phi)$ as **multiradial environment** (resp. **uniradial environment**), if Φ is multiradial (resp. uniradial).

Note that, if Φ is uniradial, then an isomoprhism in $\mathcal{C}$ does not come from an isomorphism in $\mathcal{R}$, which means that an object of $\mathcal{R}$ loses a portion of rigidity by Φ, i.e., might be subject to an additional indeterminacy (From another point of view, the liftability of isomorphism, i.e., multiradiality, makes possible doing a kind of *parallel transport* from another radial data via the associated coric data. cf. [IUTchII, Remark 1.7.1]).

(3) Let $(\mathcal{R}, \mathcal{C}, \Phi)$ be a radial environment. Let ${}^{\dagger}\mathcal{R}$ be another groupoid in which all objects are isomorphic, ${}^{\dagger}\Phi : {}^{\dagger}\mathcal{R} \to \mathcal{C}$ an essentially surjective functor, and $\Psi_{\mathcal{R}} : \mathcal{R} \to {}^{\dagger}\mathcal{R}$ a functor. We shall refer to $\Psi_{\mathcal{R}}$ as **multiradially defined** or **multiradial** (resp. **uniradially defined** or **uniradial**) if Φ is multiradial (resp. uniradial) and if the diagram

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\Psi_{\mathcal{R}}"] \arrow[d, "\Phi"'] & {}^{\dagger}\mathcal{R} \arrow[dl, "{}^{\dagger}\Phi"] \\ \mathcal{C} & \end{tikzcd}$$

is 1-commutative. We shall refer to $\Psi_{\mathcal{R}}$ as **corically defined** (or **coric**), if $\Psi_{\mathcal{R}}$ has a factorisation $\Xi_{\mathcal{R}} \circ \Phi$, where $\Xi_{\mathcal{R}} : \mathcal{C} \to {}^{\dagger}\mathcal{R}$ is a functor, and if the above diagram is 1-commutative.

(4) Let $(\mathcal{R}, \mathcal{C}, \Phi)$ be a radial environment. Let $\mathcal{E}$ be another groupoid in which all objects are isomorphic, and $\Xi : \mathcal{R} \to \mathcal{E}$ a functor. We write

$$\mathrm{Graph}(\Xi)$$

for the category whose objects are pairs $(R, \Xi(R))$ for $R \in \mathrm{Ob}(\mathcal{R})$, and whose morphisms are the pairs of morphisms $(f : R \to R', \Xi(f) : \Xi(R) \to \Xi(R'))$. We shall refer to $\mathrm{Graph}(\Xi)$ as the **graph of Ξ**. We have a commutative diagram

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\Psi_{\Xi}"] \arrow[d, "\Phi"'] & \mathrm{Graph}(\Xi) \arrow[dl, "\Phi_{\mathrm{Graph}(\Xi)}"] \\ \mathcal{C}, & \end{tikzcd}$$

of natural functors, where $\Psi_{\Xi} : R \mapsto (R, \Xi(R))$ and $\Phi_{\mathrm{Graph}(\Xi)} : (R, \Xi(R)) \mapsto \Phi(R)$.

Remark 11.1.1. ([IUTchII, Example 1.7 (iii)]) A crucial fact on the consequence of the multiradiality is the following: For a radial environment $(\mathcal{R}, \mathcal{C}, \Phi)$, we write $\mathcal{R} \times_{\mathcal{C}} \mathcal{R}$

for the category whose objects are triple (R_1, R_2, α), where $R_1, R_2 \in \mathrm{Ob}(\mathcal{R})$, and α is an isomorphism $\Phi(R_1) \xrightarrow{\sim} \Phi(R_2)$, and whose morphisms are morphisms of triples defined in an obvious manner. Then the switching functor

$$\mathcal{R} \times_{\mathcal{C}} \mathcal{R} \to \mathcal{R} \times_{\mathcal{C}} \mathcal{R} \ : \ (R_1, R_2, \alpha) \mapsto (R_2, R_1, \alpha^{-1})$$

preserves the isomorphism class of objects of $\mathcal{R} \times_{\mathcal{C}} \mathcal{R}$, if Φ is multiradial, since any object (R_1, R_2, α) in $\mathcal{R} \times_{\mathcal{C}} \mathcal{R}$ is isomorphic to the object $(R_1, R_1, \mathrm{id} : \Phi(R_1) \xrightarrow{\sim} \Phi(R_1))$. This means that, if the radial algorithm is multiradial, then we can switch two radial data up to isomorphism.

Ultimately, in the final multiradial algorithm, we can "switch", up to isomorphism, the theta values (more precisely, Θ-pilot object, up to mild indeterminacies) "$\{{}^{\ddagger}\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{1 \leq j \leq l^*}$" on the right-hand side of (the final update of) Θ-link to the theta values (more precisely, Θ-pilot object, up to mild indeterminacies) "$\{{}^{\dagger}\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{1 \leq j \leq l^*}$" on the left-hand side of (the final update of) Θ-link, which is isomorphic to ${}^{\ddagger}\underline{\underline{q}}_{\underline{v}}$ (more precisely, q-pilot object, up to mild indeterminacies) by using *the Θ-link compatibility* of the final multiradial algorithm (Theorem 13.12 (3)):

$$\{{}^{\ddagger}\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{1 \leq j \leq l^*}^{\mathbb{N}} \overset{!!}{\leftrightsquigarrow} \{{}^{\dagger}\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{1 \leq j \leq l^*}^{\mathbb{N}} \cong {}^{\ddagger}\underline{\underline{q}}_{\underline{v}}^{\mathbb{N}}$$

Then *we cannot distinguish* $\{{}^{\ddagger}\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{1 \leq j \leq l^*}$ *from* ${}^{\ddagger}\underline{\underline{q}}_{\underline{v}}$ *up to mild indeterminacies* (i.e., (Indet $\uparrow$), (Indet $\to$), and (Indet $\curvearrowright$)), which gives us a upper bound of height function (cf. also Appendix A).

Example 11.2.

(1) A classical example is holomorphic structures on $\mathbb{R}^2$:

$$\begin{tikzcd} {}^{\dagger}\mathbb{C} \arrow[d, mapsto, "\text{forget}"'] & \\ \mathbb{R}^2 & {}^{\ddagger}\mathbb{C}, \arrow[l, mapsto, "\text{forget}"] \end{tikzcd}$$

where $\mathcal{R}$ is the category of 1-dimensional $\mathbb{C}$-vector spaces and isomorphisms of $\mathbb{C}$-vector spaces, $\mathcal{C}$ is the category of 2-dimensional $\mathbb{R}$-vector spaces and isomorphisms of $\mathbb{R}$-vector spaces, and Φ sends 1-dimensional $\mathbb{C}$-vector spaces to the underlying $\mathbb{R}$-vector spaces. Then the radial environment $(\mathcal{R}, \mathcal{C}, \Phi)$ is *uniradial.* Note that the underlying $\mathbb{R}^2$ is shared (i.e., coric), and that we cannot see one holomorphic structure ${}^{\dagger}\mathbb{C}$ from another holomorphic structure ${}^{\ddagger}\mathbb{C}$.

Next, we replace $\mathcal{R}$ by the category of 1-dimensional $\mathbb{C}$-vector spaces ${}^{\dagger}\mathbb{C}$ equipped with the $\mathrm{GL}_2(\mathbb{R})$-orbit of an isomorphism ${}^{\dagger}\mathbb{C} \xrightarrow{\sim} \mathbb{R}^2$ (for a fixed $\mathbb{R}^2$). Then the resulting radial environment $(\mathcal{R}, \mathcal{C}, \Phi)$ is tautologically *multiradial*:

$$\begin{tikzcd}
({}^{\dagger}\mathbb{C} \xrightarrow{\sim} \mathbb{R}^2 \curvearrowleft \mathrm{GL}_2(\mathbb{R})) \arrow[d, mapsto, "\text{forget}"'] & \\
\mathbb{R}^2 & ({}^{\ddagger}\mathbb{C} \xrightarrow{\sim} \mathbb{R}^2 \curvearrowleft \mathrm{GL}_2(\mathbb{R})). \arrow[l, mapsto, "\text{forget}"]
\end{tikzcd}$$

Note that the underlying $\mathbb{R}^2$ is shared (i.e., coric), and that we can describe the difference between one holomorphic structure ${}^{\dagger}\mathbb{C}$ and another holomorphic structure ${}^{\ddagger}\mathbb{C}$ in terms of the underlying analytic structure $\mathbb{R}^2$.

(2) An *arithmetic analogue* of the above example is as follows: As already explained in Section 3.5, the absolute Galois group G_k of an MLF k has an automorphism which does not come from any automorphism of fields (at least in the case where the residue characteristic is $\neq 2$), and one "dimension" is rigid, and the other "dimension" is not rigid, hence we consider G_k as a mono-analytic structure. On the other hand, from the arithmetic fundamental group Π_X of hyperbolic orbicurve X of strictly Belyi type over k, we can reconstruct the field k (Theorem 3.17), hence we consider Π_X as an arithmetically holomorphic structure, and the quotient $(\Pi_X \twoheadrightarrow) G_k$ (group-theoretically reconstructable by Corollary 2.4) as the underlying mono-analytic structure. For a fixed hyperbolic orbicurve X of strictly Belyi type over an MLF k, let $\mathcal{R}$ be the category of topological groups isomorphic to Π_X and isomorphisms of topological groups, and $\mathcal{C}$ the category of topological groups isomorphic to G_k and isomorphisms of topological groups, and Φ be the functor which sends Π to the group-theoretically reconstructed quotien $(\Pi \twoheadrightarrow) G$. Then the radial environment $(\mathcal{R}, \mathcal{C}, \Phi)$ is *uniradial*:

$$\begin{tikzcd}
{}^{\dagger}\Pi \arrow[d, mapsto] & \\
{}^{\dagger}G \overset{\exists}{\cong} G_k \overset{\exists}{\cong} {}^{\ddagger}G & {}^{\ddagger}\Pi. \arrow[l, mapsto]
\end{tikzcd}$$

Next, we replace $\mathcal{R}$ by the category of topological groups isomorphic to Π_X equipped with the full-poly isomorphism $G \xrightarrow{\sim} G_k$, where $(\Pi \twoheadrightarrow) G$ is the group-theoretic reconstructed quotient. Then the resulting radial environment $(\mathcal{R}, \mathcal{C}, \Phi)$ is tautologically *multiradial*:

$$\begin{tikzcd}[column sep=small]
 & & ({}^{\dagger}\Pi \twoheadrightarrow {}^{\dagger}G \arrow[r, "\overset{\text{full poly}}{\sim}"] & G_k) \arrow[d, mapsto] & & & \\
{}^{\dagger}G \arrow[r, phantom, "\overset{\text{full poly}}{\cong}"] & G_k \arrow[r, phantom, "\overset{\text{full poly}}{\cong}"] & {}^{\ddagger}G & ({}^{\ddagger}\Pi \twoheadrightarrow {}^{\dagger}G \arrow[r, "\overset{\text{full poly}}{\sim}"] \arrow[l, mapsto] & G_k).
\end{tikzcd}$$

cf. also the following table (cf. [Pano, Fig. 2.2, Fig. 2.3]):

coric	underlying analytic str.	$\mathbb{R}^2$	G
uniradial	holomorphic str.	$\mathbb{C}$	Π
multiradial	holomorphic str. described in terms of underlying coric str.	$\mathbb{C} \overset{\sim}{\to} \mathbb{R}^2 \curvearrowleft \mathrm{GL}_2(\mathbb{R}^2)$	$\Pi/\Delta \overset{\text{full poly}}{\overset{\sim}{\longrightarrow}} G$

In the final multiradial algorithm (Theorem 13.12), which admits mild indeterminacies, we describe the arithmetically holomorphic structure on one side of (the final update of) Θ-link from the one on the other side, *in terms of shared mono-analytic structure*.

Definition 11.3. ([IUTchII, Definition 1.1, Proposition 1.5 (i), (ii)]) Let $\mathbb{M}_*^{\Theta} = (\cdots \leftarrow \mathbb{M}_M^{\Theta} \leftarrow \mathbb{M}_{M'}^{\Theta} \leftarrow \cdots)$, be a projective system of mono-theta environments determined by $\underline{\underline{X}}_{\underline{v}}$ ($\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$), where $\mathbb{M}_M^{\Theta} = (\Pi_{\mathbb{M}_M^{\Theta}}, \mathcal{D}_{\mathbb{M}_M^{\Theta}}, s^{\Theta}_{\mathbb{M}_M^{\Theta}})$. For each N, by Corollary 7.22 (3) and Lemma 7.12, we can functorially group-theoretically reconstruct, from $\mathbb{M}_N^{\Theta}$, a commutative diagram

$$\begin{tikzcd}
 & & & G_{\underline{v}}(\mathbb{M}_N^{\Theta}) & & \\
 & \Pi^{\mathrm{temp}}_{\mathbb{M}_N^{\Theta}} \arrow[r, two heads] & \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}(\mathbb{M}_N^{\Theta}) \arrow[r, hook] \arrow[ur, two heads] & \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}_N^{\Theta}) \arrow[r, hook] \arrow[u, two heads] & \Pi^{\mathrm{temp}}_{\underline{X}}(\mathbb{M}_N^{\Theta}) \arrow[r, hook] \arrow[ul, two heads] & \Pi^{\mathrm{temp}}_{C}(\mathbb{M}_N^{\Theta}) \arrow[ull, two heads] \\
\mu_N(\mathbb{M}_N^{\Theta}) \arrow[r, hook] \arrow[ur, hook] & \Delta^{\mathrm{temp}}_{\mathbb{M}_N^{\Theta}} \arrow[r, two heads] \arrow[u, hook] & \Delta^{\mathrm{temp}}_{\underline{\underline{Y}}}(\mathbb{M}_N^{\Theta}) \arrow[r, hook] \arrow[u, hook] & \Delta^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}_N^{\Theta}) \arrow[r, hook] \arrow[u, hook] & \Delta^{\mathrm{temp}}_{\underline{X}}(\mathbb{M}_N^{\Theta}) \arrow[r, hook] \arrow[u, hook] & \Delta^{\mathrm{temp}}_{C}(\mathbb{M}_N^{\Theta})
\arrow[u, hook]
\end{tikzcd}$$

of topological groups, which is an isomorph of

$$\begin{tikzcd}
 & & & G_{\underline{v}} & & \\
 & \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N] \arrow[r, two heads] & \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}} \arrow[r, hook] \arrow[ur] & \Pi^{\mathrm{temp}}_{\underline{\underline{X}}} \arrow[r, hook] \arrow[u, two heads] & \Pi^{\mathrm{temp}}_{\underline{X}} \arrow[r, hook] \arrow[ul, two heads] & \Pi^{\mathrm{temp}}_{C} \arrow[ull, two heads] \\
\mu_N \arrow[r, hook] \arrow[ur, hook] & \Delta^{\mathrm{temp}}_{\underline{\underline{Y}}}[\mu_N] \arrow[r, two heads] \arrow[u, hook] & \Delta^{\mathrm{temp}}_{\underline{\underline{Y}}} \arrow[r, hook] \arrow[u, hook] & \Delta^{\mathrm{temp}}_{\underline{\underline{X}}} \arrow[r, hook] \arrow[u, hook] & \Delta^{\mathrm{temp}}_{\underline{X}} \arrow[r, hook] \arrow[u, hook] & \Delta^{\mathrm{temp}}_{C}. \arrow[u, hook]
\end{tikzcd}$$

For each N, by Theorem 7.23 (1), we can also functorially group-theoretically reconstruct an isomorph $(l\Delta_\Theta)(\mathbb{M}^\Theta_N)$ of the internal cyclotome and the cyclotomic rigidity isomorphism

$$(l\Delta_\Theta)(\mathbb{M}^\Theta_N) \otimes (\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} \mu_N(\mathbb{M}^\Theta_N).$$

The transition morphisms of the resulting projective system $\{\cdots \leftarrow \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^\Theta_M) \leftarrow \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^\Theta_{M'}) \leftarrow \cdots\}$ are all isomorphism. We identify these topological groups via these transition morphisms, and we write $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^\Theta_*)$ for the resulting topological group. Similarly, we define $G_{\underline{v}}(\mathbb{M}^\Theta_*)$, $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}(\mathbb{M}^\Theta_*)$, $\Pi^{\mathrm{temp}}_{\underline{X}}(\mathbb{M}^\Theta_*)$, $\Pi^{\mathrm{temp}}_{C}(\mathbb{M}^\Theta_*)$, $\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}}(\mathbb{M}^\Theta_*)$, $\Delta^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^\Theta_*)$, $\Delta^{\mathrm{temp}}_{\underline{X}}(\mathbb{M}^\Theta_*)$, $\Delta^{\mathrm{temp}}_{C}(\mathbb{M}^\Theta_*)$, $(l\Delta_\Theta)(\mathbb{M}^\Theta_*)$ from $G_{\underline{v}}(\mathbb{M}^\Theta_*)$, $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}(\mathbb{M}^\Theta_N)$, $\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}}(\mathbb{M}^\Theta_N)$, $\Delta^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^\Theta_N)$, $(l\Delta_\Theta)(\mathbb{M}^\Theta_N)$ respectively. We write $\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*) := \varprojlim_N \mu_N(\mathbb{M}^\Theta_N)$, then we obtain a cyclotomic rigidity isomorphism

$$(l\Delta_\Theta)(\mathbb{M}^\Theta_*) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*).$$

Proposition 11.4. (Multiradial Mono-theta Cyclotomic Rigidity, [IUTchII, Corollary 1.10]) *Let $\Pi_{\underline{v}}$ be the tempered fundamental group of the local model objects $\underline{\underline{X}}_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ in Definition 10.2 (1), and $(\Pi_{\underline{v}} \twoheadrightarrow)G_{\underline{v}}$ the quotient group-theoretically reconstructed by Lemma 6.2.*

(1) Let $\mathcal{C}^{\vdash}$ be the category whose objects are

$$G \curvearrowright O^{\times\mu}(G),$$

where G is a topological group isomorphic to $G_{\underline{v}}$, $O^{\times\mu}(G)$ is the group-theoretically reconstructed monoid by Proposition 5.2 (Step 1) and Definition 8.5 (1), and whose morphisms $(G \curvearrowright O^{\times\mu}(G)) \xrightarrow{\sim} (G' \curvearrowright O^{\times\mu}(G'))$ are pairs of the isomorphism $G \xrightarrow{\sim} G'$ of topological groups, and an $\mathrm{Isomet}(G)$*-multiple of the functorially group-theoretically reconstructed isomorphism $O^{\times\mu}(G) \xrightarrow{\sim} O^{\times\mu}(G')$ from the isomorphism $G \xrightarrow{\sim} G'$.*

(2) Let $\mathcal{R}^{\Theta}$ be the category whose objects are triples

$$\Big(\Pi \curvearrowright \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z}\ ,\ G \curvearrowright O^{\times\mu}(G)\ ,\ \alpha_{\mu,\times\mu} : (\Pi \curvearrowright \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z}) \overset{\text{poly}}{\longrightarrow} (G \curvearrowright O^{\times\mu}(G))|_{\Pi}\Big),$$

where Π is a topological group isomorphic to $\Pi_{\underline{v}}$, the topological group $(\Pi \twoheadrightarrow)G$ is the quotient group-theoretically reconstructed by Lemma 6.2, we write $(-)|_{\Pi}$ for the restriction via $\Pi \twoheadrightarrow G$, we write $\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}(\Pi))$ for the external cyclotome (cf. just after Theorem 7.23) of the projective system of mono-theta environment $\mathbb{M}^{\Theta}_{*}(\Pi)$ group-theoretically reconstructed from Π by Corollary 7.22 (2) (Note that such a projective system is uniquely determined, up to isomorphism, by the discrete rigidity (Theorem 7.23 (2))), and $\alpha_{\mu,\times\mu}$ is the composite

$$\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z} \hookrightarrow O^{\times}(\Pi) \twoheadrightarrow O^{\times\mu}(\Pi) \overset{\text{poly}}{\overset{\sim}{\to}} O^{\times\mu}(G)$$

of ind-topological modules equipped with topological group actions, where the first arrow is given by the composite of the tautological Kummer map for $\mathbb{M}^{\Theta}_{*}(\Pi)$ and the inverse of the isomorphism induced by the cyclotomic rigidity isomorphism of mono-theta environment (cf. the diagrams in Proposition 11.7 (1), (4)), the second arrow is the natural surjection and the last arrow is the poly-isomorphism induced by the full poly-isomorphism $\Pi/\Delta \overset{\text{full poly}}{\overset{\sim}{\to}} G$ (Note that the composite of the above diagram is equal to 0), and whose morphisms are pairs (f_{Π}, f_G) of the isomorphism $f_{\Pi} : (\Pi \curvearrowright \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z}) \overset{\sim}{\to} (\Pi' \curvearrowright \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}(\Pi')) \otimes \mathbb{Q}/\mathbb{Z})$ of ind-topological modules equipped with topological group actions induced by an isomorphism $\Pi \overset{\sim}{\to} \Pi'$ of topological groups with an $\mathrm{Isomet}(G)$*-multiple of the functorially group-theoretically reconstructed isomorphism $\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z} \overset{\sim}{\to} \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}(\Pi')) \otimes \mathbb{Q}/\mathbb{Z}$, and the isomorphism $f_G : (G \curvearrowright O^{\times\mu}(G)) \overset{\sim}{\to} (G' \curvearrowright O^{\times\mu}(G'))$ of ind-topological modules equipped with topological group actions induced by an isomorphism $G \overset{\sim}{\to} G'$ of topological groups with an* $\mathrm{Isomet}(G)$*-multiple of the functorially group-theoretically reconstructed isomorphism $O^{\times\mu}(G) \overset{\sim}{\to} O^{\times\mu}(G')$ (Note that these isomorphisms are automatically compatible $\alpha_{\mu,\times\mu}$ and $\alpha'_{\mu,\times\mu}$ in an obvious sense).*

(3) Let $\Phi^{\Theta} : \mathcal{R}^{\Theta} \to \mathcal{C}^{\vdash}$ be the essentially surjective functor, which sends $(\Pi \curvearrowright \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z}, G \curvearrowright O^{\times\mu}(G), \alpha_{\mu,\times\mu})$ to $G \curvearrowright O^{\times\mu}(G)$, and (f_{Π}, f_G) to f_G.

(4) Let $\mathcal{E}^{\Theta}$ be the category whose objects are the **cyclotomic rigidity isomorphisms of mono-theta environments**

$$(l\Delta_{\Theta})(\Pi) \overset{\sim}{\to} \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}(\Pi))$$

reconstructed group-theoretically by Theorem 7.23 (1), where Π is a topological group isomorphic to $\Pi_{\underline{v}}$, the cyclomotmes $(l\Delta_{\Theta})(\Pi)$ and $\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}(\Pi))$ are the internal and external cyclotomes respectively group-theoretically reconstructed from Π

by Corollary 7.22 (1), and whose morphisms are pair of isomorphisms $(l\Delta_\Theta)(\Pi) \xrightarrow{\sim} (l\Delta_\Theta)(\Pi')$ *and* $\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*(\Pi)) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*(\Pi'))$ *which are induced functorially group-theoretically reconstructed from an isomorphism of topological groups* $\Pi \xrightarrow{\sim} \Pi'$.

(5) *Let* $\Xi^\Theta : \mathcal{R}^\Theta \to \mathcal{E}^\Theta$ *be the functor, which sends* $(\Pi \curvearrowright \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*(\Pi)) \otimes \mathbb{Q}/\mathbb{Z}, G \curvearrowright O^{\times\mu}(G), \alpha_{\mu,\times\mu})$ *to the cyclotomic rigidity isomorphisms of mono-theta environments* $(l\Delta_\Theta)(\Pi) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*(\Pi))$ *reconstructed group-theoretically by Theorem 7.23 (1), and* (f_Π, f_G) *to the isomorphism functorially group-theoretically reconstructed from* $\Pi \xrightarrow{\sim} \Pi'$.

Then the radial environment $(\mathcal{R}^\Theta, \mathcal{C}^\vdash, \Phi^\Theta)$ *is multiradial, and* Ψ_{Ξ^Θ} *is multiradially defined, where* Ψ_{Ξ^Θ} *the naturally defined functor*

$$\begin{tikzcd} \mathcal{R}^\Theta \arrow[r, "\Psi_{\Xi^\Theta}"] \arrow[d, "\Phi^\Theta"'] & \mathrm{Graph}(\Xi^\Theta) \arrow[dl, "\Phi_{\mathrm{Graph}(\Xi^\Theta)}"] \\ \mathcal{C}^\vdash & \end{tikzcd}$$

by the construction of the graph of Ξ^Θ.

Proof. By noting that the composition in the definition of $\alpha_{\mu,\times\mu}$ is 0, and that we are considering the full poly-isomorphism $\Pi/\Delta \overset{\text{full poly}}{\xrightarrow{\sim}} G$, not the tautological single isomorphism $\Pi/\Delta \xrightarrow{\sim} G$, the proposition immediately from the definitions. □

Remark 11.4.1. Let see the diagram

$$\begin{tikzcd} {}^\dagger\Pi \curvearrowright \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*({}^\dagger\Pi)) \otimes \mathbb{Q}/\mathbb{Z} \arrow[d, mapsto] & \\ ({}^\dagger G \curvearrowright O^{\times\mu}({}^\dagger G)) \cong ({}^\ddagger G \curvearrowright O^{\times\mu}({}^\ddagger G)) & {}^\ddagger\Pi \curvearrowright \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*({}^\ddagger\Pi)) \otimes \mathbb{Q}/\mathbb{Z}, \arrow[l, mapsto] \end{tikzcd}$$

by dividing into two portions:

$$\begin{tikzcd} {}^\dagger\Pi \arrow[d, mapsto, "{}^\dagger\Pi/{}^\dagger\Delta \overset{\text{full poly}}{\xrightarrow{\sim}} G"'] & \\ G & {}^\ddagger\Pi. \arrow[l, mapsto, "{}^\ddagger\Pi/{}^\ddagger\Delta \overset{\text{full poly}}{\xrightarrow{\sim}} G"] \end{tikzcd} \qquad \begin{tikzcd} {}^\dagger\mu \arrow[d, "0"'] & \\ O^{\times\mu} & {}^\ddagger\mu, \arrow[l, "0"] \end{tikzcd}$$

On the left-hand side, by "loosening" (cf. taking $\mathrm{GL}_2(\mathbb{R})$-obit in Exapmle 11.2) the natural single isomorphisms ${}^\dagger\Pi/{}^\dagger\Delta \xrightarrow{\sim} G$, ${}^\ddagger\Pi/{}^\ddagger\Delta \xrightarrow{\sim} G$ by the full poly-isomorphisms (This means that the rigidification on the underlying mono-analytic structure G by the arithmetically holomorphic structure Π *is resolved*), we make the topological group portion of the functor Φ full (i.e., multiradial).

On the right-hand side, the fact that the map $\mu \to O^{\times\mu}$ is equal to zero makes the ind-topological module portion of the functor Φ full (i.e., multiradial). This means that it makes possible to "simultaneously perform" the algorithm of the cyclotomic rigidity isomorphism of mono-theta environment *without* making harmfull effects on other radial data, since the algorithm of the cyclotomic rigidity of mono-theta environment uses only μ-portion (unlike the one via LCFT uses the value group portion as well), and the μ-portion is separated from the relation with the coric data, by the fact that the homomorphism $\mu \to O^{\times\mu}$ is zero.

For the cyclotomic rigidity via LCFT, a similarly defined radial environment is *uniradial*, since the cyclotomic rigidity via LCFT uses the value group portion as well, and the value group portion is *not* separated from the coric data, and makes harmfull effects on other radial data. Even in this case, we replace $O^{\rhd}(-)$ by $O^{\times}(-)$, and we admit $\widehat{\mathbb{Z}}^{\times}$-indeterminacy on the cyclotomic rigidity, then it is tautologically multiradial as seen in the following proposition:

Proposition 11.5. (Multiradial LCFT Cyclotomic Rigidity with Indeterminacies, [IUTchII, Corollary 1.11]) *Let $\Pi_{\underline{v}}$ be the tempered fundamental group of the local model objects $\underline{\underline{X}}_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ in Definition 10.2 (1), and $(\Pi_{\underline{v}} \twoheadrightarrow) G_{\underline{v}}$ the quotient group-theoretically reconstructed by Lemma 6.2.*

(1) Let $\mathcal{C}^{\vdash}$ be the same category as in Proposition 11.4.

(2) Let $\mathcal{R}^{\mathrm{LCFT}}$ be the category whose objects are triples

$$\Big(\Pi \curvearrowright O^{\rhd}(\Pi)\ ,\ G \curvearrowright O^{\widehat{\mathrm{gp}}}(G)\ ,\ \alpha_{\rhd,\times\mu},\Big),$$

where Π is a topological group isomorphic to $\Pi_{\underline{v}}$, the topological group $(\Pi \twoheadrightarrow)G$ is the quotient group-theoretically reconstructed by Lemma 6.2, $O^{\rhd}(\Pi)$ is the ind-topological monoid determined by the ind-topological field group-theoretically reconstructed from Π by Corollary 3.19 and $\alpha_{\mu,\times\mu}$ is the following diagram:

$$(\Pi \curvearrowright O^{\rhd}(\Pi)) \hookrightarrow (\Pi \curvearrowright O^{\widehat{\mathrm{gp}}}(\Pi)) \overset{\widehat{\mathbb{Z}}^{\times}\text{-orbit}}{\underset{\mathrm{poly}}{\overset{\sim}{\to}}} (G \curvearrowright O^{\widehat{\mathrm{gp}}}(G))|_{\Pi} \hookleftarrow (G \curvearrowright O^{\times}(G))|_{\Pi} \twoheadrightarrow (G \curvearrowright O^{\times\mu}(G))|_{\Pi}$$

of ind-topological monoids equipped with topological group actions determined by the $\widehat{\mathbb{Z}}^{\times}$-orbit of the poly-morphism determined by the full poly-morphism $\Pi/\Delta \overset{\text{full poly}}{\overset{\sim}{\longrightarrow}} G$, where $\Delta := \ker(\Pi \twoheadrightarrow G)$ and the natural homomorphisms, where $O^{\widehat{\mathrm{gp}}}(\Pi) := \varinjlim_{J\subset\Pi\,:\,open}(O^{\rhd}(\Pi)^{\mathrm{gp}})^{J}$ (resp. $O^{\widehat{\mathrm{gp}}}(G) := \varinjlim_{J\subset G\,:\,open}(O^{\rhd}(G)^{\mathrm{gp}})^{J}$), and whose *morphisms are pairs* (f_Π, f_G) *of the isomorphism* $f_\Pi : (\Pi \curvearrowright O^\rhd(\Pi)) \xrightarrow{\sim} (\Pi' \curvearrowright O^\rhd(\Pi'))$ *of ind-topological monoids equipped with topological group actions induced by an isomorphism* $\Pi \xrightarrow{\sim} \Pi'$ *of topological groups with an* $\mathrm{Isomet}(G)$*-multiple of the functorially group-theoretically reconstructed isomorphism* $O^\rhd(\Pi) \xrightarrow{\sim} O^\rhd(\Pi')$*, and the isomorphism* $f_G : (G \curvearrowright O^{\widehat{\mathrm{gp}}}(G)) \xrightarrow{\sim} (G' \curvearrowright O^{\widehat{\mathrm{gp}}}(G'))$ *of ind-topological groups equipped with topological group actions induced by an isomorphism* $G \xrightarrow{\sim} G'$ *of topological groups with an* $\mathrm{Isomet}(G)$*-multiple of the functorially group-theoretically reconstructed isomorphism* $O^{\widehat{\mathrm{gp}}}(G) \xrightarrow{\sim} O^{\widehat{\mathrm{gp}}}(G')$ *(Note that these isomorphisms are automatically compatible* $\alpha_{\rhd,\times\mu}$ *and* $\alpha'_{\rhd,\times\mu}$ *in an obvious sense).*

(3) Let $\Phi^{\mathrm{LCFT}} : \mathcal{R}^{\mathrm{LCFT}} \to \mathcal{C}^{\vdash}$ *be the essentially surjective functor, which sends* $(\Pi \curvearrowright O^\rhd(\Pi), G \curvearrowright O^{\widehat{\mathrm{gp}}}(G), \alpha_{\rhd,\times\mu})$ *to* $G \curvearrowright O^{\times\mu}(G)$*, and* (f_Π, f_G) *to the functorially group-theoretically reconstructed isomorphism* $(G \curvearrowright O^{\times\mu}(G)) \xrightarrow{\sim} (G' \curvearrowright O^{\times\mu}(G'))$*.*

(4) Let $\mathcal{E}^{\mathrm{LCFT}}$ *be the category whose objects are the pairs of the* $\widehat{\mathbb{Z}}^\times$**-orbit** *(= the full poly-isomorphism, cf. Remark 3.19.2 in the case of* $O^\times$*)*

$$\mu_{\widehat{\mathbb{Z}}}(G) \overset{\text{poly}}{\xrightarrow{\sim}} \mu_{\widehat{\mathbb{Z}}}(O^\times(G))$$

of **cyclotomic rigidity isomorphisms via LCFT** *reconstructed group-theoretically by Remark 3.19.2 (for* $M = O^\times(G)$*), and the* $\mathbf{Aut}(\boldsymbol{G})$**-orbit** *(which comes from the full poly-isomorphism* $\Pi/\Delta \overset{\text{full poly}}{\xrightarrow{\sim}} G$*)*

$$\mu_{\widehat{\mathbb{Z}}}(G) \overset{\text{poly}}{\xrightarrow{\sim}} (l\Delta_\Theta)(\Pi)$$

of the isomorphism obtained as the composite of the cyclotomic rigidity isomorphism via positive rational structure and LCFT $\mu_{\widehat{\mathbb{Z}}}(G) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(\Pi)$ *group-theoretically reconstructed by Remark 6.12.2 and the cyclotomic rigidity isomorphism* $\mu_{\widehat{\mathbb{Z}}}(\Pi) \xrightarrow{\sim} (l\Delta_\Theta)(\Pi)$ *group-theoretically reconstructed by Remark 9.4.1, where* Π *is a topological group isomorphic to* $\Pi_{\underline{v}}$*, the topological group* $(\Pi \twoheadrightarrow)G$ *is the quotient group-theoretically reconstructed by Lemma 6.2, and* $(l\Delta_\Theta)(\Pi)$ *is the internal cyclotome group-theoretically reconstructed from* Π *by Corollary 7.22 (1), and whose morphisms are triple of isomorphisms* $\mu_{\widehat{\mathbb{Z}}}(G) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(G')$*,* $\mu_{\widehat{\mathbb{Z}}}(O^\times(G)) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(O^\times(G'))$ *and* $(l\Delta_\Theta)(\Pi) \xrightarrow{\sim} (l\Delta_\Theta)(\Pi')$ *which are induced functorially group-theoretically reconstructed from an isomorphism of topological groups* $\Pi \xrightarrow{\sim} \Pi'$*.*

(5) Let $\Xi^{\mathrm{LCFT}} : \mathcal{R}^{\mathrm{LCFT}} \to \mathcal{E}^{\mathrm{LCFT}}$ *be the functor, which sends* $(\Pi \curvearrowright O^\rhd(\Pi), G \curvearrowright O^{\widehat{\mathrm{gp}}}(G), \alpha_{\rhd,\times\mu})$ *to the pair of group-theoretically reconstructed isomorphisms, and* (f_Π, f_G) *to the isomorphism functorially group-theoretically reconstructed from* $\Pi \xrightarrow{\sim} \Pi'$*.*

Then the radial environment $(\mathcal{R}^{\mathrm{LCFT}}, \mathcal{C}^{\vdash}, \Phi^{\mathrm{LCFT}})$ *is multiradial, and* $\Psi_{\Xi^{\mathrm{LCFT}}}$ *is multiradially defined, where* $\Psi_{\Xi^{\mathrm{LCFT}}}$ *the naturally defined functor*

$$\begin{tikzcd} \mathcal{R}^{\mathrm{LCFT}} \arrow[r, "\Psi_{\Xi^{\mathrm{LCFT}}}"] \arrow[d, "\Phi^{\mathrm{LCFT}}"'] & \mathrm{Graph}(\Xi^{\mathrm{LCFT}}) \arrow[dl, "\Phi_{\mathrm{Graph}(\Xi^{\mathrm{LCFT}})}"] \\ \mathcal{C}^{\vdash} & \end{tikzcd}$$

by the construction of the graph of Ξ^{LCFT}.

Definition 11.6. ([IUTchII, Remark 1.4.1 (ii)]) Recall that we have hyperbolic orbicurves $\underline{\underline{X}}_{\underline{v}} \twoheadrightarrow \underline{X}_{\underline{v}} \twoheadrightarrow \underline{C}_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, and a rational point

$$\mu_- \in \underline{X}_{\underline{v}}(K_{\underline{v}})$$

(i.e., "-1" in $\mathbb{G}_m^{\mathrm{rig}}/q_{\underline{X}_{\underline{v}}}^{\mathbb{Z}}$. cf. Definition 10.17). The unique automorphism $\iota_{\underline{\underline{X}}}$ of $\underline{\underline{X}}_{\underline{v}}$ of order 2 lying over $\iota_{\underline{X}}$ (cf. Section 7.3 and Section 7.5) corresponds to the unique $\Delta^{\mathrm{temp}}_{\underline{\underline{X}}_{\underline{v}}}$-outer automorphism of $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}_{\underline{v}}}$ over $G_{\underline{v}}$ of order 2. We also write $\iota_{\underline{\underline{X}}}$ for the latter automorphism by abuse of notation. We also have tempered coverings $\underline{\underline{\ddot{Y}}}_{\underline{v}} \twoheadrightarrow \underline{\underline{Y}}_{\underline{v}} \twoheadrightarrow \underline{\underline{X}}_{\underline{v}}$. Note that we can group-theoretically reconstruct $\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}$, $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}_{\underline{v}}}$ from $\Pi_{\underline{\underline{X}}_{\underline{v}}}$ by Corollary 7.22 (1) and the description of $\underline{\ddot{Y}} \twoheadrightarrow \underline{Y}$. We write $\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}(\Pi)$, $\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}_{\underline{v}}}(\Pi)$ for the reconstructed ones from a topological group Π isomorphic to $\Pi_{\underline{\underline{\ddot{X}}}_{\underline{v}}}$, respectively. Since $K_{\underline{v}}$ contains μ_{4l}, there exist rational points

$$(\mu_-)_{\underline{\underline{\ddot{Y}}}} \in \underline{\underline{\ddot{Y}}}_{\underline{v}}(K_{\underline{v}}), \quad (\mu_-)_{\underline{\underline{X}}} \in \underline{\underline{X}}_{\underline{v}}(K_{\underline{v}}),$$

such that $(\mu_-)_{\underline{\underline{\ddot{Y}}}} \mapsto (\mu_-)_{\underline{\underline{X}}} \to \mu_-$. Note that $\iota_{\underline{\underline{X}}}$ fixes the $\mathrm{Gal}(\underline{\underline{X}}_{\underline{v}}/\underline{X}_{\underline{v}})$-orbit of $(\mu_-)_{\underline{\underline{X}}}$, since $\iota_{\underline{X}}$ fixes μ_-, hence $\iota_{\underline{\underline{X}}}$ fixes $(\mu_-)_{\underline{\underline{X}}}$, since $\mathrm{Aut}(\underline{\underline{X}}_{\underline{v}}) \cong \mu_l \times \{\pm 1\}$ by Remark 7.12.1 (Here, $\iota_{\underline{\underline{X}}}$ corresponds to the second factor of $\mu_l \times \{\pm 1\}$ since $l \neq 2$). Then it follows that there exists an automorphism

$$\iota_{\underline{\underline{\ddot{Y}}}}$$

of $\underline{\underline{\ddot{Y}}}$ of order 2 lifting $\iota_{\underline{\underline{X}}}$, which is uniquely determined *up to* $l\mathbb{Z}$*-conjugacy and composition with an element* $\in \mathrm{Gal}(\underline{\underline{\ddot{Y}}}_{\underline{v}}/\underline{\underline{Y}}_{\underline{v}}) \cong \mu_2$, by the condition that it fixes the $\mathrm{Gal}(\underline{\underline{\ddot{Y}}}_{\underline{v}}/\underline{\underline{Y}}_{\underline{v}})$-orbit of some element ("$(\mu_-)_{\underline{\underline{\ddot{Y}}}}$" by abuse of nonation) of the $\mathrm{Gal}(\underline{\underline{\ddot{Y}}}_{\underline{v}}/\underline{X}_{\underline{v}})(\cong l\mathbb{Z} \times \mu_2)$-orbit of $(\mu_-)_{\underline{\underline{\ddot{Y}}}}$. We also write $\iota_{\underline{\underline{\ddot{Y}}}}$ for the corresponding $\Delta^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}$-outer automorphism of $\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}$ by abuse of notation. We shall refer to $\iota_{\underline{\underline{\ddot{Y}}}}$ as an **inversion automorphism** as well. We write $\iota_{\ddot{Y}}$ for the automorphism of $\ddot{Y}_{\underline{v}}$ induced by $\iota_{\underline{\underline{\ddot{Y}}}}$.

We write

$$D_{\mu_-} \subset \Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}$$

for the decomposition group of $(\mu_-)_{\underline{\underline{\ddot{Y}}}}$, which is well-defined up to $\Delta^{\text{temp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}$-conjugacy. Hence D_{μ_-} is determined by $\iota_{\underline{\underline{\ddot{Y}}}}$ up to $\Delta^{\text{temp}}_{Y_{\underline{v}}}$-conjugacy. We shall refer to the pairs

$$\left(\iota_{\underline{\underline{\ddot{Y}}}} \in \mathrm{Aut}(\underline{\underline{\ddot{Y}}}_{\underline{v}})\ ,\ (\mu_-)_{\underline{\underline{\ddot{Y}}}}\right),\quad \text{or}\quad \left(\iota_{\underline{\underline{\ddot{Y}}}} \in \mathrm{Aut}(\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})/\mathrm{Inn}(\Delta^{\text{temp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})\ ,\ D_{\mu_-}\right)$$

as a **pointed inversion automorphism**. Recall that an étale theta function of standard type is defined by the condition on the restriction to D_{μ_-} is in μ_{2l} (Definition 7.7 and Definition 7.14).

Proposition 11.7. (Multiradial Constant Multiple Rigidity, [IUTchII, Corollary 1.12]) *Let $(\mathcal{R}^{\Theta}, \mathcal{C}^{\vdash}, \Phi^{\Theta})$ be the multiradial environment defined in Proposition 11.4.*

(1) There is a functorial group-theoretic algorithm to reconstruct, from a topological group Π isomorphic to $\Pi^{\text{temp}}_{\underline{\underline{X}}_{\underline{v}}}$ ($\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$), the following commutative diagram:

$$\begin{array}{ccc} O^{\times}(\Pi) \cup O^{\times}(\Pi) \cdot {}_{\infty}\underline{\underline{\theta}}(\Pi) & \hookrightarrow & {}_{\infty}H^1(\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi), (l\Delta_{\Theta})(\Pi)) \\ \cong\downarrow & & \cong\downarrow \textbf{Cycl. Rig. Mono-Th.}\ \text{in Prop.11.4} \\ O^{\times}(\mathbb{M}^{\Theta}_*(\Pi)) \cup O^{\times}(\mathbb{M}^{\Theta}_*(\Pi)) \cdot {}_{\infty}\underline{\underline{\theta}}_{\text{env}}(\mathbb{M}^{\Theta}_*(\Pi)) & \hookrightarrow & {}_{\infty}H^1(\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_*(\Pi)), \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_*(\Pi))), \end{array}$$

where we put, for a topological group Π isomorphic to $\Pi^{\text{temp}}_{\underline{\underline{X}}_{\underline{v}}}$ (resp. for a projective system $\mathbb{M}^{\Theta}_*$ of mono-theta environments determined by $\underline{\underline{X}}_{\underline{v}}$), $\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi)$ (resp. $\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_*)$) to be the isomorph of $\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}$ reconstructed from $\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi)$ by Definition 11.6 (resp. from $\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_*)$ by Definition 11.3 and the descrption of $\underline{\underline{\ddot{Y}}} \twoheadrightarrow \underline{\underline{Y}}$), and

$${}_{\infty}H^1(\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi), (l\Delta_{\Theta})(\Pi)) := \varinjlim_{J\subset\Pi\,:\,\text{open, of fin. index}} H^1(\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi) \times_{\Pi} J, (l\Delta_{\Theta})(\Pi)),$$

$${}_{\infty}H^1(\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_*), \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_*)) := \varinjlim_{J\subset\Pi\,:\,\text{open, of fin. index}} H^1(\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_*) \times_{\Pi} J, \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_*)),$$

and

$${}_{\infty}\underline{\underline{\theta}}(\Pi)\ (\subset {}_{\infty}H^1(\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi), (l\Delta_{\Theta})(\Pi))\ \ (\textit{resp.}\ {}_{\infty}\underline{\underline{\theta}}_{\text{env}}(\mathbb{M}^{\Theta}_*)\ (\subset {}_{\infty}H^1(\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_*), \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_*))\ )$$

for the subset of elements for which some positive integer multiple (if we consider multiplicatively, some positive integer power) is, up to torsion, equal to an element of the subset

$$\underline{\underline{\theta}}(\Pi)\ (\subset H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi),(l\Delta_\Theta)(\Pi))\quad (\textit{resp. } \underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^\Theta_*)\ (\subset H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^\Theta_*),\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*))\ )$$

of the μ_l-orbit of the reciprocal of $l\underline{\mathbb{Z}}\times\mu_2$-orbit $\underline{\underline{\ddot{\eta}}}^{\Theta,l\underline{\mathbb{Z}}\times\mu_2}$ of an l-th root of the étale theta function of standard type in Section 7.3 (resp. corresponding to the μ_l-orbit of the reciprocal of $(l\underline{\mathbb{Z}}\times\mu_2)$-orbit $\underline{\underline{\ddot{\eta}}}^{\Theta,l\underline{\mathbb{Z}}\times\mu_2}$ of an l-th root of the étale theta function of standard type in Section 7.3, via the cyclotomic rigidity isomorphism $(l\Delta_\Theta)(\mathbb{M}^\Theta_*)\xrightarrow{\sim}\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*)$ group-theoretically reconstructed by Theorem 7.23 (1), where we write $(l\Delta_\Theta)(\mathbb{M}^\Theta_*)$ for the internal cyclotome of the projective system $\mathbb{M}^\Theta_*$ of mono-theta environments group-theoretically reconstructd by Theorem 7.23 (1)) (Note that these can functorially group-theoretically reconstructed by the **constant multiple rigidity** *(Proposition 11.7)), and we define*

$$O^\times(\mathbb{M}^\Theta_*(\Pi))$$

to be the submodule such that the left vertical arrow is an isomorphism. We also put

$$O^\times{}_\infty\underline{\underline{\theta}}(\Pi):=O^\times(\Pi)\cdot{}_\infty\underline{\underline{\theta}}(\Pi),\quad O^\times{}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^\Theta_*(\Pi)):=O^\times(\mathbb{M}^\Theta_*(\Pi))\cdot{}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^\Theta_*(\Pi)).$$

(2) *There is a functorial group-theoretic algorithm*

$$\Pi\quad\mapsto\quad\{(\iota,D)\}(\Pi),$$

which construct, from a topological group Π isomorphic to $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}_{\underline{v}}}$, a collection of pairs (ι,D), where ι is a $\Delta^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi)(:=\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi)\cap\Delta)$-outer automorphism of $\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi)$, and $D\subset\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi)$ is a $\Delta^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi)$-conjgacy class of closed subgroups corresponding to the pointed inversion automorphisms in Definition 11.6. We shall refer to each (ι,D) as a **pointed inversion automorphism** *as well. For a pointed inversion automorphism (ι,D), and a subset S of an abelian group A, if ι acts on $\mathrm{Im}(S\to A/A_{\mathrm{tors}})$, then we write $S^\iota:=\{s\in S\mid\iota(s \bmod A_{\mathrm{tors}})=s\bmod A_{\mathrm{tors}}\}$.*

(3) *Let (ι,D) be a pointed inversion automorphism reconstructed in (1). Then the restriction to the subgroup $D\subset\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi)$ gives us the following commutative diagram:*

$$\begin{array}{ccccc}
\{O^\times{}_\infty\underline{\underline{\theta}}(\Pi)\}^\iota & \longrightarrow & O^\times(\Pi) & & \left(\subset{}_\infty H^1(\Pi,(l\Delta_\Theta)(\Pi))\right)\\
\downarrow & & \cong\downarrow\ \textbf{Cycl. Rig. Mono-Th.}\text{ in Prop.11.4} & & \\
\{O^\times{}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^\Theta_*(\Pi))\}^\iota & \longrightarrow & O^\times(\mathbb{M}^\Theta_*(\Pi)) & & \left(\subset{}_\infty H^1(\Pi,\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*(\Pi)))\right),
\end{array}$$

where we put

$$_{\infty}H^1(\Pi,(l\Delta_\Theta)(\Pi)) := \varinjlim_{J\subset\Pi:\,\text{open, of fin. index}} H^1(J,(l\Delta_\Theta)(\Pi)),$$

$$_{\infty}H^1(\Pi,\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*(\Pi))) := \varinjlim_{J\subset\Pi:\,\text{open, of fin. index}} H^1(J,\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*(\Pi))).$$

Note that the inverse image of the torsion elements via the upper (resp. lower) horizontal arrow in the above commutative diagram is equal to $_{\infty}\underline{\underline{\theta}}(\Pi)^\iota$ *(resp.* $_{\infty}\underline{\underline{\theta}}_{\text{env}}(\mathbb{M}^\Theta_*(\Pi))^\iota$*). In particular, we obtain a functorial algorithm of constructing* **splittings**

$$O^{\times\mu}(\Pi)\times\{_{\infty}\underline{\underline{\theta}}(\Pi)^\iota/O^\mu(\Pi)\},\quad O^{\times\mu}(\mathbb{M}^\Theta_*(\Pi))\times\{_{\infty}\underline{\underline{\theta}}_{\text{env}}(\mathbb{M}^\Theta_*(\Pi))^\iota/O^\mu(\mathbb{M}^\Theta_*(\Pi))\}$$

of $\{O^\times{}_{\infty}\underline{\underline{\theta}}(\Pi)\}^\iota/O^\mu(\Pi)$ *(resp.* $\{O^\times{}_{\infty}\underline{\underline{\theta}}_{\text{env}}(\mathbb{M}^\Theta_*(\Pi))\}^\iota/O^\mu(\mathbb{M}^\Theta_*(\Pi))$ *).*

(4) For an object $(\Pi \curvearrowright \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*(\Pi))\otimes\mathbb{Q}/\mathbb{Z}, G\curvearrowright O^{\times\mu}(G),\alpha_{\mu,\times\mu})$ *of the radial category* $\mathcal{R}^\Theta$*, we assign*

- *the projective system* $\mathbb{M}^\Theta_*(\Pi)$ *of mono-theta environments,*
- *the subsets* $O^\times(\Pi)\cup O^\times{}_{\infty}\underline{\underline{\theta}}(\Pi)$ $(\subset {}_{\infty}H^1(\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi),(l\Delta_\Theta)(\Pi)))$*, and* $O^\times(\mathbb{M}^\Theta_*(\Pi))\cup O^\times{}_{\infty}\underline{\underline{\theta}}_{\text{env}}(\mathbb{M}^\Theta_*(\Pi))$ $(\subset {}_{\infty}H^1(\Pi^{\text{temp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^\Theta_*(\Pi)),\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*(\Pi))))$ *in (1),*
- *the* **splittings** $O^{\times\mu}(\Pi)\times\{_{\infty}\underline{\underline{\theta}}(\Pi)^\iota/O^\mu(\Pi)\}$*, and* $O^{\times\mu}(\mathbb{M}^\Theta_*(\Pi))\times\{_{\infty}\underline{\underline{\theta}}_{\text{env}}(\mathbb{M}^\Theta_*(\Pi))^\iota/O^\mu(\mathbb{M}^\Theta_*(\Pi))\}$ *in (3), and*
- *the diagram*

$$\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^\Theta_*(\Pi))\otimes\mathbb{Q}/\mathbb{Z}\xrightarrow{\sim}O^\mu(\mathbb{M}^\Theta_*(\Pi))\xrightarrow{\sim}O^\mu(\Pi)\hookrightarrow O^\times(\Pi)\twoheadrightarrow O^{\times\mu}(\Pi)\overset{\text{poly}}{\xrightarrow{\sim}}O^{\times\mu}(G),$$

where the first arrow is induced by the tautological Kummer map for $\mathbb{M}^\Theta_*(\Pi)$*, the second arrow is induced by the vertical arrow in (1), the third and the fourth arrow are the natural injection and surjection respectively (Note that the composite is equal to* 0*), and the last arrow is the poly-isomorphism induced by the full poly-isomorphism* $\Pi/\Delta \overset{\text{full poly}}{\xrightarrow{\sim}} G$*.*

Then this assignment determines a functor $\Xi^{\text{env}}:\mathcal{R}^\Theta\to\mathcal{E}^{\text{env}}$*, and the natural functor* $\Psi_{\Xi^{\text{env}}}:\mathcal{R}^\Theta\to\mathrm{Graph}(\Xi^{\text{env}})$ *is multiradially defined.*

Proof. Proposition immediately follows from the described algorithms. □

Remark 11.7.1. cf. also the following **étale-pictures of the étale theta functions**:

$$\boxed{_{\infty}\underline{\underline{\theta}}(^\dagger\Pi)} \dashrightarrow \boxed{G\curvearrowright O^{\times\mu}(G)\curvearrowleft \mathrm{Isomet}(G)} \dashleftarrow \boxed{_{\infty}\underline{\underline{\theta}}(^\ddagger\Pi)}$$

$$\boxed{{}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\Pi))} - - > \boxed{G \curvearrowright O^{\times\mu}(G) \curvearrowleft \mathrm{Isomet}(G)} < - - \boxed{{}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}({}^{\ddagger}\Pi))}$$

Note that the object in the center is a mono-analytic object, and the objects in the left and in the right are holomorphic objects, and that *we have a permutation symmetry in the étale-picture*, by the multiradiality of the algorithm in Proposition 11.7 (cf. also Remark 11.1.1).

Remark 11.7.2. ([IUTchII, Proposition 2.2 (ii)]) The subset

$$\underline{\underline{\theta}}^{\iota}(\Pi) \subset \underline{\underline{\theta}}(\Pi) \quad (\text{resp. } {}_\infty\underline{\underline{\theta}}^{\iota}(\Pi) \subset {}_\infty\underline{\underline{\theta}}(\Pi)\ )$$

determines a specific $\mu_{2l}(O(\Pi))$-orbit (resp. $O^{\mu}(\Pi)$-orbit) within the unique $(l\underline{\mathbb{Z}} \times \mu_{2l})$-orbit (resp. each $(l\underline{\mathbb{Z}} \times \mu)$-orbit) in the set $\underline{\underline{\theta}}(\Pi)$ (resp. ${}_\infty\underline{\underline{\theta}}(\Pi)$).

§ 11.2. Hodge-Arakelov-theoretic Evaluation and Gaussian Monoids at Bad Places.

In this subsection, we perform Hodge-Arakelov-theoretic evaluation, and construct Gaussian monoids for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ (Note that the case for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ plays a central role). Recall that Corollary 7.22 (2) reconstructs a mono-theta environment from a topological group ("$\Pi \mapsto \mathbb{M}$") and Theorem 8.14 reconstructs a mono-theta environment from a tempered Frobenioid ("$\mathcal{F} \mapsto \mathbb{M}$"). First, we transport theta classes $\underline{\underline{\theta}}$ and the theta evaluations from a group-theoretic situation to a mono-theta-theoretic situation via ("$\Pi \mapsto \mathbb{M}$") and the cyclotomic rigidity for mono-theta environments, then, via ("$\mathcal{F} \mapsto \mathbb{M}$"), a Frobenioid-theoretic situation can access to the theta evaluation (cf. also [IUTchII, Fig. 3.1]):

$$\Pi \longmapsto \mathbb{M} \longleftarrow\!\dashv \mathcal{F}$$

$$\underline{\underline{\theta}}, \mathrm{eval} \longmapsto \underline{\underline{\theta}}_{\mathrm{env}}, \mathrm{eval}_{\mathrm{env}},$$

$$\begin{array}{ccc} \mathcal{F}\text{-Theoretic Theta Monoids} & \xrightarrow{\text{Kummer}} & \mathbb{M}\text{-Theoretic Theta Monoids} \\ & & \Big\downarrow{\scriptstyle \text{Galois Evaluation}} \\ \mathcal{F}\text{-Theoretic Gaussian Monoids} & \xleftarrow[\text{(Kummer)}^{-1},\text{ or forget}]{} & \mathbb{M}\text{-Theoretic Gaussian Monoids.} \end{array}$$

Note also that, from the view point of the scheme-theoretic Hodge-Arakelov theory and p-adic Hodge theory (cf. Appendix A), the evaluation maps correspond, in some sense, to the comparison map, which sends Galois representations to filtered φ-modules in the p-adic Hodge theory.

Definition 11.8. ([IUTchII, Remark 2.1.1, Proposition 2.2, Definition 2.3])

(1) For a hyperbolic orbicurve $(-)_{\underline{v}}$ over $K_{\underline{v}}$, we write $\Gamma_{(-)}$ for the dual graph of the special fiber of a stable model. Note that each of maps

$$\begin{tikzcd}
\Gamma_{\underline{\underline{\ddot{Y}}}} \arrow[r] \arrow[d] & \Gamma_{\underline{\underline{Y}}} \arrow[d] & \Gamma_{\underline{\underline{X}}} \arrow[d] \\
\Gamma_{\ddot{Y}} \arrow[r] & \Gamma_{Y}, & \Gamma_{\underline{X}}
\end{tikzcd}$$

induces a bijection on vertices, since the covering $\underline{\underline{X}}_{\underline{v}} \twoheadrightarrow \underline{X}_{\underline{v}}$ is totally ramified at the cusps. We write

$$\Gamma^{\blacktriangleright}_{\underline{X}} \subset \Gamma_{\underline{X}}$$

for the unique connected subgraph of $\Gamma_{\underline{X}}$, which is a tree and is stabilised by $\iota_{\underline{X}}$ (cf. Section 7.3, Section 7.5, and Definition 11.6), and contains all vertices of $\Gamma_{\underline{X}}$. We write

$$\Gamma^{\bullet}_{\underline{X}} \subset \Gamma^{\blacktriangleright}_{\underline{X}}$$

for the unique connected subgraph of $\Gamma_{\underline{X}}$, which is stabilised by $\iota_{\underline{X}}$ and contains precisely one vertex and no edges. Hence if we write labels on $\Gamma_{\underline{X}}$ by $\{-l^*, \dots, -1, 0, 1, \dots, l^*\}$, where 0 is fixed by $\iota_{\underline{X}}$, then $\Gamma^{\blacktriangleright}_{\underline{X}}$ is obtained by removing, from $\Gamma_{\underline{X}}$, the edge connecting the vertices labelled by $\pm l^*$, and $\Gamma^{\bullet}_{\underline{X}}$ consists only the vertex labelled by 0. From $\Gamma^{\bullet}_{\underline{X}} \subset \Gamma^{\blacktriangleright}_{\underline{X}} (\subset \Gamma_{\underline{X}})$, by taking suitable connected components of inverse images, we obtain finite connected subgraphs

$$\Gamma^{\bullet}_{\underline{\underline{X}}} \subset \Gamma^{\blacktriangleright}_{\underline{\underline{X}}} \subset \Gamma_{\underline{\underline{X}}}, \quad \Gamma^{\bullet}_{\ddot{Y}} \subset \Gamma^{\blacktriangleright}_{\ddot{Y}} \subset \Gamma_{\ddot{Y}}, \quad \Gamma^{\bullet}_{\underline{\underline{\ddot{Y}}}} \subset \Gamma^{\blacktriangleright}_{\underline{\underline{\ddot{Y}}}} \subset \Gamma_{\underline{\underline{\ddot{Y}}}},$$

which are stabilised by respective inversion automorphisms $\iota_{\underline{\underline{X}}}$, $\iota_{\ddot{Y}}$, $\iota_{\underline{\underline{\ddot{Y}}}}$ (cf. Section 7.3, Section 7.5, and Definition 11.6). Note that each $\Gamma^{\blacktriangleright}_{(-)}$ maps isomorphically to $\Gamma^{\blacktriangleright}_{\underline{X}}$.

(2) Write

$$\Pi_{\underline{v}^{\bullet}} := \Pi^{\mathrm{temp}}_{\underline{\underline{X}}_{\underline{v}}, \Gamma^{\bullet}_{\underline{\underline{X}}}} \quad \subset \quad \Pi_{\underline{v}^{\blacktriangleright}} := \Pi^{\mathrm{temp}}_{\underline{\underline{X}}_{\underline{v}}, \Gamma^{\blacktriangleright}_{\underline{\underline{X}}}} \quad \subset \quad \Pi_{\underline{v}} \ (= \Pi^{\mathrm{temp}}_{\underline{\underline{X}}_{\underline{v}}})$$

for $\boldsymbol{\Sigma := \{l\}}$ in the notation of Corollary 6.9 (i.e., $\mathbb{H} = \Gamma^{\blacktriangleright}_{\underline{\underline{X}}}$), Note that we have $\Pi_{\underline{v}^{\blacktriangleright}} \subset \Pi^{\mathrm{temp}}_{Y_{\underline{v}}} \cap \Pi_{\underline{v}} = \Pi^{\mathrm{temp}}_{\underline{\underline{Y}}_{\underline{v}}}$. Note also that $\Pi_{\underline{v}^{\blacktriangleright}}$ is well-defined up to $\Pi_{\underline{v}}$-conjugacy, and after fixing $\Pi_{\underline{v}^{\blacktriangleright}}$, the subgroup $\Pi_{\underline{v}^{\bullet}} \subset \Pi_{\underline{v}^{\blacktriangleright}}$ is well-defined up to $\Pi_{\underline{v}^{\blacktriangleright}}$-conjugacy. Moreover, note that we may assume that $\Pi_{\underline{v}^{\bullet}}$, $\Pi_{\underline{v}^{\blacktriangleright}}$ and $\iota_{\underline{\underline{\ddot{Y}}}}$ have been chosen so that some representative of $\iota_{\underline{\underline{\ddot{Y}}}}$ stabilises $\Pi_{\underline{v}^{\bullet}}$ and $\Pi_{\underline{v}^{\blacktriangleright}}$. Finally, note also that, from $\Pi_{\underline{v}}$, we can functorially group-theoretically reconstruct the data $(\Pi_{\underline{v}^{\bullet}} \subset \Pi_{\underline{v}^{\blacktriangleright}} \subset \Pi_{\underline{v}}, \iota_{\underline{\underline{\ddot{Y}}}})$ up to $\Pi_{\underline{v}}$-conjugacy, by Remark 6.12.1.

(3) We write

$$\Delta_{\underline{v}} := \Delta^{\text{temp}}_{\underline{\underline{X}}_{\underline{v}}},\ \Delta^{\pm}_{\underline{v}} := \Delta^{\text{temp}}_{\underline{X}_{\underline{v}}},\ \Delta^{\text{cor}}_{\underline{v}} := \Delta^{\text{temp}}_{C_{\underline{v}}},\ \Pi^{\pm}_{\underline{v}} := \Pi^{\text{temp}}_{\underline{X}_{\underline{v}}},\ \Pi^{\text{cor}}_{\underline{v}} := \Pi^{\text{temp}}_{C_{\underline{v}}}$$

(Note also that we can group-theoretically reconstruct these groups from $\Pi_{\underline{v}}$ by Lemma 7.12). We also use the notation $\widehat{(-)}$ for the profinite completion in this subsection. We also put

$$\Pi^{\pm}_{\underline{v}\bullet} := N_{\Pi^{\pm}_{\underline{v}}}(\Pi_{\underline{v}\bullet}) \quad\subset\quad \Pi^{\pm}_{\underline{v}\blacktriangleright} := N_{\Pi^{\pm}_{\underline{v}}}(\Pi_{\underline{v}\blacktriangleright}) \quad\subset\quad \Pi^{\pm}_{\underline{v}}.$$

Note that we have

$$\Pi^{\pm}_{\underline{v}\bullet}/\Pi_{\underline{v}\bullet} \xrightarrow{\sim} \Pi^{\pm}_{\underline{v}\blacktriangleright}/\Pi_{\underline{v}\blacktriangleright} \xrightarrow{\sim} \Pi^{\pm}_{\underline{v}}/\Pi_{\underline{v}} \xrightarrow{\sim} \Delta^{\pm}_{\underline{v}}/\Delta_{\underline{v}} \xrightarrow{\sim} \text{Gal}(\underline{\underline{X}}_{\underline{v}}/\underline{X}_{\underline{v}}) \cong \mathbb{Z}/l\mathbb{Z},$$

and

$$\Pi^{\pm}_{\underline{v}\bullet} \cap \Pi_{\underline{v}} = \Pi_{\underline{v}\bullet},\quad \Pi^{\pm}_{\underline{v}\blacktriangleright} \cap \Pi_{\underline{v}} = \Pi_{\underline{v}\blacktriangleright},$$

since $\Pi_{\underline{v}\bullet}$ and $\Pi_{\underline{v}\blacktriangleright}$ are normally terminal in $\Pi_{\underline{v}}$, by Corollary 6.9 (6).

(4) A **$\pm$-label class of cusps of $\boldsymbol{\Pi_{\underline{v}}}$** (resp. **of $\boldsymbol{\Pi^{\pm}_{\underline{v}}}$**, resp. **of $\boldsymbol{\widehat{\Pi}_{\underline{v}}}$**, resp. **of $\boldsymbol{\widehat{\Pi}^{\pm}_{\underline{v}}}$**) is the set of $\Pi_{\underline{v}}$-conjugacy (resp. $\Pi^{\pm}_{\underline{v}}$-conjugacy, resp. $\widehat{\Pi}_{\underline{v}}$-conjugacy, resp. $\widehat{\Pi}^{\pm}_{\underline{v}}$-conjugacy) classes of cuspidal inertia subgroups of $\Pi_{\underline{v}}$ (resp. of $\Pi^{\pm}_{\underline{v}}$, resp. of $\widehat{\Pi}_{\underline{v}}$, resp. of $\widehat{\Pi}^{\pm}_{\underline{v}}$) whose commensurators in $\Pi^{\pm}_{\underline{v}}$ (resp. in $\Pi^{\pm}_{\underline{v}}$, resp. in $\widehat{\Pi}^{\pm}_{\underline{v}}$, resp. in $\widehat{\Pi}^{\pm}_{\underline{v}}$) determine a single $\Pi^{\pm}_{\underline{v}}$-conjugacy (resp. $\Pi^{\pm}_{\underline{v}}$-conjugacy, resp. $\widehat{\Pi}^{\pm}_{\underline{v}}$-conjugacy, resp. $\widehat{\Pi}^{\pm}_{\underline{v}}$-conjugacy) class of subgroups in $\Pi^{\pm}_{\underline{v}}$ (resp. in $\Pi^{\pm}_{\underline{v}}$, resp. in $\widehat{\Pi}^{\pm}_{\underline{v}}$, resp. in $\widehat{\Pi}^{\pm}_{\underline{v}}$). (Note that this is group-theoretic condition. Note also that such a set of $\Pi_{\underline{v}}$-conjugacy (resp. $\Pi^{\pm}_{\underline{v}}$-conjugacy, resp. $\widehat{\Pi}_{\underline{v}}$-conjugacy, resp. $\widehat{\Pi}^{\pm}_{\underline{v}}$-conjugacy) class is of cardinality 1, since the covering $\underline{\underline{X}}_{\underline{v}} \twoheadrightarrow \underline{X}_{\underline{v}}$ is totally ramified at cusps (or the covering $\underline{\underline{X}}_{\underline{v}} \twoheadrightarrow \underline{\underline{X}}_{\underline{v}}$ is trivial).) We write

$$\text{LabCusp}^{\pm}(\Pi_{\underline{v}})\ (\text{resp. LabCusp}^{\pm}(\Pi^{\pm}_{\underline{v}}),\ \text{resp. LabCusp}^{\pm}(\widehat{\Pi}_{\underline{v}}),\ \text{resp. LabCusp}^{\pm}(\widehat{\Pi}^{\pm}_{\underline{v}})\ )$$

for the set of $\pm$-label classes of cusps of $\Pi_{\underline{v}}$ (resp. of $\Pi^{\pm}_{\underline{v}}$, resp. of $\widehat{\Pi}_{\underline{v}}$, resp. of $\widehat{\Pi}^{\pm}_{\underline{v}}$). Note that $\text{LabCusp}^{\pm}(\Pi_{\underline{v}})$ can be naturally identified with $\text{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}})$ in Definition 10.27 (2) for ${}^{\dagger}\mathcal{D}_{\underline{v}} := \mathcal{B}^{\text{temp}}(\Pi_{\underline{v}})^0$, and admits a group-theoretically reconstructable natural action of $\mathbb{F}_l^{\times}$, a group-theoretically reconstructable zero element ${}^{\dagger}\underline{\eta}^0_{\underline{v}} \in \text{LabCusp}^{\pm}(\Pi_{\underline{v}}) = \text{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}})$, and a group-theoretically reconstructable $\pm$-canonical element ${}^{\dagger}\underline{\eta}^{\pm}_{\underline{v}} \in \text{LabCusp}^{\pm}(\Pi_{\underline{v}}) = \text{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}})$ well defined up to multiplication by ± 1.

(5) An element $t \in \text{LabCusp}^{\pm}(\Pi_{\underline{v}})$ determines a unique vertex of $\Gamma^{\blacktriangleright}_{\underline{\underline{X}}}$ (cf. Corollary 6.9 (4)). We write $\Gamma^{\bullet t}_{\underline{\underline{X}}} \subset \Gamma^{\blacktriangleright}_{\underline{\underline{X}}}$ for the connected subgraph with no edges whose unique

vertex is the vertex determined by t. Then by a functorial group-theoretic algorithm, $\Gamma^{\bullet t}_{\underline{\underline{X}}}$ gives us a decomposition group

$$\Pi_{\underline{v}\bullet t} \subset \Pi_{\underline{v}\blacktriangleright} \subset \Pi_{\underline{v}}$$

well-defined up to $\Pi_{\underline{v}\blacktriangleright}$-conjugacy. We also write

$$\Pi^{\pm}_{\underline{v}\bullet t} := N_{\Pi^{\pm}_{\underline{v}}}(\Pi_{\underline{v}\bullet t}).$$

(Note that we have a natural isomorphism $\Pi^{\pm}_{\underline{v}\bullet t}/\Pi_{\underline{v}\bullet t} \xrightarrow{\sim} \mathrm{Gal}(\underline{\underline{X}}_{\underline{v}}/\underline{X}_{\underline{v}})$ by Corollary 6.9 (6)).

(6) The images in $\mathrm{LabCusp}^{\pm}(\Pi^{\pm}_{\underline{v}})$ (resp. $\mathrm{LabCusp}^{\pm}(\widehat{\Pi}^{\pm}_{\underline{v}})$) of the $\mathbb{F}_l^{\times}$-action, the zero element ${}^{\dagger}\eta^{0}_{\underline{v}}$, and $\pm$-canonical element ${}^{\dagger}\underline{\eta}^{\pm}_{\underline{v}}$ of $\mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$ in the above (4), via the natural outer injection $\Pi_{\underline{v}} \hookrightarrow \Pi^{\pm}_{\underline{v}}$ (resp. $\Pi_{\underline{v}} \hookrightarrow \widehat{\Pi}^{\pm}_{\underline{v}}$), determine *a natural* $\mathbb{F}_l^{\pm}$-*torsor structure* (cf. Definition 10.26 (2)) on $\mathrm{LabCusp}^{\pm}(\Pi^{\pm}_{\underline{v}})$ (resp. $\mathrm{LabCusp}^{\pm}(\widehat{\Pi}^{\pm}_{\underline{v}})$). Moreover, the natural action of $\Pi^{\mathrm{cor}}_{\underline{v}}/\Pi^{\pm}_{\underline{v}}$ (resp. $\widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}/\widehat{\Pi}^{\pm}_{\underline{v}}$) on $\Pi^{\pm}_{\underline{v}}$ (resp. $\widehat{\Pi}^{\pm}_{\underline{v}}$) preserves this $\mathbb{F}_l^{\pm}$-torosr structure, thus, determines a natural outer isomorphism $\Pi^{\mathrm{cor}}_{\underline{v}}/\Pi^{\pm}_{\underline{v}} \cong \mathbb{F}_l^{\times\pm}$ (resp. $\widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}/\widehat{\Pi}^{\pm}_{\underline{v}} \cong \mathbb{F}_l^{\times\pm}$).

Here, note that, even though $\Pi_{\underline{v}}$ (resp. $\widehat{\Pi}_{\underline{v}}$) is *not* normal in $\Pi^{\mathrm{cor}}_{\underline{v}}$ (resp. $\widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$), the cuspidal inertia subgroups of $\Pi_{\underline{v}}$ (resp. $\widehat{\Pi}_{\underline{v}}$) are permuted by the conjugate action of $\Pi^{\mathrm{cor}}_{\underline{v}}$ (resp. $\widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$), since, for a cuspidal inertia subgroup I in $\Pi^{\pm}_{\underline{v}}$ (resp. $\widehat{\Pi}^{\pm}_{\underline{v}}$), we have $I \cap \Pi_{\underline{v}} = I^l$ (resp. $I \cap \widehat{\Pi}_{\underline{v}} = I^l$) (Here, we write multiplicatively in the notation I^l), and $\Pi^{\pm}_{\underline{v}}$ (resp. $\widehat{\Pi}^{\pm}_{\underline{v}}$) is normal in $\Pi^{\mathrm{cor}}_{\underline{v}}$ (resp. $\widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$) ([IUTchII, Remark 2.3.1]).

Lemma 11.9. ([IUTchII, Corollary 2.4]) *Let* $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$. *Write*

$$\Delta_{\underline{v}\bullet t} := \Delta_{\underline{v}} \cap \Pi_{\underline{v}\bullet t},\ \Delta^{\pm}_{\underline{v}\bullet t} := \Delta^{\pm}_{\underline{v}} \cap \Pi^{\pm}_{\underline{v}\bullet t},\ \Pi_{\underline{v}\ddot{\bullet} t} := \Pi_{\underline{v}\bullet t} \cap \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}},\ \Delta_{\underline{v}\ddot{\bullet} t} := \Delta_{\underline{v}} \cap \Pi_{\underline{v}\ddot{\bullet} t},$$

$$\Delta_{\underline{v}\blacktriangleright} := \Delta_{\underline{v}} \cap \Pi_{\underline{v}\blacktriangleright},\ \Delta^{\pm}_{\underline{v}\blacktriangleright} := \Delta^{\pm}_{\underline{v}} \cap \Pi^{\pm}_{\underline{v}\blacktriangleright},\ \Pi_{\underline{v}\ddot{\blacktriangleright}} := \Pi_{\underline{v}\blacktriangleright} \cap \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}},\ \Delta_{\underline{v}\ddot{\blacktriangleright}} := \Delta_{\underline{v}} \cap \Pi_{\underline{v}\ddot{\blacktriangleright}}.$$

Note that we have

$$[\Pi_{\underline{v}\bullet t} : \Pi_{\underline{v}\ddot{\bullet} t}] = [\Pi_{\underline{v}\blacktriangleright} : \Pi_{\underline{v}\ddot{\blacktriangleright}}] = [\Delta_{\underline{v}\bullet t} : \Delta_{\underline{v}\ddot{\bullet} t}] = [\Delta_{\underline{v}\blacktriangleright} : \Delta_{\underline{v}\ddot{\blacktriangleright}}] = 2,$$

$$[\Pi^{\pm}_{\underline{v}\bullet t} : \Pi_{\underline{v}\bullet t}] = [\Pi^{\pm}_{\underline{v}\blacktriangleright} : \Pi_{\underline{v}\blacktriangleright}] = [\Delta^{\pm}_{\underline{v}\bullet t} : \Delta_{\underline{v}\bullet t}] = [\Delta^{\pm}_{\underline{v}\blacktriangleright} : \Delta_{\underline{v}\blacktriangleright}] = l.$$

(1) Let $I_t \subset \Pi_{\underline{v}}$ be a cuspidal inertia subgroup which belongs to the $\pm$-label class t such that $I_t \subset \Delta_{\underline{v}\bullet t}$ (resp. $I_t \subset \Delta_{\underline{v}\blacktriangleright}$). For $\gamma \in \widehat{\Delta}^{\pm}_{\underline{v}}$, we write $(-)^{\gamma}$ for the conjugation $\gamma(-)\gamma^{-1}$ by γ. Then for $\gamma' \in \widehat{\Delta}^{\pm}_{\underline{v}}$, the following are equivalent:

(a) $\gamma' \in \Delta^{\pm}_{\underline{v}\bullet t}$ (resp. $\gamma' \in \Delta^{\pm}_{\underline{v}\blacktriangleright}$),

(b) $I_t^{\gamma\gamma'} \subset \Pi^{\gamma}_{\underline{v}\bullet t}$ (resp. $I_t^{\gamma\gamma'} \subset \Pi^{\gamma}_{\underline{v}\blacktriangleright}$),

(c) $I_t^{\gamma\gamma'} \subset (\Pi^{\pm}_{\underline{v}\bullet t})^{\gamma}$ (resp. $I_t^{\gamma\gamma'} \subset (\Pi^{\pm}_{\underline{v}\blacktriangleright})^{\gamma}$).

(2) In the situation of (1), write $\delta := \gamma\gamma' \in \widehat{\Delta}^{\pm}_{\underline{v}}$*, then any inclusion*

$$I_t^{\delta} = I_t^{\gamma\gamma'} \subset \Pi^{\gamma}_{\underline{v}\bullet t} = \Pi^{\delta}_{\underline{v}\bullet t} \text{ (resp. } I_t^{\delta} = I_t^{\gamma\gamma'} \subset \Pi^{\gamma}_{\underline{v}\blacktriangleright} = \Pi^{\delta}_{\underline{v}\blacktriangleright} \text{)}$$

as in (1) completely determines the following data:

(a) *a decomposition group* $D_t^{\delta} := N_{\Pi^{\delta}_{\underline{v}}}(I_t^{\delta}) \subset \Pi^{\delta}_{\underline{v}\ddot{\bullet} t}$ (resp. $D_t^{\delta} := N_{\Pi^{\delta}_{\underline{v}}}(I_t^{\delta}) \subset \Pi^{\delta}_{\underline{v}\ddot{\blacktriangleright}}$),

(b) *a decomposition group* $D^{\delta}_{\mu_-} \subset \Pi^{\delta}_{\underline{v}\ddot{\blacktriangleright}}$*, well-defined up to* $(\Pi^{\pm}_{\underline{v}\blacktriangleright})^{\delta}$*-conjugacy (or, equivalently* $(\Delta^{\pm}_{\underline{v}\blacktriangleright})^{\delta}$*-conjugacy), corresponding to the torsion point* μ_- *in Definition 11.6.*

(c) *a decomposition group* $D^{\delta}_{t,\mu_-} \subset \Pi^{\delta}_{\underline{v}\ddot{\bullet} t}$ (resp. $D^{\delta}_{t,\mu_-} \subset \Pi^{\delta}_{\underline{v}\ddot{\blacktriangleright}}$)*, well-defined up to* $(\Pi^{\pm}_{\underline{v}\bullet t})^{\delta}$*-conjugacy (resp.* $(\Pi^{\pm}_{\underline{v}\blacktriangleright})^{\delta}$*-conjugacy) (or equivalently,* $(\Delta^{\pm}_{\underline{v}\bullet t})^{\delta}$*-conjugacy (resp.* $(\Delta^{\pm}_{\underline{v}\blacktriangleright})^{\delta}$*-conjugacy)), that is, the image of an evaluation section corresponding to* μ_-*-translate of the cusp which gives rise to* I_t^{δ}*.*

Moreover, the construction of the above data is compatible with conjugation by arbitrary $\delta \in \widehat{\Delta}^{\pm}_{\underline{v}}$ *as well as with the natural inclusion* $\Pi_{\underline{v}\bullet t} \subset \Pi_{\underline{v}\blacktriangleright}$*, as we vary the non-resp'd case and resp'd case.*

(3) ($\mathbb{F}_l^{\rtimes\pm}$*-symmetry) The construction of the data (2a), (2c) is compatible with conjugation by arbitrary* $\delta \in \widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$*, hence we have a* $\widehat{\Delta}^{\mathrm{cor}}_{\underline{v}}/\widehat{\Delta}^{\pm}_{\underline{v}} \xrightarrow{\sim} \widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}/\widehat{\Pi}^{\pm}_{\underline{v}} \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm}$*-symmetry on the construction.*

Proof. We show (1). The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are immediately follow from the definitions. We show the implication $(c) \Rightarrow (a)$. We may assume $\gamma = 1$ without loss of generality. Then the condition $I_t^{\gamma'} \subset \Pi^{\pm}_{\underline{v}\bullet t} \subset \Pi^{\pm}_{\underline{v}}$ (resp. $I_t^{\gamma'} \subset \Pi^{\pm}_{\underline{v}\blacktriangleright} \subset \Pi^{\pm}_{\underline{v}}$) implies $\gamma' \in \Delta^{\pm}_{\underline{v}}$ by Theorem 6.11 ("profinite conjugates vs. tempered conjugates"). By Corollary 6.9 (4), we obtain $\gamma' \in \widehat{\Delta}^{\pm}_{\underline{v}\bullet t}$ (resp. $\gamma' \in \widehat{\Delta}^{\pm}_{\underline{v}\blacktriangleright}$), where we write $\widehat{(-)}$ for the closure in $\widehat{\Delta}^{\pm}_{\underline{v}}$ (which is equal to the profinite completion, by Corollary 6.9 (2)). Then we obtain $\gamma' \in \widehat{\Delta}^{\pm}_{\underline{v}\bullet t} \cap \Delta^{\pm}_{\underline{v}} = \Delta^{\pm}_{\underline{v}\bullet t}$ (resp. $\gamma' \in \widehat{\Delta}^{\pm}_{\underline{v}\blacktriangleright} \cap \Delta^{\pm}_{\underline{v}} = \Delta^{\pm}_{\underline{v}\blacktriangleright}$) by Corollary 6.9 (3).

(2) follows from Theorem 3.7 (elliptic cuspidalisation) and Remark 6.12.1 (together with Lemma 7.16, Lemma 7.12) (cf. also Definition 10.17). (3) follows immediately from the described algorithms. □

We write

$$(l\Delta_{\Theta})(\Pi_{\underline{v}\ddot{\blacktriangleright}})$$

for the subquotient of $\Pi_{\underline{v}\ddot{\blacktriangleright}}$ determined by the subquotient $(l\Delta_\Theta)(\Pi_{\underline{v}})$ of $\Pi_{\underline{v}}$ (Note that the inclusion $\Pi_{\underline{v}\ddot{\blacktriangleright}} \hookrightarrow \Pi_{\underline{v}}$ induces an isomorphism $(l\Delta_\Theta)(\Pi_{\underline{v}\ddot{\blacktriangleright}}) \xrightarrow{\sim} (l\Delta_\Theta)(\Pi_{\underline{v}})$). We write

$$\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}}), \quad \Pi_{\underline{v}\ddot{\blacktriangleright}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$$

for the quotients determined by the natural surjection $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$ (Note that we can functorially group-theoretically reconstruct these quotients by Lemma 6.2 and Definition 11.8 (2)).

Proposition 11.10. (Π-theoretic Theta Evaluation, [IUTchII, Corollary 2.5, Corollary 2.6])

(1) Let $I_t^\delta = I_t^{\gamma\gamma'} \subset \Pi^\delta_{\underline{v}\ddot{\blacktriangleright}} \subset \Pi^\gamma_{\underline{v}\blacktriangleright} = \Pi^\delta_{\underline{v}\blacktriangleright}$ be as in Lemma 11.9 (2). Then the restriction of the ι^γ-invariant sets $\underline{\underline{\theta}}^\iota(\Pi^\gamma_{\underline{v}})$, ${}_\infty\underline{\underline{\theta}}^\iota(\Pi^\gamma_{\underline{v}})$ of Remark 11.7.2 to the subgroup $\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}} \subset \Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\Pi_{\underline{v}})(\subset \Pi_{\underline{v}})$ gives us μ_{2l}-, μ-orbits of elements

$$\underline{\underline{\theta}}^\iota(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}) \subset {}_\infty\underline{\underline{\theta}}^\iota(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}) \subset {}_\infty H^1(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}, (l\Delta_\Theta)(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}})) := \varinjlim_{\widehat{J}\subset\widehat{\Pi}_{\underline{v}}\,:\,\mathrm{open}} H^1(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}} \times_{\widehat{\Pi}_{\underline{v}}} \widehat{J}, (l\Delta_\Theta)(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}})).$$

The further restriction of the decomposition groups D^δ_{t,μ_-} in Lemma 11.9 (2) gives us μ_{2l}-, μ-orbits of elements

$$\underline{\underline{\theta}}^t(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}) \subset {}_\infty\underline{\underline{\theta}}^t(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}) \subset {}_\infty H^1(G_{\underline{v}}(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}), (l\Delta_\Theta)(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}})) := \varinjlim_{J_G\subset G_{\underline{v}}(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}})\,:\,\mathrm{open}} H^1(J_G, (l\Delta_\Theta)(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}})),$$

for each $t \in \mathrm{LabCusp}^\pm(\Pi^\gamma_{\underline{v}}) \overset{\text{conj. by }\gamma}{\xrightarrow{\sim}} \mathrm{LabCusp}^\pm(\Pi_{\underline{v}})$. Since the sets $\underline{\underline{\theta}}^t(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}})$, ${}_\infty\underline{\underline{\theta}}^t(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}})$ depend only on the label $|t| \in |\mathbb{F}_l|$, we write

$$\underline{\underline{\theta}}^{|t|}(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}) := \underline{\underline{\theta}}^t(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}), \qquad {}_\infty\underline{\underline{\theta}}^{|t|}(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}) := {}_\infty\underline{\underline{\theta}}^t(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}).$$

(2) If we start with an arbitrary $\widehat{\Delta}^\pm_{\underline{v}}$-conjugate $\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}$ of $\Pi_{\underline{v}\ddot{\blacktriangleright}}$, and we consider the resulting μ_{2l}-, μ-orbits $\underline{\underline{\theta}}^{|t|}(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}})$, ${}_\infty\underline{\underline{\theta}}^{|t|}(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}})$ arising from an arbitrary $\widehat{\Delta}^\pm_{\underline{v}}$-conjugate I^δ_t of I_t contained in $\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}$, as t runs over $\mathrm{LabCusp}^\pm(\Pi^\gamma_{\underline{v}}) \overset{\text{conj. by }\gamma}{\xrightarrow{\sim}} \mathrm{LabCusp}^\pm(\Pi_{\underline{v}})$, then we obtain a group-theoretic algorithm to construct the collections of μ_{2l}-, μ-orbits

$$\left\{\underline{\underline{\theta}}^{|t|}(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}})\right\}_{|t|\in|\mathbb{F}_l|}, \quad \left\{{}_\infty\underline{\underline{\theta}}^{|t|}(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}})\right\}_{|t|\in|\mathbb{F}_l|},$$

which is functorial with respect to the isomorphisms of topological groups $\Pi_{\underline{v}}$, and compatible with the independent conjugacy actions of $\widehat{\Delta}^\pm_{\underline{v}}$ on the sets $\{I_t^{\gamma_1}\}_{\gamma_1\in\widehat{\Pi}^\pm_{\underline{v}}} = \{I_t^{\gamma_1}\}_{\gamma_1\in\widehat{\Delta}^\pm_{\underline{v}}}$ and $\{\Pi^{\gamma_2}_{\underline{v}\ddot{\blacktriangleright}}\}_{\gamma_2\in\widehat{\Pi}^\pm_{\underline{v}}} = \{\Pi^{\gamma_2}_{\underline{v}\ddot{\blacktriangleright}}\}_{\gamma_2\in\widehat{\Delta}^\pm_{\underline{v}}}$

(3) *The γ-conjugate of the quotient $\Pi_{\underline{v}\ddot{\blacktriangleright}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$ determines subsets*

$$({}_{\infty}H^1(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}), (l\Delta_\Theta)(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})) \supset) \;\; O^{\times}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \quad \subset {}_{\infty}H^1(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}, (l\Delta_\Theta)(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})),$$

$$O^{\times}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) := O^{\times}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \;\subset\; O^{\times}{}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) := O^{\times}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \;\subset\; {}_{\infty}H^1(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}, (l\Delta_\Theta)(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})),$$

which are compatible with $O^{\times}(-)$, $O^{\times}{}_{\infty}\underline{\underline{\theta}}^{\iota}(-)$ in Proposition 11.7, respectively, relative to the first restriction operation in (1). We put

$$O^{\times\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) := O^{\times}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/O^{\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}).$$

(4) *In the situation of (1), we take t to be the zero element. Then the set $\underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$ (resp. ${}_{\infty}\underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$) is equal to μ_{2l} (resp. μ). In particular, by taking quotietn by $O^{\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$, the restriction to the decomposition group D^{δ}_{t,μ_-} (where t is the zero element) gives us* **splittings**

$$O^{\times\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \times \{{}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/O^{\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})\}$$

of $O^{\times}{}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/O^{\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$, which are compatible with the splittings of Proposition 11.7 (3), relative to the first restriction operation in (1):

$$\begin{tikzcd} 0 \arrow[r] & O^{\times\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \arrow[r] & O^{\times}{}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/O^{\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \arrow[r] \arrow[l, dotted, bend right, "\text{label } 0"'] & {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/O^{\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \arrow[r] & 0. \end{tikzcd}$$

Remark 11.10.1. (principle of Galois evaluation, [IUTchII, Remark 1.12.4]) Let us consider some "mysterious evaluation algorithm" which constructs theta values from an abstract theta function, in general. It is natural to require that this algorithm is compatible with taking Kummer classes of the "abstract theta function" and the "theta values", and that this algorithm extend to coverings on both input and output data. Then by the natural requirement of functoriality with respect to the Galois groups on either side, we can conclude that the "mysterious evaluation algorithm" in fact arises from a section $G \to \Pi_{\underline{\underline{\ddot{Y}}}}(\Pi)$ of the natural surjection $\Pi_{\underline{\underline{\ddot{Y}}}}(\Pi) \twoheadrightarrow G$, as in Proposition 11.10. We shall refer to this as the **principle of Galois evaluation**. Moreover, from the point of view of Section Conjecture, we expect that this sections arise from geometric points (as in Proposition 11.10).

Remark 11.10.2. ([IUTchII, Remark 2.6.1, Remark 2.6.2]) It is important that we perform the evaluation algorithm in Proposition 11.10 (1) by using *single* base point, i.e., *connected* subgraph $\Gamma^{\blacktriangleright}_{\underline{\underline{X}}} \subset \Gamma_{\underline{\underline{X}}}$, and that the theta values

$$\underline{\underline{\theta}}^{|t|}(\Pi_{\underline{v}\ddot{\blacktriangleright}}\gamma) \quad\subset\quad H^1(G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}}\gamma), (l\Delta_\Theta)(\Pi_{\underline{v}\ddot{\blacktriangleright}}\gamma))$$

live in the cohomology of *single* Galois group $G_{\underline{v}}(\Pi_{\underline{v}\blacktriangleright}\ddot{}\,\gamma)$ with *single* cyclotome $(l\Delta_\Theta)(\Pi^{\gamma}_{\underline{v}\blacktriangleright})$ coefficient for various $|t| \in |\mathbb{F}_l|$, since we want to consider the collection of the theta values for $|t| \in |\mathbb{F}_l|$, not as separated objects, but as "connected single object", by *synchronising indeterminacies via* $\mathbb{F}_l^{\rtimes\pm}$*-symmetry*, when we construct Gaussian monoids via Kummer theory (cf. Corollary 11.17).

Remark 11.10.3. ([IUTchII, Remark 2.5.2]) Write

$$\Pi^{\circledcirc\pm} := \Pi_{\underline{X}_K}, \quad \Delta^{\circledcirc\pm} := \Delta_{\underline{X}_K}.$$

Recall that, using the global data $\Delta^{\circledcirc\pm}(\cong \widehat{\Delta}^{\pm}_{\underline{v}})$, we write $\pm$-labels on local objects in a consistent manner (Proposition 10.33), where the labels are defined in the form of conjugacy classes of I_t. Note that $\Delta^{\circledcirc\pm}(\cong \widehat{\Delta}^{\pm}_{\underline{v}})$ is a kind of "ambient container" of $\widehat{\Delta}^{\pm}_{\underline{v}}$-conjugates of both I_t and $\Delta_{\underline{v}\blacktriangleright}$. On the other hand, when we want to vary $\underline{v}$, the topological group $\Pi_{\underline{v}\blacktriangleright}$ is purely local (unlike the label t, or conjugacy classes of I_t), and cannot be globalised, hence we have *the independence of the* $\Delta^{\circledcirc\pm}(\cong \widehat{\Delta}^{\pm}_{\underline{v}})$*-conjugacy indeterminacies which act on the conjugates of* I_t *and* $\Delta_{\underline{v}\blacktriangleright}$. Moreover, since the natural surjection $\widehat{\Delta}^{\mathrm{cor}}_{\underline{v}} \twoheadrightarrow \widehat{\Delta}^{\mathrm{cor}}_{\underline{v}}/\widehat{\Delta}^{\pm}_{\underline{v}} \cong \mathbb{F}_l^{\rtimes\pm}$ deos not have a splitting, the $\widehat{\Delta}^{\mathrm{cor}}_{\underline{v}}$-outer action of $\widehat{\Delta}^{\mathrm{cor}}_{\underline{v}}/\widehat{\Delta}^{\pm}_{\underline{v}} \cong \mathbb{F}_l^{\rtimes\pm}$ in Lemma 11.9 (3) induces *independent* $\Delta^{\circledcirc\pm} \cong \widehat{\Delta}^{\pm}_{\underline{v}}$*-conjugacy indeterminacies on the subgroups* I_t *for distinct* t.

Remark 11.10.4. ([IUTchII, Remark 2.6.3]) We explain the choice of $\Gamma^{\blacktriangleright}_{\ddot{Y}} \subset \Gamma_{\ddot{Y}}$. Let $\Gamma' \subset \Gamma_{\ddot{Y}}$ be a finite subgraph. Then

(1) For the purpose of getting single base point as explained in Remark 11.10.2, the subgraph Γ' should be connected.

(2) For the purpose of getting the crucial splitting in Proposition 11.10 (4), the subgraph Γ' should contain the vertex of label 0.

(3) For the purpose of making the final height inequality sharpest (cf. the calculations in the proof of Lemma 1.10), we want to maximise the value

$$\frac{1}{\#\Gamma'} \sum_{j \in \mathbb{F}_l^{*}} \min_{\underline{\underline{j}} \in \Gamma',\ \underline{\underline{j}} \equiv j \text{ in } |\mathbb{F}_l|} \left\{ \underline{\underline{j}}^2 \right\},$$

where we identified $\Gamma_{\ddot{Y}}$ with $\mathbb{Z}$. Then we obtain $\#\Gamma' \geq l^{*}$, since the above function is non-decreasing when $\#\Gamma'$ grows, and constant for $\#\Gamma' \geq l^{*}$.

(4) For the purpose of globalising the monoids determined by theta values, via global realified Frobenioids (cf. Section 11.4), such a manner that the product formula should be satisfied, the set $\{\underline{\underline{j}} \in \Gamma',\ \underline{\underline{j}} \equiv j \text{ in } |\mathbb{F}_l|\}$ should consist of only one element

for each $j \in \mathbb{F}_l^{\circledast}$, because the independent conjugacy indeterminacies explained in Remark 11.10.3 are incompatible with the product formula, if the set has more than two elements.

Then the only subgraph satisfying (1), (2), (3), (4) is $\Gamma_{\ddot{Y}}^{\blacktriangleright}$.

For a projective system $\mathbb{M}_*^{\Theta} = (\cdots \leftarrow \mathbb{M}_M^{\Theta} \leftarrow \mathbb{M}_{M'}^{\Theta} \leftarrow \cdots)$ of mono-theta environments such that $\Pi_{\underline{\underline{X}}}^{\mathrm{temp}}(\mathbb{M}_*^{\Theta}) \cong \Pi_{\underline{v}}$, where $\mathbb{M}_M^{\Theta} = (\Pi_{\mathbb{M}_M^{\Theta}}, \mathcal{D}_{\mathbb{M}_M^{\Theta}}, s^{\Theta}_{\mathbb{M}_M^{\Theta}})$, put

$$\Pi_{\mathbb{M}_*^{\Theta}} := \varprojlim_M \Pi_{\mathbb{M}_M^{\Theta}}.$$

Note that we have a natural homomorphism $\Pi_{\mathbb{M}_*^{\Theta}} \to \Pi_{\underline{\underline{X}}}^{\mathrm{temp}}(\mathbb{M}_*^{\Theta})$ of topological groups whose kernel is equal to the external cyclotome $\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}_*^{\Theta})$, and whose image correpsonds to $\Pi_{\underline{\underline{Y}}_{\underline{v}}}^{\mathrm{temp}}$. We write

$$\Pi_{\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta}} \quad\subset\quad \Pi_{\mathbb{M}_{*\blacktriangleright}^{\Theta}} \quad\subset\quad \Pi_{\mathbb{M}_*^{\Theta}}$$

for the inverse image of $\Pi_{\underline{v}\ddot{\blacktriangleright}} \subset \Pi_{\underline{v}\blacktriangleright} \subset \Pi_{\underline{v}} \cong \Pi_{\underline{\underline{X}}}^{\mathrm{temp}}(\mathbb{M}_*^{\Theta})$ in $\Pi_{\mathbb{M}_*^{\Theta}}$ respectively, and

$$\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta}), \quad (l\Delta_{\Theta})(\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta}), \quad \Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta}), \quad G_{\underline{v}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta})$$

for the subquotients of $\Pi_{\mathbb{M}_*^{\Theta}}$ determined by the subquotient $\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}_*^{\Theta})$ of $\Pi_{\mathbb{M}_*^{\Theta}}$ and the subquotients $(l\Delta_{\Theta})(\Pi_{\underline{\underline{X}}}^{\mathrm{temp}}(\mathbb{M}_*^{\Theta}))$, $\Pi_{\underline{v}\ddot{\blacktriangleright}}$, and $G_{\underline{v}}(\Pi_{\underline{\underline{X}}}^{\mathrm{temp}}(\mathbb{M}_*^{\Theta}))$ of $\Pi_{\underline{v}} \cong \Pi_{\underline{\underline{X}}}^{\mathrm{temp}}(\mathbb{M}_*^{\Theta})$. Note that we obtain a cyclotomic rigidity isomorphism of mono-theta environment

$$(l\Delta_{\Theta})(\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta})$$

by restricting the cyclotomic rigidity isomorphism of mono-theta environment $(l\Delta_{\Theta})(\mathbb{M}_*^{\Theta}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}_*^{\Theta})$ in Proposition 11.4 to $\Pi_{\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta}}$ (Definition [IUTchII, Definition 2.7]).

Corollary 11.11. ($\mathbb{M}$-theoretic Theta Evaluation, [IUTchII, Corollary 2.8]) *Let $\mathbb{M}_*^{\Theta}$ be a projective system of mono-theta environments with $\Pi_{\underline{\underline{X}}}^{\mathrm{temp}}(\mathbb{M}_*^{\Theta}) = \Pi_{\underline{v}}$. We write*

$$(\mathbb{M}_*^{\Theta})^{\gamma}$$

for the projective system of mono-theta environments obtained via transport of structure from the isomorphism $\Pi_{\underline{v}} \xrightarrow{\sim} \Pi_{\underline{v}}^{\gamma}$ given by the conjugation by γ.

(1) Let $I_t^{\delta} = I_t^{\gamma\gamma'} \subset \Pi_{\underline{v}\ddot{\blacktriangleright}}^{\delta} \subset \Pi_{\underline{v}\blacktriangleright}^{\gamma} = \Pi_{\underline{v}\blacktriangleright}^{\delta}$ be as in Lemma 11.9 (2). Then by using **the cyclomotic rigidity isomorphisms of mono-theta environment**

$$(l\Delta_{\Theta})((\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta})^{\gamma}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}((\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta})^{\gamma}), \quad (l\Delta_{\Theta})((\mathbb{M}_*^{\Theta})^{\gamma}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}((\mathbb{M}_*^{\Theta})^{\gamma})$$

(cf. just before Corollary 11.11), we replace $H^1(-,(l\Delta_\Theta)(-))$ *by* $H^1(-,\mu_{\widehat{\mathbb{Z}}}(-))$ *in Proposition 11.10. Then the* ι^γ*-invariant subsets* $\underline{\underline{\theta}}^\iota(\Pi^\gamma_{\underline{v}}) \subset \underline{\underline{\theta}}(\Pi^\gamma_{\underline{v}})$, ${}_\infty\underline{\underline{\theta}}^\iota(\Pi^\gamma_{\underline{v}}) \subset {}_\infty\underline{\underline{\theta}}(\Pi^\gamma_{\underline{v}})$ *determines* ι^γ*-invariant subsets*

$$\underline{\underline{\theta}}^\iota_{\mathrm{env}}((\mathbb{M}^\Theta_*)^\gamma) \subset \underline{\underline{\theta}}_{\mathrm{env}}((\mathbb{M}^\Theta_*)^\gamma),\ {}_\infty\underline{\underline{\theta}}^\iota_{\mathrm{env}}((\mathbb{M}^\Theta_*)^\gamma) \subset {}_\infty\underline{\underline{\theta}}_{\mathrm{env}}((\mathbb{M}^\Theta_*)^\gamma).$$

The restriction of these subsets to $\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)$ *gives us* μ_{2l}*-,* μ*-orbits of elements*

$$\underline{\underline{\theta}}^\iota_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma) \subset {}_\infty\underline{\underline{\theta}}^\iota_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma) \subset {}_\infty H^1(\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma), \mu_{\widehat{\mathbb{Z}}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)),$$

where ${}_\infty H^1(\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma),-) := \varinjlim_{\widehat{J}\subset\widehat{\Pi}_{\underline{v}}\,;\,\mathrm{open}} H^1(\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma) \times_{\widehat{\Pi}_{\underline{v}}} \widehat{J},-)$*. The further restriction to the decomposition groups* D^δ_{t,μ_-} *in Lemma 11.9 (2) gives us* μ_{2l}*-,* μ*-orbits of elements*

$$\underline{\underline{\theta}}^t_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma) \subset {}_\infty\underline{\underline{\theta}}^t_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma) \subset {}_\infty H^1(G_{\underline{v}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma), \mu_{\widehat{\mathbb{Z}}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)),$$

where we write ${}_\infty H^1(G_{\underline{v}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma),-) := \varinjlim_{J_G\subset G_{\underline{v}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)\,:\,\mathrm{open}} H^1(J_G,-)$*, for each* $t \in \mathrm{LabCusp}^\pm(\Pi^\gamma_{\underline{v}}) \overset{\text{conj. by }\gamma}{\xrightarrow{\sim}} \mathrm{LabCusp}^\pm(\Pi_{\underline{v}})$*. Since the sets* $\underline{\underline{\theta}}^t_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)$, ${}_\infty\underline{\underline{\theta}}^t_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)$ *depend only on the label* $|t| \in |\mathbb{F}_l|$*, we write*

$$\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma) := \underline{\underline{\theta}}^t_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma), \qquad {}_\infty\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma) := {}_\infty\underline{\underline{\theta}}^t_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma).$$

(2) *If we start with an arbitrary* $\widehat{\Delta}^\pm_{\underline{v}}$*-conjugate* $\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)$ *of* $\Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})$*, and we consider the resulting* μ_{2l}*-,* μ*-orbits* $\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)$, ${}_\infty\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)$ *arising from an arbitrary* $\widehat{\Delta}^\pm_{\underline{v}}$*-conjugate* I^δ_t *of* I_t *contained in* $\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)$*, as* t *runs over* $\mathrm{LabCusp}^\pm(\Pi^\gamma_{\underline{v}}) \overset{\text{conj. by }\gamma}{\xrightarrow{\sim}} \mathrm{LabCusp}^\pm(\Pi_{\underline{v}})$*, then we obtain a group-theoretic algorithm to construct the collections of* μ_{2l}*-,* μ*-orbits*

$$\left\{\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)\right\}_{|t|\in|\mathbb{F}_l|}, \quad \left\{{}_\infty\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)\right\}_{|t|\in|\mathbb{F}_l|},$$

which is functorial with respect to the projective system $\mathbb{M}^\Theta_*$ *of mono-theta environments, and compatible with the independent conjugacy actions of* $\widehat{\Delta}^\pm_{\underline{v}}$ *on the sets* $\{I^{\gamma_1}_t\}_{\gamma_1\in\widehat{\Pi}^\pm_{\underline{v}}} = \{I^{\gamma_1}_t\}_{\gamma_1\in\widehat{\Delta}^\pm_{\underline{v}}}$ *and* $\{\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^{\gamma_2})\}_{\gamma_2\in\widehat{\Pi}^\pm_{\underline{v}}} = \{\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^{\gamma_2})\}_{\gamma_2\in\widehat{\Delta}^\pm_{\underline{v}}}$

(3) *In the situation of (1), we take* t *to be the zero element. By using the cyclomotic rigidity isomorphisms in (1) we replace* $(l\Delta_\Theta)(-)$ *by* $\mu_{\widehat{\mathbb{Z}}}(-)$ *in Proposition 11.10, then we obtain* **splittings**

$$O^{\times\mu}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma) \times \{{}_\infty\underline{\underline{\theta}}^\iota_{\mathrm{env}}((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)/O^\mu((\mathbb{M}^\Theta_{*\ddot{\blacktriangleright}})^\gamma)\}$$

of $O^{\times}\,{}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\blacktriangleright})^{\gamma})/O^{\mu}((\mathbb{M}^{\Theta}_{*\blacktriangleright})^{\gamma})$*, which are compatible with the splittings of Proposition 11.7 (3) (with respect to any isomorphism* $\mathbb{M}^{\Theta}_{*} \xrightarrow{\sim} \mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})$*), relative to the first restriction operation in (1):*

$$0 \longrightarrow O^{\times\mu}((\mathbb{M}^{\Theta}_{*\blacktriangleright})^{\gamma}) \longrightarrow O^{\times}\,{}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\blacktriangleright})^{\gamma})/O^{\mu}((\mathbb{M}^{\Theta}_{*\blacktriangleright})^{\gamma}) \longrightarrow {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\blacktriangleright})^{\gamma})/O^{\mu}((\mathbb{M}^{\Theta}_{*\blacktriangleright})^{\gamma}) \longrightarrow 0.$$

label 0

Remark 11.11.1. (Theta Evaluation via Base-field-theoretic Cyclotomes, [IUTchII, Corollary 2.9, Remark2.9.1]) If we use the cyclotomic rigidity isomorphisms

$$\mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi_{\underline{v}})) \xrightarrow{\sim} (l\Delta_{\Theta})(\Pi_{\underline{v}}), \quad \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})) \xrightarrow{\sim} (l\Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}\blacktriangleright})$$

determined by the composites of **the cyclotomic rigidity isomorphism via positive rational structure and LCFT** "$\mu_{\widehat{\mathbb{Z}}}(G) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(\Pi)$" group-theoretically reconstructed by Remark 6.12.2 and the cyclotomic rigidity isomorphism "$\mu_{\widehat{\mathbb{Z}}}(\Pi) \xrightarrow{\sim} (l\Delta_{\Theta})(\Pi)$" group-theoretically reconstructed by Remark 9.4.1 and its restriction to $\Pi^{\gamma}_{\underline{v}\blacktriangleright}$ (like Proposition 11.5; however, we allow indeterminacies in Proposition 11.5), instead of using the cyclomotic rigidity isomorphisms of mono-theta environment $(l\Delta_{\Theta})((\mathbb{M}^{\Theta}_{*\blacktriangleright})^{\gamma}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}((\mathbb{M}^{\Theta}_{*\blacktriangleright})^{\gamma})$, $(l\Delta_{\Theta})((\mathbb{M}^{\Theta}_{*})^{\gamma}) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}((\mathbb{M}^{\Theta}_{*})^{\gamma})$, then we functorially group-theoretically obtain the following similar objects with similar compatibility as in Corollary 11.11: ι^{γ}-invariant subsets

$$\underline{\underline{\theta}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}}) \subset \underline{\underline{\theta}}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}}), \ {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}}) \subset {}_{\infty}\underline{\underline{\theta}}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}}).$$

The restriction of these subsets to $\Pi^{\gamma}_{\underline{v}\blacktriangleright}$ gives us μ_{2l}-, μ-orbits of elements

$$\underline{\underline{\theta}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{*\blacktriangleright}) \subset {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{*\blacktriangleright}) \subset {}_{\infty}H^1(\Pi^{\gamma}_{\underline{v}\blacktriangleright}, \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi^{\gamma}_{*\blacktriangleright}))),$$

where ${}_{\infty}H^1(\Pi^{\gamma}_{\underline{v}\blacktriangleright}, -) := \varinjlim_{\widehat{J}\subset\widehat{\Pi}_{\underline{v}}:\,\mathrm{open}} H^1(\Pi^{\gamma}_{\underline{v}\blacktriangleright} \times_{\widehat{\Pi}_{\underline{v}}} \widehat{J}, -)$. The further restriction to the decomposition groups D^{δ}_{t,μ_-} in Lemma 11.9 (2) gives us μ_{2l}-, μ-orbits of elements

$$\underline{\underline{\theta}}^{t}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}) \subset {}_{\infty}\underline{\underline{\theta}}^{t}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}) \subset {}_{\infty}H^1(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}), \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}))),$$

where ${}_{\infty}H^1(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}), -) := \varinjlim_{J_G\subset G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}):\,\mathrm{open}} H^1(J_G, -)$, for each $t \in \mathrm{LabCusp}^{\pm}(\Pi^{\gamma}_{\underline{v}}) \overset{\text{conj. by }\gamma}{\xrightarrow{\sim}} \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$. Since the sets $\underline{\underline{\theta}}^{t}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})$, ${}_{\infty}\underline{\underline{\theta}}^{t}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})$ depend only on the label $|t| \in |\mathbb{F}_l|$, we write

$$\underline{\underline{\theta}}^{|t|}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}) := \underline{\underline{\theta}}^{t}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}), \qquad {}_{\infty}\underline{\underline{\theta}}^{|t|}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}) := {}_{\infty}\underline{\underline{\theta}}^{t}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}).$$

Hence the collections of μ_{2l}-, μ-orbits

$$\left\{\underline{\underline{\theta}}^{|t|}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})\right\}_{|t|\in|\mathbb{F}_l|}, \quad \left\{{}_{\infty}\underline{\underline{\theta}}^{|t|}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})\right\}_{|t|\in|\mathbb{F}_l|},$$

and **splittings**

$$O^{\times\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})_{\mathrm{bs}} \times \{{}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/O^{\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})_{\mathrm{bs}}\}$$

of $O^{\times}\,{}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/O^{\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})_{\mathrm{bs}}$ (Here, we write $O^{\times\mu}(-)_{\mathrm{bs}}$, $O^{\times}(-)_{\mathrm{bs}}$, $O^{\mu}(-)_{\mathrm{bs}}$ for the objects corresponding to $O^{\times\mu}(-)$, $O^{\times}(-)$, $O^{\mu}(-)$, respectively, via the cyclotomic rigidity isomorphism):

$$\begin{tikzcd}
0 \arrow[r] & O^{\times\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})_{\mathrm{bs}} \arrow[r] & O^{\times}\,{}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/O^{\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})_{\mathrm{bs}} \arrow[r] \arrow[l, dotted, bend right, "\text{label } 0"'] & {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/O^{\mu}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})_{\mathrm{bs}} \arrow[r] & 0.
\end{tikzcd}$$

Note that we use the value group portion in the construction of the cyclotomic rigidity isomorphism via positive rational structure and LCFT (cf. the final remark in Remark 6.12.2). Therefore, the algorithm in this remark (unlike Corollary 11.11) is only *uniradially defined* (cf. Proposition 11.5 and Remark 11.4.1).

On the other hand, the cyclotomic rigidity isomorphism via positive rational structure and LCFT has an advantage of having the natural surjection

$$H^1(G_{\underline{v}}(-), \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(-))) \twoheadrightarrow \widehat{\mathbb{Z}}$$

in (the proof of) Corollary 3.19 (cf. Remark 6.12.2), and we use this surjection to construct some constant monoids (cf. Definition 11.12 (2)).

Definition 11.12. ($\mathbb{M}$-theoretic Theta Monoids, [IUTchII, Proposition 3.1]) Let $\mathbb{M}^{\Theta}_{*}$ be a projective system of mono-theta environments with $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \cong \Pi_{\underline{v}}$.

(1) **(Split Theta Monoids)** We put

$$\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) := \left\{\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) := O^{\times}(\mathbb{M}^{\Theta}_{*}) \cdot \underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})^{\mathbb{N}}\,(\subset {}_{\infty}H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_{*}), \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*})))\right\}_{\iota},$$

$${}_{\infty}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) := \left\{{}_{\infty}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) := O^{\times}(\mathbb{M}^{\Theta}_{*}) \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})^{\mathbb{N}}\,(\subset {}_{\infty}H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_{*}), \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*})))\right\}_{\iota}.$$

These are functorially group-theoretically reconstructed collections of submonoids of ${}_{\infty}H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_{*}), \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}))$ equipped with natrural conjugation actions of $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$, together with the splittings up to torsion determined by Corollary 11.11 (3). We shall refer to each of $\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$, ${}_{\infty}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$ as a **mono-theta-theoretic theta monoid**.

(2) **(Constant Monoids)** By using the cyclotomic rigidity isomorphism via positive rational structure and LCFT, and taking the inverse image of $\mathbb{Z} \subset \widehat{\mathbb{Z}}$ via the surjection

$H^1(G_{\underline{v}}(-), \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(-))) \twoheadrightarrow \widehat{\mathbb{Z}}$ (cf. Remark 11.11.1) for $G_{\underline{v}}(\mathbb{M}^{\Theta}_*) := G_{\underline{v}}(\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*))$, we obtain a functorial group-theoretic reconstruction

$$\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_*) \subset {}_{\infty}H^1(\Pi^{\mathrm{temp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_*), \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_*))$$

of an isomorph of $O^{\rhd}_{\overline{F}_{\underline{v}}}$, equipped with a natural conjugate action by $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*)$. We shall refer to $\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_*)$ as a **mono-theta-theoretic constant monoid**.

Definition 11.13. ([IUTchII, Example 3.2])

(1) **(Split Theta Monoids)** Recall that, for the tempered Frobenioid $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ (cf. Example 8.8), the choice of a Frobenioid-theoretic theta function $\underline{\underline{\Theta}}_{\underline{v}} \in O^{\times}(\mathcal{O}^{\mathrm{birat}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$ (cf. Example 8.8) among the $\mu_{2l}(\mathcal{O}^{\mathrm{birat}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$-multiples of the $\mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{\underline{v}})$-conjugates of $\underline{\underline{\Theta}}_{\underline{v}}$ determines a monoid $O^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ on $\mathcal{D}^{\Theta}_{\underline{v}}$ (cf. Definition 10.5 (1)) Suppose, for simplicity, the topological group $\Pi_{\underline{v}}$ arises from a universal covering pro-object A_{∞} of $\mathcal{D}_{\underline{v}}$. Then for $A^{\Theta}_{\infty} := A_{\infty} \times \underline{\underline{\ddot{Y}}}_{\underline{v}} \in \text{pro-Ob}(\mathcal{D}^{\Theta}_{\underline{v}})$ (cf. Definition 10.5 (1)), we obtain submonoids

$$\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\mathrm{id}} := O^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) = O^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}|_{A^{\Theta}_{\infty}} \subset {}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\mathrm{id}} := O^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) \cdot \underline{\underline{\Theta}}^{\mathbb{Q}_{\geq 0}}_{\underline{v}}|_{A^{\Theta}_{\infty}} \subset O^{\times}(\mathcal{O}^{\mathrm{birat}}_{A^{\Theta}_{\infty}}).$$

For the various conjugates $\underline{\underline{\Theta}}^{\alpha}_{\underline{v}}$ of $\underline{\underline{\Theta}}_{\underline{v}}$ for $\alpha \in \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{\underline{v}})$, we also similarly obtain submonoids

$$\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha} \subset {}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha} \subset O^{\times}(\mathcal{O}^{\mathrm{birat}}_{A^{\Theta}_{\infty}}).$$

Write

$$\Psi_{\mathcal{F}^{\Theta}_{\underline{v}}} := \left\{\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}\right\}_{\alpha \in \Pi_{\underline{v}}}, \quad {}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha} := \left\{{}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}\right\}_{\alpha \in \Pi_{\underline{v}}},$$

where we use the same notation α, by abuse of notation, for the image of α via the surjection $\Pi_{\underline{v}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{\underline{v}})$. Note that we have a natural conjugation action of $\Pi_{\underline{v}}$ on the above collections of submonoids. Note also that $\underline{\underline{\Theta}}^{\mathbb{Q}_{\geq 0}}_{\underline{v}}|_{A^{\Theta}_{\infty}}$ gives us splittings up to torsion of the monoids $\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}$, ${}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}$ (cf. $\mathrm{spl}^{\Theta}_{\underline{v}}$ in Definition 10.5 (1)), which are compatible with the $\Pi_{\underline{v}}$-action. Note that, from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, we can reconstruct these collections of submonoids with $\Pi_{\underline{v}}$-actions together with the splittings up to torsion up to an indeterminacy arising from the inner automorphismsof $\Pi_{\underline{v}}$ (cf. Section 8.3. cf. also the remark given just before Theorem 8.14). We shall refer to each of $\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}$, ${}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}$ as a **Frobenioid-theoretic theta monoid**.

(2) **(Constant Monoids)** Similarly, the pre-Frobenioid structure on $\mathcal{C}_{\underline{v}} = (\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\text{base-field}} \subset \underline{\underline{\mathcal{F}}}_{\underline{v}}$ gives us a monoid $O^{\rhd}_{\mathcal{C}_{\underline{v}}}(-)$ on $\mathcal{D}_{\underline{v}}$. We put

$$\Psi_{\mathcal{C}_{\underline{v}}} := O^{\rhd}_{\mathcal{C}_{\underline{v}}}(A^{\Theta}_{\infty}),$$

which is equipped with a natural $\Pi_{\underline{v}}$-action. Note that, from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, we can reconstruct $\Pi_{\underline{v}} \curvearrowright \Psi_{\mathcal{C}_{\underline{v}}}$, up to an indeterminacy arising from the inner automorphisms of $\Pi_{\underline{v}}$. We shall refer to $\Psi_{\mathcal{C}_{\underline{v}}}$ as a **Frobenioid-theoretic constant monoid**.

Proposition 11.14. ($\mathcal{F}$-theoretic Theta Monoids, [IUTchII, Proposition 3.3]) *Let $\mathbb{M}^{\Theta}_{*}$ be a projective system of mono-theta environments with $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \cong \Pi_{\underline{v}}$. Suppose that $\mathbb{M}^{\Theta}_{*}$ arises from a tempered Frobenioid ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ in a Θ-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$ by Theorem 8.14 ("$\mathcal{F} \mapsto \mathbb{M}$"):*

$$\mathbb{M}^{\Theta}_{*} = \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}).$$

(1) **(Split Theta Monoids)** *Note that, for an object S of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ such that $\mu_{lN}(S) \cong \mathbb{Z}/lN\mathbb{Z}$, and $(l\Delta_{\Theta})_S \otimes \mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}/N\mathbb{Z}$ as abstract groups, the exterior cyclotome $\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))$ corresponds to the cyclotome $\mu_{\widehat{\mathbb{Z}}}(S) = \varprojlim_N \mu_N(S)$, where $\mu_N(S) \subset O^{\times}(S) \subset \mathrm{Aut}_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}}(S)$ (cf. [IUTchII, Proposition 1.3 (i)]). Then by the Kummer maps, we obtain collections of* **Kummer isomorphisms**

$$\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}},\alpha} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}), \qquad {}_{\infty}\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}},\alpha} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} {}_{\infty}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}),$$

of monoids, which is well-defined up to an inner automorphism and compatible with both the respective conjugation action of $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$, and the splittings up to torsion on the monoids, under a suitable bijection of $l\underline{\mathbb{Z}}$-torsors between "ι" in Definition 11.8, and the images of "α" via the natural surjection $\Pi_{\underline{v}} \twoheadrightarrow l\underline{\mathbb{Z}}$:

$$\text{“}\iota\text{”s} \quad \overset{\sim}{\longleftrightarrow} \text{“Im}(\alpha)\text{”s}.$$

(2) **(Constant Monoids)** *Similary, using the correspondence between the exterior cyclotome $\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))$ and the cyclotome $\mu_{\widehat{\mathbb{Z}}}(S) = \varprojlim_N \mu_N(S)$, we obtain* **Kummer isomorphisms**

$$\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})$$

for constant monoids, where ${}^{\dagger}\mathcal{C}_{\underline{v}} := ({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\text{base-field}}$, which is well-defined up to an inner automorphism, and compatible with the respective conjugation actions of $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$.

Proof. Proposition follows from the definitions. □

In the following, we often use the abbreviation ${}_{(\infty)}(-)$ for a description like *both* $(-)$ *and* ${}_{\infty}(-)$.

Proposition 11.15. (Π-theoretic Theta Monoids, [IUTchII, Proposition 3.4]) *Let $\mathbb{M}^{\Theta}_{*}$ be a projective system of mono-theta environments with $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \cong \Pi_{\underline{v}}$. Suppose that $\mathbb{M}^{\Theta}_{*}$ arises from a tempered Frobenioid ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ in a Θ-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$ by Theorem 8.14 ("$\mathcal{F}\mapsto\mathbb{M}$"):*

$$\mathbb{M}^{\Theta}_{*} = \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}).$$

We consider the full poly-isomorphism

$$\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \overset{\text{full poly}}{\overset{\sim}{\to}} \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

of projective systems of mono-theta environments.

(1) **(Multiradiality of Split Theta Monoids)** *Each isomorphism $\beta : \mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \overset{\sim}{\to} \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$ of projective system of mono-theta environmens induces compatible collections of isomorphisms*

$$\begin{array}{ccccc}
\Pi_{\underline{v}} \overset{\sim}{\to} \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \overset{\beta}{\overset{\sim}{\to}} & \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) & = & \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
{}_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \overset{\beta}{\overset{\sim}{\to}} & {}_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) & \overset{\mathrm{Kum}^{-1}}{\overset{\sim}{\to}} & {}_{(\infty)}\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}},
\end{array}$$

which are compatible with the respective splittings up to torsion, and

$$\begin{array}{ccccc}
G_{\underline{v}} \overset{\sim}{\to} G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \overset{\beta}{\overset{\sim}{\to}} & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) & = & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))^{\times} & \overset{\beta}{\overset{\sim}{\to}} & \Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))^{\times} & \overset{\mathrm{Kum}^{-1}}{\overset{\sim}{\to}} & \Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}}.
\end{array}$$

Moreover, the functorial algorithm

$$\Pi_{\underline{v}} \mapsto (\Pi_{\underline{v}} \curvearrowright {}_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \text{ with splittings up to torsion}),$$

which is compatible with arbitrary automorphisms of the pair

$$G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \curvearrowright (\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times\mu} := (\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times}/\text{torsions}$$

arisen as Isomet*-multiples of automorphisms induced by automorphisms of the pair $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \curvearrowright (\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times}$, relative to the above displayed diagrams, is* **multiradially defined** *in the sense of the natural functor "$\Psi_{\mathrm{Graph}(\Xi)}$" of Proposition 11.7.*

(2) **(Uniradiality of Constant Monoids)** *Each isomorphism $\beta : \mathbb{M}^{\Theta}_*(\Pi_{\underline{v}}) \xrightarrow{\sim} \mathbb{M}^{\Theta}_*({}^\dagger\mathcal{F}_{\underline{\underline{v}}})$ of projective system of mono-theta environmens induces compatible collections of isomorphisms*

$$\begin{array}{ccccc}
\Pi_{\underline{v}} \xrightarrow{\sim} \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*(\Pi_{\underline{v}})) & \overset{\beta}{\xrightarrow{\sim}} & \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*({}^\dagger\mathcal{F}_{\underline{\underline{v}}})) & = & \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*({}^\dagger\mathcal{F}_{\underline{\underline{v}}})) \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_*(\Pi_{\underline{v}})) & \overset{\beta}{\xrightarrow{\sim}} & \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_*({}^\dagger\mathcal{F}_{\underline{\underline{v}}})) & \overset{\mathrm{Kum}^{-1}}{\xrightarrow{\sim}} & \Psi_{{}^\dagger\mathcal{C}_{\underline{v}}},
\end{array}$$

and

$$\begin{array}{ccccc}
G_{\underline{v}} \xrightarrow{\sim} G_{\underline{v}}(\mathbb{M}^{\Theta}_*(\Pi_{\underline{v}})) & \overset{\beta}{\xrightarrow{\sim}} & G_{\underline{v}}(\mathbb{M}^{\Theta}_*({}^\dagger\mathcal{F}_{\underline{\underline{v}}})) & = & G_{\underline{v}}(\mathbb{M}^{\Theta}_*({}^\dagger\mathcal{F}_{\underline{\underline{v}}})) \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_*(\Pi_{\underline{v}}))^\times & \overset{\beta}{\xrightarrow{\sim}} & \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_*({}^\dagger\mathcal{F}_{\underline{\underline{v}}}))^\times & \overset{\mathrm{Kum}^{-1}}{\xrightarrow{\sim}} & \Psi^\times_{{}^\dagger\mathcal{C}_{\underline{v}}}.
\end{array}$$

Moreover, the functorial algorithm

$$\Pi_{\underline{v}} \mapsto (\Pi_{\underline{v}} \curvearrowright \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_*(\Pi_{\underline{v}}))),$$

which **fails** *to be compatible (Note that we use the cyltomic rigidity isomorphism via rational positive structure and LCFT and the surjection $H^1(G_{\underline{v}}(-), \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(-))) \twoheadrightarrow \widehat{\mathbb{Z}}$ to construct the constant monoid, which use the value group portion as well) with automorphisms of the pair*

$$G_{\underline{v}}(\mathbb{M}^{\Theta}_*({}^\dagger\mathcal{F}_{\underline{\underline{v}}})) \curvearrowright (\Psi_{{}^\dagger\mathcal{C}_{\underline{v}}})^{\times\mu} := (\Psi_{{}^\dagger\mathcal{C}_{\underline{v}}})^{\times}/\mathrm{torsions}$$

induced by automorphisms of the pair $G_{\underline{v}}(\mathbb{M}^{\Theta}_*({}^\dagger\mathcal{F}_{\underline{\underline{v}}})) \curvearrowright (\Psi_{{}^\dagger\mathcal{C}_{\underline{v}}})^\times$, relative to the above displayed diagrams, is **uniradially defined***.*

Proof. Proposition follows from the definitions. □

Corollary 11.16. ($\mathbb{M}$-theoretic Gaussian Monoids, [IUTchII, Corollary 3.5]) *Let $\mathbb{M}^{\Theta}_*$ be a projective system of mono-theta environments with $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*) \cong \Pi_{\underline{v}}$. For $t \in \mathrm{LabCusp}^{\pm}(\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*))$, we write $(-)_t$ for copies labelled by t of various objects functorially constructed from $\mathbb{M}^{\Theta}_*$ (We use this convention after this corollary as well).*

(1) **(Conjugate Synchronisation)** *If we regard the cuspidal inertia subgroups $\subset \Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*)$ corresponding to t as subgroups of cuspidal inertia subgroups of $\Pi^{\mathrm{temp}}_{\underline{X}}(\mathbb{M}^{\Theta}_*)$, then the $\Delta^{\mathrm{temp}}_{\underline{X}}(\mathbb{M}^{\Theta}_*)$-outer action of $\mathbb{F}_l^{\rtimes\pm} \cong \Delta^{\mathrm{temp}}_{C}(\mathbb{M}^{\Theta}_*)/\Delta^{\mathrm{temp}}_{\underline{X}}(\mathbb{M}^{\Theta}_*)$ on $\Pi^{\mathrm{temp}}_{\underline{X}}(\mathbb{M}^{\Theta}_*)$ induces isomorphisms between the pairs*

$$G_{\underline{v}}(\mathbb{M}^{\Theta}_*)_t \curvearrowright \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_*)_t$$

of a labelled ind-topological monoid equipped with the action of a labelled topological group for distinct $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{\underline{X}}}^{\mathrm{temp}}(\mathbb{M}_*^{\Theta}))$*. We shall refer to these isomorphisms as* $\mathbb{F}_l^{\rtimes\pm}$**-symmetrising isomorphisms***. When we identify these objects labelled by* t *and* $-t$ *via a suitable* $\mathbb{F}_l^{\rtimes\pm}$*-symmetrising isomorphism, we write* $(-)_{|t|}$ *for the resulting object labelled by* $|t| \in |\mathbb{F}_l|$*. We write*

$$(-)_{\langle|\mathbb{F}_l|\rangle}$$

for the object determined by the diagonal embedding in $\prod_{|t|\in\mathbb{F}_l}(-)_{|t|}$ *via suitable* $\mathbb{F}_l^{\rtimes\pm}$*-symmetrising isomorphisms (Note that, thanks to the* $\mathbb{F}_l^{\rtimes\pm}$*-symmetrising isomorphisms, we can construct the diagonal objects). Then by Corollary 11.11, we obtain a collection of compatible morphisms*

$$\begin{array}{ccccc}
(\Pi_{\underline{\underline{X}}}^{\mathrm{temp}}(\mathbb{M}_*^{\Theta}) \hookleftarrow) & \Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta}) & \twoheadrightarrow & G_{\underline{v}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta})_{\langle|\mathbb{F}_l|\rangle} \\
& \curvearrowright & & \curvearrowright \\
& \Psi_{\mathrm{cns}}(\mathbb{M}_*^{\Theta}) & \overset{\mathrm{diag}}{\xrightarrow{\sim}} & \Psi_{\mathrm{cns}}(\mathbb{M}_*^{\Theta})_{\langle|\mathbb{F}_l|\rangle},
\end{array}$$

which are compatible with $\mathbb{F}_l^{\rtimes\pm}$*-symmetrising isomorphisms and well-defined up to an inner automorphism of* $\Pi_{\underline{\underline{X}}}^{\mathrm{temp}}(\mathbb{M}_*^{\Theta})$ *(i.e., this inner automorphism indeterminacy, which a priori depends on* $|t| \in |\mathbb{F}_l|$*, is independent of* $|t| \in |\mathbb{F}_l|$*).*

(2) **(Gaussian Monoids)** *We shall refer to an element of the set*

$$\underline{\underline{\theta}}_{\mathrm{env}}^{\mathbb{F}_l^{\circledast}} := \prod_{|t|\in\mathbb{F}_l^{\circledast}} \underline{\underline{\theta}}_{\mathrm{env}}^{|t|} \subset \prod_{|t|\in\mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}(\mathbb{M}_*^{\Theta})_{|t|}$$

as a **value-profile** *(Note that this set has of cardinality* $(2l)^{l^{\circledast}}$*). Then by using* $\mathbb{F}_l^{\rtimes\pm}$*-symmetrising isomorphisms and Corollary 11.11, we obtain a functorial algorithm to construct, from* $\mathbb{M}_*^{\Theta}$*, two collections of submonoids*

$$\Psi_{\mathrm{gau}}(\mathbb{M}_*^{\Theta}) := \left\{ \Psi_{\xi}(\mathbb{M}_*^{\Theta}) := \Psi_{\mathrm{cns}}(\mathbb{M}_*^{\Theta})^{\times}_{\langle\mathbb{F}_l^{\circledast}\rangle} \cdot \xi^{\mathbb{N}} \subset \prod_{|t|\in\mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}(\mathbb{M}_*^{\Theta})_{|t|} \right\}_{\xi\,:\,\text{value profile}},$$

$$_{\infty}\Psi_{\mathrm{gau}}(\mathbb{M}_*^{\Theta}) := \left\{ {}_{\infty}\Psi_{\xi}(\mathbb{M}_*^{\Theta}) := \Psi_{\mathrm{cns}}(\mathbb{M}_*^{\Theta})^{\times}_{\langle\mathbb{F}_l^{\circledast}\rangle} \cdot \xi^{\mathbb{Q}_{\geq 0}} \subset \prod_{|t|\in\mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}(\mathbb{M}_*^{\Theta})_{|t|} \right\}_{\xi\,:\,\text{value profile}},$$

where each $\Pi_{\xi}(\mathbb{M}_*^{\Theta})$ *is equipped with a natural* $G_{\underline{v}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta})_{\langle\mathbb{F}_l^{\circledast}\rangle}$*-action. We shall refer to each of* $\Psi_{\xi}(\mathbb{M}_*^{\Theta})$*,* $_{\infty}\Psi_{\xi}(\mathbb{M}_*^{\Theta})$ *as a* **mono-theta-theoretic Gaussian monoid***.*

The restriction operations in Corollary 11.11 give us a collection of compatible **evaluation isomorphisms**

$$\begin{array}{ccc} (\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \hookleftarrow) \quad \Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) & \overset{D^{\delta}_{t,\mu_-}\text{'s}}{\dashleftarrow} & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_{|t|}\}_{|t|\in\mathbb{F}^{\divideontimes}_{l}} \\ \curvearrowright & & \curvearrowright \\ {}_{(\infty)}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) & \overset{\mathrm{eval}}{\overset{\sim}{\to}} & {}_{(\infty)}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}), \end{array}$$

which is well-defined up to an inner automorphism of $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$ *(Note that up to single inner automorphism by* $\mathbb{F}^{\rtimes\pm}_{l}$*-symmetrising isomorphisms), where we write* $\dashleftarrow$ *for the compatibility of the action of* $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_{|t|}$ *on the factor labelled by* $|t|$ *of the* ${}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*})$*. We write*

$${}_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) \overset{\mathrm{eval}}{\overset{\sim}{\to}} {}_{(\infty)}\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*})$$

for these collections of compatible evaluation morphisms induced by restriction.

(3) **(Constant Monoids and Splittings)** *The diagonal-in-*$|\mathbb{F}_l|$ *submonoid* $\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle|\mathbb{F}_l|\rangle}$ *can be seen as a grpah between the constant monoid* $\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{0}$ *labelled by the zero element* $0 \in |\mathbb{F}_l|$ *and the diagonal-in-*$\mathbb{F}^{\divideontimes}_{l}$ *submonoid* $\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}^{\divideontimes}_{l}\rangle}$*, hence determines an isomorphism*

$$\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{0} \overset{\mathrm{diag}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}^{\divideontimes}_{l}\rangle}$$

of monoids, which is compatible with respective labelled $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$*-actions. Moreover, the restriction operations to zero-labelled evaluation points (cf. Corollary 11.11) give us a splitting up to torsion*

$$\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}) = \Psi^{\times}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}^{\divideontimes}_{l}\rangle}\cdot\xi^{\mathbb{N}}, \quad {}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}) = \Psi^{\times}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}^{\divideontimes}_{l}\rangle}\cdot\xi^{\mathbb{Q}_{\geq 0}}$$

of each of the Gaussian monoids, which is compatible with the splitting up to torsion of Definition 11.12 (1), with respect to the restriction isomorphisms in the third display of (2).

Proof. Corollary follows from the definitions. □

Corollary 11.17. ($\mathcal{F}$-theoretic Gaussian Monoids, [IUTchII, Corollary 3.6]) *Let* $\mathbb{M}^{\Theta}_{*}$ *be a projective system of mono-theta environments with* $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \cong \Pi_{\underline{v}}$*. Suppose that* $\mathbb{M}^{\Theta}_{*}$ *arises from a tempered Frobenioid* ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ *in a* Θ*-Hodge theatre* ${}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$ *by Theorem 8.14 ("*$\mathcal{F}\mapsto\mathbb{M}$*"):*

$$\mathbb{M}^{\Theta}_{*} = \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}).$$

(1) **(Conjugate Synchronisation)** *For each* $t \in \mathrm{LabCusp}^{\pm}(\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}))$ *the Kummer isomorphism in Proposition 11.14 (2) determines a collection of compatible morphisms*

$$\begin{array}{ccc} (\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \twoheadrightarrow)\, G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})_t & \twoheadrightarrow & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_t \\ \curvearrowright & & \curvearrowright \\ (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})_t & \overset{\mathrm{Kum}}{\xrightarrow{\sim}} & \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_t, \end{array}$$

which are well-defined up to an inner automorphism (which is independent of $t \in \mathrm{LabCusp}^{\pm}(\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}))$*) of* $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$*, and* $\mathbb{F}_l^{\rtimes\pm}$*-symmetrising isomorphisms between distinct* $t \in \mathrm{LabCusp}^{\pm}(\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}))$ *induced by the* $\Delta_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$*-outer action of* $\mathbb{F}_l^{\rtimes\pm} \cong \Delta_C(\mathbb{M}^{\Theta}_{*})/\Delta_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$ *on* $\Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$*.*

(2) **(Gaussian Monoids)** *For each value-profile* ξ*, we write*

$$\Psi_{\mathcal{F}_\xi}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \subset {}_{\infty}\Psi_{\mathcal{F}_\xi}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \subset \prod_{|t|\in\mathbb{F}_l^{*}} (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})_{|t|}$$

for the submonoid determined by the monoids $\Psi_\xi(\mathbb{M}^{\Theta}_{*})$*,* ${}_{\infty}\Psi_\xi(\mathbb{M}^{\Theta}_{*})$ *in Corollary 11.16 (2), respectively, via the Kummer isomorphism* $(\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})_{|t|} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t|}$ *in (1). Write*

$$\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) := \left\{\Psi_{\mathcal{F}_\xi}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})\right\}_{\xi\,:\,\mathrm{value\ profile}},\quad {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) := \left\{{}_{\infty}\Psi_{\mathcal{F}_\xi}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})\right\}_{\xi\,:\,\mathrm{value\ profile}},$$

where each $\Pi_{\mathcal{F}_\xi}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$ *is equipped with a natural* $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}_l^{*}\rangle}$*-action. We shall refer to each of* $\Pi_{\mathcal{F}_\xi}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$*,* ${}_{\infty}\Pi_{\mathcal{F}_\xi}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$ *as a* **Frobenioid-theoretic Gaussian monoid***. Then by composing the Kummer isomorphism in (1) and Proposition 11.14 (1), (2) with the restriction isomorphism of Corollary 11.16 (2), we obtain a diagram of compatible* **evaluation ismorphisms**

$$\begin{array}{ccccccc} \Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) & = & \Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) & \overset{D^{\delta}_{t,\mu_-}\text{'s}}{\dashleftarrow} & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_{|t|}\}_{|t|\in\mathbb{F}_l^{*}} & \xrightarrow{\sim} & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})_{|t|}\}_{|t|\in\mathbb{F}_l^{*}} \\ \curvearrowright & & \curvearrowright & & \curvearrowright & & \curvearrowright \\ {}_{(\infty)}\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v},\alpha}} & \overset{\mathrm{Kum}}{\xrightarrow{\sim}} & {}_{(\infty)}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) & \overset{\mathrm{eval}}{\xrightarrow{\sim}} & {}_{(\infty)}\Psi_\xi(\mathbb{M}^{\Theta}_{*}) & \overset{\mathrm{Kum}^{-1}}{\xrightarrow{\sim}} & {}_{(\infty)}\Psi_{\mathcal{F}_\xi}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}), \end{array}$$

which is well-defined up to an inner automorphism of $\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$ *(Note that up to single inner automorphism by* $\mathbb{F}_l^{\rtimes\pm}$*-symmetrising isomorphisms), where* $\dashleftarrow$ *is the same meaning as in Corollary 11.16 (2). We write*

$${}_{(\infty)}\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} {}_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) \overset{\mathrm{eval}}{\xrightarrow{\sim}} {}_{(\infty)}\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}) \overset{\mathrm{Kum}^{-1}}{\xrightarrow{\sim}} {}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

for these collections of compatible evaluation morphisms.

(3) **(Constant Monoids and Splittings)** *By the same manner as in Corollary 11.16 (3), the diagonal submonoid* $(\Psi_{{}^\dagger\mathcal{C}_{\underline{v}}})_{\langle|\mathbb{F}_l|\rangle}$ *determines an isomorphism*

$$(\Psi_{{}^\dagger\mathcal{C}_{\underline{v}}})_0 \overset{\text{diag}}{\overset{\sim}{\to}} (\Psi_{{}^\dagger\mathcal{C}_{\underline{v}}})_{\langle\mathbb{F}_l^*\rangle}$$

of monoids, which is compatible with respective labelled $G_{\underline{v}}(\mathbb{M}_*^\Theta)$*-actions. Moreover, the splittings in Corollary 11.16 (3) give us splittings up to torsion*

$$\Psi_{\mathcal{F}_\xi}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}) = (\Psi^\times_{{}^\dagger\mathcal{C}_{\underline{v}}})_{\langle\mathbb{F}_l^*\rangle} \cdot \mathrm{Im}(\xi)^{\mathbb{N}}, \quad {}_\infty\Psi_{\mathcal{F}_\xi}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}) = (\Psi^\times_{{}^\dagger\mathcal{C}_{\underline{v}}})_{\langle\mathbb{F}_l^*\rangle} \cdot \mathrm{Im}(\xi)^{\mathbb{Q}_{\geq 0}}$$

(Here we write $\mathrm{Im}(-)$ *for the image of* $\mathrm{Kum}^{-1}\circ\mathrm{eval}\circ\mathrm{Kum}$ *in (2)) of each of the Gaussian monoids, which is compatible with the splitting up to torsion of Definition 11.12 (1), with respect to the restriction isomorphisms in the third display of (2).*

Proof. Corollary follows from the definitions. □

Remark 11.17.1. ([IUTchIII, Remark 2.3.3 (iv)]) It seems interesting to note that the cyclotomic rigidity of mono-theta environments **admits** $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, contrary to the fact that the theta functions, or the theta values $\underline{q}_{\underline{v}}^{j^2}$'s do not admit $\mathbb{F}_l^{\rtimes\pm}$-symmetry. This is because the construction of the cyclotomic rigidity of mono-theta environments only uses the commutator structure [,] (in other words, "curvature") of the theta group (i.e., Heisenberg group), not the theta function itself.

Remark 11.17.2. (Π-theoretic Gaussian Monoids, [IUTchII, Corollary 3.7, Remark 3.7.1]) If we formulate a "Gaussian analogue" of Proposition 11.15, then the resulting algorithm is only *uniradially defined*, since we use the cyclotomic rigidity isomorphism via rational positive structure and LCFT (cf. Remark 11.11.1 Proposition 11.15 (2)) to construct constant monoids. In the theta *functions* level (i.e., "env"-labelled objects), it admits multiradially defined algorithms; however, in the theta *values* level (i.e., "gau"-labelled objects), it only admits uniradially defined algorithms, since we need constant monoids as containers of theta values (Note also that this container is holomorphic container, since we need the holomorphic structures for the labels and $\mathbb{F}_l^{\rtimes\pm}$-synchronising isomorphisms). Later, by using the theory of log-shells, we will modify such a "Gaussian analogue" algorithm (cf. below) of Proposition 11.15 into a multiradially defined algorithm after admitting mild indeterminacies (i.e., (Indet ↑), (Indet →), and (Indet $\curvearrowright$)) (cf. Theorem 13.12 (1), (2)).

A precise formulation of a "Gaussian analogue" of Proposition 11.15 is as follows: Let $\mathbb{M}_*^\Theta$ be a projective system of mono-theta environments with $\Pi^{\text{temp}}_{\underline{\underline{X}}}(\mathbb{M}_*^\Theta) \cong \Pi_{\underline{v}}$. Suppose that $\mathbb{M}_*^\Theta$ arises from a tempered Frobenioid ${}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}$ in a Θ-Hodge theatre ${}^\dagger\mathcal{HT}^\Theta =$

$(\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$ by Theorem 8.14 ("$\mathcal{F} \mapsto \mathbb{M}$"):

$$\mathbb{M}^{\Theta}_{*} = \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}).$$

We consider the full poly-isomorphism

$$\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \overset{\text{full poly}}{\xrightarrow{\sim}} \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

of projective systems of mono-theta environments. We write $\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$ for $\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}$ for $\mathbb{M}^{\Theta}_{*} = \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$. For $\mathbb{M}^{\Theta}_{*} = \mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})$, we identify $\Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$ and $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$ with $\Pi_{\underline{v}\ddot{\blacktriangleright}}$ and $G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$ respectively, via the tautological isomorphisms $\Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \xrightarrow{\sim} \Pi_{\underline{v}\ddot{\blacktriangleright}}$, $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \xrightarrow{\sim} G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$.

(1) Each isomorphism $\beta : \mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \xrightarrow{\sim} \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$ of projective system of mono-theta environments induces compatible collections of **evaluation isomorphisms**

$$\begin{array}{ccccccc}
\Pi_{\underline{v}\ddot{\blacktriangleright}} & \overset{D^{\delta}_{t,\mu_{-}}\text{'s}}{\dashleftarrow} & \{G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})_{|t|}\}_{|t|\in\mathbb{F}^{\circledast}_{l}} & \overset{\beta}{\xrightarrow{\sim}} & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))_{|t|}\}_{|t|\in\mathbb{F}^{\circledast}_{l}} & \xrightarrow{\sim} & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))_{|t|}\}_{|t|\in\mathbb{F}^{\circledast}_{l}} \\
\curvearrowright & & \curvearrowright & & \curvearrowright & & \curvearrowright \\
{}_{(\infty)}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \overset{\mathrm{eval}}{\xrightarrow{\sim}} & {}_{(\infty)}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \overset{\beta}{\xrightarrow{\sim}} & {}_{(\infty)}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) & \overset{\mathrm{Kum}^{-1}}{\xrightarrow{\sim}} & {}_{(\infty)}\Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}),
\end{array}$$

and

$$\begin{array}{ccccccc}
G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}}) & \overset{\mathrm{diag}}{\xrightarrow{\sim}} & G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})_{\langle\mathbb{F}^{\circledast}_{l}\rangle} & \overset{\beta}{\xrightarrow{\sim}} & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))_{\langle\mathbb{F}^{\circledast}_{l}\rangle} & \xrightarrow{\sim} & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))_{\langle\mathbb{F}^{\circledast}_{l}\rangle} \\
\curvearrowright & & \curvearrowright & & \curvearrowright & & \curvearrowright \\
\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))^{\times} & \overset{\mathrm{eval}}{\xrightarrow{\sim}} & \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))^{\times} & \overset{\beta}{\xrightarrow{\sim}} & \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))^{\times} & \overset{\mathrm{Kum}^{-1}}{\xrightarrow{\sim}} & \Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\times},
\end{array}$$

where $\dashleftarrow$ is the same meaning as in Corollary 11.16 (2).

(2) **(Uniradiality of Gaussian Monoids)** The functorial algorithms

$$\Pi_{\underline{v}} \mapsto (G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}}) \curvearrowright \Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \text{ with splittings up to torsion}),$$

$$\Pi_{\underline{v}} \mapsto ({}_{\infty}\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \text{ with splittings up to torsion}),$$

which **fails** to be compatible (Note that we use the cyltomic rigidity isomorphism via rational positive structure and LCFT and the surjection $H^{1}(G_{\underline{v}}(-), \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(-))) \twoheadrightarrow \widehat{\mathbb{Z}}$ to construct the constant monoid, which use the value group portion as well) with automorphisms of the pair

$$G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))_{\langle\mathbb{F}^{\circledast}_{l}\rangle} \curvearrowright \Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\times\mu} := \Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\times}/\text{torsions}$$

induced by automorphisms of the pair $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \curvearrowright \Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\times}$, relative to the above displayed diagrams in (1), is **uniradially defined**.

§ 11.3. Hodge-Arakelov-theoretic Evaluation and Gaussian Monoids at Good Places.

In this subsection, we perform analogues of Hodge-Arakelov-theoretic evaluation, and construction of Gaussian monoids for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$.

Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$. For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$), put

$$\Pi_{\underline{v}} := \Pi_{\underrightarrow{X}_{\underline{v}}} \subset \Pi^{\pm}_{\underline{v}} := \Pi_{\underline{X}_{\underline{v}}} \subset \Pi^{\mathrm{cor}}_{\underline{v}} := \Pi_{C_{\underline{v}}}$$

$$(\text{resp. } \mathbb{U}_{\underline{v}} := \underrightarrow{\mathbb{X}}_{\underline{v}} \subset \mathbb{U}^{\pm}_{\underline{v}} := \underline{\mathbb{X}}_{\underline{v}} \subset \mathbb{U}^{\mathrm{cor}}_{\underline{v}} := \mathbb{C}_{\underline{v}}\ ),$$

where $\underrightarrow{\mathbb{X}}_{\underline{v}}$, $\underline{\mathbb{X}}_{\underline{v}}$, and $\mathbb{C}_{\underline{v}}$ are Aut-holomorphic orbispaces (cf. Section 4) associated to $\underrightarrow{X}_{\underline{v}}$, $\underline{X}_{\underline{v}}$, and $C_{\underline{v}}$, respectively. Note that we have $\Pi^{\mathrm{cor}}_{\underline{v}}/\Pi^{\pm}_{\underline{v}} \cong \mathbb{F}_l^{\rtimes\pm}$ (resp. $\mathrm{Gal}(\mathbb{U}^{\pm}_{\underline{v}}/\mathbb{U}^{\mathrm{cor}}_{\underline{v}}) \cong \mathbb{F}_l^{\rtimes\pm}$). We also write

$$\Delta_{\underline{v}} \subset \Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}}),\ \Delta^{\pm}_{\underline{v}} \subset \Pi^{\pm}_{\underline{v}} \twoheadrightarrow G_{\underline{v}}(\Pi^{\pm}_{\underline{v}}),\ \Delta^{\mathrm{cor}}_{\underline{v}} \subset \Pi^{\mathrm{cor}}_{\underline{v}} \twoheadrightarrow G_{\underline{v}}(\Pi^{\mathrm{cor}}_{\underline{v}})$$

$$(\text{resp.} \quad \mathcal{D}^{\vdash}_{\underline{v}}(\mathbb{U}_{\underline{v}}) \quad)$$

the natural quotients and their kernels (resp. the split monod), which can be group-theoretically reconstructed by Corollary 2.4 (resp. which can be algorithmically reconstructed by Proposition 4.5). Note that we have natural isomorphisms $G_{\underline{v}}(\Pi_{\underline{v}}) \xrightarrow{\sim} G_{\underline{v}}(\Pi^{\pm}_{\underline{v}}) \xrightarrow{\sim} G_{\underline{v}}(\Pi^{\mathrm{cor}}_{\underline{v}}) \xrightarrow{\sim} G_{\underline{v}}$.

Proposition 11.18. (Π-theoretic (resp. Aut-hol.-theoretic) Gaussian Monoids at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$ (resp. at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$), [IUTchII, Proposition 4.1, Proposition 4.3])

(1) **(Constant Monoids)** *By Corollary 3.19 (resp. by definitions), we have a functorial group-theoretic algorithm to construct, from the topological group $G_{\underline{v}}$ (resp. from the split monoid $\mathcal{D}^{\vdash}_{\underline{v}}$), the ind-topological submonoid equipped with $G_{\underline{v}}$-action (resp. the topological monoid)*

$$G_{\underline{v}} \curvearrowright \Psi_{\mathrm{cns}}(G_{\underline{v}}) \subset {}_{\infty}H^1(G_{\underline{v}}, \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}})) := \varinjlim_{J \subset G_{\underline{v}}:\,\mathrm{open}} H^1(J, \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}))$$

$$(\textit{resp.} \quad \Psi_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}}) := O^{\rhd}(\mathcal{C}^{\vdash}_{\underline{v}}) \quad),$$

which is an isomorph of ($G_{\underline{v}} \curvearrowright O^{\rhd}_{\overline{F}_{\underline{v}}}$), (resp. an isomorph of $O^{\rhd}_{F_{\underline{v}}}$). Thus, we obtain a functroial group-theoretic algorithm to construct, from the topological group $\Pi_{\underline{v}}$ (resp. from the Aut-holomorphic space $\mathbb{U}_{\underline{v}}$), the ind-topological submonoid equipped with $G_{\underline{v}}(\Pi_{\underline{v}})$-action (resp. the topological monoid)

$$\begin{aligned} G_{\underline{v}}(\Pi_{\underline{v}}) \curvearrowright \Psi_{\mathrm{cns}}(\Pi_{\underline{v}}) := \Psi_{\mathrm{cns}}(G_{\underline{v}}(\Pi_{\underline{v}})) &\subset {}_{\infty}H^1(G_{\underline{v}}(\Pi_{\underline{v}}), \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi_{\underline{v}}))) \\ &\subset {}_{\infty}H^1(\Pi^{\pm}_{\underline{v}}, \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi_{\underline{v}}))) \subset {}_{\infty}H^1(\Pi_{\underline{v}}, \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi_{\underline{v}}))) \end{aligned}$$

$$(\textit{resp.} \quad \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}}) := \Psi_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}}(\mathbb{U}_{\underline{v}})) \quad),$$

where ${}_{\infty}H^1(G_{\underline{v}}(\Pi_{\underline{v}}), -) := \varinjlim_{J \subset G_{\underline{v}}(\Pi_{\underline{v}}):\mathrm{open}} H^1(J, -)$, ${}_{\infty}H^1(\Pi^{\pm}_{\underline{v}}, -) := \varinjlim_{J \subset G_{\underline{v}}(\Pi_{\underline{v}}):\mathrm{open}} H^1(\Pi^{\pm}_{\underline{v}} \times_{G_{\underline{v}}(\Pi_{\underline{v}})} J, -)$, *and* ${}_{\infty}H^1(\Pi_{\underline{v}}, -) := \varinjlim_{J \subset G_{\underline{v}}(\Pi_{\underline{v}}):\mathrm{open}} H^1(\Pi_{\underline{v}} \times_{G_{\underline{v}}(\Pi_{\underline{v}})} J, -)$.

(2) **(Mono-analytic Semi-simplifications)** *By Definition 10.6, we have the functorial algorithm to construct, from the topological group* $G_{\underline{v}}$ *(resp. from the split monoid* $\mathcal{D}^{\vdash}_{\underline{v}}$*), the topological monoid equipped with the distinguished element*

$$\log^{G_{\underline{v}}}(p_{\underline{v}}) \in \mathbb{R}_{\geq 0}(G_{\underline{v}}) := (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}, \quad (\textit{resp.} \quad \log^{\mathcal{D}^{\vdash}}(p_{\underline{v}}) \in \mathbb{R}_{\geq 0}(\mathcal{D}^{\vdash}_{\underline{v}}) := (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}, \quad)$$

(cf. "$\log^{\mathcal{D}}_{\Phi}(p_{\underline{v}})$*" in Definition 10.6) and a natural isomorphism*

$$\Psi^{\mathbb{R}}_{\mathrm{cns}}(G_{\underline{v}}) := (\Psi_{\mathrm{cns}}(G_{\underline{v}})/\Psi_{\mathrm{cns}}(G_{\underline{v}})^{\times})^{\mathbb{R}} \xrightarrow{\sim} (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$$

$$(\textit{resp.} \quad \Psi^{\mathbb{R}}_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}}) := (\Psi_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}})/\Psi_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}})^{\times})^{\mathbb{R}} \xrightarrow{\sim} (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}} \quad)$$

of the monoids (cf. Proposition 5.2 (resp. Proposition 5.4)). Write

$$\Psi^{\mathrm{ss}}_{\mathrm{cns}}(G_{\underline{v}}) := \Psi_{\mathrm{cns}}(G_{\underline{v}})^{\times} \times (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}} \quad (\textit{resp.} \quad \Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}}) := \Psi_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}})^{\times} \times (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}} \quad),$$

which we consider as semisimplified version of $\Psi_{\mathrm{cns}}(G_{\underline{v}})$ *(resp.* $\Psi_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}})$*). We also write*

$$\Psi^{\mathrm{ss}}_{\mathrm{cns}}(\Pi_{\underline{v}}) := \Psi^{\mathrm{ss}}_{\mathrm{cns}}(G_{\underline{v}}(\Pi_{\underline{v}})),\ \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})^{\times} := \Psi_{\mathrm{cns}}(G_{\underline{v}}(\Pi_{\underline{v}}))^{\times},\ \mathbb{R}_{\geq 0}(\Pi_{\underline{v}}) := \mathbb{R}_{\geq 0}(G_{\underline{v}}(\Pi_{\underline{v}}))$$

$$(\textit{resp.} \quad \Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathbb{U}_{\underline{v}}) := \Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}}(\mathbb{U}_{\underline{v}})),\ \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})^{\times} := \Psi_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}}(\mathbb{U}_{\underline{v}}))^{\times},\ \mathbb{R}_{\geq 0}(\mathbb{U}_{\underline{v}}) := \mathbb{R}_{\geq 0}(\mathcal{D}^{\vdash}_{\underline{v}}(\mathbb{U}_{\underline{v}})) \quad),$$

just as in (1).

(3) **(Conjugate Synchronisation)** *If we regard the cuspidal inertia subgroups* $\subset \Pi_{\underline{v}}$ *corresponding to* t *as subgroups of cuspidal inertia subgroups of* $\Pi^{\pm}_{\underline{v}}$*, then the* $\Delta^{\pm}_{\underline{v}}$*-outer action of* $\mathbb{F}^{\rtimes\pm}_{l} \cong \Delta^{\mathrm{cor}}_{\underline{v}}/\Delta^{\pm}_{\underline{v}}$ *on* $\Pi^{\pm}_{\underline{v}}$ *(resp. the action of* $\mathbb{F}^{\rtimes\pm}_{l} \cong \mathrm{Gal}(\mathbb{U}^{\pm}_{\underline{v}}/\mathbb{U}^{\mathrm{cor}}_{\underline{v}})$ *on the various* $\mathrm{Gal}(\mathbb{U}_{\underline{v}}/\mathbb{U}^{\pm}_{\underline{v}})$*-orbits of cusps of* $\mathbb{U}_{\underline{v}}$*) induces isomorphisms between the pairs (resp. between the labelled topological monoids)*

$$G_{\underline{v}}(\Pi_{\underline{v}})_t \curvearrowright \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_t \quad (\textit{resp.} \quad \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_t \quad)$$

of the labelled ind-topological monoid equipped with the action of the labelled topological group for distinct $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}}) := \mathrm{LabCusp}^{\pm}(\mathcal{B}(\Pi_{\underline{v}})^0)$ *(resp.* $t \in$

$\mathrm{LabCusp}^{\pm}(\mathbb{U}_{\underline{v}})$*) (cf. Definition 10.27 (1) (resp. Definition 10.27 (2)) for the definition of* $\mathrm{LabCusp}^{\pm}(-)$*). We shall refer to these isomorphisms as* $\mathbb{F}_l^{\rtimes\pm}$**-symmetrising isomorphisms***. These symmetrising isomorphisms determine diagonal submonoids*

$$\Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_{\langle|\mathbb{F}_l|\rangle} \subset \prod_{|t|\in|\mathbb{F}_l|} \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_{|t|}, \quad \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_{\langle\mathbb{F}_l^{*}\rangle} \subset \prod_{|t|\in\mathbb{F}_l^{*}} \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_{|t|},$$

which are compatible with the respective labelled $G_{\underline{v}}(\Pi_{\underline{v}})$*-actions*

$$(\textit{resp.}\quad \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_{\langle|\mathbb{F}_l|\rangle} \subset \prod_{|t|\in|\mathbb{F}_l|} \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_{|t|}, \quad \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_{\langle\mathbb{F}_l^{*}\rangle} \subset \prod_{|t|\in\mathbb{F}_l^{*}} \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_{|t|} \quad),$$

and an isomorphism

$$\Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_0 \overset{\mathrm{diag}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_{\langle\mathbb{F}_l^{*}\rangle} \quad (\textit{resp.}\quad \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_0 \overset{\mathrm{diag}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_{\langle\mathbb{F}_l^{*}\rangle} \quad)$$

of ind-topological monoids, which is compatible with the respective labelled $G_{\underline{v}}(\Pi_{\underline{v}})$*-actions (resp. of topological monoids).*

(4) **(Theta and Gaussian Monoids)** *Write*

$$\Psi_{\mathrm{env}}(\Pi_{\underline{v}}) := \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})^{\times} \times \left\{\mathbb{R}_{\geq 0} \cdot \log^{\Pi_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\Pi_{\underline{v}}}(\underline{\underline{\Theta}})\right\}$$

$$(\textit{resp.}\quad \Psi_{\mathrm{env}}(\mathbb{U}_{\underline{v}}) := \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})^{\times} \times \left\{\mathbb{R}_{\geq 0} \cdot \log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\mathbb{U}_{\underline{v}}}(\underline{\underline{\Theta}})\right\} \quad),$$

where $\log^{\Pi_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\Pi_{\underline{v}}}(\underline{\underline{\Theta}})$ *(resp.* $\log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\mathbb{U}_{\underline{v}}}(\underline{\underline{\Theta}})$*) is just a formal symbol, and*

$$\begin{aligned}\Psi_{\mathrm{gau}}(\Pi_{\underline{v}}) &:= \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})^{\times}_{\langle\mathbb{F}_l^{*}\rangle} \times \left\{\mathbb{R}_{\geq 0} \cdot \left(j^2 \cdot \log^{\Pi_{\underline{v}}}(p_{\underline{v}})\right)_j\right\} \\ &\subset \prod_{j\in\mathbb{F}_l^{*}} \Psi^{\mathrm{ss}}_{\mathrm{cns}}(\Pi_{\underline{v}})_j = \prod_{j\in\mathbb{F}_l^{*}} \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})^{\times}_j \times \mathbb{R}_{\geq 0}(\Pi_{\underline{v}})_j\end{aligned}$$

$$\begin{aligned}(\textit{resp.}\quad \Psi_{\mathrm{gau}}(\mathbb{U}_{\underline{v}}) &:= \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})^{\times}_{\langle\mathbb{F}_l^{*}\rangle} \times \left\{\mathbb{R}_{\geq 0} \cdot \left(j^2 \cdot \log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}})\right)_j\right\} \\ &\subset \prod_{j\in\mathbb{F}_l^{*}} \Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_j = \prod_{j\in\mathbb{F}_l^{*}} \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})^{\times}_j \times \mathbb{R}_{\geq 0}(\mathbb{U}_{\underline{v}})_j \quad)\end{aligned}$$

where $\log^{\Pi_{\underline{v}}}(p_{\underline{v}})$ *(resp.* $\log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}})$*) is just a formal symbol, and* $\mathbb{R}_{\geq 0} \cdot (-)$ *is defined by the* $\mathbb{R}_{\geq 0}$*-module structures of* $\mathbb{R}_{\geq 0}(\Pi_{\underline{v}})_j$*'s (resp.* $\mathbb{R}_{\geq 0}(\mathbb{U}_{\underline{v}})_j$*'s). Note that we need the holomorphic structures for the labels and* $\mathbb{F}_l^{\rtimes\pm}$*-synchronising isomorphisms. In particular, we obtain a functorial group-theoretically algorithm to construct, from the topological group* $\Pi_{\underline{v}}$ *(from the Aut-holomorphic space* $\mathbb{U}_{\underline{v}}$*), the* *theta monoid* $\Psi_{\mathrm{env}}(\Pi_{\underline{v}})$ *(resp.* $\Psi_{\mathrm{env}}(\mathbb{U}_{\underline{v}})$*), the Gaussian monoid* $\Psi_{\mathrm{gau}}(\Pi_{\underline{v}})$ *(resp.* $\Psi_{\mathrm{gau}}(\mathbb{U}_{\underline{v}})$*) equipped with natural* $G_{\underline{v}}(\Pi_{\underline{v}})$*-actions and splittings (resp. equipped with natural splittings), and* **the formal evaluation isomorphism**

$$\Psi_{\mathrm{env}}(\Pi_{\underline{v}}) \overset{\mathrm{eval}}{\overset{\sim}{\to}} \Psi_{\mathrm{gau}}(\Pi_{\underline{v}}) \ : \ \log^{\Pi_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\Pi_{\underline{v}}}(\underline{\underline{\Theta}}) \mapsto (j^2 \cdot \log^{\Pi_{\underline{v}}}(p_{\underline{v}}))_j$$

$$(\mathit{resp.} \quad \Psi_{\mathrm{env}}(\mathbb{U}_{\underline{v}}) \overset{\mathrm{eval}}{\overset{\sim}{\to}} \Psi_{\mathrm{gau}}(\mathbb{U}_{\underline{v}}) \ : \ \log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\mathbb{U}_{\underline{v}}}(\underline{\underline{\Theta}}) \mapsto (j^2 \cdot \log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}}))_j \quad),$$

which restricts to the identity on the respective copies of $\Psi_{\mathrm{cns}}(\Pi_{\underline{v}})^{\times}$ *(resp.* $\Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})^{\times}$*), and is compatible with the respective* $G_{\underline{v}}(\Pi_{\underline{v}})$*-actions and the natural splittings (resp. compatible with the natural splittings).*

Remark 11.18.1. ([IUTchII, Remark 4.1.1 (iii)]) Similarly as in Proposition 11.15 and Remark 11.17.2, the construction of the monoids $\Psi_{\mathrm{cns}}(\Pi_{\underline{v}})$ (resp. $\Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})$) is *uniradial*, and the constructions of the monoids $\Psi^{\mathrm{ss}}_{\mathrm{cns}}(\Pi_{\underline{v}})$, $\Psi_{\mathrm{env}}(\Pi_{\underline{v}})$, and $\Psi_{\mathrm{gau}}(\Pi_{\underline{v}})$ (resp. $\Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})$, $\Psi_{\mathrm{env}}(\mathbb{U}_{\underline{v}})$, and $\Psi_{\mathrm{gau}}(\mathbb{U}_{\underline{v}})$), and the formal evaluation isomorphism $\Psi_{\mathrm{env}}(\Pi_{\underline{v}}) \overset{\mathrm{eval}}{\overset{\sim}{\to}} \Psi_{\mathrm{gau}}(\Pi_{\underline{v}})$ (resp. $\Psi_{\mathrm{env}}(\mathbb{U}_{\underline{v}}) \overset{\mathrm{eval}}{\overset{\sim}{\to}} \Psi_{\mathrm{gau}}(\mathbb{U}_{\underline{v}})$) are *multiradial*. Note that, the latter ones are constructed by using holomorphic structures; however, these can be described via the underlying mono-analytic structures (cf. also the table after Example 11.2).

Proof. Proposition follows from the definitions and described algorithms. □

Proposition 11.19. ($\mathcal{F}$-theoretic Gaussian Monoids at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$ (resp. at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$), [IUTchII, Proposition 4.2, Proposition 4.4]) *For* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$ *(resp.* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*), let* ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}} = {}^{\dagger}\mathcal{C}_{\underline{v}}$ *(resp.* ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}} = ({}^{\dagger}\mathcal{C}_{\underline{v}}, {}^{\dagger}\mathcal{D}_{\underline{v}} = {}^{\dagger}\mathbb{U}_{\underline{v}}, {}^{\dagger}\kappa_{\underline{v}})$*) be a* $p_{\underline{v}}$*-adic Frobenioid (resp. a triple) in a* Θ*-Hodge theatre* ${}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$*. We assume (for simplicity) that the base category of* ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ *is equal to* $\mathcal{B}^{\mathrm{temp}}({}^{\dagger}\Pi_{\underline{v}})^0$*). We write*

$$G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}}) \curvearrowright \Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \quad (\mathit{resp.} \quad \Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} := \mathcal{O}^{\rhd}({}^{\dagger}\mathcal{C}_{\underline{v}}) \quad)$$

for the ind-topological monoid equipped with $G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$*-action (resp. the topological monoid) determined, up to inner automorphism arising from an element of* ${}^{\dagger}\Pi_{\underline{v}}$ *by* ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$*, and*

$${}^{\dagger}G_{\underline{v}} \curvearrowright \Psi_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \quad (\mathit{resp.} \quad \Psi_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} := \mathcal{O}^{\rhd}({}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}) \quad)$$

for the ind-topological monoid equipped with ${}^{\dagger}G_{\underline{v}}$*-action (resp. the topological monoid) determined, up to inner automorphism arising from an element of* ${}^{\dagger}G_{\underline{v}}$ *by the* $\underline{v}$*-component* ${}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$ *of* $\mathcal{F}^{\vdash}$*-prime-strip* $\{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$ *determined by the* Θ*-Hodge theatre* ${}^{\dagger}\mathcal{HT}^{\Theta}$*.*

(1) **(Constant Monoids)** *By Remark 3.19.2 (resp. by the Kummer structure* ${}^{\dagger}\kappa_{\underline{v}}$*), we have a unique* **Kummer isomorphism**

$$\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \overset{\text{Kum}}{\xrightarrow{\sim}} \Psi_{\text{cns}}({}^{\dagger}\Pi_{\underline{v}}) \quad (\textit{resp.} \quad \Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \overset{\text{Kum}}{\xrightarrow{\sim}} \Psi_{\text{cns}}({}^{\dagger}\mathbb{U}_{\underline{v}}) \quad)$$

of ind-topological monoids with $G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$*-action (resp. of topological monoids).*

(2) **(Mono-analytic Semi-simplifications)** *We have a unique* $\widehat{\mathbb{Z}}^{\times}$**-orbit** *(resp. a unique* $\{\pm 1\}$**-orbit***)*

$$\Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \overset{\substack{\text{“Kum”} \\ \widehat{\mathbb{Z}}^{\times}\text{-orbit, poly}}}{\xrightarrow{\sim}} \Psi_{\text{cns}}({}^{\dagger}G_{\underline{v}})^{\times} \quad (\textit{resp.} \quad \Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \overset{\substack{\text{“Kum”} \\ \{\pm 1\}\text{-orbit, poly}}}{\xrightarrow{\sim}} \Psi_{\text{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})^{\times} \quad)$$

of isomorphisms of ind-topological groups with ${}^{\dagger}G_{\underline{v}}$*-action (resp. of topological groups), and a unique isomorphism*

$$\Psi^{\mathbb{R}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} := (\Psi_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}/\Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}})^{\mathbb{R}} \overset{\text{“Kum”}}{\xrightarrow{\sim}} \Psi^{\mathbb{R}}_{\text{cns}}({}^{\dagger}G_{\underline{v}}) \ (\textit{resp.}\ \Psi^{\mathbb{R}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} := (\Psi_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}/\Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}})^{\mathbb{R}} \overset{\text{“Kum”}}{\xrightarrow{\sim}} \Psi^{\mathbb{R}}_{\text{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \ )$$

of monoids, which sends the distinguished element of $\Psi^{\mathbb{R}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$ *determined by the unique generator (resp. by* $p_{\underline{v}} = e = 2.71828\cdots$*, i.e., the element of the complex Archimedean field which gives rise to* $\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}}$ *whose natural logarithm is equal to* 1*) of* $\Psi_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}/\Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$ *to the distinguished element of* $\Psi^{\mathbb{R}}_{\text{cns}}({}^{\dagger}G_{\underline{v}})$ *(resp.* $\Psi^{\mathbb{R}}_{\text{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$*) determined by* $\log^{G_{\underline{v}}}(p_{\underline{v}}) \in \mathbb{R}_{\geq 0}({}^{\dagger}G_{\underline{v}})$ *(resp.* $\log^{\mathcal{D}^{\vdash}_{\underline{v}}}(p_{\underline{v}}) \in \mathbb{R}_{\geq 0}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$*). In particular, we have a natural poly-isomorphism*

$$\Psi^{\text{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} := \Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \times \Psi^{\mathbb{R}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \overset{\substack{\text{“Kum”} \\ \text{poly}}}{\xrightarrow{\sim}} \Psi^{\text{ss}}_{\text{cns}}({}^{\dagger}G_{\underline{v}}) \ (\textit{resp.}\ \Psi^{\text{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} := \Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \times \Psi^{\mathbb{R}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \overset{\substack{\text{“Kum”} \\ \text{poly}}}{\xrightarrow{\sim}} \Psi^{\text{ss}}_{\text{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \ )$$

of ind-topological monoids (resp. topological monoids) which is compatible with the natural splittings (We can regard these poly-isomorphisms as analogues of Kummer isomorphism). We write $\Psi^{\text{ss}}_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} := \Psi^{\text{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$ *(resp.* $\Psi^{\text{ss}}_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} := \Psi^{\text{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$*), hence we have a tautological isomorphism*

$$\Psi^{\text{ss}}_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \overset{\text{tauto}}{\xrightarrow{\sim}} \Psi^{\text{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \quad (\textit{resp.} \quad \Psi^{\text{ss}}_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \overset{\text{tauto}}{\xrightarrow{\sim}} \Psi^{\text{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \quad).$$

(3) **(Conjugate Synchronisation)** *The Kummer isomorphism in (1) determines a collection of compatible* **Kummer isomorphisms**

$$(\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_t \overset{\text{Kum}}{\xrightarrow{\sim}} \Psi_{\text{cns}}({}^{\dagger}\Pi_{\underline{v}})_t \quad (\textit{resp.} \quad (\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_t \overset{\text{Kum}}{\xrightarrow{\sim}} \Psi_{\text{cns}}({}^{\dagger}\mathbb{U}_{\underline{v}})_t \quad),$$

which are well-defined up to an inner automorphism of ${}^\dagger\Pi_{\underline{v}}$ *(which is independent of* $t \in \mathrm{LabCusp}^\pm({}^\dagger\Pi_{\underline{v}})$*) for* $t \in \mathrm{LabCusp}^\pm({}^\dagger\Pi_{\underline{v}})$ *(resp.* $t \in \mathrm{LabCusp}^\pm({}^\dagger\mathbb{U}_{\underline{v}})$*), and* $\mathbb{F}_l^{\rtimes\pm}$*-symmetrising isomorphisms between distinct* $t \in \mathrm{LabCusp}^\pm({}^\dagger\Pi_{\underline{v}})$ *(resp.* $t \in \mathrm{LabCusp}^\pm({}^\dagger\mathbb{U}_{\underline{v}})$*) induced by the* ${}^\dagger\Delta_{\underline{v}}^\pm$*-outer action of* $\mathbb{F}_l^{\rtimes\pm} \cong {}^\dagger\Delta_{\underline{v}}^{\mathrm{cor}}/{}^\dagger\Delta_{\underline{v}}^\pm$ *on* ${}^\dagger\Pi_{\underline{v}}^\pm$ *(resp. the action of* $\mathbb{F}_l^{\rtimes\pm} \cong \mathrm{Gal}({}^\dagger\mathbb{U}_{\underline{v}}^\pm/{}^\dagger\mathbb{U}_{\underline{v}}^{\mathrm{cor}})$ *on the various* $\mathrm{Gal}({}^\dagger\mathbb{U}_{\underline{v}}/{}^\dagger\mathbb{U}_{\underline{v}}^\pm)$*-orbits of cusps of* ${}^\dagger\mathbb{U}_{\underline{v}}$*). These symmetrising isomorphisms determine an isomorphism*

$$(\Psi_{{}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}})_0 \overset{\mathrm{diag}}{\xrightarrow{\sim}} (\Psi_{{}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}})_{\langle\mathbb{F}_l^*\rangle} \quad (\textit{resp.} \quad (\Psi_{{}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}})_0 \overset{\mathrm{diag}}{\xrightarrow{\sim}} (\Psi_{{}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}})_{\langle\mathbb{F}_l^*\rangle} \quad)$$

of ind-topological monoids (resp. topological monoids), which are compatible with the respective labelled $G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$*-actions.*

(4) **(Theta and Gaussian Monoids)** *We write*

$$\Psi_{{}^\dagger\mathcal{F}_{\underline{v}}^\Theta}, \quad \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}) \quad (\textit{resp.} \quad \Psi_{{}^\dagger\mathcal{F}_{\underline{v}}^\Theta}, \quad \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}) \quad)$$

for the monoids with $G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$*-actions and natural splittings, determined by* $\Psi_{\mathrm{env}}({}^\dagger\Pi_{\underline{v}})$*,* $\Psi_{\mathrm{gau}}({}^\dagger\Pi_{\underline{v}})$ *in Proposition 11.18 (4) respectviely, via the isomorphisms in (1), (2), and (3). Then the formal evaluation isomorphism of Proposition 11.18 (4) gives us a collection of* **evaluation isomorphisms**

$$\Psi_{{}^\dagger\mathcal{F}_{\underline{v}}^\Theta} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \Psi_{\mathrm{env}}({}^\dagger\Pi_{\underline{v}}) \overset{\mathrm{eval}}{\xrightarrow{\sim}} \Psi_{\mathrm{gau}}({}^\dagger\Pi_{\underline{v}}) \overset{\mathrm{Kum}^{-1}}{\xrightarrow{\sim}} \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

$$(\textit{resp.} \quad \Psi_{{}^\dagger\mathcal{F}_{\underline{v}}^\Theta} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \Psi_{\mathrm{env}}({}^\dagger\Pi_{\underline{v}}) \overset{\mathrm{eval}}{\xrightarrow{\sim}} \Psi_{\mathrm{gau}}({}^\dagger\Pi_{\underline{v}}) \overset{\mathrm{Kum}^{-1}}{\xrightarrow{\sim}} \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}) \quad),$$

which restrict to the identity or the isomorphism of (1) or the inverse of the isomorphism of (1) on the various copies of $\Psi_{{}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}}^\times$*,* $\Psi_{\mathrm{cns}}({}^\dagger\Pi_{\underline{v}})^\times$*, and are compatible with the various natural actions of* $G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$ *and natural splittings.*

§ 11.4. Hodge-Arakelov-theoretic Evaluation and Gaussian Monoids in the Global Case.

In this subsection, we globalise the constructions in Section 11.2 ($\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$) and in Section 11.3 ($\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$) via global realified Frobenioids (cf. also Remark 10.9.1). We can globalise the local $\mathbb{F}_l^{\rtimes\pm}$-symmetries to a global $\mathbb{F}_l^{\rtimes\pm}$-symmetry, thanks to **the global $\{\pm1\}$-synchronisation** in Proposition 10.33 (cf. also Proposition 10.34 (3)). This is a ⊞-portion of constructions in ⊠⊞-Hodge theatres. In the final multiradial algorithm, we use this ⊞-portion to construct Θ-pilot object (cf. Proposition 13.7 and Definition 13.9 (1)), which gives us a ⊞-line bundle (cf. Definition 9.7) (of negative large degree) through an action on mono-analytic log-shells (cf. Corollary 13.13).

Next, we also perform NF-counterpart (cf. Section 9) of Hodge-Arakelov-theoretic evaluation. This is a $\boxtimes$-portion of constructions in $\boxtimes\boxplus$-Hodge theatres. In the final multiradial algorithm, we use this $\boxtimes$-portion to construct actions of copies of "$F^{\times}_{\mathrm{mod}}$" on mono-analytic log-shells (cf. Proposition 13.11 (2)), through which we convert $\boxtimes$-line bundles into $\boxplus$-line bundles (cf. the category equivalence (Convert) just after Definition 9.7) and vice versa (cf. Corollary 13.13).

Corollary 11.20. (Π-theoretic Monoids associated to $\mathcal{D}$-$\boxplus$-Hodge Theatres, [IUTchII, Corollary 4.5]) *Let*

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus} = ({}^{\dagger}\mathfrak{D}_{\succ} \overset{{}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}}{\longleftarrow} {}^{\dagger}\mathfrak{D}_{T} \overset{{}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}}{\longrightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

be a $\mathcal{D}$-$\boxplus$-Hodge theatre, and

$$^{\ddagger}\mathfrak{D} = \{{}^{\ddagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

a $\mathcal{D}$-prime-strip. We assume, for simplicity, that ${}^{\ddagger}\mathcal{D}_{\underline{v}} = \mathcal{B}^{\mathrm{temp}}({}^{\ddagger}\Pi_{\underline{v}})^0$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. We write ${}^{\ddagger}\mathfrak{D}^{\vdash} = \{{}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ for the associated $\mathcal{D}^{\vdash}$-prime-strip to ${}^{\ddagger}\mathfrak{D}$, and ssume that ${}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}} = \mathcal{B}^{\mathrm{temp}}({}^{\ddagger}G_{\underline{v}})^0$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$.

(1) **(Constant Monoids)** *By Definition 11.12 (2) for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ and Proposition 11.18 (1) for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, we obtain a functorial algorithm, with respect to the $\mathcal{D}$-prime-strip ${}^{\ddagger}\mathfrak{D}$, to construct the assignment*

$$\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}) : \underline{\mathbb{V}} \ni \underline{v} \mapsto \Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})_{\underline{v}} := \begin{cases} \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\ddagger}\Pi_{\underline{v}})) \curvearrowright \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}({}^{\ddagger}\Pi_{\underline{v}}))\} & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}, \\ \{G_{\underline{v}}({}^{\ddagger}\Pi_{\underline{v}}) \curvearrowright \Psi_{\mathrm{cns}}({}^{\ddagger}\Pi_{\underline{v}})\} & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}, \\ \Psi_{\mathrm{cns}}({}^{\ddagger}\mathcal{D}_{\underline{v}}) & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}, \end{cases}$$

where $\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})_{\underline{v}}$ is well-defined only up to a ${}^{\ddagger}\Pi_{\underline{v}}$-conjugacy indeterminacy for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$.

(2) **(Mono-analytic Semi-simplifications)** *By Proposition 11.18 (2) for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ and the same group-theoretic algorithm for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ (Here, we write $\Psi_{\mathrm{cns}}(\Pi_{\underline{v}}) := \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$), we obtain a functorial algorithm, with respect to the $\mathcal{D}^{\vdash}$-prime-strip ${}^{\ddagger}\mathfrak{D}^{\vdash}$, to construct the assignment*

$$\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash}) \ : \ \underline{\mathbb{V}} \ni \underline{v} \mapsto \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})_{\underline{v}} := \begin{cases} \{{}^{\ddagger}G_{\underline{v}} \curvearrowright \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}G_{\underline{v}})\} & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}, \\ \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}) & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}, \end{cases}$$

where $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})_{\underline{v}}$ is well-defined only up to a ${}^{\ddagger}G_{\underline{v}}$-conjugacy indeterminacy for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. Each $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})_{\underline{v}}$ is equipped with a splitting

$$\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})_{\underline{v}} = \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})^{\times}_{\underline{v}} \times \mathbb{R}_{\geq 0}({}^{\ddagger}\mathfrak{D}^{\vdash})_{\underline{v}}$$

and each $\mathbb{R}_{\geq 0}({}^{\ddagger}\mathfrak{D}^{\vdash})_{\underline{v}}$ *is equipped with a distinguished element*

$$\log^{{}^{\ddagger}\mathfrak{D}^{\vdash}}(p_{\underline{v}}) \in \mathbb{R}_{\geq 0}({}^{\ddagger}\mathfrak{D}^{\vdash})_{\underline{v}}.$$

If we regard ${}^{\ddagger}\mathfrak{D}^{\vdash}$ *as constructed from* ${}^{\ddagger}\mathfrak{D}$*, then we have a functorial algorithm, with respect to the* $\mathcal{D}$*-prime-strip* ${}^{\ddagger}\mathfrak{D}$*, to construct isomorphisms*

$$\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})^{\times}_{\underline{v}} \overset{\sim}{\to} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})^{\times}_{\underline{v}}$$

for each $\underline{v} \in \underline{\mathbb{V}}$*, which are compatible with* $G_{\underline{v}}({}^{\ddagger}\Pi_{\underline{v}}) \overset{\sim}{\to} {}^{\ddagger}G_{\underline{v}}$*-actions for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*. By Definition 10.6 ("*$\mathcal{D}$*-version"), we also obtain a functorial algorithm, with respect to* $\mathcal{D}^{\vdash}$*-prime-strip* ${}^{\ddagger}\mathfrak{D}^{\vdash}$*, to construct a (pre-)Frobenioid*

$$\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$$

isomorphism to the model object $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ *in Definition 10.4, equipped with a bijection*

$$\mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})) \overset{\sim}{\to} \underline{\mathbb{V}},$$

and localisation isomorphisms

$${}^{\ddagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}} : \Phi_{\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash}),\underline{v}} \overset{\text{gl. to loc.}}{\underset{\sim}{\longrightarrow}} \mathbb{R}_{\geq 0}({}^{\ddagger}\mathfrak{D}^{\vdash})_{\underline{v}}$$

of topological monoids.

(3) **(Conjugate Synchronisation)** *We put*

$${}^{\dagger}\zeta_{\succ} := {}^{\dagger}\zeta_{\pm} \circ {}^{\dagger}\zeta_0^{\Theta^{\mathrm{ell}}} \circ (\zeta_0^{\Theta^{\pm}})^{-1} : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ}) \overset{\sim}{\to} T$$

(cf. Proposition 10.33). The various local $\mathbb{F}_l^{\rtimes\pm}$*-actions in Corollary 11.16 (1) and Proposition 11.18 (3) induce isomorphisms between the labelled data*

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_t$$

for distinct $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$*. We shall refer to these isomorphisms as* $\boldsymbol{\mathbb{F}_l^{\rtimes\pm}}$**-symmetrising isomorphisms** *(Note that the* **global** $\boldsymbol{\{\pm 1\}}$**-synchronisation** *established by Proposition 10.33 is crucial here). These* $\mathbb{F}_l^{\rtimes\pm}$*-symmetrising isomorphisms are compatible with the (doubly transitive)* $\mathbb{F}_l^{\rtimes\pm}$*-action on the index set* T *of the* $\mathcal{D}$*-*Θ^{ell}*-bridge* ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$ *with respect to* ${}^{\dagger}\zeta$*, hence determine diagonal submonoids*

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{\langle|\mathbb{F}_l|\rangle} \subset \prod_{|t|\in|\mathbb{F}_l|} \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{|t|},\ \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{\langle\mathbb{F}_l^{*}\rangle} \subset \prod_{|t|\in\mathbb{F}_l^{*}} \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{|t|},$$

and an isomorphism

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_0 \overset{\mathrm{diag}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{\langle\mathbb{F}_l^{*}\rangle}$$

consisting of the local isomorphisms in Corollary 11.16 (3) and Proposition 11.18 (3).

(4) **(Local Theta and Gaussian Monoids)** *By Corollary 11.16 (2), (3) and Proposition 11.18 (4), we obtain a functorial algorithm, with respect to the $\mathcal{D}$-prime-strip ${}^{\dagger}\mathfrak{D}_{\succ}$, to construct the assignments*

$$
{}_{(\infty)}\Psi_{\text{env}}({}^{\dagger}\mathfrak{D}_{\succ}) : \underline{\mathbb{V}} \ni \underline{v} \mapsto
$$

$$
{}_{(\infty)}\Psi_{\text{env}}({}^{\dagger}\mathfrak{D}_{\succ})_{\underline{v}} := \begin{cases} \{G_{\underline{v}}(\mathbb{M}_*^{\Theta}({}^{\dagger}\Pi_{\underline{v}}))\}_{j\in\mathbb{F}_l^{*}} \curvearrowright {}_{(\infty)}\Psi_{\text{env}}(\mathbb{M}_*^{\Theta}({}^{\dagger}\Pi_{\underline{v}})) & \underline{v} \in \underline{\mathbb{V}}^{\text{bad}} \cap \underline{\mathbb{V}}^{\text{non}}, \\ \{G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})\}_{j\in\mathbb{F}_l^{*}} \curvearrowright {}_{(\infty)}\Psi_{\text{env}}({}^{\dagger}\Pi_{\underline{v}}) & \underline{v} \in \underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}, \\ {}_{(\infty)}\Psi_{\text{env}}({}^{\ddagger}\mathbb{U}_{\underline{v}}) & \underline{v} \in \underline{\mathbb{V}}^{\text{arc}}, \end{cases}
$$

and

$$
{}_{(\infty)}\Psi_{\text{gau}}({}^{\dagger}\mathfrak{D}_{\succ}) : \underline{\mathbb{V}} \ni \underline{v} \mapsto
$$

$$
{}_{(\infty)}\Psi_{\text{gau}}({}^{\dagger}\mathfrak{D}_{\succ})_{\underline{v}} := \begin{cases} \{G_{\underline{v}}(\mathbb{M}_*^{\Theta}({}^{\dagger}\Pi_{\underline{v}}))\}_{j\in\mathbb{F}_l^{*}} \curvearrowright {}_{(\infty)}\Psi_{\text{gau}}(\mathbb{M}_*^{\Theta}({}^{\dagger}\Pi_{\underline{v}})) & \underline{v} \in \underline{\mathbb{V}}^{\text{bad}} \cap \underline{\mathbb{V}}^{\text{non}}, \\ \{G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})\}_{j\in\mathbb{F}_l^{*}} \curvearrowright {}_{(\infty)}\Psi_{\text{gau}}({}^{\dagger}\Pi_{\underline{v}}) & \underline{v} \in \underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}, \\ {}_{(\infty)}\Psi_{\text{gau}}({}^{\ddagger}\mathbb{U}_{\underline{v}}) & \underline{v} \in \underline{\mathbb{V}}^{\text{arc}}, \end{cases}
$$

where we write ${}_{\infty}\Psi_{\text{env}}({}^{\dagger}\Pi_{\underline{v}}) := \Psi_{\text{env}}({}^{\dagger}\Pi_{\underline{v}})$ (resp. ${}_{\infty}\Psi_{\text{env}}({}^{\dagger}\mathbb{U}_{\underline{v}}) := \Psi_{\text{env}}({}^{\dagger}\mathbb{U}_{\underline{v}})$) and ${}_{\infty}\Psi_{\text{gau}}({}^{\dagger}\Pi_{\underline{v}}) := \Psi_{\text{gau}}({}^{\dagger}\Pi_{\underline{v}})$ (resp. ${}_{\infty}\Psi_{\text{gau}}({}^{\dagger}\mathbb{U}_{\underline{v}}) := \Psi_{\text{gau}}({}^{\dagger}\mathbb{U}_{\underline{v}})$) for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \cap \underline{\mathbb{V}}^{\text{non}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$) and ${}_{(\infty)}\Psi_{\text{env}}({}^{\dagger}\mathfrak{D}_{\succ})_{\underline{v}}$'s, ${}_{(\infty)}\Psi_{\text{gau}}({}^{\dagger}\mathfrak{D}_{\succ})_{\underline{v}}$'s are equipped with natural splittings, and compatible **evaluation isomorphisms**

$$
{}_{(\infty)}\Psi_{\text{env}}({}^{\dagger}\mathfrak{D}_{\succ}) \overset{\text{eval}}{\overset{\sim}{\to}} {}_{(\infty)}\Psi_{\text{gau}}({}^{\dagger}\mathfrak{D}_{\succ})
$$

constructed by Corollary 11.16 (2) and Proposition 11.18 (4).

(5) **(Global Realified Theta and Gaussian Monoids)** *We have a functorial algorithm, with respect to the $\mathcal{D}^{\vdash}$-prime-strip ${}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}$, to construct a (pre-)Frobenioid*

$$
\mathcal{D}^{\Vdash}_{\text{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})
$$

as a coply of the Frobenioid $\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$ of (2) above, multiplied a formal symbol $\log^{{}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}}(\underline{\underline{\Theta}})$, equipped with a bijection

$$
\text{Prime}(\mathcal{D}^{\Vdash}_{\text{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})) \overset{\sim}{\to} \underline{\mathbb{V}},
$$

and localisation isomorphisms

$$
\Phi_{\mathcal{D}^{\Vdash}_{\text{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}),\underline{v}} \overset{\text{gl. to loc.}}{\overset{\sim}{\longrightarrow}} \Psi_{\text{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})^{\mathbb{R}}_{\underline{v}}
$$

of topological monoids. We have a functorial algorithm, with respect to the $\mathcal{D}^{\vdash}$-prime-strip ${}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}$ to construct a (pre-)Frobenioid

$$
\mathcal{D}^{\Vdash}_{\text{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}) \subset \prod_{j\in\mathbb{F}_l^{*}} \mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})_j
$$

whose divisor and rational function monoids are determined by the weighted diagonal $(\ j^2\ )_{j\in\mathbb{F}_l^*}$*, equipped with a bijection*

$$\mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})) \stackrel{\sim}{\to} \underline{\mathbb{V}},$$

and localisation isomorphisms

$$\Phi_{\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}),\underline{v}} \stackrel{\text{gl. to loc.}}{\stackrel{\sim}{\longrightarrow}} \Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})^{\mathbb{R}}_{\underline{v}}$$

of topological monoids for each $\underline{v}\in\underline{\mathbb{V}}$*. We also have a functorial algorithm, with respect to the* $\mathcal{D}^{\vdash}$*-prime-strip* ${}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}$ *to construct a* **global formal evaluation isomorphism**

$$\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}) \stackrel{\text{eval}}{\stackrel{\sim}{\to}} \mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$$

of (pre-)Frobenioids, which is compatible with local evaluation isomorphisms of (4), with respect to the localisation isomorphisms for each $\underline{v}\in\underline{\mathbb{V}}$ *and the bijections* $\mathrm{Prime}(-)\stackrel{\sim}{\to}\underline{\mathbb{V}}$.

Proof. Corollary follows from the definitions. □

Corollary 11.21. ($\mathcal{F}$-theoretic Monoids associated to $\boxplus$-Hodge Theatres, [IUTchII, Corollary 4.6]) *Let*

$${}^{\dagger}\mathcal{HT}^{\boxplus} = \left({}^{\dagger}\mathfrak{F}_{\succ} \stackrel{{}^{\dagger}\psi^{\Theta^{\pm}}_{\pm}}{\longleftarrow} {}^{\dagger}\mathfrak{F}_{T} \stackrel{{}^{\dagger}\psi^{\Theta^{\mathrm{ell}}}_{\pm}}{\longrightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}\right)$$

be a $\boxplus$*-Hodge theatre, and*

$${}^{\ddagger}\mathfrak{F} = \{{}^{\ddagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

an $\mathcal{F}$*-prime-strip. We assume, for simplicity, that the* $\mathcal{D}$*-*$\boxplus$*-Hodge theatre associated to* ${}^{\dagger}\mathcal{HT}^{\boxplus}$ *is equal to* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$ *in Corollary 11.20, and that the* $\mathcal{D}$*-prime-strip associated to* ${}^{\ddagger}\mathfrak{F}$ *is equal to* ${}^{\ddagger}\mathfrak{D}$ *in Corollary 11.20. We write* ${}^{\ddagger}\mathfrak{F}^{\vdash}=\{{}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ *for the associated* $\mathcal{F}^{\vdash}$*-prime-strip to* ${}^{\ddagger}\mathfrak{F}$.

(1) **(Constant Monoids)** *By Proposition 11.19 (1) for* $\underline{\mathbb{V}}^{\mathrm{good}}$*, and the same group-theoretic algorithm for* $\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}$*, we have a functorial algorithm, with respect to the* $\mathcal{F}$*-prime-strip* ${}^{\ddagger}\mathfrak{F}$*, to construct the assignment*

$$\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F})\ :\ \underline{\mathbb{V}}\ni\underline{v}\mapsto\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F})_{\underline{v}} := \begin{cases} \left\{G_{\underline{v}}({}^{\ddagger}\Pi_{\underline{v}}) \curvearrowright \Psi_{{}^{\ddagger}\mathcal{F}_{\underline{v}}}\right\} & \underline{v}\in\underline{\mathbb{V}}^{\mathrm{non}},\\ \Psi_{{}^{\ddagger}\mathcal{F}_{\underline{v}}} & \underline{v}\in\underline{\mathbb{V}}^{\mathrm{arc}},\end{cases}$$

where $\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F})_{\underline{v}}$ *is well-defined only up to a* ${}^{\ddagger}\Pi_{\underline{v}}$*-conjugacy indeterminacy for* $\underline{v}\in\underline{\mathbb{V}}^{\mathrm{non}}$*. By Proposition 11.14 (2) for* $\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}$ *(where we take "*${}^{\dagger}\mathcal{C}_{\underline{v}}$*" to be* ${}^{\ddagger}\mathcal{F}_{\underline{v}}$*) and*

Proposition 11.19 (1) for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$, we obtain a collection of **Kummer isomorphism**

$$\Psi_{\text{cns}}({}^{\ddagger}\mathfrak{F}) \overset{\text{Kum}}{\xrightarrow{\sim}} \Psi_{\text{cns}}({}^{\ddagger}\mathfrak{D}).$$

(2) **(Mono-analytic Semi-simplifications)** *By Proposition 11.19 (2) for $\underline{\mathbb{V}}^{\text{good}}$, and the same group-theoretic algorithm for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, we have a functorial algorithm, with respect to the $\mathcal{F}^{\vdash}$-prime-strip ${}^{\ddagger}\mathfrak{F}^{\vdash}$, to construct the assignment*

$$\Psi^{\text{ss}}_{\text{cns}}({}^{\ddagger}\mathfrak{F}^{\vdash}) \ : \ \underline{\mathbb{V}} \ni \underline{v} \mapsto \Psi^{\text{ss}}_{\text{cns}}({}^{\ddagger}\mathfrak{F}^{\vdash})_{\underline{v}} := \Psi^{\text{ss}}_{{}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}}$$

where $\Psi^{\text{ss}}_{\text{cns}}({}^{\ddagger}\mathfrak{F}^{\vdash})_{\underline{v}}$ is well-defined only up to a ${}^{\ddagger}G_{\underline{v}}$-conjugacy indeterminacy for $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$. Each $\Psi^{\text{ss}}_{\text{cns}}({}^{\ddagger}\mathfrak{F}^{\vdash})_{\underline{v}}$ is equipped with its natural splitting, and for $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, with a distinguished element (Note that the distinguished element in $\Psi^{\text{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$ is not preserved by automorphism of ${}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$. cf. also the first table in Section 4.3 cf. [IUTchII, Remark 4.6.1]). By Proposition 11.19 (2) for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$ and the same group-theoretic algorithm for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, we have a functorial algorithm, with respect to $\mathcal{F}^{\vdash}$-prime-strip ${}^{\ddagger}\mathfrak{F}^{\vdash}$, to construct the collection of poly-isomorphisms (analogues of Kummer isomorphism)

$$\Psi^{\text{ss}}_{\text{cns}}({}^{\ddagger}\mathfrak{F}^{\vdash}) \overset{\text{“Kum”}}{\underset{\text{poly}}{\xrightarrow{\sim}}} \Psi^{\text{ss}}_{\text{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash}).$$

Let

$${}^{\ddagger}\mathfrak{F}^{\Vdash} = ({}^{\ddagger}\mathcal{C}^{\Vdash}, \text{Prime}({}^{\ddagger}\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^{\ddagger}\mathfrak{F}^{\vdash}, \{{}^{\ddagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

be the $\mathcal{F}^{\Vdash}$-prime-strip associated to ${}^{\ddagger}\mathfrak{F}$. We also have a functorial algorithm, with respect to $\mathcal{F}^{\Vdash}$-prime-strip ${}^{\ddagger}\mathfrak{F}^{\Vdash}$, to construct an isomorphism

$${}^{\ddagger}\mathcal{C}^{\Vdash} \overset{\text{“Kum”}}{\xrightarrow{\sim}} \mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$$

(We can regard this isomorphism as an analogue of Kummer isomorphism), where $\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$ is constructed in Corollary 11.20 (2), which is uniquely determined by the condition that it is compatible with the respective bijections $\text{Prime}(-) \xrightarrow{\sim} \underline{\mathbb{V}}$ *and the localisation isomorphisms of topological monoids for each $\underline{v} \in \underline{\mathbb{V}}$, with respect to the above collection of poly-isomorphisms $\Psi^{\text{ss}}_{\text{cns}}({}^{\ddagger}\mathfrak{F}^{\vdash}) \overset{\text{“Kum”}}{\underset{\text{poly}}{\xrightarrow{\sim}}} \Psi^{\text{ss}}_{\text{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})$ (Note that, if we reconstruct both $\Psi^{\text{ss}}_{\text{cns}}({}^{\ddagger}\mathfrak{F}^{\vdash}) \overset{\text{“Kum”}}{\underset{\text{poly}}{\xrightarrow{\sim}}} \Psi^{\text{ss}}_{\text{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})$ and ${}^{\ddagger}\mathcal{C}^{\Vdash} \overset{\text{“Kum”}}{\xrightarrow{\sim}} \mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$ in a compatible manner, then the distinguished elements in $\Psi^{\text{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$ at $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$ can be computed from the distinguished elements at $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$ and the structure (e.g.. using rational function monoids) of the global realified Frobenioids ${}^{\ddagger}\mathcal{C}^{\Vdash}$, $\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$. cf. [IUTchII, Remark 4.6.1]).*

(3) **(Conjugate Synchronisation)** *For each* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$*, the collection of isomorphisms in (1) determine a collection of compatible* **Kummer isomorphisms**

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_t \overset{\mathrm{Kum}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_t,$$

where $\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_t$ *is the labelled data constructed in Corollary 11.20 (3), and the* ${}^{\dagger}\Pi_{\underline{v}}$*-conjugacy indeterminacy at each* $\underline{v} \in \underline{\mathbb{V}}$ *is independent of* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$*, and* $\mathbb{F}_l^{\rtimes\pm}$*-symmerising isomorphisms induced by the various local* $\mathbb{F}_l^{\rtimes\pm}$*-actions in Corollary 11.17 (1) and Proposition 11.19 (3) between the data labelled by distinct* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$*. These* $\mathbb{F}_l^{\rtimes\pm}$*-symmetrising isomorphisms are compatible with the (doubly transitive)* $\mathbb{F}_l^{\rtimes\pm}$*-action on the index set* T *of the* $\mathcal{D}$*-*Θ^{ell}*-bridge* ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$ *with respect to* ${}^{\dagger}\zeta$ *in Corollary 11.20 (3), hence determine (diagonal submonoids and) an isomorphism*

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_0 \overset{\mathrm{diag}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_{\langle\mathbb{F}_l^{*}\rangle}$$

consisting of the local isomorphisms in Corollary 11.17 (3) and Proposition 11.19 (3).

(4) **(Local Theta and Gaussian Monoids)** *Let*

$${}^{\dagger}\mathfrak{F}_J \xrightarrow{{}^{\dagger}\psi_{*}^{\Theta}} {}^{\dagger}\mathfrak{D}_{>} \dashrightarrow {}^{\dagger}\mathcal{HT}^{\Theta}$$

be a Θ*-bridge which is glued to the* $\Theta^{\pm}$*-bridge associate to the* $\boxplus$*-Hodge theatre* ${}^{\dagger}\mathcal{HT}^{\boxplus}$ *via the algorithm in Lemma 10.38 (Hence* $J = T^{*}$*). By Corollary 11.17 (2), (3) and Proposition 11.19 (4), we have a functorial algorithm, with respect to the above* Θ*-bridge with its gluing to the* $\Theta^{\pm}$*-bridge associated to* ${}^{\dagger}\mathcal{HT}^{\boxplus}$*, to construct assignments*

$$
\begin{aligned}
&{}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta}) : \underline{\mathbb{V}} \ni \underline{v} \mapsto \\
&{}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} := \begin{cases} \{G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})\}_{j\in\mathbb{F}_l^{*}} \curvearrowright {}_{(\infty)}\Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}^{\Theta}} & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}, \\ {}_{(\infty)}\Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}^{\Theta}} & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}, \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&{}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta}) : \underline{\mathbb{V}} \ni \underline{v} \mapsto \\
&{}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} := \begin{cases} \{G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})\}_{j\in\mathbb{F}_l^{*}} \curvearrowright {}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}} \\ {}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}} \end{cases}
\end{aligned}
$$

(Here the notation $(-)({}^{\dagger}\mathcal{HT}^{\Theta})$ *is slightly abuse of notation), where we write* ${}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} := \Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$*, and* ${}_{\infty}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} := \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$ *for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$*, and*

${}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$*'s,* ${}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$*'s are equipped with natural splittings, and compatible* **evaluation isomorphisms**

$${}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta}) \overset{\mathrm{Kum}}{\stackrel{\sim}{\to}} {}_{(\infty)}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{\succ}) \overset{\mathrm{eval}}{\stackrel{\sim}{\to}} {}_{(\infty)}\Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}_{\succ}) \overset{\mathrm{Kum}^{-1}}{\stackrel{\sim}{\to}} {}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})$$

constructed by Corollary 11.17 (2) and Proposition 11.19 (4).

(5) **(Global Realified Theta and Gaussian Monoids)** *By Proposition 11.19 (4) for labelled and non-labelled versions of the isomorphism* ${}^{\ddagger}\mathcal{C}^{\Vdash} \overset{\text{“Kum”}}{\stackrel{\sim}{\to}} \mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$ *of (2) to the global realified Frobenioids* $\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$*,* $\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$ *constructed in Corollary 11.20 (5), we obtain a functorial algorithm, with respect to the above* Θ*-bridge, to construct (pre-)Frobenioids*

$$\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}), \quad \mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})$$

(Here the notation $(-)({}^{\dagger}\mathcal{HT}^{\Theta})$ *is slightly abuse of notation. Note also that the construction of* $\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta})$ *is similar to the one of* $\mathcal{C}^{\Vdash}_{\mathrm{theta}}$ *in Definition 10.5 (4)) with equipped with bijections*

$$\mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta})) \stackrel{\sim}{\to} \underline{\mathbb{V}}, \quad \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})) \stackrel{\sim}{\to} \underline{\mathbb{V}},$$

localisation isomorphisms

$$\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}),\underline{v}} \overset{\text{gl. to loc.}}{\stackrel{\sim}{\longrightarrow}} \Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}}_{\underline{v}}, \quad \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}),\underline{v}} \overset{\text{gl. to loc.}}{\stackrel{\sim}{\longrightarrow}} \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}}_{\underline{v}}$$

of topological monoids for each $\underline{v} \in \underline{\mathbb{V}}$*, and* **evaluation isomrphisms**

$$\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}) \overset{\text{“Kum”}}{\stackrel{\sim}{\to}} \mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}) \overset{\mathrm{eval}}{\stackrel{\sim}{\to}} \mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}) \overset{\text{“Kum}^{-1}\text{”}}{\stackrel{\sim}{\to}} \mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})$$

of (pre-)Frobenioids constructed by Proposition 11.19 (4) and Corollary 11.20 (5), which are compatible with local evaluation isomorphisms of (4), with respect to the localisation isomorphisms for each $\underline{v} \in \underline{\mathbb{V}}$ *and the bijections* $\mathrm{Prime}(-) \stackrel{\sim}{\to} \underline{\mathbb{V}}$*.*

Proof. Corollary follows from the definitions. □

Next, we consider ⊠-portion.

Corollary 11.22. (Π-theoretic Monoids associated to $\mathcal{D}$-⊠-Hodge Theatres, [IUTchII, Corollary 4.7]) *Let*

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes} = ({}^{\dagger}\mathcal{D}^{\circledcirc} \overset{{}^{\dagger}\phi^{\mathrm{NF}}_{*}}{\longleftarrow} {}^{\dagger}\mathfrak{D}_{J} \overset{{}^{\dagger}\phi^{\Theta}_{*}}{\longrightarrow} {}^{\dagger}\mathfrak{D}_{>})$$

be a $\mathcal{D}$*-⊠-Hodge theatre, which is* **glued** *to the* $\mathcal{D}$*-⊞-Hodge theatre* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$ *of Corollary 11.20 via the algorithm in Lemma 10.38 (Hence* $J = T^{*}$*).*

(1) **(Global Non-realified Structures)** *By Example 9.5, we have a functorial algorithm, with respect to the category* $^{\dagger}\mathcal{D}^{\circledcirc}$*, to construct the morphism*

$$^{\dagger}\mathcal{D}^{\circledcirc} \to {}^{\dagger}\mathcal{D}^{\circledast},$$

the monoid/field/pseudo-monoid

$$\pi_1({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}),\ \pi_1({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}),\ \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

with $\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$*-/*$\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})$*-actions (Here, we use the notation* $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$*,* $\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$ *and* $\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})$*, not* ${}^{\dagger}\Pi^{\circledcirc}$*,* ${}^{\dagger}\Pi^{\circledast}$*,* $({}^{\dagger}\Pi^{\circledast})^{\mathrm{rat}}$ *in Example 9.5, respectively, for making clear the dependence of objects), which is well-defined up to* $\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$*-/*$\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})$*-conjugay indeterminacies, the submooid/subfield/subset*

$$\mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subset \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}),\ \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subset \overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}),\ \mathbb{M}^{\circledast}_{\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subset \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}),$$

of $\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$*-/*$\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})$*-invariant parts, the Frobeniods*

$$\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \ \subset\ \mathcal{F}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \ \supset\ \mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

(Here, we write $\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})$*,* $\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$ *for* ${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}$*,* ${}^{\dagger}\mathcal{F}^{\circledcirc}$ *in Example 9.5, respectively) with a natural bijection (by abuse of notation)*

$$\mathrm{Prime}(\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})) \xrightarrow{\sim} \underline{\mathbb{V}},$$

and the natural realification functor

$$\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \to \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}).$$

(2) **($\mathbb{F}_l^{\circledast}$-symmetry)** *By Definition 10.22, for* $j \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$*, we have a functorial algorithm, with respect to the category* ${}^{\dagger}\mathcal{D}^{\circledcirc}$*, to construct an* $\mathcal{F}$*-prime-strip*

$$\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j,$$

which **is only well-defined up to isomorphism***, Moreover, the natural poly-action of* $\mathbb{F}_l^{\circledast}$ *on* ${}^{\dagger}\mathcal{D}^{\circledcirc}$ *induces isomorphisms between the labelled data*

$$\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j,\quad \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j,\quad \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j,$$

$$\{\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})\}_j,\quad \mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j \to \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$$

for distinct $j \in \mathrm{LabCusp}({}^\dagger\mathcal{D}^\circledcirc)$. *We shall refer to these isomorphisms as* $\mathbb{F}_l^{\circledast}$**-symmetrising isomorphisms**. *These* $\mathbb{F}_l^{\circledast}$*-symmetrising isomorphisms are compatbile with the (simply transitive)* $\mathbb{F}_l^{\circledast}$*-action on the index set* J *of the* $\mathcal{D}$-NF*-bridge* ${}^\dagger\phi_{\circledast}^{\mathrm{NF}}$ *with respect to* ${}^\dagger\zeta_{\circledast} : \mathrm{LabCusp}({}^\dagger\mathcal{D}^\circledcirc) \xrightarrow{\sim} J(\xrightarrow{\sim} \mathbb{F}_l^{\circledast})$ *in Proposition 10.19 (3), and determine diagonal objects*

$$\mathbb{M}_{\mathrm{mod}}^{\circledast}({}^\dagger\mathcal{D}^\circledcirc)_{\langle\mathbb{F}_l^{\circledast}\rangle} \subset \prod_{j\in\mathbb{F}_l^{\circledast}} \mathbb{M}_{\mathrm{mod}}^{\circledast}({}^\dagger\mathcal{D}^\circledcirc)_j, \quad \overline{\mathbb{M}}_{\mathrm{mod}}^{\circledast}({}^\dagger\mathcal{D}^\circledcirc)_{\langle\mathbb{F}_l^{\circledast}\rangle} \subset \prod_{j\in\mathbb{F}_l^{\circledast}} \overline{\mathbb{M}}_{\mathrm{mod}}^{\circledast}({}^\dagger\mathcal{D}^\circledcirc)_j.$$

We also write

$$\mathcal{F}^\circledcirc({}^\dagger\mathcal{D}^\circledcirc)|_{\langle\mathbb{F}_l^{\circledast}\rangle}, \ \{\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \curvearrowright \mathbb{M}_{\infty\kappa}^{\circledast}({}^\dagger\mathcal{D}^\circledcirc)\}_{\langle\mathbb{F}_l^{\circledast}\rangle}, \ \mathcal{F}_{\mathrm{mod}}^{\circledast}({}^\dagger\mathcal{D}^\circledcirc)_{\langle\mathbb{F}_l^{\circledast}\rangle} \to \mathcal{F}_{\mathrm{mod}}^{\circledast\mathbb{R}}({}^\dagger\mathcal{D}^\circledcirc)_{\langle\mathbb{F}_l^{\circledast}\rangle}$$

for a purely formal notational shorthand for the above $\mathbb{F}_l^{\circledast}$*-symmetrising isomorphisms for the respective objects (cf. also Remark 11.22.1 below).*

(3) **(Localisations and Global Realified Structures)** *For simplicity, we write* ${}^\dagger\mathfrak{D}_j = \{{}^\dagger\mathcal{D}_{\underline{v}_j}\}_{\underline{v}\in\underline{\mathbb{V}}}$ *(resp.* ${}^\dagger\mathfrak{D}_j^{\vdash} = \{{}^\dagger\mathcal{D}_{\underline{v}_j}^{\vdash}\}_{\underline{v}\in\underline{\mathbb{V}}}$*) for the* $\mathcal{D}$*-(resp.* $\mathcal{D}^{\vdash}$*-)prime-strip associated to the* $\mathcal{F}$*-prime-strip* $\mathcal{F}^\circledcirc({}^\dagger\mathcal{D}^\circledcirc)|_j$ *(cf. Definition 10.22 (2)). By Definition 10.22 (2), Definition 9.6 (2), (3), and Definition 10.23 (3), we have a functorial algorithm, with respect to the category* ${}^\dagger\mathcal{D}^\circledcirc$*, to construct (*1*-)compatible collections of "localisation" functors/poly-morphisms*

$$\mathcal{F}_{\mathrm{mod}}^{\circledast}({}^\dagger\mathcal{D}^\circledcirc)_j \xrightarrow{\mathrm{gl.\ to\ loc.}} \mathcal{F}^\circledcirc({}^\dagger\mathcal{D}^\circledcirc)|_j, \quad \mathcal{F}_{\mathrm{mod}}^{\circledast\mathbb{R}}({}^\dagger\mathcal{D}^\circledcirc)_j \xrightarrow{\mathrm{gl.\ to\ loc.}} (\mathcal{F}^\circledcirc({}^\dagger\mathcal{D}^\circledcirc)|_j)^{\mathbb{R}},$$

$$\Big\{\{\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \curvearrowright \mathbb{M}_{\infty\kappa}^{\circledast}({}^\dagger\mathcal{D}^\circledcirc)\}_j \xrightarrow{\mathrm{gl.\ to\ loc.}} \mathbb{M}_{\infty\kappa v}^{\circledast}({}^\dagger\mathcal{D}_{\underline{v}_j}) \subset \mathbb{M}_{\infty\kappa\times v}^{\circledast}({}^\dagger\mathcal{D}_{\underline{v}_j})\Big\}_{\underline{v}\in\underline{\mathbb{V}}}$$

up to isomorphism, together with a natural isomorphism

$$\mathcal{D}^{\Vdash}({}^\dagger\mathfrak{D}_j^{\vdash}) \xrightarrow[\sim]{\mathrm{gl.\ real'd\ to\ gl.\ non\text{-}real'd}\otimes\mathbb{R}} \mathcal{F}_{\mathrm{mod}}^{\circledast\mathbb{R}}({}^\dagger\mathcal{D}^\circledcirc)_j$$

of global realified Frobenioids (global side), and a natural isomorphism

$$\mathbb{R}_{\geq 0}({}^\dagger\mathfrak{D}_j^{\vdash})_{\underline{v}} \xrightarrow[\sim]{\mathrm{localised\ (gl.\ real'd\ to\ gl.\ non\text{-}real'd}\otimes\mathbb{R})} \Psi_{(\mathcal{F}^\circledcirc({}^\dagger\mathcal{D}^\circledcirc)|_j)^{\mathbb{R}},\underline{v}}$$

of topological monoids for each $\underline{v} \in \underline{\mathbb{V}}$ *(local side), which are compatible with the respective bijections* $\mathrm{Prime}(-) \xrightarrow{\sim} \underline{\mathbb{V}}$ *and the localisation isomorphisms* $\{\Phi_{\mathcal{D}^{\Vdash}({}^\dagger\mathfrak{D}_j^{\vdash}),\underline{v}} \xrightarrow[\sim]{\mathrm{gl.\ to\ loc.}} \mathbb{R}_{\geq 0}({}^\dagger\mathfrak{D}_j^{\vdash})_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ *constructed by Corollary 11.20 (2) and the above* $\mathcal{F}_{\mathrm{mod}}^{\circledast\mathbb{R}}({}^\dagger\mathcal{D}^\circledcirc)_j \xrightarrow{\mathrm{gl.\ to\ loc.}} (\mathcal{F}^\circledcirc({}^\dagger\mathcal{D}^\circledcirc)|_j)^{\mathbb{R}}$. *Finally, all of these structures are compatible with the respective* $\mathbb{F}_l^{\circledast}$*-symmetrising isomorphisms of (2).*

Remark 11.22.1. ([IUTchII, Remark 4.7.2]) Recall that $\mathbb{F}_l^{\circledast}$, in the context of $\mathbb{F}_l^{\circledast}$-symmetry, is a subquotient of $\mathrm{Gal}(K/F)$ (cf. Definition 10.29), hence we cannot perform the kind of conjugate synchronisations in Corollary 11.20 (3) for $\mathbb{F}_l^{\circledast}$-symmetry (for example, it nontrivially acts on the number field $\overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$). Therefore, we have to work with

(1) $\mathcal{F}$-prime-strips, instead of the corresponding ind-topological monoids with Galois actions as in Corollary 11.20 (3),

(2) the objects labelled by $(-)_{\mathrm{mod}}$ (Note that the natural action of Galois group $\mathrm{Gal}(K/F)$ on them is trival, since they are in the Galois invariant parts), and

(3) the objects labelled by $(-)_{\infty\kappa}$,

because we can *ignore* the conjugacy indeterminacies for them (In the case of (2), there is no conjugacy indeterminacy). cf. also Remark 9.6.2 (4) (in the second numeration).

Proof. Corollary follows from the definitions. $\square$

Corollary 11.23. ($\mathcal{F}$-theoretic Monoids associated to $\boxtimes$-Hodge Theatres, [IUTchII, Corollary 4.8]) *Let*

$$ {}^{\dagger}\mathcal{HT}^{\boxtimes} = \left({}^{\dagger}\mathcal{F}^{\circledast} \dashleftarrow {}^{\dagger}\mathcal{F}^{\circledcirc} \xleftarrow{{}^{\dagger}\psi^{\mathrm{NF}}_{\circledast}} {}^{\dagger}\mathfrak{F}_J \xrightarrow{{}^{\dagger}\psi^{\Theta}_{\circledast}} {}^{\dagger}\mathfrak{F}_{>} \dashrightarrow {}^{\dagger}\mathcal{HT}^{\Theta} \right) $$

be a $\boxtimes$-Hodge theatre, which lifts the $\mathcal{D}$-$\boxtimes$-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$ of Corollary 11.22, and is **glued** *to the $\boxplus$-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\boxplus}$ of Corollary 11.21 via the algorithm in Lemma 10.38 (Hence $J = T^{\circledast}$).*

(1) **(Global Non-realified Structures)** *By Definition 9.6 (1) (the Kummer isomorphism by the cyclotomic rigidity isomorphism via $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^{\times} = \{1\}$* (Cyc. Rig. NF1)*), we have a functorial algorithm, with respect to the pre-Frobenioid ${}^{\dagger}\mathcal{F}^{\circledcirc}$, to construct* **Kummer isomorphism**

$$ \left\{ \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa} \right\} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \left\{ \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) \right\}, \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\kappa} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathbb{M}^{\circledast}_{\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) $$

of pseudo-monoids with group actions, which is well-defined up to conjugacy indeterminacies, and by restricting Kummer classes (cf. Definition 9.6 (1)), natural **Kummer isomorphisms**

$$ \left\{ \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast} \right\} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \left\{ \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \right\}, \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}), $$

$$ \left\{ \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\overline{\mathbb{M}}^{\circledast} \right\} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \left\{ \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \right\}, \quad {}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}). $$

These isomorphisms can be interpreted as a compatible collection of isomorphisms

$$^\dagger\mathcal{F}^{\circledcirc} \overset{\text{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledcirc}(^\dagger\mathcal{D}^{\circledcirc}),\ ^\dagger\mathcal{F}^{\circledast} \overset{\text{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledast}(^\dagger\mathcal{D}^{\circledcirc}),\ ^\dagger\mathcal{F}^{\circledast}_{\text{mod}} \overset{\text{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledast}_{\text{mod}}(^\dagger\mathcal{D}^{\circledcirc}),\ ^\dagger\mathcal{F}^{\circledast\mathbb{R}}_{\text{mod}} \overset{\text{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledast\mathbb{R}}_{\text{mod}}(^\dagger\mathcal{D}^{\circledcirc})$$

of (pre-)Frobenioids (cf. Definition 9.6 (1), and Example 9.5).

(2) **($\mathbb{F}_l^{\divideontimes}$-symmetry)** *The collection of isomorphisms of Corollary 11.21 (1) for the capsule $^\dagger\mathfrak{F}_J$ of the $\mathcal{F}$-prime-strips and the isomorphism in (1) give us, for each $j \in \mathrm{LabCusp}(^\dagger\mathcal{D}^{\circledcirc})(\xrightarrow{\sim} J)$, a collection of* **Kummer isomorphisms**

$$^\dagger\mathfrak{F}_j \xrightarrow{\sim} {}^\dagger\mathcal{F}^{\circledcirc}|_j \overset{\text{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledcirc}(^\dagger\mathcal{D}^{\circledcirc})|_j,\ \left\{\pi_1^{\text{rat}}(^\dagger\mathcal{D}^{\circledast}) \curvearrowright {}^\dagger\mathbb{M}^{\circledast}_{\infty\kappa}\right\}_j \overset{\text{Kum}}{\xrightarrow{\sim}} \left\{\pi_1^{\text{rat}}(^\dagger\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}(^\dagger\mathcal{D}^{\circledcirc})\right\}_j,$$

$$(^\dagger\mathbb{M}^{\circledast}_{\text{mod}})_j \overset{\text{Kum}}{\xrightarrow{\sim}} \mathbb{M}^{\circledast}_{\text{mod}}(^\dagger\mathcal{D}^{\circledcirc})_j,\ (^\dagger\overline{\mathbb{M}}^{\circledast}_{\text{mod}})_j \overset{\text{Kum}}{\xrightarrow{\sim}} \overline{\mathbb{M}}^{\circledast}_{\text{mod}}(^\dagger\mathcal{D}^{\circledcirc})_j,$$

$$(^\dagger\mathcal{F}^{\circledast}_{\text{mod}})_j \overset{\text{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledast}_{\text{mod}}(^\dagger\mathcal{D}^{\circledcirc})_j,\ \ (^\dagger\mathcal{F}^{\circledast\mathbb{R}}_{\text{mod}})_j \overset{\text{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledast\mathbb{R}}_{\text{mod}}(^\dagger\mathcal{D}^{\circledcirc})_j,$$

and $\mathbb{F}_l^{\divideontimes}$-symmetrising isomorphisms between the data indexed by distinct $j \in \mathrm{LabCusp}(^\dagger\mathcal{D}^{\circledcirc})$, induced by the natural poly-action of $\mathbb{F}_l^{\divideontimes}$ on $^\dagger\mathcal{F}^{\circledcirc}$. These $\mathbb{F}_l^{\divideontimes}$-symmetrising isomorphisms are compatbile with the (simply transitive) $\mathbb{F}_l^{\divideontimes}$-action on the index set J of the $\mathcal{D}$-NF-bridge $^\dagger\phi_{\divideontimes}^{\mathrm{NF}}$ with respect to $^\dagger\zeta_{\divideontimes} : \mathrm{LabCusp}(^\dagger\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} J(\xrightarrow{\sim} \mathbb{F}_l^{\divideontimes})$ in Proposition 10.19 (3), and determine various diagonal objects

$$(^\dagger\mathbb{M}^{\circledast}_{\text{mod}})_{\langle\mathbb{F}_l^{\divideontimes}\rangle} \subset \prod_{j\in\mathbb{F}_l^{\divideontimes}} (^\dagger\mathbb{M}^{\circledast}_{\text{mod}})_j,\ \ (^\dagger\overline{\mathbb{M}}^{\circledast}_{\text{mod}})_{\langle\mathbb{F}_l^{\divideontimes}\rangle} \subset \prod_{j\in\mathbb{F}_l^{\divideontimes}} (^\dagger\overline{\mathbb{M}}^{\circledast}_{\text{mod}})_j,$$

and formal notational "diagonal objects" (cf. Corollary 11.22 (2))

$$^\dagger\mathcal{F}^{\circledcirc}|_{\langle\mathbb{F}_l^{\divideontimes}\rangle},\ \ \{\pi_1^{\text{rat}}(^\dagger\mathcal{D}^{\circledast}) \curvearrowright {}^\dagger\mathbb{M}^{\circledast}_{\infty\kappa}\}_{\langle\mathbb{F}_l^{\divideontimes}\rangle},\ \ (^\dagger\mathcal{F}^{\circledast}_{\text{mod}})_{\langle\mathbb{F}_l^{\divideontimes}\rangle},\ \ (^\dagger\mathcal{F}^{\circledast\mathbb{R}}_{\text{mod}})_{\langle\mathbb{F}_l^{\divideontimes}\rangle}.$$

(3) **(Localisations and Global Realified Structures)** *By Definition 10.22 (2) and Definition 10.23 (3), we have a functorial algorithm, with respect to the* NF*-bridge $^\dagger\mathfrak{F}_J \overset{^\dagger\psi_{\divideontimes}^{\mathrm{NF}}}{\to} {}^\dagger\mathcal{F}^{\circledcirc} \dashrightarrow {}^\dagger\mathcal{F}^{\circledast}$, to construct mutually (1-)compatible collections of localisation functors/poly-morphisms,*

$$(^\dagger\mathcal{F}^{\circledast}_{\text{mod}})_j \overset{\text{gl. to loc.}}{\longrightarrow} {}^\dagger\mathfrak{F}_j,\ \ (^\dagger\mathcal{F}^{\circledast\mathbb{R}}_{\text{mod}})_j \overset{\text{gl. to loc.}}{\longrightarrow} {}^\dagger\mathfrak{F}^{\mathbb{R}}_j,$$

$$\left\{\left\{\pi_1^{\text{rat}}(^\dagger\mathcal{D}^{\circledast}) \curvearrowright {}^\dagger\mathbb{M}^{\circledast}_{\infty\kappa}\right\}_j \overset{\text{gl. to loc.}}{\longrightarrow} {}^\dagger\mathbb{M}_{\infty\kappa v_j} \subset {}^\dagger\mathbb{M}_{\infty\kappa\times v_j}\right\}_{\underline{v}\in\underline{\mathbb{V}}},$$

up to isomorphism, which is compatible with the collections of functors/poly-morphisms of Corollary 11.22 (3), with respect to the various Kummer isomorphisms of (1), (2), together with a natural isomorphism

$$ {}^{\dagger}\mathcal{C}_j^{\Vdash} \overset{\text{gl. real'd to gl. non-real'd}\otimes\mathbb{R}}{\xrightarrow{\sim}} ({}^{\dagger}\mathcal{F}_{\mathrm{mod}}^{\circledast\mathbb{R}})_j $$

of global realified Frobenioids (global side), which is compatible with respective bijections $\mathrm{Prime}(-) \xrightarrow{\sim} \underline{\mathbb{V}}$, *and a natural isomorphism*

$$ \Psi_{{}^{\dagger}\mathfrak{F}_j^{\vdash},\underline{v}} \overset{\text{localised (gl. real'd to gl. non-real'd}\otimes\mathbb{R})}{\xrightarrow{\sim}} \Psi_{{}^{\dagger}\mathcal{F}_j^{\mathbb{R}},\underline{v}} $$

of topological monoids for each $\underline{v} \in \underline{\mathbb{V}}$ *(local side), which are compatible with the respective bijections* $\mathrm{Prime}(-) \xrightarrow{\sim} \underline{\mathbb{V}}$, *the localisation isomorphisms* $\{\Phi_{{}^{\dagger}\mathcal{C}_j^{\Vdash},\underline{v}} \overset{\text{gl. to loc.}}{\xrightarrow{\sim}} \Psi^{\mathbb{R}}_{{}^{\dagger}\mathfrak{F}_j^{\vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ *constructed by Corollary 11.20 (2) and the above* $({}^{\dagger}\mathcal{F}_{\mathrm{mod}}^{\circledast\mathbb{R}})_j \overset{\text{gl. to loc.}}{\longrightarrow} {}^{\dagger}\mathfrak{F}_j^{\mathbb{R}}$, *the isomorphisms of Corollary 11.22 (3), and various (Kummer) isomorphisms of (1), (2). Finally, all of these structures are compatible with the respective* $\mathbb{F}_l^{\circledast}$*-symmetrising isomorphisms of (2).*

Proof. Corollary follows from the definitions. □

Write the results of this Chapter together, we obtain the following:

Corollary 11.24. (Frobenius-picture of $\boxtimes\boxplus$-Hodge Theatres, [IUTchII, Corollary 4.10]) *Let* ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$, ${}^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus}$ *be* $\boxtimes\boxplus$*-Hodge theatres with respect to the fixed initial* Θ*-data. We write* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$, ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$ *for the assosiated* $\mathcal{D}$*-* $\boxtimes\boxplus$*-Hodge theatres respectively.*

(1) **(Constant Prime-strips)** *Apply the constructions of Corollary 11.21 (1), (3) for the underlying* $\boxplus$*-Hodge theatre of* ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$. *Then the collection* $\Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\succ})_t$ *of data determines an* $\mathcal{F}$*-prime-strip for each* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$. *We identify the collections*

$$ \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\succ})_0, \quad \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\succ})_{\langle\mathbb{F}_l^{\circledast}\rangle} $$

of data, via the isomorphisms $\overset{\text{diag}}{\xrightarrow{\sim}}$ *in Corollary 11.21 (3), and we write*

$$ {}^{\dagger}\mathfrak{F}_{\Delta}^{\Vdash} = ({}^{\dagger}\mathcal{C}_{\Delta}^{\Vdash}, \mathrm{Prime}({}^{\dagger}\mathcal{D}_{\Delta}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^{\dagger}\mathfrak{F}_{\Delta}^{\vdash}, \{{}^{\dagger}\rho_{\Delta,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) \quad (i.e., \text{“}\Delta = \{\ 0\ ,\ \langle\mathbb{F}_l^{\circledast}\rangle\ \}\text{”}) $$

for the resulting $\mathcal{F}^{\Vdash}$*-prime-strip determined by the algorithm “*$\mathfrak{F} \mapsto \mathfrak{F}^{\Vdash}$*”. Note that we have a natural isomorphism* ${}^{\dagger}\mathfrak{F}_{\Delta}^{\Vdash} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}_{\mathrm{mod}}^{\Vdash}$ *of* $\mathcal{F}^{\Vdash}$*-prime-strips, where* ${}^{\dagger}\mathfrak{F}_{\mathrm{mod}}^{\Vdash}$ *is the data contained in the* Θ*-Hodge theatre of* ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$.

(2) **(Theta and Gaussian Prime-strips)** *Apply Corollary 11.21 (4), (5) to the underlying* Θ*-bridge and* $\boxplus$*-Hodge theatre of* ${}^\dagger\mathcal{HT}^{\boxtimes\boxplus}$. *Then the collection* $\Psi_{\mathcal{F}_{\mathrm{env}}}({}^\dagger\mathcal{HT}^{\Theta})$ *of data, the global realified Frobenioid* ${}^\dagger\mathcal{C}_{\mathrm{env}} := \mathcal{C}_{\mathrm{env}}({}^\dagger\mathcal{HT}^{\Theta})$*, localisation isomorphisms* $\Phi_{{}^\dagger\mathcal{C}_{\mathrm{env}},\underline{v}} \overset{\text{gl. to loc.}}{\xrightarrow{\sim}} \Psi_{\mathcal{F}_{\mathrm{env}}}({}^\dagger\mathcal{HT}^{\Theta})^{\mathbb{R}}_{\underline{v}}$ *for* $\underline{v} \in \underline{\mathbb{V}}$ *give rise to an* $\mathcal{F}^{\Vdash}$*-prime-strip*

$$ {}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{env}} = ({}^\dagger\mathcal{C}^{\Vdash}_{\mathrm{env}}, \mathrm{Prime}({}^\dagger\mathcal{D}^{\Vdash}_{\mathrm{env}}) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^\dagger\mathfrak{F}^{\vdash}_{\mathrm{env}}, \{{}^\dagger\rho_{\mathrm{env},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$

(Note that ${}^\dagger\mathfrak{F}^{\vdash}_{\mathrm{env}}$ *is the* $\mathcal{F}^{\vdash}$*-prime-strip determined by* $\Psi_{\mathcal{F}_{\mathrm{env}}}({}^\dagger\mathcal{HT}^{\Theta})$*). Thus, there is a natural identification isomorphism* ${}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{env}} \xrightarrow{\sim} {}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{theta}}$*, where* ${}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{env}}$ *is associated to data in* ${}^\dagger\mathcal{HT}^{\Theta}$ *(cf. Definition 10.5 (4) for* ${}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{env}}$*).*

Similarly, the collection $\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^\dagger\mathcal{HT}^{\Theta})$ *of data, the global realified Frobenioid* ${}^\dagger\mathcal{C}_{\mathrm{gau}} := \mathcal{C}_{\mathrm{gau}}({}^\dagger\mathcal{HT}^{\Theta})$*, localisation isomorphisms* $\Phi_{{}^\dagger\mathcal{C}_{\mathrm{gau}},\underline{v}} \overset{\text{gl. to loc.}}{\xrightarrow{\sim}} \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^\dagger\mathcal{HT}^{\Theta})^{\mathbb{R}}_{\underline{v}}$ *for* $\underline{v} \in \underline{\mathbb{V}}$ *give rise to an* $\mathcal{F}^{\Vdash}$*-prime-strip*

$$ {}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{gau}} = ({}^\dagger\mathcal{C}^{\Vdash}_{\mathrm{env}}, \mathrm{Prime}({}^\dagger\mathcal{D}^{\Vdash}_{\mathrm{gau}}) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^\dagger\mathfrak{F}^{\vdash}_{\mathrm{gau}}, \{{}^\dagger\rho_{\mathrm{gau},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$

(Note that ${}^\dagger\mathfrak{F}^{\vdash}_{\mathrm{gau}}$ *is the* $\mathcal{F}^{\vdash}$*-prime-strip determined by* $\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^\dagger\mathcal{HT}^{\Theta})$*). Finally, the evaluation isomorphisms of Corollary 11.21 (4), (5) determine an* **evaluation isomorphism**

$$ {}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{env}} \overset{\text{eval}}{\xrightarrow{\sim}} {}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{gau}} $$

of $\mathcal{F}^{\Vdash}$*-prime-strips.*

(3) **($\Theta^{\times\mu}$- and $\Theta^{\times\mu}_{\mathrm{gau}}$-Links)** *Let*

$$ {}^\ddagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta} \quad (\mathit{resp.}\ \ {}^\dagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{env}},\ \ \mathit{resp.}\ \ {}^\dagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{gau}}\ ) $$

donote $\mathcal{F}^{\Vdash\blacktriangleright\times\mu}$*-prime-strip associated to the* $\mathcal{F}^{\Vdash}$*-prime-strip* ${}^\ddagger\mathfrak{F}^{\Vdash}_{\Delta}$ *(resp.* ${}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{env}}$*, resp.* ${}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$*) (cf. Definition 10.12 (3) for* $\mathcal{F}^{\Vdash\blacktriangleright\times\mu}$*-prime-strips). Then the functriality of this algorithm induces maps*

$$ \mathrm{Isom}_{\mathcal{F}^{\Vdash}}({}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{env}}, {}^\ddagger\mathfrak{F}^{\Vdash}_{\Delta}) \ \to\ \mathrm{Isom}_{\mathcal{F}^{\Vdash\blacktriangleright\times\mu}}({}^\dagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{env}}, {}^\ddagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta}), $$

$$ \mathrm{Isom}_{\mathcal{F}^{\Vdash}}({}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{gau}}, {}^\ddagger\mathfrak{F}^{\Vdash}_{\Delta}) \ \to\ \mathrm{Isom}_{\mathcal{F}^{\Vdash\blacktriangleright\times\mu}}({}^\dagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{gau}}, {}^\ddagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta}). $$

Note that the second map is equal to the composition of the first map with the evaluation isomorphism ${}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{env}} \overset{\text{eval}}{\xrightarrow{\sim}} {}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$ *and the functorially obtained isomorphism* ${}^\dagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{env}} \overset{\text{eval}}{\xrightarrow{\sim}} {}^\dagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{gau}}$ *from this isomorphism. We shall refer to the full poly-isomorphism*

$$ {}^\dagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{env}} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta} \quad (\mathit{resp.}\ \ {}^\dagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{gau}} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta}\ ) $$

as the $\boldsymbol{\Theta}^{\times\boldsymbol{\mu}}$**-link** *(resp.* $\boldsymbol{\Theta}^{\times\boldsymbol{\mu}}_{\mathbf{gau}}$**-link***) from* ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$ *to* ${}^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus}$ *(cf. Definition 10.8), and we write it as*

$$ {}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus} \stackrel{\Theta^{\times\mu}}{\longrightarrow} {}^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus} \quad (resp.\ \ {}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus} \stackrel{\Theta^{\times\mu}_{\mathrm{gau}}}{\longrightarrow} {}^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus}\ ) $$

and we shall refer to this diagram as the **Frobenius-picture of** $\boxtimes\boxplus$**-Hodge theatres** *(This is an enhanced version of Definition 10.8). Note that the essential meaning of the above link is*

$$ \text{“}\ \underline{\underline{\Theta}}_{\underline{v}}^{\mathbb{N}} \stackrel{\sim}{\longrightarrow} \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}}\ \text{”} \quad (resp.\ \ \text{“}\ \{\underline{\underline{q}}_{\underline{v}}^{j^2}\}^{\mathbb{N}}_{1\leq j\leq l^{*}} \stackrel{\sim}{\longrightarrow} \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}}\ \text{”}\ ) $$

for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$.

(4) **(Horizontally Coric** $\boldsymbol{\mathcal{F}}^{\vdash\times\boldsymbol{\mu}}$**-Prime-strips)** *By the definition of the unit portion of the theta monoids and the Gaussian monoids, we have natural isomorphisms*

$$ {}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \stackrel{\sim}{\to} {}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}} \stackrel{\sim}{\to} {}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\mathrm{gau}}, $$

where ${}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\Delta}$, ${}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}}$, ${}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\mathrm{gau}}$ *are the* $\mathcal{F}^{\vdash\times\mu}$*-prime-strips associated to the* $\mathcal{F}^{\vdash}$*-prime-strips* ${}^{\dagger}\mathfrak{F}^{\vdash}_{\Delta}$, ${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}}$, ${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{gau}}$*, respectively. Then the composite*

$$ {}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \stackrel{\sim}{\to} {}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}} \stackrel{\mathrm{poly}}{\stackrel{\sim}{\to}} {}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \quad (resp.\ \ {}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \stackrel{\sim}{\to} {}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\mathrm{gau}} \stackrel{\mathrm{poly}}{\stackrel{\sim}{\to}} {}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\Delta}\ ) $$

with the poly-isomorphism induced by the full poly-isomorphism ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{env}} \stackrel{\text{full poly}}{\stackrel{\sim}{\longrightarrow}} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta}$ *(resp.* ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{gau}} \stackrel{\text{full poly}}{\stackrel{\sim}{\longrightarrow}} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta}$*) in the definition of* $\Theta^{\times\mu}$*-link (resp.* $\Theta^{\times\mu}_{\mathrm{gau}}$*-link) is equal to the full poly-isomorphism of* $\mathcal{F}^{\vdash\times\mu}$*-prime-strips. This means that* ${}^{(-)}\mathfrak{F}^{\vdash\times\mu}_{\Delta}$ *is preserved (or "shared") under both the* $\Theta^{\times\mu}$*-link and* $\Theta^{\times\mu}_{\mathrm{gau}}$*-link (This is an enhanced version of Remark 10.8.1 (2)). Note that the value group portion is not shared under the* $\Theta^{\times\mu}$*-link and the* $\Theta^{\times\mu}_{\mathrm{gau}}$*-link. Finally, this full poly-isomorphism induces the full poly-isomorphism*

$$ {}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta} \stackrel{\text{full poly}}{\stackrel{\sim}{\longrightarrow}} {}^{\ddagger}\mathfrak{D}^{\vdash}_{\Delta} $$

of the associated $\mathcal{D}^{\vdash}$*-prime-strips. We shall refer to this as the* $\boldsymbol{\mathcal{D}}$**-** $\boxtimes\boxplus$**-link** *from* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$ *to* ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$*, and we write it as*

$$ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \stackrel{\mathcal{D}}{\longrightarrow} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}. $$

This means that ${}^{(-)}\mathfrak{D}^{\vdash}_{\Delta}$ *is preserved (or "shared") under both the* $\Theta^{\times\mu}$*-link and* $\Theta^{\times\mu}_{\mathrm{gau}}$*-link (This is an enhanced version of Remark 10.8.1 (1), Definition 10.21 and*

Definition 10.35). Note that the holomorphic base "$\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$" is not shared under the $\Theta^{\times\mu}$-link and the $\Theta^{\times\mu}_{\mathrm{gau}}$-link (i.e., $\Theta^{\times\mu}$-link and $\Theta^{\times\mu}_{\mathrm{gau}}$-link share the underlying mono-analytic base structures, but not the arithmetically holomorphic base structures).

(5) **(Horizontally Coric Global Realified Frobenioids)** *The full poly-isomorphism $^{\dagger}\mathfrak{D}^{\vdash}_{\Delta} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{D}^{\vdash}_{\Delta}$ in (4) induces an isomorphism*

$$(\mathcal{D}^{\Vdash}(^{\dagger}\mathfrak{D}^{\vdash}_{\Delta}), \mathrm{Prime}(\mathcal{D}^{\Vdash}(^{\dagger}\mathfrak{D}^{\vdash}_{\Delta})) \xrightarrow{\sim} \underline{\mathbb{V}}, \{^{\dagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) \xrightarrow{\sim} (\mathcal{D}^{\Vdash}(^{\ddagger}\mathfrak{D}^{\vdash}_{\Delta}), \mathrm{Prime}(\mathcal{D}^{\Vdash}(^{\ddagger}\mathfrak{D}^{\vdash}_{\Delta})) \xrightarrow{\sim} \underline{\mathbb{V}}, \{^{\ddagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

of triples. This isomorphism is compatible with the $\mathbb{R}_{>0}$**-orbits**

$$(^{\dagger}\mathcal{C}^{\Vdash}_{\Delta}, \mathrm{Prime}(^{\dagger}\mathcal{C}^{\Vdash}_{\Delta}) \xrightarrow{\sim} \underline{\mathbb{V}}, \{^{\dagger}\rho_{\Delta,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) \overset{\text{“Kum” poly}}{\xrightarrow{\sim}} (\mathcal{D}^{\Vdash}(^{\dagger}\mathfrak{D}^{\vdash}_{\Delta}), \mathrm{Prime}(\mathcal{D}^{\Vdash}(^{\dagger}\mathfrak{D}^{\vdash}_{\Delta})) \xrightarrow{\sim} \underline{\mathbb{V}}, \{^{\dagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

and

$$(^{\ddagger}\mathcal{C}^{\Vdash}_{\Delta}, \mathrm{Prime}(^{\ddagger}\mathcal{C}^{\Vdash}_{\Delta}) \xrightarrow{\sim} \underline{\mathbb{V}}, \{^{\ddagger}\rho_{\Delta,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) \overset{\text{“Kum” poly}}{\xrightarrow{\sim}} (\mathcal{D}^{\Vdash}(^{\ddagger}\mathfrak{D}^{\vdash}_{\Delta}), \mathrm{Prime}(\mathcal{D}^{\Vdash}(^{\ddagger}\mathfrak{D}^{\vdash}_{\Delta})) \xrightarrow{\sim} \underline{\mathbb{V}}, \{^{\ddagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

of isomorphisms of triples obtained by the functorial algorithm in Corollary 11.21 (2), with respect to the $\Theta^{\times\mu}$-link and the $\Theta^{\times\mu}_{\mathrm{gau}}$-link. Here, the $\mathbb{R}_{>0}$-orbits are naturally defined by the diagonal (with respect to Prime$(-)$*) $\mathbb{R}_{>0}$-action on the divisor monoids.*

Proof. Corollary follows from the definitions. □

Remark 11.24.1. (Étale picture of $\mathcal{D}$-$\boxtimes\boxplus$-Hodge Theatres, [IUTchII, Corollary 4.11]) We can visualise the "shared" and "non-shared" relation in Corollary 11.24 as follows:

$$\boxed{^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}} \dashrightarrow \boxed{^{\dagger}\mathfrak{D}^{\vdash}_{\Delta} \cong {}^{\ddagger}\mathfrak{D}^{\vdash}_{\Delta}} \dashleftarrow \boxed{^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}}$$

We shall refer to this diagram as the **étale-picture of $\boxtimes\boxplus$-Hodge theatres** (This is an enhanced version of Remark 10.8.1, Remark 10.21.1 and Remark 10.35.1). Note that, *there is the notion of the order in the Frobenius-picture* (i.e., $^{\dagger}(-)$ is on the left, and $^{\ddagger}(-)$ is on the right), on the other hand, there is no such an order and *it has a permutation symmetry in the étale-picture* (cf. also the last table in Section 4.3). Note that these constructions are compatible, in an obvious sense, with Definition 10.21 and Definition 10.35, with respect to the natural identification $^{(-)}\mathfrak{D}^{\vdash}_{\Delta} \xrightarrow{\sim} {}^{(-)}\mathfrak{D}^{\vdash}_{>}$.

§ 12. Log-links — An Arithmetic Analogue of Analytic Continuation.

§ 12.1. Log-links and Log-theta-lattices.

Definition 12.1. ([IUTchIII, Definition 1.1]) Let ${}^{\dagger}\mathfrak{F} = \{{}^{\dagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ be an $\mathcal{F}$-prime-strip with the associated $\mathcal{F}^{\vdash}$-prime-strip (resp. $\mathcal{F}^{\vdash\times\mu}$-prime-strip, resp. $\mathcal{D}$-prime-strip) ${}^{\dagger}\mathfrak{F}^{\vdash} = \{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ (resp. ${}^{\dagger}\mathfrak{F}^{\vdash\times\mu} = \{{}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, resp. ${}^{\dagger}\mathfrak{D} = \{{}^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$).

(1) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. We write

$$(\Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \supset \Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow)\ \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} := (\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})^{\mathrm{pf}}$$

for the perfection of $\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ (cf. Section 5.1). By the Kummer isomorphism of Remark 3.19.2, we can construct an ind-topological field structure on $\Psi^{\underline{\mathrm{gp}}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$, which is an isomorph of $\overline{K_{\underline{v}}}$ (cf. Section 5.1 for the notation $(-)^{\underline{\mathrm{gp}}}$). Then we can define the $p_{\underline{v}}$-adic logarithm on $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$, and this gives us an isomorphism $\log_{\underline{v}} : \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \xrightarrow{\sim} \Psi^{\underline{\mathrm{gp}}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ of ind-topological groups. Thus, we can transport the ind-topological field structure of $\Psi^{\underline{\mathrm{gp}}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ into $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$. Hence we can consider the multiplicative monoid "$O^{\rhd}$" of non-zero integers of $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$, and we write $\Psi_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$ for it. Note that $\Psi^{\underline{\mathrm{gp}}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})} = \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$. The pair ${}^{\dagger}\Pi_{\underline{v}} \curvearrowright \Psi_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$ determines a pre-Frobenioid

$$\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}}).$$

The resulting ${}^{\dagger}\Pi_{\underline{v}}$-equivariant diagram

$$(\text{Log-link } \underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}) \qquad \Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \supset \Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} = \Psi^{\underline{\mathrm{gp}}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$$

is called the **tautological $\mathfrak{log}$-link** associated to ${}^{\dagger}\mathcal{F}_{\underline{v}}$ (This is a review, in our setting, of constructions of the diagram (Log-link (non-Arch)) in Section 5.1), and we write it as

$${}^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} \mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}}).$$

For any (poly-)isomorphism (resp. the full poly-isomorphism) $\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}}) \overset{(\mathrm{poly})}{\xrightarrow{\sim}} {}^{\ddagger}\mathcal{F}_{\underline{v}}$ (resp. $\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}}) \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathcal{F}_{\underline{v}}$) of pre-Frobenioids, we shall refer to the composite ${}^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} \mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}}) \overset{(\mathrm{poly})}{\xrightarrow{\sim}} {}^{\ddagger}\mathcal{F}_{\underline{v}}$ as a **$\mathfrak{log}$-link** (resp. the **full $\mathfrak{log}$-link**) **from ${}^{\dagger}\mathcal{F}_{\underline{v}}$ to ${}^{\ddagger}\mathcal{F}_{\underline{v}}$** and we write it as

$${}^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{F}_{\underline{v}} \quad (\text{resp. } {}^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\text{full } \mathfrak{log}} {}^{\ddagger}\mathcal{F}_{\underline{v}} \ ).$$

Finally, put

$$\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} := \frac{1}{2p_{\underline{v}}}\mathrm{Im}\left((\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})^{G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})} \to \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}\right) \subset \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} = \Psi^{\underline{\mathrm{gp}}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})},$$

and we shall refer to this as the **Frobenius-like holomorphic log-shell associated to $^\dagger\mathcal{F}_{\underline{v}}$** (This is a review of Definition 5.1 in our setting). By the reconstructible ind-topological field structure on $\Psi_{{}^\dagger\mathcal{F}^{\sim}_{\underline{v}}} = \Psi^{\mathrm{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$, we can regard $\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ as an object associated to the *codomain* of *any* $\mathfrak{log}$-link $^\dagger\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathcal{F}_{\underline{v}}$.

(2) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Recall that $^\dagger\mathcal{F}_{\underline{v}} = ({}^\dagger\mathcal{C}_{\underline{v}}, {}^\dagger\mathcal{D}_{\underline{v}}, {}^\dagger\kappa_{\underline{v}})$ is a triple of a pre-Frobenioid $^\dagger\mathcal{C}_{\underline{v}}$, an Aut-holomorphic space $^\dagger\mathbb{U}_{\underline{v}} := {}^\dagger\mathcal{D}_{\underline{v}}$, and a Kummer structure $^\dagger\kappa_{\underline{v}} : \Psi_{{}^\dagger\mathcal{F}_{\underline{v}}} := O^{\rhd}({}^\dagger\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}^{{}^\dagger\mathcal{D}_{\underline{v}}}$, which is isomorphic to the model triple $(\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}})$ of Definition 10.2 (3). For $N \geq 1$, we write $\Psi^{\mu_N}_{{}^\dagger\mathcal{F}_{\underline{v}}} \subset \Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}} \subset \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ for the subgroup of N-th roots of unity, and $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ for the universal covering of the topological group $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ (Recall that $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ is an isomorph of "$\mathbb{C} \xrightarrow{\exp} \mathbb{C}^{\times}$"). Then the composite

$$\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\mu_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$$

is also a universal covering of $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\mu_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$. We can regard $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ as constructed from $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\mu_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ (cf. also Remark 10.12.1, Proposition 12.2, (4) in this definition, Proposition 13.7, and Proposition 13.11). By the Kummer structure $^\dagger\kappa_{\underline{v}}$, we can construct a topological field structure on $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$. Then we can define the Archimedean logarithm on $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$, and this gives us an isomorphism $\log_{\underline{v}} : \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} \xrightarrow{\sim} \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ of topological groups. Thus, we can transport the topological field structure of $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ into $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$, and the Kummer structure $\Psi_{{}^\dagger\mathcal{F}_{\underline{v}}} \hookrightarrow \mathcal{A}^{{}^\dagger\mathcal{D}_{\underline{v}}}$ into a Kummer structure $^\dagger\kappa^{\sim}_{\underline{v}} : \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} \hookrightarrow \mathcal{A}^{{}^\dagger\mathcal{D}_{\underline{v}}}$. Hence we can consider the multiplicative monoid "$O^{\rhd}$" of non-zero elements of absolute values ≤ 1 of $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$, and we write $\Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$ for it. Note that $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})} = \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$. The triple of topological monoid $\Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$, the Aut-holomorphic space $^\dagger\mathbb{U}_{\underline{v}}$, and the Kummer structure $^\dagger\kappa^{\sim}_{\underline{v}}$ determines a triple

$$\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}}).$$

The resulting co-holomorphicisation-compatible-diagram

$$(\text{Log-link } \underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}) \qquad \Psi_{{}^\dagger\mathcal{F}_{\underline{v}}} \subset \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}} \twoheadleftarrow \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} = \Psi^{\mathrm{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$$

is called the **tautological $\mathfrak{log}$-link** associated to $^\dagger\mathcal{F}_{\underline{v}}$ (This is a review, in our setting, of constructions of the diagram (Log-link (Arch)) in Section 5.2), and we write it as

$$^\dagger\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} \mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}}).$$

For any (poly-)isomorphism (resp. the full poly-isomorphism) $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}}) \overset{\text{(poly)}}{\xrightarrow{\sim}} {}^\ddagger\mathcal{F}_{\underline{v}}$ (resp. $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}}) \overset{\text{full poly}}{\xrightarrow{\sim}} {}^\ddagger\mathcal{F}_{\underline{v}}$) of triples, we shall refer to the composite $^\dagger\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}}$

$\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}}) \overset{(\text{poly})}{\xrightarrow{\sim}} {}^{\ddagger}\mathcal{F}_{\underline{v}}$ as a **$\mathfrak{log}$-link** (resp. the **full $\mathfrak{log}$-link**) **from ${}^{\dagger}\boldsymbol{\mathcal{F}}_{\underline{v}}$ to ${}^{\ddagger}\boldsymbol{\mathcal{F}}_{\underline{v}}$** and we write it as

$$ {}^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{F}_{\underline{v}} \quad (\text{resp. } {}^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\text{full } \mathfrak{log}} {}^{\ddagger}\mathcal{F}_{\underline{v}}). $$

Finally, we write

$$ \mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} $$

for the $\Psi^{\times}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$-orbit of the uniquely determined closed line segment of $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ which is preserved by multiplication by ± 1 and whose endpoints differ by a generator of the kernel of the natural surjection $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\text{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ (i.e., "the line segment $[-\pi, +\pi]$"), or (when we regard $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ as constructed from $\Psi^{\text{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}/\Psi^{\mu_N}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$) equivalently, the $\Psi^{\times}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$-orbit of the result of multiplication by N of the uniquely determined closed line segment of $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ which is preserved by multiplication by ± 1 and whose endpoints differ by a generator of the kernel of the natural surjection $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\text{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}/\Psi^{\mu_N}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ (i.e., "the line segment $N[-\frac{\pi}{N}, +\frac{\pi}{N}] = [-\pi, +\pi]$"), and we shall refer to this as the **Frobenius-like holomorphic log-shell associated to ${}^{\dagger}\boldsymbol{\mathcal{F}}_{\underline{v}}$** (This is a review of Definition 5.3 in our setting). By the reconstructible topological field structure on $\Psi_{{}^{\dagger}\mathcal{F}^{\sim}_{\underline{v}}} = \Psi^{\text{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$, we can regard $\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ as an object associated to the *codomain* of *any* $\mathfrak{log}$-link ${}^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{F}_{\underline{v}}$.

(3) We put

$$ \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}) := \left\{ \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}) := \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \right\}_{\underline{v}\in\underline{\mathbb{V}}} $$

for the collection of ind-topological modules (i.e., we forget the field structure on $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$), where the group structure arises from the *additive* portion of the field structures on $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$. For $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, we regard $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ as equipped with natural $G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$-action. Write also

$$ \mathfrak{log}({}^{\dagger}\mathfrak{F}) := \{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}} $$

for the $\mathcal{F}_{\underline{v}}$-prime-strip determined by $\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})$'s, and we write

$$ {}^{\dagger}\mathfrak{F} \xrightarrow{\mathfrak{log}} \mathfrak{log}({}^{\dagger}\mathfrak{F}) $$

for the collection $\{{}^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} \mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$ of diagrams, and we shall refer to this as the **tautological $\mathfrak{log}$-link** associated to ${}^{\dagger}\mathfrak{F}$. For any (poly-)isomorphism (resp. the full poly-isomorphism) $\mathfrak{log}({}^{\dagger}\mathfrak{F}) \overset{(\text{poly})}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{F}$ (resp. $\mathfrak{log}({}^{\dagger}\mathfrak{F}) \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{F}$) of $\mathcal{F}$-prime-strips, we shall refer to the composite ${}^{\dagger}\mathfrak{F} \xrightarrow{\mathfrak{log}} \mathfrak{log}({}^{\dagger}\mathfrak{F}) \overset{(\text{poly})}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{F}$ as a **$\mathfrak{log}$-link** (resp. the **full $\mathfrak{log}$-link**) **from ${}^{\dagger}\mathfrak{F}$ to ${}^{\ddagger}\mathfrak{F}$** and we write it as

$$ {}^{\dagger}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F} \quad (\text{resp. } {}^{\dagger}\mathfrak{F} \xrightarrow{\text{full } \mathfrak{log}} {}^{\ddagger}\mathfrak{F}). $$

Finally, we put

$$\mathcal{I}_{{}^\dagger\mathfrak{F}} := \{\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}},$$

and we shall refer to this as the **Frobenius-like holomorphic log-shell associated to ${}^\dagger\mathfrak{F}$**. We also write

$$\mathcal{I}_{{}^\dagger\mathfrak{F}} \subset \underline{\mathfrak{log}}({}^\dagger\mathfrak{F})$$

for $\{\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}} \subset \underline{\mathfrak{log}}({}^\dagger\mathcal{F}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$. We can regard $\mathcal{I}_{{}^\dagger\mathfrak{F}}$ as an object associated to the *codomain* of *any* $\mathfrak{log}$-link ${}^\dagger\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathfrak{F}$.

(4) For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$), the ind-topological modules with $G_{\underline{v}}({}^\dagger\Pi)$-action (resp. the topological module and the closed subspace) $\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}} \subset \underline{\mathfrak{log}}({}^\dagger\mathcal{F}_{\underline{v}})$ can be constructed *only* from the $\underline{v}$-component ${}^\dagger\mathcal{F}_{\underline{v}}^{\vdash\times\mu}$ of the associated $\mathcal{F}^{\vdash\times\mu}$-prime-strip, by the $\times\mu$-Kummer structure, since these constructions only use the perfection $(-)^{\mathrm{pf}}$ of the units and are unaffected by taking the quotient by $O^\mu(-)$ (cf. (Step 2) of Proposition 5.2) (resp. *only* from the $\underline{v}$-component ${}^\dagger\mathcal{F}_{\underline{v}}^{\vdash}$ of the associated $\mathcal{F}^{\vdash}$-prime-strip, by (Step 3) of Proposition 5.4, hence *only* from the $\underline{v}$-component ${}^\dagger\mathcal{F}_{\underline{v}}^{\vdash\times\mu}$ of the associated $\mathcal{F}^{\vdash\times\mu}$-prime-strip, by regarding this functorial algorithm as an algorithm which only makes us of the quotien of this unit portion by μ_N for $N \geq 1$ with a universal covering of this quotient). We write

$$\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}^{\vdash\times\mu}} \subset \underline{\mathfrak{log}}({}^\dagger\mathcal{F}_{\underline{v}}^{\vdash\times\mu})$$

for the resulting ind-topological modules with $G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$-action (resp. the resulting topological module and a closed subspace). We shall refer to this as the **Frobenius-like mono-analytic log-shell associated to ${}^\dagger\mathcal{F}_{\underline{v}}^{\vdash\times\mu}$**. Finally, we put

$$\mathcal{I}_{{}^\dagger\mathfrak{F}^{\vdash\times\mu}} := \{\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}^{\vdash\times\mu}}\}_{\underline{v}\in\underline{\mathbb{V}}} \subset \underline{\mathfrak{log}}({}^\dagger\mathfrak{F}^{\vdash\times\mu}) := \{\underline{\mathfrak{log}}({}^\dagger\mathcal{F}_{\underline{v}}^{\vdash\times\mu})\}_{\underline{v}\in\underline{\mathbb{V}}}$$

for the collections constructed from the $\mathcal{F}^{\vdash\times\mu}$-prime-strip ${}^\dagger\mathfrak{F}^{\vdash\times\mu}$ (*not* from ${}^\dagger\mathfrak{F}$). We shall refer to this as the **Frobenius-like mono-analytic log-shell associated to ${}^\dagger\mathfrak{F}^{\vdash\times\mu}$**.

Proposition 12.2. ($\mathfrak{log}$-Links Between $\mathcal{F}$-Prime-strips, [IUTchIII, Proposition 1.2]) *Let ${}^\dagger\mathfrak{F} = \{{}^\dagger\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, ${}^\ddagger\mathfrak{F} = \{{}^\ddagger\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ be $\mathcal{F}$-prime-strips with associated $\mathcal{F}^{\vdash\times\mu}$-prime-strips (resp. $\mathcal{D}$-prime-strips, resp. $\mathcal{D}^{\vdash}$-prime-strips) ${}^\dagger\mathfrak{F}^{\vdash\times\mu} = \{{}^\dagger\mathcal{F}_{\underline{v}}^{\vdash\times\mu}\}_{\underline{v}\in\underline{\mathbb{V}}}$, ${}^\dagger\mathfrak{F}^{\vdash\times\mu} = \{{}^\ddagger\mathcal{F}_{\underline{v}}^{\vdash\times\mu}\}_{\underline{v}\in\underline{\mathbb{V}}}$ (resp. ${}^\dagger\mathfrak{D} = \{{}^\dagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, ${}^\ddagger\mathfrak{D} = \{{}^\ddagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, resp. ${}^\dagger\mathfrak{D}^{\vdash} = \{{}^\dagger\mathcal{D}_{\underline{v}}^{\vdash}\}_{\underline{v}\in\underline{\mathbb{V}}}$, ${}^\ddagger\mathfrak{D}^{\vdash} = \{{}^\ddagger\mathcal{D}_{\underline{v}}^{\vdash}\}_{\underline{v}\in\underline{\mathbb{V}}}$), respectively, and ${}^\dagger\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathfrak{F}$ a $\mathfrak{log}$-link from ${}^\dagger\mathfrak{F}$ to ${}^\ddagger\mathfrak{F}$. We recall the $\mathfrak{log}$-link diagrams*

$$(\mathfrak{log}_{\mathrm{non}}) \qquad \Psi_{{}^\dagger\mathcal{F}_{\underline{v}}} \supset \Psi_{{}^\dagger\mathcal{F}_{\underline{v}}}^{\times} \twoheadrightarrow \underline{\mathfrak{log}}({}^\dagger\mathcal{F}_{\underline{v}}) = \Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}^{\mathrm{gp}} \overset{(\mathrm{poly})}{\xrightarrow{\sim}} \Psi_{{}^\ddagger\mathcal{F}_{\underline{v}}}^{\mathrm{gp}},$$

$$(\mathfrak{log}_{\mathrm{arc}}) \qquad \Psi_{^\dagger\mathcal{F}_{\underline{v}}} \subset \Psi^{\mathrm{gp}}_{^\dagger\mathcal{F}_{\underline{v}}} \twoheadleftarrow \underline{\mathfrak{log}}(^\dagger\mathcal{F}_{\underline{v}}) = \Psi^{\mathrm{gp}}_{\mathfrak{log}(^\dagger\mathcal{F}_{\underline{v}})} \overset{(\mathrm{poly})}{\overset{\sim}{\to}} \Psi^{\mathrm{gp}}_{^\ddagger\mathcal{F}_{\underline{v}}}.$$

for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ and $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, respectively.

(1) **(Vertically Coric $\mathcal{D}$-Prime-strips)** *The $\mathfrak{log}$-link $^\dagger\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathfrak{F}$ induces (poly-)isomorphisms*

$$^\dagger\mathfrak{D} \overset{(\mathrm{poly})}{\overset{\sim}{\to}} {}^\ddagger\mathfrak{D}, \quad ^\dagger\mathfrak{D}^{\vdash} \overset{(\mathrm{poly})}{\overset{\sim}{\to}} {}^\ddagger\mathfrak{D}^{\vdash}$$

of $\mathcal{D}$-prime-strips and $\mathcal{D}^{\vdash}$-prime-strips, respectively. In particular, the (poly-)isomorphism $^\dagger\mathfrak{D} \overset{(\mathrm{poly})}{\overset{\sim}{\to}} {}^\ddagger\mathfrak{D}$ induces a (poly-)isomorphism

$$\Psi_{\mathrm{cns}}(^\dagger\mathfrak{D}) \overset{(\mathrm{poly})}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}(^\ddagger\mathfrak{D}).$$

(2) **(Compatibility with Log-volumes)** *For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$), the diagram $\mathfrak{log}_{\mathrm{non}}$ (resp. the diagram $\mathfrak{log}_{\mathrm{arc}}$) is compatible with the natural $p_{\underline{v}}$-adic log-volumes on $(\Psi^{\mathrm{gp}}_{^\dagger\mathcal{F}_{\underline{v}}})^{^\dagger\Pi_{\underline{v}}}$, and $(\Psi^{\mathrm{gp}}_{\mathfrak{log}(^\dagger\mathcal{F}_{\underline{v}})})^{^\dagger\Pi_{\underline{v}}}$ (resp. the natural angular log-volume on $\Psi^{\times}_{^\dagger\mathcal{F}_{\underline{v}}}$ and the natural radial log-volume on $\Psi^{\mathrm{gp}}_{\mathfrak{log}(^\dagger\mathcal{F}_{\underline{v}})}$) in the sense of the formula* (5.1) *of Proposition 5.2 (resp. in the sense of the formula* (5.2) *of Proposition 5.4). When we regard $\Psi^{\sim}_{^\dagger\mathcal{F}_{\underline{v}}}$ as constructed from $\Psi^{\mathrm{gp}}_{^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\mu_N}_{^\dagger\mathcal{F}_{\underline{v}}}$ (cf. Definition 12.1 (2)), then we equip $\Psi^{\mathrm{gp}}_{^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\mu_N}_{^\dagger\mathcal{F}_{\underline{v}}}$ the metric obtained by descending the metric of $\Psi^{\mathrm{gp}}_{^\dagger\mathcal{F}_{\underline{v}}}$; however, we regard the object $\Psi^{\mathrm{gp}}_{^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\mu_N}_{^\dagger\mathcal{F}_{\underline{v}}}$ (or $\Psi^{\times}_{^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\mu_N}_{^\dagger\mathcal{F}_{\underline{v}}}$) as being equipped with a "weight N", that is, the log-volume of $\Psi^{\times}_{^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\mu_N}_{^\dagger\mathcal{F}_{\underline{v}}}$ is equal to the log-volume of $\Psi^{\mathrm{gp}}_{^\dagger\mathcal{F}_{\underline{v}}}$ ([IUTchIII, Remark1.2.1 (i)]) (cf. also Remark 10.12.1, Definition 12.1 (2), (4), Proposition 13.7, and Proposition 13.11).*

(3) **((Frobenius-like) Holomorphic Log-shells)** *For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$), we have*

$$\Psi^{^\dagger\Pi_{\underline{v}}}_{\mathfrak{log}(^\dagger\mathcal{F}_{\underline{v}})},\ \mathrm{Im}\left((\Psi^{\times}_{^\dagger\mathcal{F}_{\underline{v}}})^{^\dagger\Pi_{\underline{v}}} \to \underline{\mathfrak{log}}(^\dagger\mathcal{F}_{\underline{v}})\right) \subset \mathcal{I}_{^\dagger\mathcal{F}_{\underline{v}}} \ \left(\subset \underline{\mathfrak{log}}(^\dagger\mathcal{F}_{\underline{v}})\right)$$

(cf. the inclusions (Upper Semi-Compat. (non-Arch)) $O^{\times}_{k}$, $\log(O^{\times}_{k}) \subset \mathcal{I}_k$ *in Section 5.1) (resp.*

$$\Psi_{\mathfrak{log}(^\dagger\mathcal{F}_{\underline{v}})} \subset \mathcal{I}_{^\dagger\mathcal{F}_{\underline{v}}} \ \left(\subset \underline{\mathfrak{log}}(^\dagger\mathcal{F}_{\underline{v}})\right), \quad \Psi^{\times}_{^\dagger\mathcal{F}_{\underline{v}}} \subset \mathrm{Im}\left(\mathcal{I}_{^\dagger\mathcal{F}_{\underline{v}}} \to \Psi^{\mathrm{gp}}_{^\dagger\mathcal{F}_{\underline{v}}}\right)$$

(cf. the inclusions (Upper Semi-Compat. (Arch)) $O^{\triangleright}_{k^{\sim}} \subset \mathcal{I}_k$, $O^{\times}_{k} \subset \exp_k(\mathcal{I}_k)$ *in Section 5.2)).*

(4) **((Frobenius-like and Étale-like) Mono-analytic Log-shells)** *For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$), by Proposition 5.2 (resp. Proposition 5.4), we have a functorial algorithm, with respect to the category $^\dagger\mathcal{D}^{\vdash}_{\underline{v}}(= \mathcal{B}(^\dagger G_{\underline{v}})^0)$ (resp. the split monoid*

${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$*), to construct an ind-topological module equipped with a continuous* ${}^{\dagger}G_{\underline{v}}$*-action (resp. a topological module)*

$$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) := \left\{{}^{\dagger}G_{\underline{v}} \curvearrowright k^{\sim}({}^{\dagger}G_{\underline{v}})\right\} \quad (\textit{resp.}\ \ \underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) := k^{\sim}({}^{\dagger}G_{\underline{v}})\ \ )$$

and a topological submodule (resp. a topological subspace)

$$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}} := \mathcal{I}({}^{\dagger}G_{\underline{v}}) \ \subset\ k^{\sim}({}^{\dagger}G_{\underline{v}})$$

(which is called the **étale-like mono-analytic log-shell associated to** ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$*) equipped with a* $p_{\underline{v}}$*-adic log-volume (resp. an angular log-volume and a radial log-volume). Moreover, we have a natural functorial algorithm, with respect to the split-*$\times\mu$*-Kummer pre-Frobenioid* ${}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}$ *(resp. the triple* ${}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}$*), to construct an* **Isomet-orbit** *(resp.* $\{\pm 1\}\times\{\pm 1\}$**-orbit** *arising from the independent* $\{\pm1\}$*-actions on each of the direct factors "*$k^{\sim}(G) = C^{\sim}\times C^{\sim}$*" in the notation of Proposition 5.4)*

$$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}) \ \overset{\text{"Kum"}}{\underset{\text{poly}}{\overset{\sim}{\to}}}\ \underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$$

of isomorphisms of ind-topological modules (resp. topological modules) (cf. the poly-isomorphism $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F}^{\vdash}) \overset{\text{"Kum" poly}}{\xrightarrow{\sim}} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})$ *of Corollary 11.21 (2)). We also have a natural functorial algorithm, with respect to the* $p_{\underline{v}}$*-adic Frobenioid* ${}^{\dagger}\mathcal{F}_{\underline{v}}$ *(resp. the triple* ${}^{\dagger}\mathcal{F}_{\underline{v}}$*), to construct isomorphisms (resp. poly-isomorphisms of the* $\{\pm1\}\times\{\pm1\}$**-orbit** *arising from the independent* $\{\pm1\}$*-actions on each of the direct factors "*$k^{\sim}(G) = C^{\sim}\times C^{\sim}$*" in the notation of Proposition 5.4)*

$$(\Psi^{\mathrm{gp}}_{{}^{\ddagger}\mathcal{F}_{\underline{v}}} \overset{(\text{poly})}{\cong})\ \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \ \overset{\text{tauto}}{\overset{\sim}{\to}}\ \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}) \ \overset{\text{induced by Kum}}{\xrightarrow{\sim}}\ \underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$$

$$(\textit{resp.}\ \ (\Psi^{\mathrm{gp}}_{{}^{\ddagger}\mathcal{F}_{\underline{v}}} \overset{(\text{poly})}{\cong})\ \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \ \overset{\text{tauto}}{\overset{\sim}{\to}}\ \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}) \ \overset{\substack{\text{induced by Kum} \\ \text{poly}, \{\pm1\}\times\{\pm1\}}}{\xrightarrow{\sim}}\ \underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})\ \ )$$

of isomorphisms of ind-topological modules (resp. topological modules) (cf. the isomorphism $\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})^{\times}_{\underline{v}} \overset{\sim}{\to} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})^{\times}_{\underline{v}}$ *of Corollary 11.20 (2) and the Kummer isomorphism* $\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F}) \underset{\text{Kum}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})$ *of Corollary 11.21), which is compatible with the respective* ${}^{\dagger}G_{\underline{v}}$ *and* $G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$*-actions, the respective log-shells, and the respective log-volumes on these log-shells (resp. compatible with the respective log-shells, and the respective angular and radial log-volumes on these log-shells).*

The above (poly-)isomorphisms induce collections of (poly-)isomorphisms

$$\underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}^{\vdash\times\mu}) := \{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}} \ \overset{\text{"Kum"}}{\underset{\text{poly}}{\overset{\sim}{\to}}}\ \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{D}^{\vdash}) := \{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}},$$

$$\mathcal{I}_{{}^\dagger\mathfrak{F}^{\vdash\times\mu}} := \{\mathcal{I}_{{}^\dagger\mathcal{F}^{\vdash\times\mu}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}} \overset{\substack{\text{“Kum”}\\ \text{poly}}}{\overset{\sim}{\to}} \mathcal{I}_{{}^\dagger\mathfrak{D}^{\vdash}} := \{\mathcal{I}_{{}^\dagger\mathcal{D}^{\vdash}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}},$$

$$(\Psi^{\mathrm{gp}}_{\mathrm{cns}}({}^\ddagger\mathfrak{F}) := \{\Psi^{\mathrm{gp}}_{{}^\ddagger\mathcal{F}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}} \overset{\text{(poly)}}{\cong}) \ \underline{\mathfrak{log}}({}^\dagger\mathfrak{F}) := \{\underline{\mathfrak{log}}({}^\dagger\mathcal{F}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}} \overset{\text{tauto}}{\overset{\sim}{\to}} \underline{\mathfrak{log}}({}^\dagger\mathfrak{F}^{\vdash\times\mu}) \overset{\substack{\text{induced by Kum}\\ \text{poly}}}{\overset{\sim}{\to}} \underline{\mathfrak{log}}({}^\dagger\mathfrak{D}^{\vdash}),$$

$$\mathcal{I}_{{}^\dagger\mathfrak{F}} := \{\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}} \overset{\text{tauto}}{\overset{\sim}{\to}} \mathcal{I}_{{}^\dagger\mathfrak{F}^{\vdash\times\mu}} \overset{\substack{\text{induced by Kum}\\ \text{poly}}}{\overset{\sim}{\to}} \mathcal{I}_{{}^\dagger\mathfrak{D}^{\vdash}}$$

(Here, we regard each $\Psi^{\mathrm{gp}}_{{}^\ddagger\mathcal{F}_{\underline{v}}}$ *as equipped with* $G_{\underline{v}}({}^\ddagger\Pi_{\underline{v}})$*-action in the definition of* $\Psi^{\mathrm{gp}}_{\mathrm{cns}}({}^\ddagger\mathfrak{F})$*).*

(5) **((Étale-like) Holomorphic Vertically Coric Log-shells)** *Let* ${}^*\mathfrak{D}$ *be a* $\mathcal{D}$*-prime-strip with associated* $\mathcal{D}^{\vdash}$*-prime-strip* ${}^*\mathfrak{D}^{\vdash}$*. We write*

$$\mathfrak{F}({}^*\mathfrak{D})$$

for the $\mathcal{F}$*-prime-strip determined by* $\Psi_{\mathrm{cns}}({}^*\mathfrak{D})$*. Assume that* ${}^\dagger\mathfrak{F} = {}^\ddagger\mathfrak{F} = \mathfrak{F}({}^*\mathfrak{D})$*, and that the given* $\mathfrak{log}$*-link is the full* $\mathfrak{log}$*-link* ${}^\dagger\mathfrak{F} \overset{\text{full }\mathfrak{log}}{\longrightarrow} {}^\ddagger\mathfrak{F} = \mathfrak{F}({}^*\mathfrak{D})$*. We have a functorial algorithm, with respect to the* $\mathcal{D}$*-prime-strip* ${}^*\mathfrak{D}$*, to construct a collection of topological subspaces*

$$\mathcal{I}_{{}^*\mathfrak{D}} := \mathcal{I}_{{}^\dagger\mathfrak{F}}$$

(which is called a collection of **vertically coric étale-like holomorphic log-shell** *associated to* ${}^*\mathfrak{D}$*) of the collection* $\Psi^{\mathrm{gp}}_{\mathrm{cns}}({}^*\mathfrak{D}) = \Psi^{\mathrm{gp}}_{\mathrm{cns}}({}^*\mathfrak{F})$*,and a collection of isomorphisms*

$$\mathcal{I}_{{}^*\mathfrak{D}} \overset{\sim}{\to} \mathcal{I}_{{}^*\mathfrak{D}^{\vdash}}$$

(cf. the isomorphism $\Psi_{\mathrm{cns}}({}^\ddagger\mathfrak{D})^{\times}_{\underline{v}} \overset{\sim}{\to} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\ddagger\mathfrak{D}^{\vdash})^{\times}_{\underline{v}}$ *of Corollary 11.20 (2)).*

Remark 12.2.1. (Kummer Theory, [IUTchIII, Proposition 1.2 (iv)]) Note that the **Kummer isomorphisms**

$$\Psi_{\mathrm{cns}}({}^\dagger\mathfrak{F}) \overset{\text{Kum}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^\dagger\mathfrak{D}), \quad \Psi_{\mathrm{cns}}({}^\ddagger\mathfrak{F}) \overset{\text{Kum}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^\ddagger\mathfrak{D})$$

of Corollary 11.21 (1) are *not* compatible with the (poly-)isomorphism $\Psi_{\mathrm{cns}}({}^\dagger\mathfrak{D}) \overset{\text{(poly)}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^\ddagger\mathfrak{D})$ of (1), with respect to the diagrams ($\mathfrak{log}_{\mathrm{non}}$) and ($\mathfrak{log}_{\mathrm{arc}}$).

Remark 12.2.2. (Frobenius-picture, [IUTchIII, Proposition 1.2 (x)]) Let $\{{}^n\mathfrak{F}\}_{n\in\mathbb{Z}}$ be a collection of $\mathcal{F}$-prime-strips indexed by $\mathbb{Z}$ with associated collection of $\mathcal{D}$-prime-strips (resp. $\mathcal{D}^{\vdash}$-prime-strips) $\{{}^n\mathfrak{D}\}_{n\in\mathbb{Z}}$ (resp. $\{{}^n\mathfrak{D}^{\vdash}\}_{n\in\mathbb{Z}}$). Then the chain of full $\mathfrak{log}$-links

$$\cdots \xrightarrow{\text{full } \mathfrak{log}} {}^{(n-1)}\mathfrak{F} \xrightarrow{\text{full } \mathfrak{log}} {}^{n}\mathfrak{F} \xrightarrow{\text{full } \mathfrak{log}} {}^{(n+1)}\mathfrak{F} \xrightarrow{\text{full } \mathfrak{log}} \cdots$$

of $\mathcal{F}$-prime-strips (which is called the **Frobenius-picture of $\mathfrak{log}$-links for $\mathcal{F}$-prime-strips**) induces chains of full poly-isomorphisms

$$\cdots \overset{\text{full poly}}{\overset{\sim}{\to}} {}^{(n-1)}\mathfrak{D} \overset{\text{full poly}}{\overset{\sim}{\to}} {}^{n}\mathfrak{D} \overset{\text{full poly}}{\overset{\sim}{\to}} {}^{(n+1)}\mathfrak{D} \overset{\text{full poly}}{\overset{\sim}{\to}} \cdots,$$

$$\cdots \overset{\text{full poly}}{\overset{\sim}{\to}} {}^{(n-1)}\mathfrak{D}^{\vdash} \overset{\text{full poly}}{\overset{\sim}{\to}} {}^{n}\mathfrak{D}^{\vdash} \overset{\text{full poly}}{\overset{\sim}{\to}} {}^{(n+1)}\mathfrak{D}^{\vdash} \overset{\text{full poly}}{\overset{\sim}{\to}} \cdots$$

of $\mathcal{D}$-prime-strips and $\mathcal{D}^{\vdash}$-prime-strips respectively. We identify ${}^{(-)}\mathfrak{D}$'s by these full poly-isomorphisms, then we obtain a diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\text{full } \mathfrak{log}"] \arrow[drr, "\text{Kum}"'] & \Psi_{\text{cns}}({}^{(n-1)}\mathfrak{F}) \arrow[r, "\text{full } \mathfrak{log}"] \arrow[dr, "\text{Kum}"] & \Psi_{\text{cns}}({}^{n}\mathfrak{F}) \arrow[r, "\text{full } \mathfrak{log}"] \arrow[d, "\text{Kum}"'] & \Psi_{\text{cns}}({}^{(n+1)}\mathfrak{F}) \arrow[r, "\text{full } \mathfrak{log}"] \arrow[dl, "\text{Kum}"'] & \cdots \arrow[dll, "\text{Kum}"] \\
& & \Psi_{\text{cns}}({}^{(-)}\mathfrak{D}). & &
\end{tikzcd}$$

This diagram expresses the vertical coricity of $\Psi_{\text{cns}}({}^{(-)}\mathfrak{D})$. Note that Remark 12.2.1 says that this diagram is *not* commutative.

Proof. Proposition follows from the definitions. □

Definition 12.3. ($\mathfrak{log}$-Links Between $\boxtimes\boxplus$-Hodge Theatres, [IUTchIII, Proposition 1.3 (i)]) Let

$$^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}, \quad ^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus}$$

be $\boxtimes\boxplus$-Hodge theatres with associated $\mathcal{D}$-$\boxtimes\boxplus$-Hodge theatres $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$, $^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$ respectively. We write $^{\dagger}\mathfrak{F}_{>}$, $^{\dagger}\mathfrak{F}_{\succ}$, $^{\dagger}\mathfrak{F}_{j}$ (in $^{\dagger}\mathfrak{F}_{J}$), $^{\dagger}\mathfrak{F}_{t}$ (in $^{\dagger}\mathfrak{F}_{T}$) (resp. $^{\ddagger}\mathfrak{F}_{>}$, $^{\ddagger}\mathfrak{F}_{\succ}$, $^{\ddagger}\mathfrak{F}_{j}$ (in $^{\ddagger}\mathfrak{F}_{J}$), $^{\ddagger}\mathfrak{F}_{t}$ (in $^{\ddagger}\mathfrak{F}_{T}$)) for $\mathcal{F}$-prime-strips in the $\boxtimes\boxplus$-Hodge theatre $^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$ (resp. $^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus}$). For an isomorphism

$$\Xi : {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \overset{\sim}{\to} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$$

of $\mathcal{D}$-$\boxtimes\boxplus$-Hodge theatres, the poly-isomorphisms determined by Ξ between the $\mathcal{D}$-prime-strips associated to $^{\dagger}\mathfrak{F}_{>}$, $^{\ddagger}\mathfrak{F}_{>}$ (resp. $^{\dagger}\mathfrak{F}_{\succ}$, $^{\ddagger}\mathfrak{F}_{\succ}$, resp. $^{\dagger}\mathfrak{F}_{j}$, $^{\ddagger}\mathfrak{F}_{j}$, resp. $^{\dagger}\mathfrak{F}_{t}$, $^{\ddagger}\mathfrak{F}_{t}$) uniquely determines a poly-isomorphism $\mathfrak{log}(^{\dagger}\mathfrak{F}_{>}) \overset{\text{poly}}{\overset{\sim}{\to}} {}^{\ddagger}\mathfrak{F}_{>}$ (resp. $\mathfrak{log}(^{\dagger}\mathfrak{F}_{\succ}) \overset{\text{poly}}{\overset{\sim}{\to}} {}^{\ddagger}\mathfrak{F}_{\succ}$, resp. $\mathfrak{log}({}^\dagger\mathfrak{F}_j) \overset{\text{poly}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{F}_j$, resp. $\mathfrak{log}({}^\dagger\mathfrak{F}_t) \overset{\text{poly}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{F}_t$), hence a $\mathfrak{log}$-link ${}^\dagger\mathfrak{F}_> \overset{\mathfrak{log}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{F}_>$ (resp. ${}^\dagger\mathfrak{F}_\succ \overset{\mathfrak{log}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{F}_\succ$, resp. ${}^\dagger\mathfrak{F}_j \overset{\mathfrak{log}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{F}_j$, resp. ${}^\dagger\mathfrak{F}_t \overset{\mathfrak{log}}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{F}_t$), by Lemma 10.10 (2). We write

$$ {}^\dagger\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathcal{HT}^{\boxtimes\boxplus} $$

for the collection of data $\Xi : {}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \xrightarrow{\sim} {}^\ddagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$, ${}^\dagger\mathfrak{F}_> \overset{\mathfrak{log}}{\to} {}^\ddagger\mathfrak{F}_>$, ${}^\dagger\mathfrak{F}_\succ \overset{\mathfrak{log}}{\to} {}^\ddagger\mathfrak{F}_\succ$, $\{{}^\dagger\mathfrak{F}_j \overset{\mathfrak{log}}{\to} {}^\ddagger\mathfrak{F}_j\}_{j\in J}$, and $\{{}^\dagger\mathfrak{F}_t \overset{\mathfrak{log}}{\to} {}^\ddagger\mathfrak{F}_t\}_{t\in T}$, and we shall refer to it as a **$\mathfrak{log}$-link from ${}^\dagger\mathcal{HT}^{\boxtimes\boxplus}$ to ${}^\ddagger\mathcal{HT}^{\boxtimes\boxplus}$**. When Ξ is replaced by a poly-isomorphism ${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \overset{\text{poly}}{\xrightarrow{\sim}} {}^\ddagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$ (resp. the full poly-isomorphism ${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^\ddagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$), then we shall refer to the resulting collection of $\mathfrak{log}$-links constructed from each constituent isomorphism of the poly-isomorphism (resp. full poly-isomorphism) as a **$\mathfrak{log}$-link** (resp. the **full $\mathfrak{log}$-link from ${}^\dagger\mathcal{HT}^{\boxtimes\boxplus}$ to ${}^\ddagger\mathcal{HT}^{\boxtimes\boxplus}$**, and we also write it

$$ {}^\dagger\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathcal{HT}^{\boxtimes\boxplus} \quad (\text{resp. } {}^\dagger\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\text{full } \mathfrak{log}} {}^\ddagger\mathcal{HT}^{\boxtimes\boxplus}). $$

Note that we have to carry out the construction of the $\mathfrak{log}$-link first for single Ξ for the purpose of maintaining the compatibility with the crucial **global $\{\pm 1\}$-synchronisation** in the $\boxplus$-Hodge theatre ([IUTchIII, Remark 1.3.1]) (cf. Proposition 10.33 and Corollary 11.20 (3)) (For a given poly-isomorphism of $\boxtimes\boxplus$-Hodge theatres, if we considered the uniquely determined poly-isomorphisms on $\mathcal{F}$-prime-strips induced by the poly-isomorphisms on $\mathcal{D}$-prime-strips by the given poly-isomorphism of $\boxtimes\boxplus$-Hodge theatres, not the "constituent-isomorphism-wise" manner, then the crucial global $\{\pm 1\}$-synchronisation would collapse (cf. [IUTchI, Remark 6.12.4 (iii)], [IUTchII, Remark 4.5.3 (iii)])).

Remark 12.3.1. (Frobenius-picture and Vertical Coricity of $\mathcal{D}$-$\boxtimes\boxplus$-Hodge theatres, [IUTchIII, Proposition 1.3 (ii), (iv)]) Let $\{{}^n\mathcal{HT}^{\boxtimes\boxplus}\}_{n\in\mathbb{Z}}$ be a collection of $\boxtimes\boxplus$-Hodge theatres indexed by $\mathbb{Z}$ with associated collection of $\mathcal{D}$-$\boxtimes\boxplus$-Hodge theatres $\{{}^n\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}\}_{n\in\mathbb{Z}}$. Then the chain of full $\mathfrak{log}$-links

$$ \cdots \xrightarrow{\text{full } \mathfrak{log}} {}^{(n-1)}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\text{full } \mathfrak{log}} {}^{n}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\text{full } \mathfrak{log}} {}^{(n+1)}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\text{full } \mathfrak{log}} \cdots $$

of $\boxtimes\boxplus$-Hodge theatres (which is called the **Frobenius-picture of $\mathfrak{log}$-links for $\boxtimes\boxplus$-Hodge theatres**) induces chains of full poly-isomorphisms

$$ \cdots \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{(n-1)}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{(n+1)}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \overset{\text{full poly}}{\xrightarrow{\sim}} \cdots, $$

of $\mathcal{D}$-$\boxtimes\boxplus$-Hodge theatres. We identify ${}^{(-)}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$'s by these full poly-isomorphisms, then we obtain a diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\text{full }\mathfrak{log}"] \arrow[drr, "\mathrm{Kum}"'] & {}^{(n-1)}\mathcal{HT}^{\boxtimes\boxplus} \arrow[r, "\text{full }\mathfrak{log}"] \arrow[dr, "\mathrm{Kum}"] & {}^{n}\mathcal{HT}^{\boxtimes\boxplus} \arrow[r, "\text{full }\mathfrak{log}"] \arrow[d, "\mathrm{Kum}"'] & {}^{(n+1)}\mathcal{HT}^{\boxtimes\boxplus} \arrow[r, "\text{full }\mathfrak{log}"] \arrow[dl, "\mathrm{Kum}"'] & \cdots \arrow[dll, "\mathrm{Kum}"] \\
& & {}^{(-)}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}, & &
\end{tikzcd}$$

where Kum expresses the Kummer isomorphisms in Remark 12.2.1. This diagram expresses the vertical coricity of ${}^{(-)}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$. Note that Remark 12.2.1 says that this diagram is *not* commutative.

Definition 12.4. ([IUTchIII, Definition 1.4]) Let $\{{}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}\}_{n,m\in\mathbb{Z}}$ be a collection of $\boxtimes\boxplus$-Hodge theatres indexed by pairs of integers. We shall refer to either of the diagrams

$$\begin{tikzcd}
& \vdots & \vdots & \\
\cdots \arrow[r, "\Theta^{\times\mu}"] & {}^{n,m+1}\mathcal{HT}^{\boxtimes\boxplus} \arrow[r, "\Theta^{\times\mu}"] \arrow[u, "\text{full }\mathfrak{log}"'] & {}^{n+1,m+1}\mathcal{HT}^{\boxtimes\boxplus} \arrow[r, "\Theta^{\times\mu}"] \arrow[u, "\text{full }\mathfrak{log}"'] & \cdots \\
\cdots \arrow[r, "\Theta^{\times\mu}"] & {}^{n,m}\mathcal{HT}^{\boxtimes\boxplus} \arrow[r, "\Theta^{\times\mu}"] \arrow[u, "\text{full }\mathfrak{log}"'] & {}^{n+1,m}\mathcal{HT}^{\boxtimes\boxplus} \arrow[r, "\Theta^{\times\mu}"] \arrow[u, "\text{full }\mathfrak{log}"'] & \cdots, \\
& \vdots \arrow[u, "\text{full }\mathfrak{log}"'] & \vdots \arrow[u, "\text{full }\mathfrak{log}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \vdots & \vdots & \\
\cdots \arrow[r, "\Theta^{\times\mu}_{\mathrm{gau}}"] & {}^{n,m+1}\mathcal{HT}^{\boxtimes\boxplus} \arrow[r, "\Theta^{\times\mu}_{\mathrm{gau}}"] \arrow[u, "\text{full }\mathfrak{log}"'] & {}^{n+1,m+1}\mathcal{HT}^{\boxtimes\boxplus} \arrow[r, "\Theta^{\times\mu}_{\mathrm{gau}}"] \arrow[u, "\text{full }\mathfrak{log}"'] & \cdots \\
\cdots \arrow[r, "\Theta^{\times\mu}_{\mathrm{gau}}"] & {}^{n,m}\mathcal{HT}^{\boxtimes\boxplus} \arrow[r, "\Theta^{\times\mu}_{\mathrm{gau}}"] \arrow[u, "\text{full }\mathfrak{log}"'] & {}^{n+1,m}\mathcal{HT}^{\boxtimes\boxplus} \arrow[r, "\Theta^{\times\mu}_{\mathrm{gau}}"] \arrow[u, "\text{full }\mathfrak{log}"'] & \cdots \\
& \vdots \arrow[u, "\text{full }\mathfrak{log}"'] & \vdots \arrow[u, "\text{full }\mathfrak{log}"'] &
\end{tikzcd}$$

as the **log-theta-lattice**. We shall refer to the former diagram (resp. the latter diagram) as **non-Gaussian** (resp. **Gaussian**).

Remark 12.4.1. For the proof of the main Theorem 0.1, we need only two adjacent columns in the (final update version of) log-theta-lattice. In the analogy with p-adic

Teichmüller theory, this means that we need only "lifting to modulo p^2" (cf. the last table in Section 3.5).

Theorem 12.5. (Bi-Cores of the Log-Theta-Lattice, [IUTchIII, Theorem 1.5]) *Fix an initial Th-data*

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon}).$$

For any Gaussian log-theta-lattice corresponding to this initial Θ-data, we write ${}^{n,m}\mathfrak{D}_{\succ}$ (resp. ${}^{n,m}\mathfrak{D}_{>}$) for the $\mathcal{D}$-prime-strip labelled "$\succ$" (resp. "$>$") of the $\boxtimes\boxplus$-Hodge theatre.

(1) **(Vertical Coricity)** *The vertical arrows of the Gaussian log-theta-lattice induce the full poly-isomorphisms between the associated $\mathcal{D}$-$\boxtimes\boxplus$-Hodge theatres*

$$\cdots \overset{\text{full poly}}{\overset{\sim}{\to}} {}^{n,m}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \overset{\text{full poly}}{\overset{\sim}{\to}} {}^{n,m+1}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \overset{\text{full poly}}{\overset{\sim}{\to}} \cdots,$$

where n is fixed (cf. Remark 12.3.1).

(2) **(Horizontal Coricity)** *The horizontal arrows of the Gaussian log-theta-lattice induce the full poly-isomorphisms between the associated $\mathcal{F}^{\vdash\times\mu}$-prime-strips*

$$\cdots \overset{\text{full poly}}{\overset{\sim}{\to}} {}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \overset{\text{full poly}}{\overset{\sim}{\to}} {}^{n+1,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \overset{\text{full poly}}{\overset{\sim}{\to}} \cdots$$

where m is fixed (cf. Corollary 11.24 (4)).

(3) **(Bi-coric $\mathcal{F}^{\vdash\times\mu}$-Prime-strips)** *Let ${}^{n,m}\mathfrak{D}^{\vdash}_{\Delta}$ for the $\mathcal{D}^{\vdash}$-prime-strip associated to the $\mathcal{F}^{\vdash}$-prime-strip ${}^{n,m}\mathfrak{F}^{\vdash}_{\Delta}$ of Corollary 11.24 (1) for the $\boxtimes\boxplus$-Hodge theatre ${}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}$. We idenfity the collections $\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{\succ})_0$, $\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{\succ})_{\langle\mathbb{F}^{\circledast}_l\rangle}$ of data via the isomorphism $\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{\succ})_0 \overset{\text{diag}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{\succ})_{\langle\mathbb{F}^{\circledast}_l\rangle}$ constructed in Corollary 11.20 (3), and we write*

$$\mathfrak{F}^{\vdash}_{\Delta}({}^{n,m}\mathfrak{D}_{\succ})$$

for the resulting $\mathcal{F}^{\vdash}$-prime-strip (Recall that "$\Delta = \{0, \langle\mathbb{F}^{\circledast}_l\rangle\}$") Note also we have a natural identification isomorphism $\mathfrak{F}^{\vdash}_{\Delta}({}^{n,m}\mathfrak{D}_{\succ}) \overset{\sim}{\to} \mathfrak{F}^{\vdash}_{>}({}^{n,m}\mathfrak{D}_{>})$, where we write $\mathfrak{F}^{\vdash}_{>}({}^{n,m}\mathfrak{D}_{>})$ for the $\mathcal{F}^{\vdash}$-prime-strip determined by $\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{>})$ (Recall that "$>= \{0, \succ\}$". cf. Lemma 10.38). We write

$$\mathfrak{F}^{\vdash\times}_{\Delta}({}^{n,m}\mathfrak{D}_{\succ}),\quad \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m}\mathfrak{D}_{\succ})$$

for the associated $\mathcal{F}^{\vdash\times}$-prime-strip and $\mathcal{F}^{\vdash\times\mu}$-prime-strip to $\mathfrak{F}^{\vdash}_{\Delta}({}^{n,m}\mathfrak{D}_{\succ})$, respectively. By the isomorphism "$\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})^{\times}_{\underline{v}} \overset{\sim}{\to} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})^{\times}_{\underline{v}}$" of Corollary 11.20 (2), we have a functorial algorithm, with respect to the $\mathcal{D}^{\vdash}$-prime-strip ${}^{n,m}\mathfrak{D}^{\vdash}_{\Delta}$, to construct *an $\mathcal{F}^{\vdash\times}$-prime-strip $\mathfrak{F}^{\vdash\times}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})$. We also have a functorial algorithm, with respect to the $\mathcal{D}$-prime-strip ${}^{n,m}\mathfrak{D}_{\succ}$, to construct an isomorphism*

$$\mathfrak{F}^{\vdash\times}_{\Delta}({}^{n,m}\mathfrak{D}_{\succ}) \overset{\text{tauto}}{\overset{\sim}{\to}} \mathfrak{F}^{\vdash\times}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta}),$$

by definitions. Then the poly-isomorphisms of (1) and (2) induce poly-isomorphisms

$$\cdots \overset{\text{poly}}{\overset{\sim}{\to}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m}\mathfrak{D}_{\succ}) \overset{\text{poly}}{\overset{\sim}{\to}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m+1}\mathfrak{D}_{\succ}) \overset{\text{poly}}{\overset{\sim}{\to}} \cdots,$$

$$\cdots \overset{\text{poly}}{\overset{\sim}{\to}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta}) \overset{\text{poly}}{\overset{\sim}{\to}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n+1,m}\mathfrak{D}^{\vdash}_{\Delta}) \overset{\text{poly}}{\overset{\sim}{\to}} \cdots$$

of $\mathcal{F}^{\vdash\times\mu}$-prime-strips, respectively. Note that the poly-isomorphisms (as sets of isomorphisms) of $\mathcal{F}^{\vdash\times\mu}$-prime-strips in the first line is strictly smaller than the poly-isomorphisms (as sets of isomorphisms) of $\mathcal{F}^{\vdash\times\mu}$-prime-strips in the second line in general, with respect to the above isomorphism $\mathfrak{F}^{\vdash\times}_{\Delta}({}^{n,m}\mathfrak{D}_{\succ}) \overset{\text{tauto}}{\overset{\sim}{\to}} \mathfrak{F}^{\vdash\times}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})$, by the existence of non-scheme-theoretic automorphisms of absolute Galois groups of MLF*'s (cf. the inclusion* (nonGC for MLF) *in Section 3.5), and that the poly-morphisms in the second line are not full by Remark 8.5.1. In particular, by composing these isomorphisms, we obtain poly-isomorphisms*

$$\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta}) \overset{\text{poly}}{\overset{\sim}{\to}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n',m'}\mathfrak{D}^{\vdash}_{\Delta})$$

of $\mathcal{F}^{\vdash\times\mu}$-prime-strips for any $n', m' \in \mathbb{Z}$. This means that the $\mathcal{F}^{\vdash\times\mu}$-prime-strip $\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})$ is coric both horizontally and vertically, i.e., it is **bi-coric***. Finally, the Kummer isomorphism "$\Psi_{\text{cns}}({}^{\ddagger}\mathfrak{F}) \overset{\text{Kum}}{\overset{\sim}{\to}} \Psi_{\text{cns}}({}^{\ddagger}\mathfrak{D})$" of Corollary 11.21 (1) determines* **Kummer isomorphism**

$${}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \overset{\text{induced by Kum}}{\overset{\sim}{\longrightarrow}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})$$

which is compatible with the poly-isomorphisms of (2), and the $\times\mu$-Kummer structures at $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$ and a similar compatibility for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$ (cf. Definition10.12 (1)).

(4) **(Bi-coric Mono-analytic Log-shells)** *The poly-isomorphisms in the bi-coricity in (3) induce poly-isomorphisms*

$$\left\{\mathcal{I}_{{}^{n,m}\mathfrak{D}^{\vdash}_{\Delta}} \subset \underline{\mathfrak{log}}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})\right\} \overset{\text{poly}}{\overset{\sim}{\to}} \left\{\mathcal{I}_{{}^{n',m'}\mathfrak{D}^{\vdash}_{\Delta}} \subset \underline{\mathfrak{log}}({}^{n',m'}\mathfrak{D}^{\vdash}_{\Delta})\right\},$$

$$\left\{\mathcal{I}_{\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})} \subset \underline{\mathfrak{log}}(\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta}))\right\} \overset{\text{poly}}{\overset{\sim}{\to}} \left\{\mathcal{I}_{\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n',m'}\mathfrak{D}^{\vdash}_{\Delta})} \subset \underline{\mathfrak{log}}(\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n',m'}\mathfrak{D}^{\vdash}_{\Delta}))\right\}$$

for any $n, m, n', m, \in \mathbb{Z}$, which are compatible with the natural poly-isomorphisms

$$\left\{\mathcal{I}_{\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})} \subset \underline{\mathfrak{log}}(\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta}))\right\} \overset{\text{“Kum”}}{\underset{}{\overset{\text{poly}}{\xrightarrow{\sim}}}} \left\{\mathcal{I}_{{}^{n,m}\mathfrak{D}^{\vdash}_{\Delta}} \subset \underline{\mathfrak{log}}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})\right\}$$

of Proposition 12.2 (4). On the other hand, by Definition 12.1 (1) for “$\Psi_{\text{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_0$” and “$\Psi_{\text{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_{\langle\mathbb{F}^{\circledast}_{l}\rangle}$” in Corollary 11.24 (1) (which construct ${}^{n,m}\mathfrak{F}^{\vdash}_{\Delta}$), we obtain

$$\mathcal{I}_{{}^{n,m}\mathfrak{F}_{\Delta}} \subset \underline{\mathfrak{log}}({}^{n,m}\mathfrak{F}_{\Delta})$$

(This is a slight abuse of notation since no $\mathcal{F}$-prime-strip “${}^{n,m}\mathfrak{F}_{\Delta}$” has been defined). Then we have natural poly-isomorphisms

$$\left\{\mathcal{I}_{{}^{n,m}\mathfrak{F}_{\Delta}} \subset \underline{\mathfrak{log}}({}^{n,m}\mathfrak{F}_{\Delta})\right\} \overset{\text{tauto}}{\xrightarrow{\sim}} \left\{\mathcal{I}_{{}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta}} \subset \underline{\mathfrak{log}}({}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta})\right\} \overset{\substack{\text{induced by Kum}\\ \text{poly}}}{\xrightarrow{\sim}} \left\{\mathcal{I}_{{}^{n,m}\mathfrak{D}^{\vdash}_{\Delta}} \subset \underline{\mathfrak{log}}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})\right\}$$

(cf. Proposition 12.2 (4)), where the last poly-isomorphism is compatible with the poly-isomorphisms induced by the poly-isomorphisms of (2).

(5) **(Bi-coric Mono-analytic Global Realified Frobenioids)** *The poly-isomorphisms ${}^{n,m}\mathfrak{D}^{\vdash}_{\Delta} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{n',m'}\mathfrak{D}^{\vdash}_{\Delta}$ of $\mathcal{D}^{\vdash}$-prime-strips induced by the full poly-isomorphisms of (1) and (2) for n, m, n', m' induce an isomorphism*

$$\begin{aligned}&(\mathcal{D}^{\Vdash}({}^{n,m}(\mathfrak{D}^{\vdash}_{\Delta}), \text{Prime}(\mathcal{D}^{\Vdash}({}^{n,m}(\mathfrak{D}^{\vdash}_{\Delta})) \xrightarrow{\sim} \underline{\mathbb{V}}, \{{}^{n,m}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})\\ &\xrightarrow{\sim} (\mathcal{D}^{\Vdash}({}^{n',m'}(\mathfrak{D}^{\vdash}_{\Delta}), \text{Prime}(\mathcal{D}^{\Vdash}({}^{n',m'}(\mathfrak{D}^{\vdash}_{\Delta})) \xrightarrow{\sim} \underline{\mathbb{V}}, \{{}^{n',m'}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})\end{aligned}$$

of triples (cf. Corollary 11.20 (2), and Corollary 11.24 (5)). Moreover, this isomorphism of triples is compatible, with respect to the horizontal arrows of the Gaussian log-theta-lattice, with the $\mathbb{R}_{>0}$-orbits of the isomorphisms

$$\begin{aligned}&({}^{n,m}\mathcal{C}^{\Vdash}_{\Delta}, \text{Prime}({}^{n,m}\mathcal{C}^{\Vdash}_{\Delta}) \xrightarrow{\sim} \underline{\mathbb{V}}, \{{}^{n,m}\rho_{\Delta,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})\\ &\overset{\text{“Kum”}}{\xrightarrow{\sim}} (\mathcal{D}^{\Vdash}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta}), \text{Prime}(\mathcal{D}^{\Vdash}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})) \xrightarrow{\sim} \underline{\mathbb{V}}, \{{}^{n,m}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})\end{aligned}$$

of triples, obtained by the functorial algorithm in Corollary 11.21 (2) (cf. also Corollary 11.24 (1), (5)).

Proof. Theorem follows from the definitions. □

§ 12.2. Kummer Compatible Multiradial Theta Monoids.

In this subsection, we globalise the multiradiality of local theta monoids (Proposition 11.7, and Proposition 11.15) to cover the theta monoids and the global realified theta monoids in Corollary 11.20 (4), (5) Corollary 11.21 (4), (5), in the setting of log-theta-lattice.

In this subsection, let ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$ be a $\boxtimes\boxplus$-Hodge theatre with respect to the fixed initial Θ-data, and ${}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}$ a collection of $\boxtimes\boxplus$-Hodge theatres arising from a Gaussian log-theta-lattice.

Proposition 12.6. (Vertical Coricity and Kummer Theory of Theta Monoids, [IUTchIII, Proposition 2.1]) *We summarise the theta monoids and their Kummer theory as follows:*

(1) **(Vertically Coric Theta Monoids)** *By Corollary 11.20 (4) (resp. Corollary 11.20 (5)), each isomorphism of the full poly-isomorphism induced by a vertical arrow of the Gaussian log-theta-lattice induces a compatible collection*

$$_{(\infty)}\Psi_{\mathrm{env}}({}^{n,m}\mathfrak{D}_{>}) \xrightarrow{\sim} {}_{(\infty)}\Psi_{\mathrm{env}}({}^{n,m+1}\mathfrak{D}_{>}) \quad (\textit{resp.}\ \ \mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{n,m}\mathfrak{D}^{\vdash}_{>}) \xrightarrow{\sim} \mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{n,m+1}\mathfrak{D}^{\vdash}_{>})\ \ )$$

of isomorphisms, where the last isomorphism is compatible with the respective bijection $\mathrm{Prime}(-) \xrightarrow{\sim} \underline{\mathbb{V}}$*, and localisation isomorphisms.*

(2) **(Kummer Isomorphisms)** *By Corollary 11.21 (4) (resp. Corollary 11.21 (5)), we have a functorial algorithm, with respect to the* $\boxtimes\boxplus$*-Hodge theatre* ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$*, to construct the Kummer isomorphism*

$$_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta}) \overset{\mathrm{Kum}}{\xrightarrow{\sim}} {}_{(\infty)}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \quad (\textit{resp.}\ \ \mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}) \overset{\text{“Kum”}}{\xrightarrow{\sim}} \mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})\ \ ).$$

Here, the resp'd isomorphism is compatible with the respective $\mathrm{Prime}(-) \xrightarrow{\sim} \underline{\mathbb{V}}$ *and the respective localisation isomorphisms. Note that the collection* $\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$ *of data gives us an* $\mathcal{F}^{\vdash}$*-prime-strip* $\mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$*, and an* $\mathcal{F}^{\Vdash}$*-prime-strip* $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) = (\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>}), \mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})) \xrightarrow{\sim} \underline{\mathbb{V}}, \mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}), \{\rho_{\mathcal{D}^{\Vdash}_{\mathrm{env}},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ *and that the non-resp'd (resp. the resp'd) Kummer isomorphism in the above can be interpreted as an isomorphism*

$${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}} \overset{\text{induced by Kum}}{\xrightarrow{\sim}} \mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \quad (\textit{resp.}\ \ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \overset{\text{“Kum”}}{\xrightarrow{\sim}} \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})\ \ )$$

of $\mathcal{F}^{\vdash}$*-prime-strips (resp.* $\mathcal{F}^{\Vdash}$*-prime-strips).*

(3) **(Compatibility with Constant Monoids)** *By the definition of the unit portion of the theta monoids (cf. Corollary 11.24 (4)), we have natural isomorphisms*

$${}^{\dagger}\mathfrak{F}^{\vdash\times}_{\Delta} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\vdash\times}_{\mathrm{env}}, \quad \mathfrak{F}^{\vdash\times}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}),$$

which are compatible with the Kummer isomorphisms ${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}} \overset{\text{induced by Kum}}{\xrightarrow{\sim}} \mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$, ${}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \overset{\text{induced by Kum}}{\xrightarrow{\sim}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta})$ *of (2) and Theorem 12.5 (3).*

Proof. Proposition follows from the definitions. □

Theorem 12.7. (Kummer-Compatible Multiradiality of Theta Monoids, [IUTchIII, Theorem 2.2]) *Fix an initial Th-data*

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon}).$$

Let ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$ *be a* $\boxtimes\boxplus$*-Hodge theatre with respect to the fixed initial* Θ*-data.*

(1) The natural functors which send an $\mathcal{F}^{\Vdash}$*-prime-strip to the associated* $\mathcal{F}^{\Vdash\blacktriangleright\times\mu}$*- and* $\mathcal{F}^{\vdash\times\mu}$*-prime-strips and composing with the natural isomorphisms of Proposition 12.6 (3) give us natural homomorphisms*

$$\mathrm{Aut}_{\mathcal{F}^{\Vdash}}(\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})) \to \mathrm{Aut}_{\mathcal{F}^{\Vdash\blacktriangleright\times\mu}}(\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})) \twoheadrightarrow \mathrm{Aut}_{\mathcal{F}^{\vdash\times\mu}}(\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta})),$$

$$\mathrm{Aut}_{\mathcal{F}^{\Vdash}}({}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}) \to \mathrm{Aut}_{\mathcal{F}^{\Vdash\blacktriangleright\times\mu}}({}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{env}}) \twoheadrightarrow \mathrm{Aut}_{\mathcal{F}^{\vdash\times\mu}}({}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\Delta})$$

(Note that the second homomorphisms in each line are surjective), which are compatible with the Kummer isomorphisms ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \overset{\text{“Kum”}}{\xrightarrow{\sim}} \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$, ${}^{\dagger}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \overset{\text{induced by Kum}}{\xrightarrow{\sim}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta})$ *of Proposition 12.6 (2), and Theorem 12.5 (3)*

(2) **(Kummer Aspects of Multiradiality at Bad Primes)** *For* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*, we write*

$${}_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}} \ \subset\ {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}},\quad {}_{\infty}\Psi^{\perp}_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} \ \subset\ {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}},$$

for the submonoids corresponding to the respective splittings (i.e., the submonoids generated by “$\infty\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$*” and the respective torsion subgroups). We have a commutative diagram*

$$\begin{array}{ccccccccc}
{}_{\infty}\Psi^{\perp}_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} & \supset & {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mu}_{\underline{v}} & \subset & {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\times}_{\underline{v}} & \twoheadrightarrow & {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\times\mu}_{\underline{v}} & \overset{\text{poly}}{\xrightarrow{\sim}} & \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}^{\vdash}_{\Delta})^{\times\mu}_{\underline{v}} \\
\mathrm{Kum}\downarrow\cong & & \mathrm{Kum}\downarrow\cong & & \mathrm{Kum}\downarrow\cong & & \mathrm{Kum}\downarrow\cong & & \text{“Kum”}\downarrow\overset{\text{poly}}{\cong} \\
{}_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}} & \supset & {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\mu}_{\underline{v}} & \subset & {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\times}_{\underline{v}} & \twoheadrightarrow & {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\times\mu}_{\underline{v}} & \overset{\text{poly}}{\xrightarrow{\sim}} & \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta})^{\times\mu}_{\underline{v}},
\end{array}$$

where ${}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta}$ *and* ${}^{\dagger}\mathfrak{F}^{\vdash}_{\Delta}$ *are as in Theorem 12.5 (3), and Corollary 11.24 (1), respectively, the most right vertical arrow is the poly-isomorphism of Corollary 11.21 (2), the most right lower horizontal arrow is the poly-isomorphism obtained by composing the inverse of the isomorphism* $\mathfrak{F}^{\vdash\times}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \xleftarrow{\sim} \mathfrak{F}^{\vdash\times}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta})$ *of Proposition 12.6 (3) and the poly-automorphism of* $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta})^{\times\mu}_{\underline{v}}$ *induced by the full poly-automorphism of the* $\mathcal{D}^{\vdash}$*-prime-strip* ${}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta}$*, and the most right upper horizontal arrow is the poly-isomorphism defined such a manner that the diagram is commutative. This commutative diagram is compatible with the various group actions with respect to the diagram*

$$\Pi^{\mathrm{temp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}})) \twoheadrightarrow G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}})) = G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}})) = G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}})) \overset{\text{full poly}}{\xrightarrow{\sim}} G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}})).$$

Finally, each of the various composite ${}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\mu}_{\underline{v}} \to \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}^{\vdash}_{\Delta})^{\times\mu}_{\underline{v}}$ *is equal to the* **zero map***, hence the identity automorphism on the following objects is compatible* *(with respect to the various natural morphisms) with the collection of automorphisms of* $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\dagger\mathfrak{F}^{\vdash}_{\Delta})^{\times\mu}_{\underline{v}}$ *induced by any automorphism in* $\mathrm{Aut}_{\mathcal{F}^{\vdash\times\mu}}({}^\dagger\mathfrak{F}^{\vdash\times\mu}_{\Delta})$*:*

$(\perp,\mu)^{\text{ét}}_{\underline{v}}$ *the submonoid and the subgroup* $_\infty\Psi^{\perp}_{\mathrm{env}}({}^\dagger\mathfrak{D}_{>})_{\underline{v}} \supset {}_\infty\Psi_{\mathrm{env}}({}^\dagger\mathfrak{D}_{>})^{\mu}_{\underline{v}}$*,*

$(\mu_{\widehat{\mathbb{Z}}})^{\text{ét}}_{\underline{v}}$ *the cyclotome* $\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}({}^\dagger\mathcal{D}_{>,\underline{v}}))\otimes\mathbb{Q}/\mathbb{Z}$ *with respect to the natural isomorphism* $\mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}({}^\dagger\mathcal{D}_{>,\underline{v}}))\otimes\mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} {}_\infty\Psi_{\mathrm{env}}({}^\dagger\mathfrak{D}_{>})^{\mu}_{\underline{v}}$

$(\mathbb{M})^{\text{ét}}_{\underline{v}}$ *the projective system* $\mathbb{M}^{\Theta}_{*}({}^\dagger\mathcal{D}_{>,\underline{v}})$ *of mono-theta environments*

$(\mathrm{spl})^{\text{ét}}_{\underline{v}}$ *the splittings* $_\infty\Psi^{\perp}_{\mathrm{env}}({}^\dagger\mathfrak{D}_{>})_{\underline{v}} \twoheadrightarrow {}_\infty\Psi_{\mathrm{env}}({}^\dagger\mathfrak{D}_{>})^{\mu}_{\underline{v}}$ *by the restriction to the zero-labelled evaluation points (cf. Corollary 11.11 (3) and Definition 11.12 (1)).*

Proof. Theorem follows from the definitions. □

Remark 12.7.1. ([IUTchIII, Remark 2.2.2 (iii)]) Note that the Galois evaluation

$$(\mathcal{O}^{\times}{}_\infty\underline{\underline{\theta}} \supset)_\infty\underline{\underline{\theta}} \mapsto 1 \in \mathcal{O}^{\times\mu}$$

by which we obtain the multiradial algorithm of constructing the splittings of the theta monoid $\mathcal{O}^{\times}{}_\infty\underline{\underline{\theta}}$ is compatible with the respective Kummer theories of $\mathcal{O}^{\times}$ ($\subset \mathcal{O}^{\times}{}_\infty\underline{\underline{\theta}}$) and the coric $\mathcal{O}^{\times\mu}$.

If we replace the Θ-link by a naive correspondence $\underline{\underline{q}} \mapsto \underline{\underline{q}}^{\lambda}$ with $\lambda \in \mathbb{Z}_{>0}$, then the analoguous map

$$\underline{\underline{q}}^{\lambda} \mapsto 1 \in \mathcal{O}^{\times\mu}$$

is **not** compatible with the Kummer theories, since the Kummer class of $\underline{\underline{q}}^{\lambda}$ in a cohomology group of Π/Δ is never sent to the trivial element of the related cohomology group of G via the poly-isomorphism $\Pi/\Delta \overset{\text{poly}}{\xrightarrow{\sim}} G$.

Corollary 12.8. ([IUTchIII, Étale Picture of Multiradial Theta Monoids, Corollary 2.3]) *Let* $\{{}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}\}_{n,m\in\mathbb{Z}}$ *be a collection of* ⊠⊞*-Hodge theatres arising from a Gaussian log-theta-lattice, with associated* $\mathcal{D}$*-* ⊠ ⊞*-Hodge theatres* ${}^{n,m}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$*. We consider the following radial environment. We define a radial datum*

$${}^\dagger\mathfrak{R} = ({}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus},\ \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^\dagger\mathfrak{D}_{>}),\ {}^\dagger\mathfrak{R}^{\mathrm{bad}},\ \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^\dagger\mathfrak{D}^{\vdash}_{\Delta}),\ \mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}}({}^\dagger\mathfrak{D}_{>}) \overset{\text{full poly}}{\xrightarrow{\sim}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^\dagger\mathfrak{D}^{\vdash}_{\Delta}))$$

to be a quintuple of

$(\mathcal{HT}^{\mathcal{D}})^{\text{ét}}_{\mathfrak{R}}$ *a* $\mathcal{D}$*-* ⊠ ⊞*-Hodge theatre* ${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$*,*

$(\mathcal{F}^{\Vdash})^{\text{ét}}_{\mathfrak{R}}$ *the* $\mathcal{F}^{\Vdash}$*-prime-strip* $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^\dagger\mathfrak{D}_{>})$ *associated to* ${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$*,*

$(\mathrm{bad})^{\text{ét}}_{\mathfrak{R}}$ *the quadruple* ${}^{\dagger}\mathfrak{R}^{\mathrm{bad}} = ((\perp,\mu)^{\text{ét}}_{\underline{v}},\ (\mu_{\widehat{\mathbb{Z}}})^{\text{ét}}_{\underline{v}},\ (\mathbb{M})^{\text{ét}}_{\underline{v}},\ (\mathrm{spl})^{\text{ét}}_{\underline{v}})$ *of Theorem 12.7 (2) for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$,

$(\mathcal{F}^{\vdash\times\mu})^{\text{ét}}_{\mathfrak{R}}$ *the* $\mathcal{F}^{\vdash\times\mu}$*-prime-strip* $\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta})$ *associated to* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$, *and*

$(\mathrm{env}\Delta)^{\text{ét}}_{\mathfrak{R}}$ *the full poly-isomorphim* $\mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \overset{\text{full poly}}{\xrightarrow{\sim}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta})$.

We define a morphism from a radial datum ${}^{\dagger}\mathfrak{R}$ *to another radial datum* ${}^{\ddagger}\mathfrak{R}$ *to be a quintuple of*

$(\mathcal{HT}^{\mathcal{D}})^{\text{ét}}_{\mathrm{Mor}_{\mathfrak{R}}}$ *an isomorphism* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$ *of* $\mathcal{D}$*-* $\boxtimes\boxplus$*-Hodge theatres,*

$(\mathcal{F}^{\Vdash})^{\text{ét}}_{\mathrm{Mor}_{\mathfrak{R}}}$ *the isomorphism* $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\ddagger}\mathfrak{D}_{>})$ *of* $\mathcal{F}^{\Vdash}$*-prime-strips induced by the isomorphism* $(\mathcal{HT}^{\mathcal{D}})^{\text{ét}}_{\mathrm{Mor}}$,

$(\mathrm{bad})^{\text{ét}}_{\mathrm{Mor}_{\mathfrak{R}}}$ *the isomorphism* ${}^{\dagger}\mathfrak{R}^{\mathrm{bad}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{R}^{\mathrm{bad}}$ *of quadruples induced by the isomorphism* $(\mathcal{HT}^{\mathcal{D}})^{\text{ét}}_{\mathrm{Mor}}$, *and*

$(\mathcal{F}^{\vdash\times\mu})^{\text{ét}}_{\mathrm{Mor}_{\mathfrak{R}}}$ *an isomorphism* $\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{\ddagger}\mathfrak{D}^{\vdash}_{\Delta})$ *of* $\mathcal{F}^{\vdash\times\mu}$*-prime-strips*

(Note that the isomorphisms of $(\mathcal{F}^{\Vdash})^{\text{ét}}_{\mathrm{Mor}}$ *and* $(\mathcal{F}^{\vdash\times\mu})^{\text{ét}}_{\mathrm{Mor}}$ *are automatically compatible with* $(\mathrm{env}\Delta)^{\text{ét}}$*).*

We define a coric datum

$$ {}^{\dagger}\mathfrak{C} = ({}^{\dagger}\mathfrak{D}^{\vdash},\ \mathfrak{F}^{\vdash\times\mu}({}^{\dagger}\mathfrak{D}^{\vdash})) $$

to be a pair of

$(\mathfrak{D}^{\vdash})^{\vdash\text{ét}}_{\mathfrak{C}}$ *a* $\mathcal{D}^{\vdash}$*-prime-strip* ${}^{\dagger}\mathfrak{D}^{\vdash}$, *and*

$(\mathfrak{F}^{\vdash\times\mu})^{\vdash\text{ét}}_{\mathfrak{C}}$ *the* $\mathcal{F}^{\vdash\times\mu}$*-prime-strip* $\mathfrak{F}^{\vdash\times\mu}({}^{\dagger}\mathfrak{D}^{\vdash})$ *associated to* ${}^{\dagger}\mathfrak{D}^{\vdash}$.

We define a morphism from a coric datum ${}^{\dagger}\mathfrak{C}$ *to another coric datum* ${}^{\ddagger}\mathfrak{C}$ *to be a pair of*

$(\mathfrak{D}^{\vdash})^{\vdash\text{ét}}_{\mathrm{Mor}_{\mathfrak{C}}}$ *an isomorphism* ${}^{\dagger}\mathfrak{D}^{\vdash} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}^{\vdash}$ *of* $\mathcal{D}^{\vdash}$*-prime-strips, and*

$(\mathfrak{F}^{\vdash\times\mu})^{\vdash\text{ét}}_{\mathrm{Mor}_{\mathfrak{C}}}$ *an isomorphism* $\mathfrak{F}^{\vdash\times\mu}({}^{\dagger}\mathfrak{D}^{\vdash}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\mu}({}^{\ddagger}\mathfrak{D}^{\vdash})$ *of* $\mathcal{F}^{\vdash\times\mu}$*-prime-strips which induces the isomorphism* $(\mathfrak{D}^{\vdash})^{\vdash\text{ét}}_{\mathrm{Mor}_{\mathfrak{C}}}$ *on the associated* $\mathcal{D}^{\vdash}$*-prime-strips.*

We define the radial algorithm to be the assignment

$$\begin{aligned}&{}^{\dagger}\mathfrak{R} = ({}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus},\ \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}),\ {}^{\dagger}\mathfrak{R}^{\mathrm{bad}},\ \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta}),\ \mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \overset{\text{full poly}}{\xrightarrow{\sim}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta}))\\ &\mapsto {}^{\dagger}\mathfrak{C} = ({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta},\ \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta}))\end{aligned}$$

and the assignment on morphisms determined by the data $(\mathcal{F}^{\vdash\times\mu})^{\text{ét}}_{\mathrm{Mor}_{\mathfrak{R}}}$.

(1) **(Multiradiality)** *The functor defined by the above radial algorithm is full and essentially surjective, hence the above radial environment is* **multiradial**.

(2) **(Étale Picture)** *For each* $\mathcal{D}$-$\boxtimes\boxplus$*-Hodge theatre* ${}^{n,m}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$ *with* $n, m \in \mathbb{Z}$*, we can associate a radial datum* ${}^{n,m}\mathfrak{R}$*. The poly-isomorphisms induced by the vertical arrows of the Gaussian log-theta-lattice induce poly-isomorphisms* $\cdots \overset{\text{poly}}{\xrightarrow{\sim}} {}^{n,m}\mathfrak{R} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{n,m+1}\mathfrak{R} \overset{\text{poly}}{\xrightarrow{\sim}} \cdots$ *of radial data by Theorem 12.5 (1). We write*

$${}^{n,\circ}\mathfrak{R}$$

for the radial datum obtained by idenfitying ${}^{n,m}\mathfrak{R}$ *for* $m \in \mathbb{Z}$ *via these poly-isomorphisms, and we write*

$${}^{n,\circ}\mathfrak{C}$$

for the coric datum obtained by applying the radial algorithm to ${}^{n,\circ}\mathfrak{R}$*. Similarly, the poly-isomorphisms induced by the horizontal arrows of the Gaussian log-theta-lattice induce full poly-isomorphisms* $\cdots \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{n,m}\mathfrak{D}^{\vdash}_{\Delta} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{n+1,m}\mathfrak{D}^{\vdash}_{\Delta} \overset{\text{full poly}}{\xrightarrow{\sim}} \cdots$ *of* $\mathcal{D}^{\vdash}$*-prime-strips Theorem 12.5 (2). We write*

$${}^{\circ,\circ}\mathfrak{C}$$

for the coric datum obtained by idenfitying ${}^{n,\circ}\mathfrak{C}$ *for* $n \in \mathbb{Z}$ *via these full poly-isomorphisms. We can visualise the "shared" and "non-shared" relation in Corollary 12.8 (2) as follows:*

$$\boxed{\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>}) + {}^{n,\circ}\mathfrak{R}^{\mathrm{bad}} + \cdots} -- > \boxed{\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{\circ,\circ}\mathfrak{D}^{\vdash}_{\Delta})} <-- \boxed{\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n',\circ}\mathfrak{D}_{>}) + {}^{n',\circ}\mathfrak{R}^{\mathrm{bad}} + \cdots}$$

We shall refer to this diagram as the **étale-picture of multiradial theta monoids**. *Note that it has a permutation symmetry in the étale-picture (cf. also the last table in Section 4.3). Note also that these constructions are compatible, in an obvious sense, with Definition 11.24.1.*

(3) **(Kummer Compatibility of $\Theta^{\times\mu}_{\mathbf{gau}}$-Link, env $\to$ Δ)** *The (poly-)isomorphisms of $\mathcal{F}^{\vdash\times\mu}$-prime-strips of/induced by* $(\mathrm{env}\Delta)^{\text{ét}}_{\mathfrak{R}}$, $(\mathcal{F}^{\Vdash})^{\text{ét}}_{\mathrm{Mor}_{\mathfrak{R}}}$, *and* $(\mathcal{F}^{\vdash\times\mu})^{\text{ét}}_{\mathrm{Mor}_{\mathfrak{R}}}$ *are compatible with the poly-isomorphisms* ${}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{n+1,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta}$ *of Theorem 12.5 (2) arising from the horizontal arrows of Gaussian log-theta-lattice, with respect to the Kummer isomorphisms* ${}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \overset{\text{induced by Kum}}{\xrightarrow{\sim}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})$, ${}^{n,m}\mathfrak{F}^{\vdash}_{\mathrm{env}} \overset{\text{induced by Kum}}{\xrightarrow{\sim}} \mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{n,m}\mathfrak{D}_{>})$ *of Theorem 12.5 (3) and Proposition 12.6 (2). In particular, we have a commutative diagram*

$$\begin{tikzcd}
{}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \arrow[r, "\overset{\text{full poly}}{\sim}"] \arrow[d, "\cong"', "\text{induced by Kum \& “}\Delta\mapsto\text{env”}"'] & {}^{n+1,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \arrow[d, "\cong", "\text{induced by Kum \& “}\Delta\mapsto\text{env”}"] \\
\mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}^{\vdash}_{>}) \arrow[r, "\overset{\text{full poly}}{\sim}"] & \mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}}({}^{n+1,\circ}\mathfrak{D}^{\vdash}_{>}).
\end{tikzcd}$$

(4) **(Kummer Compatibility of $\Theta^{\times\mu}_{\mathbf{gau}}$-Link, $\perp$ & $\Vdash$)** *The isomorphisms* $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n,m}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n+1,m}\mathfrak{D}_{>})$, ${}^{n,m}\mathfrak{R}^{\mathrm{bad}} \xrightarrow{\sim} {}^{n+1,m}\,\mathfrak{R}^{\mathrm{bad}}$ *of* $(\mathcal{F}^{\Vdash})^{\text{ét}}_{\mathrm{Mor}_{\mathfrak{R}}}$, $(\mathrm{bad})^{\text{ét}}_{\mathrm{Mor}_{\mathfrak{R}}}$ *are compatible with the poly-isomorphisms* ${}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{n+1,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta}$ *of Theorem 12.5 (2) arising from the horizontal arrows of Gaussian log-theta-lattice, with respect to the Kummer isomorphisms* ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \overset{\text{“Kum”}}{\xrightarrow{\sim}} \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n,m}\mathfrak{D}_{>})$, ${}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \overset{\text{induced by Kum}}{\xrightarrow{\sim}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})$, *and* $({}^{n,m}\mathcal{C}^{\Vdash}_{\Delta}, \mathrm{Prime}({}^{n,m}\mathcal{C}^{\Vdash}_{\Delta}) \xrightarrow{\sim} \underline{\mathbb{V}}, \{{}^{n,m}\rho_{\Delta,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) \underset{\text{“Kum”}}{\xrightarrow{\sim}} (\mathcal{D}^{\Vdash}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta}), \mathrm{Prime}\,(\mathcal{D}^{\Vdash}({}^{n,m}\mathfrak{D}^{\vdash}_{\Delta})) \xrightarrow{\sim} \underline{\mathbb{V}}, \{{}^{n,m}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ *of Proposition 12.6 (2), Theorem 12.5 (3), (5) and their* ${}^{n+1,m}(-)$*-labelled versions, and the full poly-isomorphism of projective system of mono-theta environments* “$\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}) \overset{\text{full poly}}{\xrightarrow{\sim}} \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$” *of Proposition 11.15.*

Proof. Corollary follows from the definitions. □

Remark 12.8.1. ([IUTchIII, Remark 2.3.3]) In this remark, we explain similarities and differences between theta evaluations and NF evaluations. Similarities are as follows: For the theta case, the theta functions are multiradial in two-dimensional geometric containers, where we use the cyclotomic rigidity of mono-theta environments in the Kummer theory, which uses only μ-portion (unlike the cyclotomic rigidity via LCFT), and the evaluated theta values (in the evaluation, which depends on a holomorphic structure, the elliptic cuspidalisation is used), in $\mathfrak{log}$-Kummer correspondence later (cf. Proposition 13.7 (2)), has a crucial non-interference property by the constant multiple rigidity (cf. Proposition 13.7 (2c)). For the NF case, the κ-coric functions are multiradial in two-dimensional geometric containers, where we use the cyclotomic rigidity of via $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^{\times} = \{1\}$ in the Kummer theory, which uses only $\{1\}$-portion (unlike the cyclotomic rigidity via LCFT), and the evaluated number fields (in the evaluation, which depends on a holomorphic structure, the Beyli cuspidalisation is used), in $\mathfrak{log}$-Kummer correspondence later (cf. Proposition 13.11 (2)), has a crucial non-interference property by $F_{\mathrm{mod}}^{\times} \cap \prod_{v\leq\infty} O_v = \mu(F_{\mathrm{mod}}^{\times})$ (cf. Proposition 13.11 (2)). cf. also the following table:

	multirad. geom. container	in mono-an. container	cycl. rig.	$\mathfrak{log}$-Kummer
theta	theta fct. $\overset{\text{eval}}{\rightsquigarrow}$ theta values $\underline{\underline{q}}^{j^2}$(ell. cusp'n)		mono-theta	no interf. by
	(depends on labels&hol. str.)			const. mul. rig.
NF	${}_\infty\kappa$-coric fct. $\overset{\text{eval}}{\rightsquigarrow}$ NF $F_{\mathrm{mod}}^{\times}$ (up to $\{\pm 1\}$)(Belyi cusp'n)		via $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^{\times}$	no interf. by
	(indep. of labels, dep. on hol. str.)		$= \{1\}$	$F_{\mathrm{mod}}^{\times} \cap \prod_{v\leq\infty} O_v = \mu$

The differences are as follows: The output theta values $\underline{\underline{q}}^{j^2}$ depend on the labels $j \in \mathbb{F}_l^{\circledast}$ (Recall that the labels depend on a holomorphic structure), and the evaluation is compatible with the labels, on the other hand, the output number field $F_{\mathrm{mod}}^{\times}$ (up to $\{\pm 1\}$) does not depend on the labels $j \in \mathbb{F}_l^{\circledast}$ (Note also that, in the final multiradial algorithm, we also use global realified monoids, and these are of mono-analytic nature (since units are killed) and do not depend of holomorphic structure). We continue to explain the differences of the theta case and the NF case. The theta function is *transcendental* and of *local* nature, and the cyclotomic rigidity of mono-theta environments, which is *compatible with profinite topology* (cf. Remark 9.6.2), comes from the fact that the order of zero at each cusp is equal to one (Such "*only one valuation*" phenomenon corresponds precisely to the notion of "local"). Note that such a function only exists as a transcendental function. (Note also that the theta functions and theta values do not have $\mathbb{F}_l^{\rtimes\pm}$-symmetry; however, the cyclotomic rigidity of mono-theta enrionments have $\mathbb{F}_l^{\rtimes\pm}$-symmetry. cf. Remark 11.17.1). On the other hand, the rational functions used in Belyi cuspidalisation are *algebraic* and of *global* nature, and the cyclotomic rigidity via $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^{\times} = \{1\}$, which is obtained by *sacrificing the compatibility with profinite topology* (cf. Remark 9.6.2). Algebraic rational function never satisfy the property like "the order of zero at each cusp is equal to one" (Such "many valuations" phenomenon corresponds precisely to the notion of "global"). cf. also the following table (cf. [IUTchIII, Fig. 2.7]):

theta	$\boxplus$ (0 is permuted)	transcendental	local	compat. w/prof. top.	"one valuation"
NF	$\boxtimes$ (0 is isolated)	algebraic	global	incompat. w/prof. top.	"many valuations"

We also explain the "vicious circles" in Kummer theory. In the mono-anabelian reconstruction algorithm, we use various cyclotomes $\mu^{*}_{\text{ét}}$ arising from cuspidal inertia subgroups (cf. Theorem 3.17), these are naturally identified by the cyclotomic rigidity isomorphism for inertia subgroups (cf. Proposition 3.14 and Remark 3.14.1). We write $\mu^{\forall}_{\text{ét}}$ for the cyclotome resulting from the natural identifications. In the context of $\mathfrak{log}$-Kummer correspondence, the Frobenius-like cyclotomes μ_{Fr}'s are related to $\mu^{\forall}_{\text{ét}}$, via cyclotomic rigidity isomorphisms:

$$\begin{tikzcd}
\bullet\, {}^{\dagger}\mu_{\text{Fr}} \arrow[rd, "\text{Kum}"] & \\
\bullet\, {}^{\ddagger}\mu_{\text{Fr}} \arrow[u, "\mathfrak{log}"] \arrow[r, "\text{Kum}"] & \circ\, \mu^{\forall}_{\text{ét}} \\
{} \arrow[u, "\mathfrak{log}"] \arrow[ru, "\text{Kum}"'] &
\end{tikzcd}$$

If we consider these various Frobenius-like μ_{Fr}'s and the vertically coric étale-like $\mu^{\forall}_{\text{ét}}$ as distinct labelled objects, then the diagram does not result in any "vicious circles" or "loops". On the other hand, ultimately in Theorem 13.12, we will construct algorithms to describe objects of one holomorphic structure on one side of Θ-link, in terms of another alien arithmetic holomorphic structure on another side of Θ-link by means of multiradial containers. These multiradial containers arise from étale-like versions of objects, but are ultimately applied as containers for Frobenius-like versions of objects. Hence we need to contend with the consequences of identifying the Frobenius-like μ_{Fr}'s and the étale-like $\mu^{\forall}_{\text{ét}}$, which gives us possible "vicious circles" or "loops". We consider the indeterminacies arising from possible "vicious circles". The cyclotome $\mu^{\forall}_{\text{ét}}$ is subject to indeterminacies with respect to multiplication by elements of the submonoid

$$\mathbb{I}^{\text{ord}} \subset \mathbb{N}_{\geq 1} \times \{\pm 1\}$$

generated by the orders of the zeroes of poles of the rational functions appearing the cyclotomic rigidity isomorphism under consideration (Recall that constructing cyclotomic rigidity isomorphisms associated to rational functions via the Kummer-theoretic approach of Definition 9.6 amounts to identifying various $\mu^{*}_{\text{ét}}$'s with various sub-cyclotomes of μ_{Fr}'s via morphisms which differ from the usual natural identification precisely by multiplication by the order $\in \mathbb{Z}$ at a cusp "$*$" of the zeroes/poles of the rational function). In the theta case, we have

$$\mathbb{I}^{\mathrm{ord}} = \{1\}$$

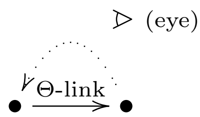
as a consequence of the fact that the order of the zeros/poles of the theta function at any cusp is equal to 1. On the other hand, for the NF case, such a phenomenon never happens for algebraic rational functions, and we have

$$\mathrm{Im}(\mathbb{I}^{\mathrm{ord}} \to \mathbb{N}_{\geq 1}) = \{1\}$$

by the fact $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^{\times} = \{1\}$. Note also that the indeterminacy arising from $\mathrm{Im}(\mathbb{I}^{\mathrm{ord}} \to \{\pm 1\})\,(\subset \{\pm 1\})$ is avoided in Definition 9.6, by the fact that the inverse of a non-constand κ-coric rational function is never κ-coric, and that this thechnique is incompatible with the identification of μ_{Fr} and $\mu_{\text{ét}}^{\forall}$ discussed above. Hence in the final multiradial algorithm, a possible $\mathrm{Im}(\mathbb{I}^{\mathrm{ord}} \to \{\pm 1\})\,(\subset \{\pm 1\})$-indeterminacy arises. However, the totality $F_{\mathrm{mod}}^{\times}$ of the non-zero elements is invariant under $\{\pm 1\}$, and this indeterminacy is harmless (Note that, in the theta case, the theta values $\underline{\underline{q}}^{j^2}$ have no $\{\pm 1\}$-invariance).

§ 13. Multiradial Representation Algorithms.

In this section, we construct the main multiradial algorithm to describe objects of one holomorphic structure on one side of Θ-link, in terms of another *alien* arithmetic holomorphic structure on another side of Θ-link by means of multiradial containers. We briefly explain the ideas. We want to "see" (=multiradiality) the *alien* ring structure on the left-hand side of Θ-link (more precisely, $\Theta^{\times\mu}_{\mathrm{LGP}}$-link) from the right-hand side of Θ-link:



As explained in Section 4.3, after constructing link (or wall) by using Frobenius-like objects, we relate Frobenius-like objects to étale-like objects via Kummer theory (**Kummer detachment**). Then étale-like objects can penetrate the wall (**étale transport**) (cf. Remark 9.6.1). We also have another step to go from holomorphic structure to the underlying mono-analytic structure for the purpose of using the horizontally coric (i.e., shared) objects in the final multiradial algorithm. This is a fundamental strategy:

arith.-holomorphic Frobenius-like obj's	data assoc. to $\mathcal{F}$-prime-strips
	$\downarrow$ Kummer theory
arith.-holomorphic étale-like obj's	data assoc. to $\mathcal{D}$-prime-strips
	$\downarrow$ forget arith.-hol. str.
mono-analytic étale-like obj's	data assoc. to $\mathcal{D}^{\vdash}$-prime-strips.

We look more. The Θ-link only concerns the multiplicative structure ($\boxtimes$), hence it seems difficult to see the additive structure ($\boxplus$) on the left-hand side, from the right-hand side. First, we try to overcome this difficulty by using a $\mathfrak{log}$-link (Note that $\mathbb{F}_l^{\rtimes\pm}$-symmetrising isomorphisms are compatible with $\mathfrak{log}$-links, hence we can pull back Ψ_{gau} via $\mathfrak{log}$-link to construct Ψ_{LGP}):

$$\begin{tikzcd} & & & & \triangleright \text{ (eye)} \\ \boxplus & \log(O^{\times\mu}) & \bullet \arrow[r, "\Theta\text{-link}"] & \bullet & \\ \boxtimes \arrow[u] & O^{\times\mu} \arrow[u] & \bullet \arrow[u, "\mathfrak{log}\text{-link}"'] & & \end{tikzcd}$$

However, the square

$$\begin{tikzcd} & & & & \triangleright \text{ (eye)} \\ \boxplus & \log(O^{\times\mu}) & \bullet \arrow[r, "\Theta\text{-link}"] & \bullet & \\ \boxtimes \arrow[u] & O^{\times\mu} \arrow[u] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"'] & \bullet \arrow[u, "\mathfrak{log}"'] & \end{tikzcd}$$

is non-commutative (cf. $\log(a^N) \neq (\log a)^N$), hence we cannot describe the left vertical arrow in terms of the right vertical arrow. We overcome this difficulty by considering the infinite chain of $\mathfrak{log}$-links:

$$\begin{tikzcd} \vdots & & \\ & & \triangleright \text{ (eye)} \\ \bullet \arrow[uu, "\mathfrak{log}"] \arrow[r, "\Theta"] & \bullet & \\ \bullet \arrow[u, "\mathfrak{log}"] & & \\ \vdots \arrow[u, "\mathfrak{log}"] & & \end{tikzcd}$$

Then the infinite chain of $\mathfrak{log}$-links is invariant under the vertical shift, and we can describe the infinite chain of $\mathfrak{log}$-links on the left-hand side, in terms of the infinite chain of $\mathfrak{log}$-links on the right-hand side. This is a rough explanation of the idea.

§ 13.1. Local and Global Packets.

Here, we introduce a notion of processions.

Definition 13.1. ([IUTchI, Definition 4.10]) Let $\mathcal{C}$ be a category. A ***n*-procession** of $\mathcal{C}$ is a diagram of the form

$$P_1 \overset{\text{all capsule-full poly}}{\hookrightarrow} P_2 \overset{\text{all capsule-full poly}}{\hookrightarrow} \cdots \overset{\text{all capsule-full poly}}{\hookrightarrow} P_n,$$

where P_j is a j-capsule of $\mathrm{Ob}(\mathcal{C})$ for $1 \leq j \leq n$, and each $\hookrightarrow$ is the set of all capsule-full poly-morphisms. A **morphism** from an n-procession of $\mathcal{C}$ to an m-procession of $\mathcal{C}$

$$\left(P_1 \overset{\text{all capsule-full poly}}{\hookrightarrow} \cdots \overset{\text{all capsule-full poly}}{\hookrightarrow} P_n\right) \to \left(Q_1 \overset{\text{all capsule-full poly}}{\hookrightarrow} \cdots \overset{\text{all capsule-full poly}}{\hookrightarrow} Q_m\right)$$

consists of an order-preserving injection $\iota : \{1, \dots, n\} \hookrightarrow \{1, \dots, m\}$ together with a capsule-full poly-morphism $P_j \overset{\text{capsule-full poly}}{\hookrightarrow} Q_{\iota(j)}$ for $1 \leq j \leq n$.

Ultimately, l^*-processions of $\mathcal{D}^{\vdash}$-prime-strips corresponding to the subsets $\{1\} \subset \{1, 2\} \subset \cdots \subset \mathbb{F}_l^*$ will be important.

Remark 13.1.1. As already seen, the labels (LabCusp$(-)$) depend on the arithmetically holomorphic structures (cf. also Section 3.5), i.e., $\Delta_{(-)}$'s or $\Pi_{(-)}$'s (Recall that $\Pi_{(-)}$ for hyperbolic curves of strictly Belyi type over an MLF has the information of the field structure of the base field, and can be considered as arithmetically holomorphic, on the other hand, the Galois group of the base field $(\Pi_{(-)} \twoheadrightarrow) G_{(-)}$ has no information of the field structure of the base field, and can be considered as mono-analytic). In inter-universal Teichmüller theory, we will reconstruct an alien ring structure on one side of (the updated version of) Θ-link from the other side of (the updated version of) Θ-link (cf. also the primitive form of Θ-link shares the mono-analytic structure ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$, but *not* the arithmetically holomorphic structures ${}^{\dagger}\mathcal{D}_{\underline{v}}$, ${}^{\ddagger}\mathcal{D}_{\underline{v}}$ (Remark 10.8.1)), and we *cannot* send arithmetically holomorphic structures from one side to the other side of (the updated version of) Θ-link. In particular, *we cannot send the labels (*LabCusp$(-)$*) from one side to the other side of (the updated version of)* Θ*-link*, i.e., we cannot see the labels on one side from the other side:

$$1,\ 2,\ \dots,\ l^* \quad \longmapsto \quad ?,\ ?,\ \dots,\ ?.$$

Then we have $(l^*)^{l^*}$-indeterminacies in total. However, *we can send processions*:

$$\{1\} \hookrightarrow \{1,2\} \hookrightarrow \{1,2,3\} \hookrightarrow \cdots \hookrightarrow \{1,2,\dots,l^*\} \quad \longmapsto \quad \{?\} \hookrightarrow \{?,?\} \hookrightarrow \cdots \hookrightarrow \{?,?,\dots,?\}.$$

In this case, we can reduce the indeterminacies from $(l^*)^{l^*}$ to $(l^*)!$. If we did not use this reduction of indeterminacies, then the final inequality of height function would be weaker (More precisely, it would be ht $\lesssim (2+\epsilon)(\text{log-diff} + \text{log-cond})$, not ht $\lesssim (1+\epsilon)(\text{log-diff} + \text{log-cond})$). More concretely, in the calculations of Lemma 1.10, if we did not use the processions, then the calculation $\frac{1}{l^*}\sum_{1 \leq j \leq l^*}(j+1) = \frac{l^*+1}{2} + 1$ would be changed into $\frac{1}{l^*}\sum_{1 \leq j \leq l^*}(l^*+1) = l^*+1$, whose coefficient of l would be twice.

For $j = 1, \ldots, l^{\pm}$ (Recall that $l^{\pm} = l^{\circledast} + 1 = \frac{l+1}{2}$ (cf. Section 0.2)), we write

$$\mathbb{S}^{\circledast}_{j} := \{1, \ldots, j\}, \quad \mathbb{S}^{\pm}_{j} := \{0, \ldots, j-1\}.$$

Note that we have

$$\mathbb{S}^{\circledast}_{1} \subset \mathbb{S}^{\circledast}_{2} \subset \cdots \subset \mathbb{S}^{\circledast}_{l^{\circledast}} = \mathbb{F}^{\circledast}_{l}, \quad \mathbb{S}^{\pm}_{1} \subset \mathbb{S}^{\pm}_{2} \subset \cdots \subset \mathbb{S}^{\pm}_{l^{\pm}} = |\mathbb{F}_{l}|.$$

We also consider $\mathbb{S}^{\circledast}_{j}$ as a subset of $\mathbb{S}^{\pm}_{j+1}$.

Definition 13.2. ([IUTchI, Proposition 4.11, Proposition 6.9]) For a $\mathcal{D}$-Θ-bridge ${}^{\dagger}\mathfrak{D}_{J} \xrightarrow{{}^{\dagger}\phi^{\Theta}_{\circledast}} {}^{\dagger}\mathfrak{D}_{>}$ (resp. $\mathcal{D}$-$\Theta^{\pm}$-bridge ${}^{\dagger}\mathfrak{D}_{T} \xrightarrow{{}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}} {}^{\dagger}\mathfrak{D}_{\succ}$), we write

$$\mathrm{Proc}({}^{\dagger}\mathfrak{D}_{J}) \quad (\text{resp. } \mathrm{Proc}({}^{\dagger}\mathfrak{D}_{T}) \ )$$

for the $l^{\circledast}$-processin (resp. $l^{\pm}$-procession) of $\mathcal{D}$-prime-strips determined by the sub-capsules of ${}^{\dagger}\mathfrak{D}_{J}$ (resp. ${}^{\dagger}\mathfrak{D}_{T}$) corresponding to the subsets $\mathbb{S}^{\circledast}_{1} \subset \mathbb{S}^{\circledast}_{2} \subset \ldots \subset \mathbb{S}^{\circledast}_{l^{\circledast}} = \mathbb{F}^{\circledast}_{l}$ (resp. $\mathbb{S}^{\pm}_{1} \subset \mathbb{S}^{\pm}_{2} \subset \ldots \subset \mathbb{S}^{\pm}_{l^{\pm}} = |\mathbb{F}_{l}|$), with respect to the bijection ${}^{\dagger}\chi : J \xrightarrow{\sim} \mathbb{F}^{\circledast}_{l}$ of Proposition 10.19 (1) (resp. the bijection $|T| \xrightarrow{\sim} |\mathbb{F}_{l}|$ determined by the $\mathbb{F}^{\pm}_{l}$-group structure of T). For the capsule ${}^{\dagger}\mathfrak{D}^{\vdash}_{J}$ (resp. ${}^{\dagger}\mathfrak{D}^{\vdash}_{T}$) of $\mathcal{D}^{\vdash}$-prime-strips associated to ${}^{\dagger}\mathfrak{D}_{J}$ (resp. ${}^{\dagger}\mathfrak{D}_{T}$), we similarly define the $l^{\circledast}$-processin (resp. $l^{\pm}$-procession)

$$\mathrm{Proc}({}^{\dagger}\mathfrak{D}^{\vdash}_{J}) \quad (\text{resp. } \mathrm{Proc}({}^{\dagger}\mathfrak{D}^{\vdash}_{T}) \ )$$

of $\mathcal{D}^{\vdash}$-prime-strips. If the $\mathcal{D}$-Θ-bridge ${}^{\dagger}\phi^{\Theta}_{\circledast}$ (resp. the $\mathcal{D}$-$\Theta^{\pm}$-bridge ${}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}$) arises from a capsule Θ-bridge (resp. $\Theta^{\pm}$-bridge), we similarly define the $l^{\circledast}$-processin (resp. $l^{\pm}$-procession)

$$\mathrm{Proc}({}^{\dagger}\mathfrak{F}_{J}) \quad (\text{resp. } \mathrm{Proc}({}^{\dagger}\mathfrak{F}_{T}) \ )$$

of $\mathcal{F}$-prime-strips.

Proposition 13.3. (Local Holomorphic Tensor Packets, [IUTchIII, Proposition 3.1]) *Let*

$$\{{}^{\alpha}\mathfrak{F}\}_{\alpha \in \mathbb{S}^{\pm}_{j}} = \left\{\{{}^{\alpha}\mathcal{F}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}\right\}_{\alpha \in \mathbb{S}^{\pm}_{j}}$$

be a j-capsule of $\mathcal{F}$-prime-strips with index set $\mathbb{S}^{\pm}_{j}$. For $\underline{\mathbb{V}} \ni \underline{v}(\mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}} := \mathbb{V}(\mathbb{Q}))$, we regard $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ as an inductive limit of finite-dimensional topological modules over $\mathbb{Q}_{v_{\mathbb{Q}}}$, by $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}}) = \varinjlim_{J \subset {}^{\alpha}\Pi_{\underline{v}} : \mathrm{open}} (\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}}))^{J}$. We shall refer to the assignment

$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \ \mapsto \ \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) := \bigoplus_{\underline{\mathbb{V}} \ni \underline{v} | v_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$$

as the **1-tensor packet associated to the $\mathcal{F}$-prime-strip** ${}^{\alpha}\mathfrak{F}$*, and the assignment*

$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \ \mapsto \ \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{F}_{v_{\mathbb{Q}}}) := \bigotimes_{\alpha \in \mathbb{S}^{\pm}_{j}} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$$

the ***j*-tensor packet associated to the collection** $\{{}^{\alpha}\mathfrak{F}\}_{\alpha \in \mathbb{S}^{\pm}_{j}}$ **of $\mathcal{F}$-prime-strips**, *where the tensor product is taken as a tensor product of ind-topological modules.*

(1) **(Ring Structures)** *The ind-topological field structures on* $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ *for* $\alpha \in \mathbb{S}^{\pm}_{j}$ *determine an ind-topological ring structure on* $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{F}_{v_{\mathbb{Q}}})$ *as an inductive limit of direct sums of ind-topological fields. Such decompositions are compatible with the natural action of the topological group* ${}^{\alpha}\Pi_{\underline{v}}$ *on the direct summand with subscript* $\underline{v}$ *of the factor labelled* α*.*

(2) **(Integral Structures)** *Fix* $\alpha \in \mathbb{S}^{\pm}_{j+1}$, $\underline{v} \in \underline{\mathbb{V}}$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ *with* $\underline{v} \mid v_{\mathbb{Q}}$*. Write*

$$\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}}) \ := \ \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}}) \otimes \left\{ \bigotimes_{\beta \in \mathbb{S}^{\pm}_{j+1} \setminus \{\alpha\}} \underline{\mathfrak{log}}({}^{\beta}\mathcal{F}_{v_{\mathbb{Q}}}) \right\} \ \subset \ \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}}).$$

Then the ind-topological submodule $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$ *forms a direct summand of the ind-topological ring* $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}})$*. Note that* $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$ *is also an inductive limit of direct sums of ind-topological fields. Moreover, by forming the tensor product with* 1*'s in the factors labelled by* $\beta \in \mathbb{S}^{\pm}_{j+1} \setminus \{\alpha\}$*, we obtain a natural injective homomorphism*

$$\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}}) \ \hookrightarrow \ \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$$

of ind-topological rings, which, for suitable (cofinal) choices of objects in the inductive limit descriptions for the domain and codomain, induces an isomorphism of such an object in the domain onto each of the direct summand ind-topological fields of the object in the codomain. In particular, the integral structure

$$\overline{\Psi}_{\mathfrak{log}({}^{\alpha}\mathcal{F}_{\underline{v}})} \ := \ \Psi_{\mathfrak{log}({}^{\alpha}\mathcal{F}_{\underline{v}})} \cup \{0\} \ \subset \ \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$$

determines integral structures on each of the direct summand ind-topological fields appearing in the inductive limit descriptions of $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$, $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}})$*.*

Note that $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ *is an isomorph of* $\log(\overline{K_{\underline{v}}}^{\times}) \cong \overline{K_{\underline{v}}}$*, the integral structure* $\overline{\Psi}_{\mathfrak{log}({}^{\alpha}\mathcal{F}_{\underline{v}})}$ *is an isomorph of* $O_{\overline{K_{\underline{v}}}}$*, and* $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$ *is an isomorph of* $\bigotimes \overline{K_{\underline{v}}} \xrightarrow{\sim} \varinjlim \bigoplus \overline{K_{\underline{v}}}$*.*

Proof. Proposition follows from the definitions. □

Remark 13.3.1. ([IUTchIII, Remark 3.1.1 (ii)]) From the point of view of "analytic section" $\mathbb{V}_{\mathrm{mod}} \xrightarrow{\sim} \underline{\mathbb{V}}(\subset \mathbb{V}(K))$ of $\operatorname{Spec} K \twoheadrightarrow \operatorname{Spec} F_{\mathrm{mod}}$, we need to consider the log-volumes on the portion of $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ corresponding to $K_{\underline{v}}$ relative to the **weight**

$$\frac{1}{[K_{\underline{v}} : (F_{\mathrm{mod}})_{v}]},$$

where we write $v \in \mathbb{V}_{\mathrm{mod}}$ for the valuation corresponding to $\underline{v}$ via the bijection $\mathbb{V}_{\mathrm{mod}} \xrightarrow{\sim} \underline{\mathbb{V}}$ (cf. also Definition 10.4). When we consider $\bigoplus_{\underline{\mathbb{V}} \ni \underline{v} | v_{\mathbb{Q}}}$ as in case of $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, we use the **normalised weight**

$$\frac{1}{[K_{\underline{v}} : (F_{\mathrm{mod}})_v] \cdot \left(\sum_{\mathbb{V}_{\mathrm{mod}} \ni w | v_{\mathbb{Q}}} [(F_{\mathrm{mod}})_w : \mathbb{Q}_{v_{\mathbb{Q}}}] \right)}$$

so that the multiplication by $p_{v_{\mathbb{Q}}}$ affects log-volumes as $+\log(p_{v_{\mathbb{Q}}})$ (resp. by $-\log(p_{v_{\mathbb{Q}}})$) for $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$ (resp. $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$) (cf. also Section 1.2). Similarly, when we consider log-volumes on the portion of $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}})$ corresponding to the tensor product of $K_{\underline{v}_i}$ with $\underline{\mathbb{V}} \ni \underline{v}_i \mid v_{\mathbb{Q}}$ for $0 \leq i \leq j$, we have to consider these log-volumes relative to the **weight**

$$\frac{1}{\prod_{0 \leq i \leq j} [K_{\underline{v}_i} : (F_{\mathrm{mod}})_{v_i}]},$$

where $v_i \in \mathbb{V}_{\mathrm{mod}}$ corresponds to $\underline{v}_i$. Moreover, when we consider direct sums over all possible choices for the data $\{\underline{v}_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$, we use the **normalised weight**

$$\frac{1}{\left(\prod_{0 \leq i \leq j} [K_{\underline{v}_i} : (F_{\mathrm{mod}})_{v_i}] \right) \cdot \left\{ \sum_{\{w_i\}_{0 \leq i \leq j} \in ((\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}})^{j+1}} \left(\prod_{0 \leq i \leq j} [(F_{\mathrm{mod}})_{w_i} : \mathbb{Q}_{v_{\mathbb{Q}}}] \right) \right\}}$$

(cf. also Section 1.2) so that the multiplication by $p_{v_{\mathbb{Q}}}$ affects log-volumes as $+\log(p_{v_{\mathbb{Q}}})$ (resp. by $-\log(p_{v_{\mathbb{Q}}})$) for $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$ (resp. $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$) (cf. Section 0.2 for the notation $(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}$).

Proposition 13.4. (Local Mono-analytic Tensor Packets, [IUTchIII, Proposition 3.2]) *Let*

$$\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha \in \mathbb{S}^{\pm}_j} = \left\{ \{{}^{\alpha}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}} \right\}_{\alpha \in \mathbb{S}^{\pm}_j}$$

be a j-capsule of $\mathcal{D}^{\vdash}$-prime-strips with index set $\mathbb{S}^{\pm}_j$. We shall refer to the assignment

$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \mapsto \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) := \bigoplus_{\underline{\mathbb{V}} \ni \underline{v} | v_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$$

as the **1-tensor packet associated to the $\mathcal{D}^{\vdash}$-prime-strip ${}^{\alpha}\mathfrak{D}$**, *and the assignment*

$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \mapsto \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_j}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) := \bigotimes_{\alpha \in \mathbb{S}^{\pm}_j} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$$

the **j-tensor packet associated to the collection $\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha \in \mathbb{S}^{\pm}_j}$ of $\mathcal{D}^{\vdash}$-prime-strips**, *where the tensor product is taken as a tensor product of ind-topological modules. For $\alpha \in \mathbb{S}^{\pm}_{j+1}$, $\underline{v} \in \underline{\mathbb{V}}$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ with $\underline{v} \mid v_{\mathbb{Q}}$, put*

$$\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) := \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \otimes \left\{ \bigotimes_{\beta \in \mathbb{S}^{\pm}_{j+1} \setminus \{\alpha\}} \underline{\mathfrak{log}}({}^{\beta}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \right\} \subset \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}).$$

If $\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha\in\mathbb{S}^{\pm}_{j}}$ *arises from a* j*-capsule*

$$\{{}^{\alpha}\mathfrak{F}^{\vdash\times\mu}\}_{\alpha\in\mathbb{S}^{\pm}_{j}} = \Big\{\{{}^{\alpha}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\Big\}_{\alpha\in\mathbb{S}^{\pm}_{j}}$$

of $\mathcal{F}^{\vdash\times\mu}$*-prime-strips, then we put*

$$\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}^{\vdash\times\mu}_{v_{\mathbb{Q}}}) := \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}),\ \ \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{F}^{\vdash\times\mu}_{v_{\mathbb{Q}}}) := \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}),\ \ \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}) := \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}}),$$

and we shall refer to the first two of them as the **1-tensor packetassociated to the** $\boldsymbol{\mathcal{F}^{\vdash\times\mu}}$**-prime-strip** ${}^{\alpha}\mathfrak{F}^{\vdash\times\mu}$*, and the* **$j$-tensor packet associated to the collection** $\{{}^{\alpha}\mathfrak{F}^{\vdash\times\mu}\}_{\alpha\in\mathbb{S}^{\pm}_{j}}$ **of $\boldsymbol{\mathcal{F}^{\vdash\times\mu}}$-prime-strips**, *respectively.*

(1) **(Mono-analytic/Holomorphic Compatibility)** *Assume that* $\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha\in\mathbb{S}^{\pm}_{j}}$ *arises from a* j*-capsule*

$$\{{}^{\alpha}\mathfrak{F}\}_{\alpha\in\mathbb{S}^{\pm}_{j}} = \big\{\{{}^{\alpha}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\big\}_{\alpha\in\mathbb{S}^{\pm}_{j}}$$

of $\mathcal{F}$*-prime-strips. We write* $\{{}^{\alpha}\mathfrak{F}^{\vdash\times\mu}\}_{\alpha\in\mathbb{S}^{\pm}_{j}}$ *for the* j*-capsule of* $\mathcal{F}^{\vdash\times\mu}$*-prime-strips associated to* $\{{}^{\alpha}\mathfrak{F}\}_{\alpha\in\mathbb{S}^{\pm}_{j}}$*. Then the (poly-)isomorphisms* $\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \overset{\text{tauto}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}) \overset{\text{“Kum” poly}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$ *of Proposition 12.2 (4) induce natural poly-isomorphisms*

$$\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \overset{\text{tauto}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}^{\vdash\times\mu}_{v_{\mathbb{Q}}}) \overset{\text{“Kum” poly}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}),\ \ \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{F}_{v_{\mathbb{Q}}}) \overset{\text{tauto}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{F}^{\vdash\times\mu}_{v_{\mathbb{Q}}}) \overset{\text{“Kum” poly}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}),$$

$$\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}}) \overset{\text{tauto}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}) \overset{\text{“Kum” poly}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$$

of ind-topological modules.

(2) **(Integral Structures)** *For* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ *the étale-like mono-analytic log-shells “*$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}$*” of Proposition 12.2 (4) determine topological submodules*

$$\mathcal{I}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subset \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subset \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \subset \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}}),$$

which can be regarded as integral structures on the $\mathbb{Q}$*-spans of these submodules. For* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$ *by regarding the étale-like mono-analytic log-shells “*$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}$*” of Proposition 12.2 (4) as the “closed unit ball” of a Hermitian metric on “*$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$*”, and putting the induced direct sum Hermitian metric on* $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*, and the induced tensor product Hermitian metric on* $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*, we obtain Hemitian metrics on* $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*,* $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*, and* $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*, whose associated closed unit balls*

$$\mathcal{I}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subset \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subset \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \subset \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}}),$$

can be regarded as integral structures on $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$, $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$, *and* $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$, *respectively. For any* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, *we put*

$$\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) := \mathbb{Q}\text{-span of } \mathcal{I}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subset \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) := \mathbb{Q}\text{-span of } \mathcal{I}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subset \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}),$$

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) := \mathbb{Q}\text{-span of } \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \subset \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}}).$$

If $\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha\in\mathbb{S}^{\pm}_{j}}$ *arises from a* j*-capsule* $\{{}^{\alpha}\mathfrak{F}\}_{\alpha\in\mathbb{S}^{\pm}_{j}}$ *of* $\mathcal{F}$*-prime-strips then, the objects* $\mathcal{I}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$, $\mathcal{I}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$, $\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$, $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$ *determine*

$$\mathcal{I}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{F}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{F}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}}),\ \ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}}),$$

and

$$\mathcal{I}({}^{\alpha}\mathcal{F}^{\vdash\times\mu}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}^{\vdash\times\mu}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{F}^{\vdash\times\mu}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{F}^{\vdash\times\mu}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}),\ \ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}^{\vdash\times\mu}_{\underline{v}})$$

via the above natural poly-isomorphisms $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \overset{\text{tauto}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}^{\vdash\times\mu}_{v_{\mathbb{Q}}}) \overset{\substack{\text{“Kum”}\\ \text{poly}}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$, $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{F}_{v_{\mathbb{Q}}}) \overset{\text{tauto}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{F}^{\vdash\times\mu}_{v_{\mathbb{Q}}}) \overset{\substack{\text{“Kum”}\\ \text{poly}}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$, $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}}) \overset{\text{tauto}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}) \overset{\substack{\text{“Kum”}\\ \text{poly}}}{\xrightarrow{\sim}} \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$ *of ind-topological modules.*

Proof. Proposition follows from the definitions. □

Proposition 13.5. (Global Tensor Packets, [IUTchIII, Proposition 3.3]) *Let*

$${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$$

be a $\boxtimes\boxplus$*-Hodge theatre with associated* $\boxtimes$*- and* $\boxplus$*-Hodge theatres* ${}^{\dagger}\mathcal{HT}^{\boxtimes}$, ${}^{\dagger}\mathcal{HT}^{\boxplus}$ *respectively. Let* $\{{}^{\alpha}\mathfrak{F}\}_{\alpha\in\mathbb{S}^{*}_{j}}$ *be a* j*-capsule of* $\mathcal{F}$*-prime-strips. We consider* $\mathbb{S}^{*}_{j}$ *as a subset of the index set* J *appearing the* $\boxtimes$*-Hodge theatre* ${}^{\dagger}\mathcal{HT}^{\boxtimes}$ *via the isomorphism* ${}^{\dagger}\chi : J \xrightarrow{\sim} \mathbb{F}^{*}_{l}$ *of Proposition 10.19 (1). We assume that for each* $\alpha \in \mathbb{S}^{*}_{j}$, *a* $\mathfrak{log}$*-link*

$${}^{\alpha}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathfrak{F}_{\alpha}$$

(i.e., a poly-morphism $\mathfrak{log}({}^{\alpha}\mathfrak{F}) \overset{\text{poly}}{\xrightarrow{\sim}} {}^{\dagger}\mathfrak{F}_{\alpha}$ *of* $\mathcal{F}$*-prime-strips) is given. Recall that we have a labelled version* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{j}$ *of the field* ${}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}$ *(cf. Corollary 11.23 (1), (2)). We call*

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\mathbb{S}^{*}_{j}} \ :=\ \bigotimes_{\alpha\in\mathbb{S}^{*}_{j}} ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha}$$

the **global j-tensor packet associated to $\mathbb{S}^{*}_{j}$ and the $\boxtimes\boxplus$-Hodge theatre ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$**.

(1) **(Ring Structures)** *The field structures on* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha}$ *for* $\alpha \in \mathbb{S}^{*}_{j}$ *determine a ring structure on* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\mathbb{S}^{*}_{j}}$*, which decomposes uniquely as a direct sum of number fields. Moreover, by composing with the given* $\mathfrak{log}$*-links, the various localisation functors "*$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{j} \to {}^{\dagger}\mathfrak{F}_{j}$*" of Corollary 11.23 (3) give us a natural injective localisation ring homomorphism*

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\mathbb{S}^{*}_{j}} \overset{\text{gl. to loc.}}{\hookrightarrow} \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) := \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}})$$

to the product of the local holomorphic tensor packets of Proposition 13.3, where we consider $\mathbb{S}^{*}_{j}$ *as a subset of* $\mathbb{S}^{\pm}_{j+1}$*, and the component labelled by* 0 *in* $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}})$ *of the localisation homomorphism is defined to be* 1*.*

(2) **(Integral Structures)** *For* $\alpha \in \mathbb{S}^{*}_{j}$*, by taking the tensor product with* 1*'s in the factors labelled by* $\beta \in \mathbb{S}^{*}_{j} \setminus \{\alpha\}$*, we obtain a natural injective ring homomorphism*

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha} \hookrightarrow ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\mathbb{S}^{*}_{j}}$$

which induces an isomorphism of the domain onto a subfield of each of the direct summand number fields of the codomain. For each $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$*, this homomorphism is compatible, in the obvious sense, with the natural injective homomorphism* $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}}) \hookrightarrow \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$ *of ind-topological rings of Proposition 13.3 (2), with respect to the localisation homomorphisms of (1). Moreover, for each* $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ *(resp.* $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$*), the composite*

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha} \hookrightarrow ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\mathbb{S}^{*}_{t}} \overset{\text{gl. to loc.}}{\hookrightarrow} \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) \twoheadrightarrow \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}})$$

of the above displayed homomorphism with the $v_{\mathbb{Q}}$*-component of the localisation homomorphism of (1) sends the ring of integers (resp. the set of elements of absolute value* ≤ 1 *for all Archimedean primes) of the number field* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha}$ *into the submodule (resp. the direct product of subsets) constituted by the integral structures on* $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}})$ *(resp. on various direct summand ind-topological fields of* $\underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}})$*) of Proposition 13.3 (2).*

Proof. Proposition follows from the definitions. □

§ 13.2. Log-Kummer Correspondences and Multiradial Representation Algorithms.

Proposition 13.6. (Local Packet-Theoretic Frobenioids, [IUTchIII, Proposition 3.4])

(1) **(Single Packet Monoids)** *In the situation of Proposition 13.3, for* $\alpha \in \mathbb{S}^{\pm}_{j+1}$, $\underline{v} \in \underline{\mathbb{V}}$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ *with* $\underline{v} \mid v_{\mathbb{Q}}$, *the* **image** *of the monoid* $\Psi_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})}$, *its submonoid* $\Psi^{\times}_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})}$ *of units, and realification* $\Psi^{\mathbb{R}}_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})}$, *via the natural homomorphism* $\underline{\mathfrak{log}}(^{\alpha}\mathcal{F}_{\underline{v}}) \hookrightarrow \underline{\mathfrak{log}}(^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$ *of Proposition 13.3 (2), determines monoids*

$$\Psi_{\mathfrak{log}(^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})},\quad \Psi^{\times}_{\mathfrak{log}(^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})},\quad \Psi^{\mathbb{R}}_{\mathfrak{log}(^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})}$$

which are equipped with $G_{\underline{v}}(^{\alpha}\Pi_{\underline{v}})$*-actions when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, *and for the first monoid, with a pair of an Aut-holomorphic orbispace and a Kummer structure when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. *We regard these monoids as (possibly realified) subquotients of* $\underline{\mathfrak{log}}(^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$ *which act on appropriate (possibly realified) subquotients of* $\underline{\mathfrak{log}}(^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$. *(For the purpose of equipping* $\Psi_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})}$ *etc. with the action on subquotients of* $\underline{\mathfrak{log}}(^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$, *in the algorithmical outputs, we define* $\Psi_{\mathfrak{log}(^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})}$ *etc. by using the* **image** *of the natural homomorphism* $\underline{\mathfrak{log}}(^{\alpha}\mathcal{F}_{\underline{v}}) \hookrightarrow \underline{\mathfrak{log}}(^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$*).*

(2) **(Local Logarithmic Gaussian Procession Monoids)** *Let*

$$^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$$

be a $\mathfrak{log}$*-link of* $\boxtimes\boxplus$*-Hodge theatres. Consider the* $\mathcal{F}$*-prime-strip processions* $\mathrm{Proc}(^{\dagger}\mathfrak{F}_T)$. *Recall that the Frobenius-like Gaussian monoid* $_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}(^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$ *of Corollary 11.21 (4) is defined by the submonoids in the product* $\prod_{j\in\mathbb{F}^{*}_{l}}(\Psi_{^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_j$ *(cf. Corollary 11.17 (2), Proposition 11.19 (4)). Consider the following diagram:*

$$\begin{array}{ccccc}
 & & \prod_{j\in\mathbb{F}^{*}_{l}}\underline{\mathfrak{log}}(^{j;\ddagger}\mathcal{F}_{\underline{v}}) & \subset & \prod_{j\in\mathbb{F}^{*}_{l}}\underline{\mathfrak{log}}(^{\mathbb{S}^{\pm}_{j+1}j;\ddagger}\mathcal{F}_{\underline{v}}) \\
 & & \cup & & \cup \\
\prod_{j\in\mathbb{F}^{*}_{l}}(\Psi_{^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_j & \overset{\mathrm{poly}}{\underset{\sim}{\longleftarrow}} & \prod_{j\in\mathbb{F}^{*}_{l}}\Psi_{\mathfrak{log}(^{j;\ddagger}\mathcal{F}_{\underline{v}})} & \overset{\text{by (1)}}{\underset{\sim}{\to}} & \prod_{j\in\mathbb{F}^{*}_{l}}\Psi_{\mathfrak{log}(^{\mathbb{S}^{\pm}_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})}, \\
\cup & & & & \\
\Psi_{\mathcal{F}_{\mathrm{gau}}}(^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) & & & &
\end{array}$$

where, in the last line, we write $\Psi_{\mathcal{F}_{\mathrm{gau}}}(^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$, *by abuse of notation, for* $\Psi_{\mathcal{F}_{\xi}}(^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$ *for a value profie* ξ *in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. *We take the pull-backs of* $\Psi_{\mathcal{F}_{\mathrm{gau}}}(^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$ *via the poly-isomorphism given by* $\mathfrak{log}$*-link* $^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$, *and send them to the isomorphism* $\prod_{j\in\mathbb{F}^{*}_{l}}\Psi_{\mathfrak{log}(^{j;\ddagger}\mathcal{F}_{\underline{v}})} \overset{\sim}{\to} \prod_{j\in\mathbb{F}^{*}_{l}}\Psi_{\mathfrak{log}(^{\mathbb{S}^{\pm}_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})}$ *constructed in (1). By this construction, we obtain a functorial algorithm, with respect to the* $\mathfrak{log}$*-link* $^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$ *of* $\boxtimes\boxplus$*-Hodge theatres, to construct collections of monoids*

$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto\ \Psi_{\mathcal{F}_{\mathrm{LGP}}}((^{\ddagger}\xrightarrow{\mathfrak{log}})^{\dagger}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}},\quad {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{LGP}}}((^{\ddagger}\xrightarrow{\mathfrak{log}})^{\dagger}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}},$$

equipped with splittings up to torsion when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *(resp. splittings when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$*). We shall refer to them as* **Frobenius-like local LGP-monoids** *or* **Frobenius-like local logarithmic Gaussian procession monoids**. *Note that we are able to perform this construction, thanks to the* **compatibility of $\mathfrak{log}$-link with the $\mathbb{F}_l^{\rtimes\pm}$-symmetrising isomorphisms**.

Note that, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*, we have*

$$\left(j\text{-labelled component of } \Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{(\ddagger \xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}}^{G_{\underline{v}}({}^{\ddagger}\Pi_{\underline{v}})}\right) \subset \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})$$

(i.e., "$(\widetilde{K_{\underline{v}}} \supset)\, O^{\times}_{K_{\underline{v}}} \cdot \underline{\underline{q}}_{\underline{v}}^{j^2} \subset \mathbb{Q}\log(O^{\times}_{K_{\underline{v}}})$*"), where we write* $(-)^{G_{\underline{v}}({}^{\ddagger}\Pi_{\underline{v}})}$ *for the invariant part, and the above* j*-labelled component of Galois invariant part acts multiplicatively on* $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})$*. For any* $\underline{v} \in \underline{\mathbb{V}}$*, we also have*

$$\left(j\text{-labelled component of } (\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{(\ddagger \xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})^{\times}_{\underline{v}})^{G_{\underline{v}}({}^{\ddagger}\Pi_{\underline{v}})}\right) \subset \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})$$

(i.e., "$(\widetilde{K_{\underline{v}}} \supset)\, O^{\times}_{K_{\underline{v}}} \subset \mathbb{Q}\log(O^{\times}_{K_{\underline{v}}})$*" for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$*), where* ${}^{\ddagger}\Pi_{\underline{v}} = \{1\}$ *for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*, and the above* j*-labelled component of Galois invariant part of the unit portion acts multiplicatively on* $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})$*.*

Proof. Proposition follows from the definitions. □

Proposition 13.7. (Kummer Theory and Upper Semi-Compatibility for Vertically Coric Local LGP-Monoids, [IUTchIII, Proposition 3.5]) *Let* $\{{}^{n,m}\mathcal{HT}^{\boxtimes\mathfrak{p}}\}_{n,m\in\mathbb{Z}}$ *be a collection of* $\boxtimes\boxplus$*-Hodge theatres arising from a Gaussian log-theta-lattice. For each* n *inZ, we write*

$${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$$

for the $\mathcal{D}$*-*$\boxtimes\boxplus$*-Hodge theatre determined, up to isomorphism, by* ${}^{n,m}\mathcal{HT}^{\boxtimes\mathfrak{p}}$ *for* $m \in \mathbb{Z}$*, via the vertical coricity of Theorem 12.5 (1).*

(1) **(Vertically Coric Local LGP-Monoids and Associated Kummer Theory)** *We write*

$$\mathfrak{F}({}^{n,\circ}\mathfrak{D}_{\succ})_t$$

for the $\mathcal{F}$*-prime-strip associated to the labelled collection of monoids "*$\Psi_{\mathrm{cns}}({}^{n,\circ}\mathfrak{D}_{\succ})_t$*" of Corollary 11.20 (3). Then by applying the constructions of Proposition 13.6 (2) to the full* $\mathfrak{log}$*-links associated these (étale-like)* $\mathcal{F}$*-prime-strips (cf. Proposition 12.2 (5)), we obtain a functorial algorithm, with respect to the* $\mathcal{D}$*-*$\boxtimes\boxplus$*-Hodge theatre* ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$*, to construct collections of monoids*

$$\underline{\mathbb{V}} \in \underline{v} \;\mapsto\; \Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\underline{v}}, \;\; {}_{\infty}\Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\underline{v}}$$

equipped with splittings up to torsion when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *(resp. splittings when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$*). We shall refer to them as* **vertically coric étale-like local LGP-monoids** *or* **vertically coric étale-like local logarithmic Gaussian procession monoids***. Note again that we are able to perform this construction, thanks to the* **compatibility of $\mathfrak{log}$-link with the $\mathbb{F}_l^{\rtimes\pm}$-symmetrising isomorphisms***. For each* $n, m \in \mathbb{Z}$*, this functorial algorithm is compatible, in the obvious sende, with the functorial alogrithm of Proposition 13.6 (2) for* ${}^{\dagger}(-) = {}^{n,m}(-)$*, and* ${}^{\ddagger}(-) = {}^{n,m-1}(-)$*, with respect to the Kummer isomorphism*

$$\Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{F}_{\succ})_t \overset{\mathrm{Kum}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^{n,\circ}\mathfrak{D}_{\succ})_t$$

of labelled data of Corollary 11.21 (3) and the identification of ${}^{n,m'}\mathfrak{F}_t$ *with the* $\mathcal{F}$*-prime-strip associated to* $\Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{F}_{\succ})_t$ *for* $m' = m-1, m$*. In particular, for each* $n, m \in \mathbb{Z}$*, we obtain* **Kummer isomorphisms**

$${}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{(n,m-1 \xrightarrow{\mathfrak{log}}) n,m}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}} \overset{\mathrm{Kum}}{\overset{\sim}{\to}} {}_{(\infty)}\Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\underline{v}}$$

for local LGP-monoids for $\underline{v} \in \underline{\mathbb{V}}$*.*

(2) **(Upper Semi-Compatibility)** *The Kummer isomorphisms of the above (1) are* **upper semi-compatible** *with the* $\mathfrak{log}$*-links* ${}^{n,m-1}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\mathfrak{log}} {}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}$ *of* $\boxtimes\boxplus$*-Hodge theatres in the Gaussian log-theta-lattice in the following sense:*

(a) (non-Archimedean Primes) For $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$*, (and* $n \in \mathbb{Z}$*) by Proposition 13.6 (2), we obtain a vertically coric topological module*

$$\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{v_{\mathbb{Q}}}).$$

Then for any $j = 0, \ldots, l^*$*,* $m \in \mathbb{Z}$*,* $\underline{v} \mid v_{\mathbb{Q}}$*, and* $m' \geq 0$*, we have*

$$\bigotimes_{|t| \in \mathbb{S}^{\pm}_{j+1}} \mathrm{Kum} \circ \mathfrak{log}^{m'} \left(\left(\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})^{\times}_{|t|} \right)^{{}^{n,m}\Pi_{\underline{v}}} \right) \subset \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{v_{\mathbb{Q}}}),$$

where we write Kum *for the Kummer isomorphism of (1), and* $\mathfrak{log}^{m'}$ *for the* m'*-th iteration of* $p_{\underline{v}}$*-adic logarithm part of the* $\mathfrak{log}$*-link (Here we consider the* m'*-th iteration only for the elements whose* $(m'-1)$*-iteration lies in the unit group). cf. also the inclusion* (Upper Semi-Compat. (non-Arch)) *in Section 5.1.*

(b) (Archimedean Primes) For $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$*, (and* $n \in \mathbb{Z}$*) by Proposition 13.6 (2), we obtain a vertically coric closed unit ball*

$$\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{v_{\mathbb{Q}}}).$$

Then for any $j = 0, \dots, l^{*}$, $m \in \mathbb{Z}$, $\underline{v} \mid v_{\mathbb{Q}}$, we have

$$\bigotimes_{|t| \in \mathbb{S}^{\pm}_{j+1}} \mathrm{Kum}\left(\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})^{\times}_{|t|}\right) \subset \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{v_{\mathbb{Q}}}),$$

$$\bigotimes_{|t| \in \mathbb{S}^{\pm}_{j+1}} \mathrm{Kum}\left(\text{closed ball of radius } \pi \text{ inside } \Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})^{\mathrm{gp}}_{|t|}\right) \subset \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{v_{\mathbb{Q}}}),$$

and, for $m' \geq 1$,

$$\left(\text{closed ball of radius } \pi \text{ inside } \Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})^{\mathrm{gp}}_{|t|}\right) \supset (\text{a subset}) \overset{\mathfrak{log}^{m'}}{\twoheadrightarrow} \Psi_{\mathrm{cns}}({}^{n,m-m'}\mathfrak{F}_{\succ})^{\times}_{|t|},$$

where we write Kum *for the Kummer isomorphism of (1), and $\mathfrak{log}^{m'}$ for the m'-th iteration of the Archimedean exponential part of the $\mathfrak{log}$-link (Here we consider the m'-th iteration only for the elements whose $(m'-1)$-iteration lies in the unit group). cf. also the inclusion* (Upper Semi-Compat. (Arch)) *in Section 5.2.*

(c) *(Bad Primes) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, and $j \neq 0$. Recall that the monoids ${}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{(\ddagger \xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}}$, and ${}_{(\infty)}\Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\underline{v}}$ are equipped with natural splitting up to torsion in the case of ${}_{\infty}\Psi(-)$, and up to $2l$-torsion in the case of $\Psi(-)$. We write*

$${}_{(\infty)}\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{(n,m-1 \xrightarrow{\mathfrak{log}})n,m}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}} \ \subset \ {}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{(n,m-1 \xrightarrow{\mathfrak{log}})n,m}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}},$$

$${}_{(\infty)}\Psi^{\perp}_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\underline{v}} \ \subset \ {}_{(\infty)}\Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\underline{v}}$$

for the submonoids defined by these splittings. Then the actions of the monoids

$$\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{(n,m-1 \xrightarrow{\mathfrak{log}})n,m}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}} \quad (m \in \mathbb{Z})$$

on the ind-topological modules

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{\underline{v}}) \ \subset \ \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},j}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{\underline{v}}) \quad (j = 1, \dots, l^{*}),$$

via the Kummer isomorphisms of (1) is **mutually compatible***, with respect to the $\mathfrak{log}$-links of the n-th column of the Gaussian log-theta-lattice, in the following sense: The only portions of these actions which are possibly related to each other via these $\mathfrak{log}$-links are the indeterminacies with respect to multiplication by roots of unity in the domains of the $\mathfrak{log}$-links (since $\Psi^{\perp}(-) \cap \Psi^{\times}(-) = \mu_{2l}$). Then the $p_{\underline{v}}$-adic logarithm portion of the $\mathfrak{log}$-link sends the indeterminacies at m (i.e., multiplication by μ_{2l}) to addition by zero, i.e., no indeterminacy! at $m+1$ (cf. also Remark 10.12.1, Definition 12.1 (2), (4), and Proposition 12.2 (2) for the discussion on quotients by $\Psi^{\mu_N}_{\dagger\mathcal{F}_{\underline{v}}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$).*

Now, we consider the groups

$$((\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})_{|t|})^{\times}_{\underline{v}})^{G_{\underline{v}}({}^{n,m}\Pi_{\underline{v}})}, \quad \Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{(n,m-1\xrightarrow{\mathfrak{log}})n,m}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}}^{G_{\underline{v}}({}^{n,m-1}\Pi_{\underline{v}})}$$

of units for $\underline{v} \in \underline{\mathbb{V}}$*, and the splitting monoids*

$$\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{(n,m-1\xrightarrow{\mathfrak{log}})n,m}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}}$$

for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *as acting on the modules*

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{v_{\mathbb{Q}}})$$

not via a single Kummer isomorphism of (1), which fails to be compatible with the $\mathfrak{log}$*-links, but rather via the* **totality** *of the pre-composites of Kummer isomorphisms with iterates of the* $p_{\underline{v}}$*-adic logarithmic part/Archimedean exponential part of* $\mathfrak{log}$*-links as in the above (2). In this way, we obtain a* **local $\mathfrak{log}$-Kummer correspondence** *between the* **totality** *of the various groups of units and splitting monoids for* $m \in \mathbb{Z}$*, and their actions on the "*$\mathcal{I}^{\mathbb{Q}}(-)$*" labelled by "*$n, \circ$*"*

$$\{\ \mathrm{Kum} \circ \mathfrak{log}^{m'}(\text{groups of units, splitting monoids at } (n,m)) \curvearrowright \mathcal{I}^{\mathbb{Q}}({}^{n,\circ}(-))\ \}_{m\in\mathbb{Z},\, m'\geq 0},$$

which is invariant with respect to the translation symmetries $m \mapsto m+1$ *of the* n*-th column of the Gaussian log-theta-latice.*

Proof. Proposition follows from the definitions. □

Proposition 13.8. (Global Packet-Theoretic Frobenioids, [IUTchIII, Proposition 3.7])

(1) **(Single Packet Global non-Realified Frobenioid, ⊠-Line Bundle Version)** *In the situation of Proposition 13.5, for each* $\alpha \in \mathbb{S}^{*}_{j}$*, by the construction of Definition 9.7 (1), we have a functorial algorithm, from the* **image**

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_{\alpha} := \mathrm{Im}\left(({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha} \hookrightarrow ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\mathbb{S}^{*}_{j}} \hookrightarrow \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})\right)$$

of the number field, via the homomorphisms of Proposition 13.5 (1), (2) to construct a (pre-)Frobenioid

$$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$$

with a natural isomorphism

$$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$$

of (pre-)Frobenioids (cf. Corollary 11.23 (2) for $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha}$*), which induces the tautological isomorphism* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ *on the associated rational function monoids. We ofthen identify* $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha}$ *with* $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$*, via the above isomorphism. We write* $({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_{\alpha}$ *for the realification of* $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$*.*

(2) **(Single Packet Global non-Realified Frobenioid, $\boxplus$-Line Bundle Version)** *For each $\alpha \in \mathbb{S}^{\circledast}_{j}$, by the construction of Definition 9.7 (2), we have a functorial algorithm, from the number field $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_{\alpha} := ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ and the Galois invariant local monoids*

$$(\Psi_{\mathfrak{log}(^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})})^{G_{\underline{v}}(^{\alpha}\Pi_{\underline{v}})}$$

of Proposition 13.6 (1) for $\underline{v} \in \underline{\mathbb{V}}$, to construct a (pre-)Frobenioid

$$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$$

(Note that, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ (resp. $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$), the corresponding local fractional ideal $J_{\underline{v}}$ of Definition 9.7 (2) is a submodule (resp. subset) of $\mathcal{I}^{\mathbb{Q}}(^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$ whose $\mathbb{Q}$-span is equal to $\mathcal{I}^{\mathbb{Q}}(^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$) with natural isomorphisms

$$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha}, \quad ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$$

of (pre-)Frobenioids, which induces the tautological isomorphisms $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha}$, $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ on the associated rational function monoids, respectively. We write $({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_{\alpha}$ for the realification of $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$.

(3) **(Global Realified Logarithmic Gaussian Procession Frobenioids, $\boxtimes$-Line Bundle Version)** *Let ${}^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$ a $\mathfrak{log}$-link. In this case, in the construction of the above (1), (2), the target $\underline{\mathfrak{log}}(^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})$ of the injection is $\ddagger$-labbeled object $\underline{\mathfrak{log}}(^{\mathbb{S}^{\pm}_{j+1},j;\ddagger}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})$, thus, we write $(^{(\ddagger\to)\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_{\alpha}$, $(^{(\ddagger\to)\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_{\alpha}$, $(^{(\ddagger\to)\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$, $(^{(\ddagger\to)\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$ for $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_{\alpha}$, $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_{\alpha}$, $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$, $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$, respectively, in order to specify the dependence. Consider the diagram*

$$\begin{array}{ccc} \prod_{j\in\mathbb{F}^{\circledast}_{l}} {}^{\dagger}\mathcal{C}^{\Vdash}_{j} & \overset{\text{gl. real'd to gl. non-real'd}\otimes\mathbb{R}}{\xrightarrow{\sim}} & \prod_{j\in\mathbb{F}^{\circledast}_{l}} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_{j} \xrightarrow{\sim} \prod_{j\in\mathbb{F}^{\circledast}_{l}} (^{(\ddagger\to)\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_{j}, \\ \cup & & \\ {}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{gau}} & & \end{array}$$

where the isomorphisms in the upper line are Corollary 11.23 (3) and the realification of the isomorphism in (1). Then by sending the global realified portion ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{gau}}$ of the $\mathcal{F}^{\Vdash}$-prime-strip ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$ of Corolllary 11.24 (2) via the isomorphisms of the upper line, we obtain a functorial algorithm, with respect to the $\mathfrak{log}$-link ${}^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$ of Proposition 13.6 (2), to construct a (pre-)Frobenioid

$$\mathcal{C}^{\Vdash}_{\mathrm{LGP}}((^{\ddagger}\xrightarrow{\mathfrak{log}})^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}).$$

We shall refer to $^{(\ddagger\to)\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} := \mathcal{C}^{\Vdash}_{\mathrm{LGP}}((^{\ddagger}\xrightarrow{\mathfrak{log}})^{\dagger}\mathcal{HT}^{\boxtimes\boxplus})$ *as a* **Frobenius-like global realified LGP-monoid** *or* **Frobenius-like global realified $\boxtimes$-logarithmic Gaussian procession monoids**. *The combination of it with the collection $\Psi_{\mathcal{F}_{\mathrm{LGP}}}((^{\ddagger}\xrightarrow{\mathfrak{log}})^{\dagger}\mathcal{HT}^{\boxtimes\boxplus})$*

of data constructed by Proposition 13.6 (2) gives rise to an $\mathcal{F}^{\Vdash}$*-prime-strip*

$$^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} = ({}^{(\ddagger\to)\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}},\ \mathrm{Prime}({}^{(\ddagger\to)\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}}) \xrightarrow{\sim} \underline{\mathbb{V}},\ {}^{(\ddagger\to)\dagger}\mathfrak{F}^{\vdash}_{\mathrm{LGP}},\ \{{}^{(\ddagger\to)\dagger}\rho_{\mathrm{LGP},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

with a natural isomorphism

$$^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}} \xrightarrow{\sim} {}^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$$

of $\mathcal{F}^{\Vdash}$*-prime-strips.*

(4) **(Global Realified Logarithmic Gaussian Procession Frobenioids, ⊞-Line Bundle Version)** *Write*

$$\Psi_{\mathcal{F}_{\mathfrak{lgp}}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus}) := \Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus}), \quad {}^{(\ddagger\to)\dagger}\mathfrak{F}^{\vdash}_{\mathfrak{lgp}} := {}^{(\ddagger\to)\dagger}\mathfrak{F}^{\vdash}_{\mathrm{LGP}}.$$

In the construction of (3), by replacing $({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_j$ *by* $({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_j$*, we obtain a functorial algorithm, with respect to the* $\mathfrak{log}$*-link* ${}^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus}$ *of Proposition 13.6 (2), to construct a (pre-)Frobenioid*

$$^{(\ddagger\to)\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}} := \mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus}).$$

and an $\mathcal{F}^{\Vdash}$*-prime-strip*

$$^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}} = ({}^{(\ddagger\to)\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}},\ \mathrm{Prime}({}^{(\ddagger\to)\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}) \xrightarrow{\sim} \underline{\mathbb{V}},\ {}^{(\ddagger\to)\dagger}\mathfrak{F}^{\vdash}_{\mathfrak{lgp}},\ \{{}^{(\ddagger\to)\dagger}\rho_{\mathfrak{lgp},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

with tautological isomorphisms

$$^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}} \xrightarrow{\sim} {}^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} \xrightarrow{\sim} {}^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$$

of $\mathcal{F}^{\Vdash}$*-prime-strips. We shall refer to* ${}^{(\ddagger\to)\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}} := \mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})$ *as a* **Frobenius-like global realified** $\mathfrak{lgp}$**-monoid** *or* **Frobenius-like global realified ⊞-logarithmic Gaussian procession monoids***.*

(5) **(Global Realified to Global non-Realified⊗ℝ)** *By the constructions of global realified Frobenioids* $\mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})$ *and* $\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})$ *of (3), (4), we have a commutative diagram*

$$\begin{tikzcd}
\mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus}) \arrow[r, hook] \arrow[d, "\cong"] & \prod_{j\in\mathbb{F}^{*}_{l}}({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_j \arrow[d, "\cong"] \\
\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus}) \arrow[r, hook] & \prod_{j\in\mathbb{F}^{*}_{l}}({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_j.
\end{tikzcd}$$

In particular, by the definition of $({}^\dagger\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j$ *in terms of local fractional ideals, and the product of the realification functors* $\prod_{j\in\mathbb{F}_l^*}({}^\dagger\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j \to \prod_{j\in\mathbb{F}_l^*}({}^\dagger\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_j$*, we obtain an algorithm, which is compatible, in the obvious sense, with the localisation isomorphisms* $\{{}^\dagger\rho_{\mathfrak{lgp},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ *and* $\{{}^\dagger\rho_{\mathrm{LGP},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$*, to construct objects of the (global) categories* $\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})$*,* $\mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})$*, from the local fractional ideals generated by elements of the monoid* $\Psi_{\mathcal{F}_{\mathfrak{lgp}}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}}$ *for* $\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}$.

Proof. Proposition follows from the definitions. □

Definition 13.9. ([IUTchIII, Definition 3.8])

(1) Write $\Psi^{\perp}_{\mathcal{F}_{\mathfrak{lgp}}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}} := \Psi_{\mathcal{F}_{\mathfrak{lgp}}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}}$ for $\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}$. When we regard the object of

$$\prod_{j\in\mathbb{F}_l^*}({}^\dagger\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j$$

and its realification determined by any collection, indexed by $\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}$, of generators up to μ_{2l} of the monoids $\Psi^{\perp}_{\mathcal{F}_{\mathfrak{lgp}}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}}$, as an object of the global realified Frobenioid ${}^{(\ddagger\to)\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} = \mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})$ or ${}^{(\ddagger\to)\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}} = \mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{(\ddagger\xrightarrow{\mathfrak{log}})\dagger}\mathcal{HT}^{\boxtimes\boxplus})$, then we shall refer to it as a **Θ-pilot object**.

We shall refer to the object of the global realified Frobenioid ${}^\dagger\mathcal{C}^{\Vdash}_{\Delta}$ of Corollary 11.24 (1) determined by any collection, indexed by $\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}$, of generators up to torsion of the splitting monoid associated to the split Frobenioid ${}^\dagger\mathcal{F}^{\vdash}_{\Delta,\underline{v}}$ in the $\underline{v}$-component of the $\mathcal{F}^{\vdash}$-prime-strip ${}^\dagger\mathfrak{F}^{\vdash}_{\Delta}$ of Corollary 11.24 (1), as a ***q*-pilot object**.

(2) Let ${}^\ddagger\mathcal{HT}^{\boxtimes\boxplus}\xrightarrow{\mathfrak{log}}{}^\dagger\mathcal{HT}^{\boxtimes\boxplus}$ be a $\mathfrak{log}$-link of ⊠⊞-Hodge theatres, and

$${}^*\mathcal{HT}^{\boxtimes\boxplus}$$

a ⊠⊞-Hodge theatre. Let

$${}^*\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta}\quad(\text{resp. }{}^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{LGP}},\quad\text{resp. }{}^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathfrak{lgp}}\quad)$$

be the $\mathcal{F}^{\Vdash\blacktriangleright\times\mu}$-prime-strip associated to the $\mathcal{F}^{\Vdash}$-prime strip ${}^*\mathfrak{F}^{\Vdash}_{\Delta}$ of Corollary 11.24 (1) (resp. ${}^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, resp. ${}^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$). We shall refer to the full poly-isomorphism

$${}^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{LGP}}\overset{\text{full poly}}{\xrightarrow{\sim}}{}^*\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta}\quad(\text{resp. }{}^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathfrak{lgp}}\overset{\text{full poly}}{\xrightarrow{\sim}}{}^*\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta}\quad)$$

as the $\mathbf{\Theta^{\times\mu}_{LGP}}$**-link** (resp. $\Theta^{\times\mu}_{\mathfrak{lgp}}$**-link**) from ${}^\dagger\mathcal{HT}^{\boxtimes\boxplus}$ to ${}^*\mathcal{HT}^{\boxtimes\boxplus}$, relative to the $\mathfrak{log}$-link ${}^\ddagger\mathcal{HT}^{\boxtimes\boxplus}\xrightarrow{\mathfrak{log}}{}^\dagger\mathcal{HT}^{\boxtimes\boxplus}$, and we write it as

$${}^\dagger\mathcal{HT}^{\boxtimes\boxplus}\xrightarrow{\Theta^{\times\mu}_{\mathrm{LGP}}}{}^*\mathcal{HT}^{\boxtimes\boxplus}\quad(\text{resp. }{}^\dagger\mathcal{HT}^{\boxtimes\boxplus}\xrightarrow{\Theta^{\times\mu}_{\mathfrak{lgp}}}{}^*\mathcal{HT}^{\boxtimes\boxplus}\quad).$$

(3) Let $\{{}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}\}_{n,m\in\mathbb{Z}}$ be a collection of $\boxtimes\boxplus$-Hodge theatres indexed by pairs of integers. We shall refer to the diagram

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r, "\Theta^{\times\mu}_{\mathrm{LGP}}"] & {}^{n,m+1}\mathcal{HT}^{\boxtimes\boxplus} \arrow[u, "\text{full } \mathfrak{log}"'] \arrow[r, "\Theta^{\times\mu}_{\mathrm{LGP}}"] & {}^{n+1,m+1}\mathcal{HT}^{\boxtimes\boxplus} \arrow[u, "\text{full } \mathfrak{log}"'] \arrow[r, "\Theta^{\times\mu}_{\mathrm{LGP}}"] & \cdots \\
\cdots \arrow[r, "\Theta^{\times\mu}_{\mathrm{LGP}}"] & {}^{n,m}\mathcal{HT}^{\boxtimes\boxplus} \arrow[u, "\text{full } \mathfrak{log}"'] \arrow[r, "\Theta^{\times\mu}_{\mathrm{LGP}}"] & {}^{n+1,m}\mathcal{HT}^{\boxtimes\boxplus} \arrow[u, "\text{full } \mathfrak{log}"'] \arrow[r, "\Theta^{\times\mu}_{\mathrm{LGP}}"] & \cdots, \\
 & \vdots \arrow[u, "\text{full } \mathfrak{log}"'] & \vdots \arrow[u, "\text{full } \mathfrak{log}"'] &
\end{tikzcd}$$

(resp.

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r, "\Theta^{\times\mu}_{\mathfrak{lgp}}"] & {}^{n,m+1}\mathcal{HT}^{\boxtimes\boxplus} \arrow[u, "\text{full } \mathfrak{log}"'] \arrow[r, "\Theta^{\times\mu}_{\mathfrak{lgp}}"] & {}^{n+1,m+1}\mathcal{HT}^{\boxtimes\boxplus} \arrow[u, "\text{full } \mathfrak{log}"'] \arrow[r, "\Theta^{\times\mu}_{\mathfrak{lgp}}"] & \cdots \\
\cdots \arrow[r, "\Theta^{\times\mu}_{\mathfrak{lgp}}"] & {}^{n,m}\mathcal{HT}^{\boxtimes\boxplus} \arrow[u, "\text{full } \mathfrak{log}"'] \arrow[r, "\Theta^{\times\mu}_{\mathfrak{lgp}}"] & {}^{n+1,m}\mathcal{HT}^{\boxtimes\boxplus} \arrow[u, "\text{full } \mathfrak{log}"'] \arrow[r, "\Theta^{\times\mu}_{\mathfrak{lgp}}"] & \cdots \\
 & \vdots \arrow[u, "\text{full } \mathfrak{log}"'] & \vdots \arrow[u, "\text{full } \mathfrak{log}"'] &
\end{tikzcd}$$

) as the LGP-**Gaussian log-theta-lattice** (resp. $\mathfrak{lgp}$-**Gaussian log-theta-lattice**), where the $\Theta^{\times\mu}_{\mathrm{LGP}}$-link (resp. $\Theta^{\times\mu}_{\mathfrak{lgp}}$-link) from ${}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}$ to ${}^{n+1,m}\mathcal{HT}^{\boxtimes\boxplus}$ is taken relative to the full $\mathfrak{log}$-link ${}^{n,m-1}\mathcal{HT}^{\boxtimes\boxplus} \stackrel{\text{full } \mathfrak{log}}{\longrightarrow} {}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}$. Note that both $\Theta^{\times\mu}_{\mathrm{LGP}}$-link and $\Theta^{\times\mu}_{\mathfrak{lgp}}$-link send Θ-pilot objects to q-pilot objects.

Proposition 13.10. (Log-volume for Packets and Processions, [IUTchIII, Proposition 3.9])

(1) **(Local Holomorphic Packets)** *In the situation of Proposition 13.4 (1), (2), for $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ (resp. $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$), $\alpha \in \mathbb{S}^{\pm}_{j+1}$, the $p_{v_{\mathbb{Q}}}$-adic log-volume (resp. the radial log-volume) on each of the direct summand $p_{v_{\mathbb{Q}}}$-adic fields (resp. complex Archimedean fields) of $\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}})$, and $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j}\mathcal{F}_{v_{\mathbb{Q}}})$ with the normalised weights of Remark 13.3.1 determines log-volumes*

$$\mu^{\log}_{\alpha,v_{\mathbb{Q}}} : \mathbb{M}(\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})) \to \mathbb{R}, \quad \mu^{\log}_{\mathbb{S}^{\pm}_{j+1},v_{\mathbb{Q}}} : \mathbb{M}(\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}})) \to \mathbb{R},$$

$$\mu^{\log}_{\mathbb{S}^{\pm}_{j+1},\alpha,\underline{v}} : \mathbb{M}(\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})) \to \mathbb{R},$$

where we write $\mathbb{M}(-)$ *for the set of compact open non-empty subsets of* $(-)$ *(resp. the set of compact closures of open subsets of* $(-)$*), such that the log-volume of each of the local holomorphic integral structures*

$$\mathcal{O}_{{}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}} \subset \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}), \quad \mathcal{O}_{{}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}}} \subset \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}}), \quad \mathcal{O}_{{}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}}} \subset \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}}),$$

given by the integral structures of Proposition 13.3 (2) on each of the direct summand, is equal to zero. Here, we assume that these log-volumes are normalised in such a manner that multiplication by $p_{\underline{v}}$ *corresponds to* $-\log(p_{\underline{v}})$ *(resp.* $+\log(p_{\underline{v}})$*) on the log-volume (cf. Remark 13.3.1) (cf. Section 0.2 for* $p_{\underline{v}}$ *with Archimedean* $\underline{v}$*). We shall refer to this normalisation as the* **packet-normalisation***. Note that "*$\mu^{\log}_{\mathbb{S}^{\pm}_{j+1},v_{\mathbb{Q}}}$*" is invaariant by permutations of* $\mathbb{S}^{\pm}_{j+1}$*. When we are working with collections of capsules in a procession, we normalise log-volumes on the products of "*$\mathbb{M}(-)$*" associated to the various capsules by taking the average over the various capsules. We shall refer to this normalisation as the* **procession-normalisation***.*

(2) **(Mono-analytic Compatibility)** *In the situation of Proposition 13.4 (1), (2), for* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ *(resp.* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$*),* $\alpha \in \mathbb{S}^{\pm}_{j+1}$*, by applying the* $p_{v_{\mathbb{Q}}}$*-adic log-volume (resp. the radial log-volume) on the mono-analytic log-shells "*$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}$*" of Proposition 12.2 (4), and adjusting appropriately the discrepancy between the local holomorphic integral structures of Proposition 13.3 (2) and the mono-analytic integral structures of Proposition 13.4 (2), we obtain log-volumes*

$$\mu^{\log}_{\alpha,v_{\mathbb{Q}}} : \mathbb{M}(\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})) \to \mathbb{R}, \quad \mu^{\log}_{\mathbb{S}^{\pm}_{j+1},v_{\mathbb{Q}}} : \mathbb{M}(\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})) \to \mathbb{R},$$

$$\mu^{\log}_{\mathbb{S}^{\pm}_{j+1},\alpha,\underline{v}} : \mathbb{M}(\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{D}^{\vdash}_{\underline{v}})) \to \mathbb{R},$$

where we write $\mathbb{M}(-)$ *for the set of compact open non-empty subsets of* $(-)$ *(resp. the set of compact closures of open subsets of* $(-)$*), which are compatible with the log-volumes of (1), with respect to the natural poly-isomorphisms of Proposition 13.4 (1). In particular, these log-volumes can be constructed via a functorial alogrithm from the* $\mathcal{D}^{\vdash}$*-prime-strips. If we consider the mono-analyticisation of an* $\mathcal{F}$*-prime-strip procession as in Proposition 13.6 (2), then taking the average of the packet-normalised log-volumes gives rise to procession-normalised log-volumes, which are compatible with the procession-normalised log-volumes of (1), with respect to the natural poly-isomorphisms of Proposition 13.4 (1). By replacing "*$\mathcal{D}^{\vdash}$*" by* $\mathcal{F}^{\vdash\times\mu}$*,* *we obtain a similar theory of log-volumes for the various objects associated to the mono-analytic log-shells "$\mathcal{I}_{\dagger \mathcal{F}_{\underline{v}}^{\vdash \times \mu}}$"*

$$\mu^{\log}_{\alpha, v_{\mathbb{Q}}} : \mathbb{M}(\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}^{\vdash \times \mu}_{v_{\mathbb{Q}}})) \to \mathbb{R}, \quad \mu^{\log}_{\mathbb{S}^{\pm}_{j+1}, v_{\mathbb{Q}}} : \mathbb{M}(\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}^{\vdash \times \mu}_{v_{\mathbb{Q}}})) \to \mathbb{R},$$

$$\mu^{\log}_{\mathbb{S}^{\pm}_{j+1}, \alpha, \underline{v}} : \mathbb{M}(\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}, \alpha}\mathcal{F}^{\vdash \times \mu}_{\underline{v}})) \to \mathbb{R},$$

which is compatible with the "$\mathcal{D}^{\vdash}$"-version, with respect to the natural poly-isomorphisms of Proposition 13.4 (1).

(3) **(Global Compatibility)** *In the situation of Proposition 13.8 (1), (2), write*

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) := \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}}) \ \subset \ \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) = \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}})$$

and we write

$$\mathbb{M}(\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})) \ \subset \ \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \mathbb{M}(\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}}))$$

for the subset of elements whose components have zero log-volume for all but finitely many $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. Then by adding the log-volumes of (1) for $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, we obtain a **global log-volume**

$$\mu^{\log}_{\mathbb{S}^{\pm}_{j+1}, \mathbb{V}_{\mathbb{Q}}} : \mathbb{M}(\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})) \ \to \ \mathbb{R}$$

which is invariant by multiplication by elements of

$$({}^{\dagger}\mathbb{M}^{\circledast}_{\mathfrak{mod}})_{\alpha} \ = \ ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_{\alpha} \ \subset \ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})$$

*(***product formula***), and permutations of $\mathbb{S}^{\pm}_{j+1}$. The global log-volume $\mu^{\log}_{\mathbb{S}^{\pm}_{j+1}, \mathbb{V}_{\mathbb{Q}}}(\{J_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}})$ of an object $\{J_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ of $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$ (cf. Definition 9.7 (2)) is equal to the degree of the arithmetic line bundle determined by $\{J_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ (cf. the natural isomorphism $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha}$ of Proposition 13.8 (2)), with respect to a suitable normalisation.*

(4) **($\mathfrak{log}$-Link Compatibility)** *Let $\{{}^{n,m}\mathcal{HT}^{\boxtimes \boxplus}\}_{n,m \in \mathbb{Z}}$ be a collection of $\boxtimes\boxplus$-Hodge theatres arising from an* LGP*-Gaussian log-theta-lattice.*

(a) For $n, m \in \mathbb{Z}$, the log-volumes of the above (1), (2), (3) determine log-volumes on the various "$\mathcal{I}^{\mathbb{Q}}(-)$" appearing in the construction of the local/global LGP*-/$\mathfrak{lgp}$-monoids/Frobenioids in the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$ of Proposition 13.8 (3), (4), relative to the $\mathfrak{log}$-link ${}^{n,m-1}\mathcal{HT}^{\boxtimes \boxplus} \overset{\text{full } \mathfrak{log}}{\longrightarrow} {}^{n,m}\mathcal{HT}^{\boxtimes \boxplus}$.*

(b) At the level of the $\mathbb{Q}$-spans of log-shells "$\mathcal{I}^{\mathbb{Q}}(-)$" arising from the various $\mathcal{F}$-prime-strips involved, the log-volumes of (a) indexed by (n, m) are compatible, in the sense of Proposition 12.2 (2) (i.e., in the sense of the formula (5.1) *of Proposition 5.2 and the formula* (5.2) *of Proposition 5.4), with the log-volumes indexed by $(n, m-1)$ with respect to the $\mathfrak{log}$-link ${}^{n,m-1}\mathcal{HT}^{\boxtimes\boxplus} \stackrel{\text{full } \mathfrak{log}}{\longrightarrow} {}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}$ (This means that we* **do not** *need to be worried about* **how many times $\mathfrak{log}$-links are applied** *in the $\mathfrak{log}$-Kummer correspondence, when we take values of the log-volumes).*

Proof. Proposition follows from the definitions. $\square$

Proposition 13.11. (Global Kummer Theory and Non-Interference with Local Integers, [IUTchIII, Proposition 3.10]) *Let $\{{}^{n,m}\mathcal{HT}^{\boxtimes\mathfrak{p}}\}_{n,m\in\mathbb{Z}}$ be a collection of $\boxtimes\boxplus$-Hodge theatres arising from an* LGP*-Gaussian log-theta-lattice. For each n inZ, we write*

$${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$$

for the $\mathcal{D}$-$\boxtimes\boxplus$-Hodge theatre determined, up to isomorphism, by ${}^{n,m}\mathcal{HT}^{\boxtimes\mathfrak{p}}$ for $m \in \mathbb{Z}$, via the vertical coricity of Theorem 12.5 (1).

(1) **(Vertically Coric Global** LGP**- $\mathfrak{lgp}$-Frobenioids and Assosiated Kummer Theory)** *By applying the construcions of Proposition 13.8 to the (étale-like) $\mathcal{F}$-prime-strips "$\mathfrak{F}({}^{n,\circ}\mathfrak{D}_{\succ})_t$" and to the full $\mathfrak{log}$-links associated to these (étale-like) $\mathcal{F}$-prime-strips (cf. Proposition 12.2 (5)), we obtain functorial algorithms, with respect to the $\mathcal{D}$-$\boxtimes\boxplus$-Hodge theatre ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$, to construct* **vertically coric étale-like number fields, monoids, and (pre-)Frobenioids equipped with natural isomorphisms**

$$\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\alpha} = \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\alpha} \supset \mathbb{M}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\alpha} = \mathbb{M}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\alpha},$$

$$\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\alpha} \supset \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\alpha},$$

$$\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\alpha} \stackrel{\sim}{\to} \mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\alpha} \stackrel{\sim}{\to} \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\alpha}$$

for $\alpha \in \mathbb{S}^{*}_{j} \stackrel{\text{via } {}^{\dagger}\chi}{\subset} J$, and **vertically coric étale-like $\mathcal{F}^{\Vdash}$-prime-strips equipped with natural isomorphisms**

$$\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\mathrm{gau}} \stackrel{\sim}{\to} \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\mathrm{LGP}} \stackrel{\sim}{\to} \mathcal{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\mathfrak{lgp}}.$$

Note again that we are able to perform this construction, thanks to the **compatibility of $\mathfrak{log}$-link with the $\mathbb{F}_l^{\rtimes\pm}$-symmetrising isomorphisms***. For each*

$n, m \in \mathbb{Z}$, these functorial algorithms are compatible, in the obvious sense, with the (non-vertically coric Frobenius-like) functorial algorithms of Proposition 13.8 for $^{\dagger}(-) = {}^{n,m}(-)$, and $^{\ddagger}(-) = {}^{n,m-1}(-)$, with respect to the **Kummer isomorphisms**

$$\Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{F}_{\succ})_t \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{D}_{\succ})_t,$$

$$({}^{n,m'}\mathbb{M}^{\circledast}_{\mathrm{mod}})_j \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{n,m'}\mathcal{D}^{\circledcirc})_j, \quad ({}^{n,m'}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_j \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{n,m'}\mathcal{D}^{\circledcirc})_j$$

of labelled data (cf. Corollary 11.21 (3), and Corollary 11.23 (2)), and the evident identification of ${}^{n,m'}\mathfrak{F}_t$ with the $\mathcal{F}$-primes-strip associated to $\Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{F}_{\succ})_t$ for $m' = m-1, m$. In particular, for each $n, m \in \mathbb{Z}$, we obtain **Kummer isomorphisms**

$$({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_\alpha \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_\alpha, \quad ({}^{(n,m-1\to)n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}})_\alpha \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_\alpha,$$

$$({}^{n,m}\mathbb{M}^{\circledast}_{\mathrm{mod}})_\alpha \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_\alpha, \quad ({}^{(n,m-1\to)n,m}\mathbb{M}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}})_\alpha \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathbb{M}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_\alpha,$$

$$({}^{n,m}\mathcal{F}^{\circledast}_{\mathrm{mod}})_\alpha \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_\alpha, \quad ({}^{(n,m-1\to)n,m}\mathcal{F}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}})_\alpha \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_\alpha,$$

$${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{gau}} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\mathrm{gau}}, \quad ({}^{(n,m-1\to)n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}/\mathfrak{lgp}}) \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\mathrm{LGP}/\mathfrak{lgp}},$$

(Here $(-)_{\mathrm{MOD}/\mathfrak{mod}}$ is the shorthand for "$(-)_{\mathrm{MOD}}$ (resp. $(-)_{\mathfrak{mod}}$)", and $(-)_{\mathrm{LGP}/\mathfrak{lgp}}$ is the shorthand for "$(-)_{\mathrm{LGP}}$ (resp. $(-)_{\mathfrak{lgp}}$)") of fields, monoids, Frobenioids, and $\mathcal{F}^{\Vdash}$-prime-strips, which are compatible with the above various equalities, natural inclusions, and natural isomorphisms.

(2) **(Non-Interference with Local Integers)** *In the notation of Proposition 13.4 (2), Proposition 13.6 (1), Proposition 13.8 (1), (2), and Proposition 13.10 (3), we have*

$$({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_\alpha \cap \prod_{\underline{v}\in\underline{\mathbb{V}}} \Psi_{\mathfrak{log}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})} = \mu(({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_\alpha) \left(\subset \prod_{\underline{v}\in\underline{\mathbb{V}}} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}}) = \prod_{v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}}) = \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) \right)$$

(i.e., "$F^{\times}_{\mathrm{mod}} \cap \prod_{v\le\infty} O^{\rhd}_{(F_{\mathrm{mod}})_v} = \mu(F^{\times}_{\mathrm{mod}})$") (Here, we identify $\prod_{\underline{\mathbb{V}}\ni\underline{v}|v_{\mathbb{Q}}} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},\alpha}\mathcal{F}_{\underline{v}})$ with $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}_{v_{\mathbb{Q}}})$). Now, we consider the multiplicative groups

$$({}^{(n,m-1\to)n,m}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_j$$

of non-zero elements of number fields as acting on the modules

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{\mathbb{V}_{\mathbb{Q}}})$$

not via a single Kummer isomorphism of (1), which fails to be compatible with the $\mathfrak{log}$*-links, but rather via the* **totality** *of the pre-composites of Kummer isomorphisms with iterates of the* $p_{\underline{v}}$*-adic logarithmic part/Archimedean exponential part of* $\mathfrak{log}$*-links, where we observe that these actions are* **mutually compatible***, with respect to the* $\mathfrak{log}$*-links of the* n*-th column of the* LGP*-Gaussian log-theta-lattice, in the following sense: The only portions of these actions which are possibly related to each other via these* $\mathfrak{log}$*-links are the indeterminacies with respect to multiplication by roots of unity in the domains of the* $\mathfrak{log}$*-links (by the above displayed equality). Then the* $p_{\underline{v}}$*-adic logarithm portion of the* $\mathfrak{log}$*-link sends the indeterminacies at* m *(i.e., multiplication by* $\mu({}^{(n,m-1\to)n,m}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_j$*) to addition by zero, i.e., no indeterminacy! at* $m+1$ *(cf. also Remark 10.12.1, Definition 12.1 (2), (4), and Proposition 12.2 (2) for the discussion on quotients by* $\Psi^{\mu_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ *for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*). In this way, we obtain a* **global** $\mathfrak{log}$**-Kummer correspondence** *between the* **totality** *of the various multiplicative groups of non-zero elements of number fields for* $m \in \mathbb{Z}$*, and their actions on the "*$\mathcal{I}^{\mathbb{Q}}(-)$*" labelled by "*$n, \circ$*"*

$$\{ \ \mathrm{Kum} \circ \mathfrak{log}^{m'}(({}^{(n,m-1\to)n,m}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_j) \curvearrowright \mathcal{I}^{\mathbb{Q}}({}^{n,\circ}(-)) \ \}_{m\in\mathbb{Z},\, m'\geq 0},$$

which is invariant with respect to the translation symmetries $m \mapsto m+1$ *of the* n*-th column of the* LGP*-Gaussian log-theta-latice.*

(3) **(Frobenioid-theoretic** $\mathfrak{log}$**-Kummer Correspondences)** *The Kummer isomorphisms of (1) induce, via the* $\mathfrak{log}$*-Kummer correspondence of (2), isomorphisms of (pre-)Frobenioids*

$$({}^{(n,m-1\to)n,m}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_\alpha \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_\alpha, \ \ ({}^{(n,m-1\to)n,m}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_\alpha \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_\alpha$$

which are mutually compatible with the $\mathfrak{log}$*-links of the* LGP*-Gaussian log-theta-lattice, as* m *rus over the elements of* $\mathbb{Z}$*. These compatible isomorphisms of (pre-)Frobenioids with the Kummer isomorphisms of (1) induce, via the global* $\mathfrak{log}$*-Kummer correspondence of (2) and the splitting monoid portion of the the local* $\mathfrak{log}$*-Kummer correspondence of Proposition 13.7 (2), a* **Kummer isomorphism**

$${}^{(n,m-1\to)n,m}\mathfrak{F}^{\Vdash\perp}_{\mathrm{LGP}} \overset{\mathrm{Kum}}{\xrightarrow{\sim}} \mathfrak{F}^{\Vdash\perp}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\mathrm{LGP}}$$

of associated $\mathcal{F}^{\Vdash\perp}$*-prime-strips, which are* **mutually compatible** *with the* $\mathfrak{log}$*-links of the* LGP*-Gaussian log-theta-lattice, as* m *rus over the elements of* $\mathbb{Z}$*.*

Note that we use only MOD*-/*LGP*-labelled objects in (2) and (3), since these are defined only in terms of multiplicative operations (*$\boxtimes$*), and that the compatibility of Kummer isomorphisms with* $\mathfrak{log}$*-links does not hold for* $\mathfrak{mod}$*-/*$\mathfrak{lgp}$*-labelled objects, since these are* *defined in terms of both multiplicative and additive operaions ($\boxtimes$ and $\boxplus$), where we only expect only a upper semi-compatibility (cf. Definition 9.7, and Proposition 13.7 (2)).*

Proof. Proposition follows from the definitions. □

The following the **Main Theorem** of inter-universal Teichmüller theory:

Theorem 13.12. (Multiradial Algorithms via LGP-Monoids/Frobenioids, [IUTchIII, Theorem 3.11]) *Fix an initial* Θ*-data*

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon}).$$

Let

$$\{{}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}\}_{n,m\in\mathbb{Z}}$$

be a collection of $\boxtimes\boxplus$*-Hodge theatres, with respect to the fixed initial* Θ*-date, arising from an* LGP*-Gaussian log-theta-lattice. For each* $n\in\mathbb{Z}$*, we write*

$${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$$

for the $\mathcal{D}$*-*$\boxtimes\boxplus$*-Hodge theatre determined, up to isomorphism, by* ${}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}$ *for* $m\in\mathbb{Z}$*, via the vertical coricity of Theorem 12.5 (1).*

(1) **(Multiradial Representation)** *Consider the procession of* $\mathcal{D}^{\vdash}$*-prime-strips* $\mathrm{Proc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$

$$\{{}^{n,\circ}\mathfrak{D}^{\vdash}_0\}\hookrightarrow\{{}^{n,\circ}\mathfrak{D}^{\vdash}_0,{}^{n,\circ}\mathfrak{D}^{\vdash}_1\}\hookrightarrow\dots\hookrightarrow\{{}^{n,\circ}\mathfrak{D}^{\vdash}_0,{}^{n,\circ}\mathfrak{D}^{\vdash}_1,\dots,{}^{n,\circ}\mathfrak{D}^{\vdash}_{l^*}\}.$$

Consider also the following data:

(Shells) *(Unit portion — Mono-anaytic Containers) For* $\underline{\mathbb{V}}\ni\underline{v}\mid v_{\mathbb{Q}}$*,* $j\in|\mathbb{F}_l|$*, the* **topological modules and mono-analytic integral structures**

$$\mathcal{I}(^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})\ \subset\ \mathcal{I}^{\mathbb{Q}}(^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}),\ \ \mathcal{I}(^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}})\ \subset\ \mathcal{I}^{\mathbb{Q}}(^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}}),$$

which we regard as equipped with the procession-normalised mono-analytic log-volumes of Proposition 13.10 (2),

(ThVals) *(Value Group Portion — Theta Values) For* $\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}$*, the* **splitting monoid**

$$\Psi^{\perp}_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\underline{v}}$$

of Proposition 13.7 (2c), which we regard as a subset of

$$\prod_{j\in\mathbb{F}^*_l}\mathcal{I}^{\mathbb{Q}}(^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}}),$$

equipped with a multiplicative action on $\prod_{j\in\mathbb{F}_l^*}\mathcal{I}^{\mathbb{Q}}(\mathbb{S}^{\pm}_{j+1},{}^{j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}})$*, via the natural poly-isomorphisms*

$$\mathcal{I}^{\mathbb{Q}}(\mathbb{S}^{\pm}_{j+1},{}^{j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}}) \overset{\text{“Kum”}^{-1}}{\underset{\text{poly}}{\stackrel{\sim}{\to}}} \mathcal{I}^{\mathbb{Q}}(\mathbb{S}^{\pm}_{j+1},{}^{j;n,\circ}\mathcal{F}^{\vdash\times\mu}(\mathfrak{D}_{\succ})_{\underline{v}}) \overset{\text{tauto}^{-1}}{\stackrel{\sim}{\to}} \mathcal{I}^{\mathbb{Q}}(\mathbb{S}^{\pm}_{j+1},{}^{j;n,\circ}\mathcal{F}(\mathfrak{D}^{\vdash}_{\succ})_{\underline{v}})$$

of Proposition 13.4 (2), and

(NFs) (Global Portion — Number Fields) For $j\in\mathbb{F}_l^*$*, the* **number field**

$$\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j = \overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j \subset \mathcal{I}^{\mathbb{Q}}(\mathbb{S}^{\pm}_{j+1};{}^{n,\circ}\mathcal{D}^{\vdash}_{\mathbb{V}_{\mathbb{Q}}}) := \prod_{v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}}\mathcal{I}^{\mathbb{Q}}(\mathbb{S}^{\pm}_{j+1};{}^{n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$$

with natural isomorphisms

$$\mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j \stackrel{\sim}{\to} \mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j,\ \ \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j \stackrel{\sim}{\to} \mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j$$

(cf. Proposition 13.11 (1)) between the associated global non-realified/realified Frobenioids, whose associated global degrees can be computed by means of the log-volumes of (a).

We write

$${}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}}$$

for the collection of data (Shells), (ThVals), (NFs) regarded up to indeterminacies of the following two types:

(Indet $\curvearrowright$) *the indeterminacies induced by the automorphisms of the procession of* $\mathcal{D}^{\vdash}$*-prime-strip* $\mathrm{Proc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$*, and*

(Indet $\to$) *for each* $v_{\mathbb{Q}}\in\mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ *(resp.* $v_{\mathbb{Q}}\in\mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$*), the indeterminacies induced by the action of independent copies of* Isomet *(resp. copies of* $\{\pm1\}\times\{\pm1\}$*-orbit arising from the independent* $\{\pm1\}$*-actions on each of the direct factors “*$k^{\sim}(G)=C^{\sim}\times C^{\sim}$*” of Proposition 12.2 (4)) on each of the direct summands of the* $j+1$ *factors appearing in the tensor product used to define* $\mathcal{I}^{\mathbb{Q}}(\mathbb{S}^{\pm}_{j+1};{}^{n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$

Then we have a functorial algorithm, with respect to $\mathrm{Proc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$*, to construct* ${}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}}$ *(from the given initial* Θ*-data). For* $n,n'\in\mathbb{Z}$*, the permutation symmetries of the étale picture of Corollary 12.8 (2) induce compatible poly-isomorphisms*

$$\mathrm{Proc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T) \overset{\text{poly}}{\stackrel{\sim}{\to}} \mathrm{Proc}({}^{n',\circ}\mathfrak{D}^{\vdash}_T),\quad {}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}} \overset{\text{poly}}{\stackrel{\sim}{\to}} {}^{n',\circ}\mathfrak{R}^{\mathrm{LGP}}$$

which are, moreover, compatible with the poly-isomorphisms ${}^{n,\circ}\mathfrak{D}^{\vdash}_0 \overset{\text{poly}}{\xrightarrow{\sim}} {}^{n',\circ}\mathfrak{D}^{\vdash}_0$ *induced by the bi-coricity of the poly-isomorphisms of Theorem 12.5 (3). We shall refer to the switching poly-isomorphism* ${}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{n',\circ}\mathfrak{R}^{\mathrm{LGP}}$ *as an* **étale-transport poly-isomorphism** *(cf. also Remark 11.1.1), and we also shall refer to (Indet* $\curvearrowright$*) as the* **étale-transport indeterminacies***.*

(2) **($\mathfrak{log}$-Kummer Correspondence)** *For* $n, m \in \mathbb{Z}$*, the* **Kummer isomorphisms**

$$\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})_t \overset{\text{Kum}}{\xrightarrow{\sim}} \Psi_{\mathrm{cns}}({}^{n,\circ}\mathfrak{D}_{\succ})_t, \quad ({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_j \overset{\text{Kum}}{\xrightarrow{\sim}} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{D}^{\circledcirc})_j,$$

$$\{\pi_1^{\mathrm{rat}}({}^{n,m}\mathcal{D}^{\circledast}) \curvearrowright {}^{n,m}\mathbb{M}^{\circledast}_{\infty\kappa}\}_j \overset{\text{Kum}}{\xrightarrow{\sim}} \{\pi_1^{\mathrm{rat}}({}^{n,\circ}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{n,\circ}\mathcal{D}^{\circledcirc})\}_j$$

(where $t \in \mathrm{LabCusp}^{\pm}({}^{n,\circ}\mathfrak{D}_{\succ})$*) of labelled data of Corollary 11.21 (3), Corollary 11.23 (1), (2) (cf. Proposition 13.7 (1), Proposition 13.11 (1)) induce isomorphisms between the vertically coric étale-like data* (Shells), (ThVals)*, and* (NFs) *of (1), and the corresponding Frobenius-like data arising from each* $\boxtimes\boxplus$*-Hodge theatre* ${}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}$*:*

(a) for $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$*,* $j \in |\mathbb{F}_l|$*, isomorphisms*

$$\mathcal{I}^{(\mathbb{Q})}({}^{\mathbb{S}^{\pm}_{j+1};n,m}\mathcal{F}_{v_{\mathbb{Q}}}) \overset{\text{tauto}}{\xrightarrow{\sim}} \mathcal{I}^{(\mathbb{Q})}({}^{\mathbb{S}^{\pm}_{j+1};n,m}\mathcal{F}^{\vdash\times\mu}_{v_{\mathbb{Q}}}) \overset{\substack{\text{“Kum”}\\ \text{poly}}}{\xrightarrow{\sim}} \mathcal{I}^{(\mathbb{Q})}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}_{v_{\mathbb{Q}}}),$$

$$\mathcal{I}^{(\mathbb{Q})}({}^{\mathbb{S}^{\pm}_{j+1},j;n,m}\mathcal{F}_{\underline{v}}) \overset{\text{tauto}}{\xrightarrow{\sim}} \mathcal{I}^{(\mathbb{Q})}({}^{\mathbb{S}^{\pm}_{j+1},j;n,m}\mathcal{F}^{\vdash\times\mu}_{\underline{v}}) \overset{\substack{\text{“Kum”}\\ \text{poly}}}{\xrightarrow{\sim}} \mathcal{I}^{(\mathbb{Q})}({}^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}_{\underline{v}})$$

of local mono-analytic tensor packets and their $\mathbb{Q}$*-spans (cf. Proposition 13.4 (2)), all of which are* **compatible with the respective log-volumes** *by Proposition 13.10 (2) (Here,* $\mathcal{I}^{(\mathbb{Q})}(-)$ *is a shorthand for “*$\mathcal{I}(-)$ *(resp.* $\mathcal{I}^{\mathbb{Q}}(-)$*)”),*

(b) for $\underline{\mathbb{V}}^{\mathrm{bad}} \ni \underline{v}$*, isomorphisms*

$$\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{(n,m-1\to)n,m}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}} \overset{\text{Kum}}{\xrightarrow{\sim}} \Psi^{\perp}_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\underline{v}}$$

of splitting monoids (cf. Proposition 13.7 (1)),

(c) for $j \in \mathbb{F}_l^{\circledast}$*, isomorphisms*

$$({}^{(n,m-1\to)n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}})_j \overset{\text{Kum}}{\xrightarrow{\sim}} \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j,$$

$$({}^{(n,m-1\to)n,m}\mathcal{F}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}})_j \overset{\text{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j,$$

$$({}^{(n,m-1\to)n,m}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}/\mathfrak{mod}})_j \overset{\text{Kum}}{\xrightarrow{\sim}} \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}/\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j,$$

of number fields and global non-realified/realified Frobenioids (cf. Proposition 13.11 (1)), which are compatible with the respective natural isomorphisms between "$(-)_{\mathrm{MOD}}$" and "$(-)_{\mathfrak{mod}}$" (Here, $(-)_{\mathrm{MOD}/\mathfrak{mod}}$ is a shorthand for "$(-)_{\mathrm{MOD}}$ (resp. $(-)_{\mathfrak{mod}}$)"), here, the last isomorphisms induce isomorphisms

$$ {}^{(n,m-1\to)n,m}\mathcal{C}^{\Vdash}_{\mathrm{LGP}/\mathfrak{lgp}} \overset{\mathrm{Kum}}{\overset{\sim}{\to}} \mathcal{C}^{\Vdash}_{\mathrm{LGP}/\mathfrak{lgp}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}) $$

(Here, $(-)_{\mathrm{LGP}/\mathfrak{lgp}}$ is a shorthand for "$(-)_{\mathrm{LGP}}$ (resp. $(-)_{\mathfrak{lgp}}$)") of the global realified Frobenioid portions of the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{(n,m-1\to)n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\mathrm{LGP}}$, ${}^{(n,m-1\to)n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$, and $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\mathfrak{lgp}}$ (cf. Proposition 13.11 (1)).

Moreover, the various isomorphisms $\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{(n,m-1\to)n,m}\mathcal{HT}^{\boxtimes\boxplus})_{\underline{v}} \overset{\mathrm{Kum}}{\overset{\sim}{\to}} \Psi^{\perp}_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\underline{v}}$'s, and $({}^{(n,m-1\to)n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}})_j \overset{\mathrm{Kum}}{\overset{\sim}{\to}} \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j$'s in (b), (c) are **mutually compatible** *with each other, as m runs over $\mathbb{Z}$, with respect to the $\mathfrak{log}$-links of the n-th column of the* LGP-*Gaussian log-theta-lattice, in the sense that the only portions of the domains of these isomorphisms which are possibly related to each other via the $\mathfrak{log}$-links consist of μ in the domains of the $\mathfrak{log}$-links at (n,m), and these indeterminacies at (n,m) (i.e., multiplication by μ) are sent to addition by zero, i.e., no indeterminacy! at $(n,m+1)$ (cf. Proposition 13.7 (2c), Proposition 13.11 (2)). This mutual compatibility of $({}^{(n,m-1\to)n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}})_j \overset{\mathrm{Kum}}{\overset{\sim}{\to}} \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}/\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j$'s implies mutual compatibilities of $({}^{(n,m-1\to)n,m}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_j \overset{\mathrm{Kum}}{\overset{\sim}{\to}} \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j$'s, and $({}^{(n,m-1\to)n,m}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_j \overset{\mathrm{Kum}}{\overset{\sim}{\to}} \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_j$'s (Note that the mutual compatibility does not hold for $(-)_{\mathfrak{mod}}$-labelled objects, since these are defined in terms of both multiplicative and additive operaions ($\boxtimes$ and $\boxplus$), where we only expect only a upper semi-compatibility (cf. Definition 9.7, Proposition 13.7 (2), and Proposition 13.11 (3)). On the other hand, the isomorphisms of (a) are subject to the following indeterminacy:*

(Indet $\uparrow$) the isomorphisms of (a) are **upper semi-compatible**, *with respect to the $\mathfrak{log}$-links of the n-th column of the* LGP-*Gaussian log-theta-lattice, as m runs over $\mathbb{Z}$, in a sense of Proposition 13.7 (2a), (2b).*

(We shall refer to (Indet $\to$) and (Indet $\uparrow$) as the **Kummer detachment indeterminacies**.*) Finally, the isomorphisms of (a) are* **compatible with the respective log-volumes**, *with respect to the $\mathfrak{log}$-links of the n-th column of the* LGP-*Gaussian log-theta-lattice, as m runs over $\mathbb{Z}$ (This means that we* **do not** *need*

to be worried about **how many times $\mathfrak{log}$-links are applied** *in the $\mathfrak{log}$-Kummer correspondence, when we take values of the log-volumes).*

(3) **($\Theta^{\times\mu}_{\mathrm{LGP}}$-Link Compatibility)** *The various Kummer isomorphisms of (2) are compatible with the $\Theta^{\times\mu}_{\mathrm{LGP}}$-links in the following sense:*

(a) *(Kummer on Δ) By applying the $\mathbb{F}_l^{\rtimes\pm}$-symmetry of the $\boxtimes\boxplus$-Hodge theatre ${}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}$, the Kummer isomorphism $\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})_t \overset{\mathrm{Kum}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^{n,\circ}\mathfrak{D}_{\succ})_t$ induces a* **Kummer isomorphism** ${}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \overset{\text{induced by Kum}}{\overset{\sim}{\to}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,\circ}\mathfrak{D}^{\vdash}_{\Delta})$ *(cf. Theorem 12.5 (3)). Then we have a commutative diagram*

$$\begin{array}{ccc} {}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} & \xrightarrow[\sim]{\text{full poly}} & {}^{n+1,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \\ \text{induced by Kum } \cong \downarrow & & \downarrow \cong \text{ induced by Kum} \\ \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,\circ}\mathfrak{D}^{\vdash}_{\Delta}) & \xrightarrow[\sim]{\text{full poly}} & \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n+1,\circ}\mathfrak{D}^{\vdash}_{\Delta}), \end{array}$$

where the upper horizontal arrow is induced (cf. Theorem 12.5 (2)) by the $\Theta^{\times\mu}_{\mathrm{LGP}}$-link between (n,m) and $(n+1,m)$ by Theorem 12.5 (3).

(b) *($\Delta \to \mathrm{env}$) The $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$, $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>})$ appearing implicitly in the construction of the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{(n,m-1\to)n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\mathrm{LGP}}$, ${}^{(n,m-1\to)n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\mathfrak{lgp}}$, admit natural isomorphisms ${}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \overset{\sim}{\to} {}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}}$, $\mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,\circ}\mathfrak{D}^{\vdash}_{\Delta}) \overset{\sim}{\to} \mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}^{\vdash}_{>})$ of associated $\mathcal{F}^{\vdash\times\mu}$-prime-strips (cf. Proposition 12.6 (3)). Then we have a commutative diagram*

$$\begin{array}{ccc} {}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} & \xrightarrow[\sim]{\text{full poly}} & {}^{n+1,m}\mathfrak{F}^{\vdash\times\mu}_{\Delta} \\ \text{induced by Kum \& ``}\Delta\mapsto\text{env''} \cong \downarrow & & \downarrow \cong \text{ induced by Kum \& ``}\Delta\mapsto\text{env''} \\ \mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}^{\vdash}_{>}) & \xrightarrow[\sim]{\text{full poly}} & \mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}}({}^{n+1,\circ}\mathfrak{D}^{\vdash}_{>}), \end{array}$$

where the upper horizontal arrow is induced (cf. Theorem 12.5 (2)) by the $\Theta^{\times\mu}_{\mathrm{LGP}}$-link between (n,m) and $(n+1,m)$ by Corollary 12.8 (3).

(c) *($\mathrm{env} \to \mathrm{gau}$) Recall that the (vertically coric étale-like) data "${}^{n,\circ}\mathfrak{R}$" i.e.,*

$$\Big({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus},\ \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>}),\ \Big[{}_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>})_{\underline{v}} \supset {}_{\infty}\Psi_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>})^{\mu}_{\underline{v}},\ \mu_{\widehat{\mathbb{Z}}}(\mathbb{M}^{\Theta}_{*}({}^{n,\circ}\mathfrak{D}_{>,\underline{v}}))\otimes\mathbb{Q}/\mathbb{Z},\ \mathbb{M}^{\Theta}_{*}({}^{n,\circ}\mathfrak{D}_{>,\underline{v}}),$$

$${}_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>})_{\underline{v}} \twoheadrightarrow {}_{\infty}\Psi_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>})^{\mu}_{\underline{v}}\Big]_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}},\ \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,\circ}\mathfrak{D}^{\vdash}_{\Delta}),\ \mathfrak{F}^{\vdash\times\mu}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>}) \overset{\text{full poly}}{\overset{\sim}{\to}} \mathfrak{F}^{\vdash\times\mu}_{\Delta}({}^{n,\circ}\mathfrak{D}^{\vdash}_{\Delta})\Big)$$

of Corollary 12.8 (2) implicitly appears in the construction of the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{(n,m-1\to)n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\mathrm{LGP}}$, ${}^{(n,m-1\to)n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus})_{\mathfrak{lgp}}$.

This (vertically coric étale-like) data arising from ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$ *is related to corresponding (Frobenius-like) data arising from the projective system of the mono-theta environments associated to the tempered Frobenioids of the* $\boxtimes\boxplus$*-Hodge theatre* ${}^{n,m}\mathcal{HT}^{\boxtimes\boxplus}$ *at* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *via the Kummer isomorphisms and poly-isomorphisms of projective systems of mono-theta environments of Proposition 12.6 (2), (3) and Theorem 12.5 (3). With respect to these Kummer isomorphisms and poly-isomorphisms of projective systems of mono-theta environments, the poly-isomorphism*

$$ {}^{n,\circ}\mathfrak{R} \overset{\text{poly}}{\xrightarrow{\sim}} {}^{n+1,\circ}\mathfrak{R} $$

induced by the permutation symmetry of the étale picture ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{n+1,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$ *is compatible with the full poly-isomorphism*

$$ {}^{n,m}\mathfrak{F}_{\Delta}^{\vdash\times\mu} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{n+1,m}\mathfrak{F}_{\Delta}^{\vdash\times\mu} $$

of $\mathcal{F}^{\vdash\times\mu}$*-prime-strips induced by* $\Theta^{\times\mu}_{\mathrm{LGP}}$*-link between* (n,m) *and* $(n+1,m)$ *and so on. Finally, the above two displayed poly-isomorphisms and the various related Kummer isomorphisms are compatible with the various* **evaluation** *map implicit in the portion of the* $\mathfrak{log}$*-Kummer correspondence of (2b), up to indeterminacies* (Indet $\curvearrowright$), (Indet $\to$), (Indet $\uparrow$) *of (1), (2).*

(d) (κ-coric $\to$ NF) *With respect to the Kummer isomorphisms of (2) and the gluing of Corollary 11.21, the poly-isomorphism*

$$ \Big[\big\{\pi_1^{\mathrm{rat}}({}^{n,\circ}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{n,\circ}\mathcal{D}^{\circledcirc})\big\}_j \overset{\text{gl. to loc.}}{\to} \mathbb{M}_{\infty\kappa v}({}^{n,\circ}\mathcal{D}_{\underline{v}_j}) \subset \mathbb{M}_{\infty\kappa\times v}({}^{n,\circ}\mathcal{D}_{\underline{v}_j})\Big]_{\underline{v}\in\underline{\mathbb{V}}} $$

$$ \overset{\text{poly}}{\xrightarrow{\sim}} \Big[\big\{\pi_1^{\mathrm{rat}}({}^{n+1,\circ}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{n+1,\circ}\mathcal{D}^{\circledcirc})\big\}_j \overset{\text{gl. to loc.}}{\to} \mathbb{M}_{\infty\kappa v}({}^{n+1,\circ}\mathcal{D}_{\underline{v}_j}) \subset \mathbb{M}_{\infty\kappa\times v}({}^{n+1,\circ}\mathcal{D}_{\underline{v}_j})\Big]_{\underline{v}\in\underline{\mathbb{V}}} $$

(cf. Corollary 11.22 (3)) induced by the permutation symmetry of the étale picture ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{n+1,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$ *is compatible with the full poly-isomorphism*

$$ {}^{n,m}\mathfrak{F}_{\Delta}^{\vdash\times\mu} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{n+1,m}\mathfrak{F}_{\Delta}^{\vdash\times\mu} $$

of $\mathcal{F}^{\vdash\times\mu}$*-prime-strips induced by* $\Theta^{\times\mu}_{\mathrm{LGP}}$*-link between* (n,m) *and* $(n+1,m)$*. Finally, the above two displayed poly-isomorphisms and the various related Kummer isomorphisms are compatible with the various* **evaluation** *map implicit in the portion of the* $\mathfrak{log}$*-Kummer correspondence of (2b), up to indeterminacies* (Indet $\curvearrowright$), (Indet $\to$), (Indet $\uparrow$) *of (1), (2).*

Proof. Theorem follows from the definitions. □

A rough picture of the final multiradial representation is as follows:

$$
\begin{tikzcd}
 & (F^{\times}_{\mathrm{mod}})_1 \arrow[d, phantom, "\curvearrowright"] & \cdots & & (F^{\times}_{\mathrm{mod}})_{l^{\divideontimes}} \arrow[d, phantom, "\curvearrowright"] \\
\{\mathcal{I}^{\mathbb{Q}}_0\} \subset \{\mathcal{I}^{\mathbb{Q}}_0, & \mathcal{I}^{\mathbb{Q}}_1\} & \subset\cdots & \subset\{\mathcal{I}^{\mathbb{Q}}_0,\dots, & \mathcal{I}^{\mathbb{Q}}_{l^{\divideontimes}}\} \\
 & \Psi^{\perp}_{\mathrm{LGP}}, \arrow[u, hook, "\underline{\underline{q}}^{1^2}"] \arrow[ur, hook, "\underline{\underline{q}}^{2^2}", "\text{acts}"'] \arrow[urrr, hook, "\underline{\underline{q}}^{(l^{\divideontimes})^2}"'] & & &
\end{tikzcd}
$$

where the multiplicative group $(F^{\times}_{\mathrm{mod}})_j$ of non-zero elements of a j-labelled number field acts on $\mathcal{I}^{\mathbb{Q}}_j$, and $\Psi^{\perp}_{\mathrm{LGP}}$ acts on $\mathcal{I}^{\mathbb{Q}}_j$ in the $(j+1)$-capsule by multiplication by $\underline{\underline{q}}^{j^2}$. Note that $\Psi^{\perp}_{\mathrm{LGP}}$ does not act on other components $\mathcal{I}^{\mathbb{Q}}_0, \dots, \mathcal{I}^{\mathbb{Q}}_{j-1}$ of the $(j+1)$-capsule. Note also that the 0-labelled objects (together with the diagonal labelled objects) are used to form horizontally coric objects (Recall that "$\Delta = \{0, \langle \mathbb{F}^{\divideontimes}_l \rangle\}$"), and $(F^{\times}_{\mathrm{mod}})_j$'s or $\Psi^{\perp}_{\mathrm{LGP}}$ do not act on 0-labelled ($\mathbb{Q}$-span of) log-shell $\mathcal{I}^{\mathbb{Q}}_0$.

The following table is a summary of Theorem 13.12 and related topics:

	(temp. conj. vs. prof. conj. $\to \mathbb{F}_l^{\rtimes\pm}$-conj. synchro.$\to$ diag.$\to$hor. core$\to$ $\Theta^{\times\mu}_{\mathrm{LGP}}$-link$\downarrow$)		
	(1) (Objects)	(2) ($\mathfrak{log}$-Kummer)	(3) (Comp'ty with $\Theta^{\times\mu}_{\mathrm{LGP}}$-link)
$\mathbb{F}_l^{\rtimes\pm}$-sym. $\boxplus$	$\mathcal{I}$ ($\leftrightsquigarrow$ units)	inv. after admitting (Indet $\uparrow$)	inv. after admitting (Indet $\to$) ($\rightsquigarrow \widehat{\mathbb{Z}}^\times$-indet.)
$\mathbb{F}_l^{\rtimes\pm}$-sym. $\boxplus$	$\Psi^{\perp}_{\mathrm{LGP}}$ val. gp. ($\leftarrow$compat. of $\mathfrak{log}$-link w/ $\mathbb{F}_l^{\rtimes\pm}$-sym.)	**no interf.** by const. mult. rig. (ell. cusp'n$\leftarrow$pro-p anab. +hidden. endom.)	**protected from $\widehat{\mathbb{Z}}^\times$-indet.** by mono-theta cycl. rig. ($\leftarrow$quad. str. of Heis. gp.)
$\mathbb{F}_l^{\circledast}$-sym. $\boxtimes$	$\overline{\mathbb{M}}_{\mathrm{mod}}$ NF Belyi cusp'n($\leftarrow$pro-p anab. +hidden endom.)	**no interf.** by $F^\times_{\mathrm{mod}} \cap \prod_{v\le\infty} O_v = \mu$	**protected from $\widehat{\mathbb{Z}}^\times$-indet.** by $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\}$
others: (compat. of log.-vol. w/ $\mathfrak{log}$-links), (Arch. theory:Aut-hol. space (ell. cusp'n is used)) (disc. rig. of mono-theta), (étale pic.: permutable after admitting (Indet $\curvearrowright$) (autom. of proc. incl.))			

Corollary 13.13. (Log-volume Estimates for Θ-Pilot Objects, [IUTchIII, Corollary 3.12]) *We write*

$$-|\log(\underline{\underline{\Theta}})| \ \in\ \mathbb{R}\cup\{+\infty\}$$

for the procession-normalised mono-analytic log-volume (where the average is taken over $j\in\mathbb{F}_l^{\circledast}$) of **the holomorphic hull** *(cf. the definiton after Lemma 1.6)* **of the union of the possible image of a Θ-pilog object**, *with respect to the relevant Kummer isomorphisms in the multiradial representation of Theorem 13.13 (1), which we regard as subject to the indeterminacies (Indet $\uparrow$), (Indet $\to$), and (Indet $\curvearrowright$) of Theorem 13.13 (1), (2). We write*

$$-|\log(\underline{\underline{q}})| \ \in\ \mathbb{R}$$

for the procession-normalised mono-analytic log-volume of the image of a q-pilot object, with respect to the relevant Kummer isomorphisms in the multiradial representation of

Theorem 13.13 (1), which we **do not** *regard as subject to the indeterminacies (Indet $\uparrow$), (Indet $\rightarrow$), and (Indet $\curvearrowright$) of Theorem 13.13 (1), (2) (Note that we have* $|\log(\underline{\underline{q}})| > 0$*). Then we obtain*

$$-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})|$$

(i.e., "$0 \lesssim -(\text{large number}) + (\text{mild indeterminacies})$*". cf. also § A.4). Note also that the explicit computations of the indeterminacies in Proposition 1.12, in fact, shows that* $-|\log(\underline{\underline{\Theta}})| < \infty$*.*

Proof. The $\Theta^{\times\mu}_{\mathrm{LGP}}$-link ${}^{0,0}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\Theta^{\times\mu}_{\mathrm{LGP}}} {}^{1,0}\mathcal{HT}^{\boxtimes\boxplus}$ induces the full poly-isomorphism ${}^{0,0}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{LGP}} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{1,0}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta}$ of $\mathcal{F}^{\Vdash\blacktriangleright\times\mu}$-prime-strips, which sends Θ-pilot objects to a q-pilot objects. By the Kummer isomorphisms, the $(0,0)$-labelled Frobenius-like objects corresponding to the objects in the multiradial representaion of Theorem 13.12 (1) are isomorphically related to the $(0,\circ)$-labelled vertically coric étale-like objects (i.e., mono-analytic containers with actions by theta values, and number fields) in the multiradial representaion of Theorem 13.12 (1). After admitting the indeterminacies (Indet $\curvearrowright$), (Indet $\rightarrow$), and (Indet $\uparrow$), these $(0,\circ)$-labelled vertically coric étale-like objects are isomorphic (cf. Remark 11.1.1) to the $(1,\circ)$-labelled vertically coric étale-like objects. Then Corollary follows by comparing the log-volumes (Note that log-volumes are invariant under (Indet $\curvearrowright$), (Indet $\rightarrow$), and also compatible with $\mathfrak{log}$-Kummer correspondence of Theorem 13.12 (2)) of $(1,0)$-labelled q-pilot objects (by the compatibility with $\Theta^{\times\mu}_{\mathrm{LGP}}$-link of Theorem 13.12 (3)) and $(1,\circ)$-labelled Θ-pilot objects, since, in the mono-analytic containers (i.e., $\mathbb{Q}$-spans of log-shells), the holomorphic hull of the union of possible images of Θ-pilot objects subject to indeterminacies (Indet $\curvearrowright$), (Indet $\rightarrow$), (Indet $\uparrow$) contains a region which is isomorphic (*not* equal) to the region determined by the q-pilot objects (This means that "very small region with indeterminacies" contains "almost unit region"). $\square$

Then Theorem 0.1 (hence Corollary 0.2 as well) is proved, by combining Proposition 1.2, Proposition 1.15, and Corollary 13.13.

Remark 13.13.1. By admitting (Indet $\curvearrowright$), (Indet $\rightarrow$), and (Indet $\uparrow$), we obtain objects which are invariant under the $\Theta^{\times\mu}_{\mathrm{LGP}}$-link. On the other hand, the $\Theta^{\times\mu}_{\mathrm{LGP}}$-link can be considered as "**absolute Frobenius" over** $\mathbb{Z}$, since it relates (non-ring-theoretically) $\underline{\underline{q}}_{\underline{v}}$ to $\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{1\leq j\leq l^*}$. Therefore, we can consider

(Indet $\curvearrowright$) the permutation indeterminacy in the étale transport:

$$\begin{tikzcd} \bullet \arrow[r] & \circ & \bullet \arrow[l] \arrow[ll, bend right] \end{tikzcd} \qquad \text{"}{}^{\dagger}G_{\underline{v}} \cong {}^{\ddagger}G_{\underline{v}}\text{" (and autom's of processions)}$$

(Indet $\rightarrow$) the horizontal indeterminacy in the Kummer detachment:

$$\bullet \xrightarrow{\Theta} \bullet \qquad {}^{\dagger}\mathcal{O}^{\times\mu} \cong {}^{\ddagger}\mathcal{O}^{\times\mu} \text{ with integral structures,}$$

and

(Indet $\uparrow$) the vertical indeterminacy in the Kummer detachment:

$$\begin{tikzcd} \bullet & \log(\mathcal{O}^{\times}) \arrow[r, hook] & \frac{1}{2p}\log(\mathcal{O}^{\times}) \\ \bullet \arrow[u, "\mathfrak{log}"] & \mathcal{O}^{\times} \arrow[u, "\log"] \arrow[ur, hook] & \end{tikzcd}$$

as "**descent data from $\mathbb{Z}$ to $\mathbb{F}_1$**".

Remark 13.13.2. The following diagram (cf. [IUTchIII, Fig. 3.8]) expresses the **tautological two ways of computations of log-volumes of q-pilot objects** in the proof of Corollary 13.13:

$$\begin{tikzcd}[column sep=small] \left(\begin{array}{c} \boxplus\text{-line bdls.}_{1\le j\le l^{*}} \text{ assoc. to} \\ \{{}^{0,\circ}\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{\underline{v}\in\underline{\mathbb{V}}} \text{ up to Indet.'s} \end{array}\right) \arrow[rr, "\overset{\text{étale transport}}{\cong}"', "\text{suited to } \mathcal{F}_{\mathfrak{mod}}"', dash] & & \left(\begin{array}{c} \boxplus\text{-line bdls.}_{1\le j\le l^{*}} \text{ assoc. to} \\ \{{}^{1,\circ}\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{\underline{v}\in\underline{\mathbb{V}}} \text{ up to Indet.'s} \end{array}\right) \arrow[d, leftrightarrow, "\text{compare log-vol.'s}"] \\ \left(\begin{array}{c} \boxtimes\text{-line bdl. assoc. to} \\ \{{}^{0,0}\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{\underline{v}\in\underline{\mathbb{V}}} \end{array}\right) \arrow[u, "\substack{\text{Kummer detach.} \\ \text{via } \mathfrak{log}\text{-Kummer corr.}}"] \arrow[r, dash, "\overset{\Theta^{\times\mu}_{\mathrm{LGP}}\text{-link}}{\cong}", "\text{suited to } \mathcal{F}_{\mathrm{MOD}}"'] & \left(\begin{array}{c} \boxtimes\text{-line bdl. assoc. to} \\ \{{}^{1,0}\underline{\underline{q}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \end{array}\right) \arrow[ul, "\text{compatibility with } \Theta^{\times\mu}_{\mathrm{LGP}}\text{-link}"] \arrow[r, phantom, "\cong"] & \left(\begin{array}{c} \boxplus\text{-line bdl. assoc. to} \\ \{{}^{1,0}\underline{\underline{q}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \end{array}\right). \end{tikzcd}$$

These tautological two ways of computations of log-volumes of q-pilot objects can be considered as computations of self-intersection numbers "$\Delta.\Delta$" of the diagonal "$\Delta \subset \mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$" from point view of Remark 13.13.1. This observation is compatible with the analogy with p-adic Teichmüller theory (cf. last table in Section 3.5), where the computation of the global degree of line bundles arising from the derivative of the canonical Frobenius lifting ($\leftrightarrow$ Θ-link) gives us an inequality $(1-p)(2g-2) \le 0$ (Recall that self-intersection numbers give us Euler numbers). This inequality $(1-p)(2g-2) \le 0$ essentially means the hyperbolicity of hyperbolic curves. Analogously, the inequality

$$|\log(\underline{\underline{\Theta}})| \le |\log(\underline{\underline{q}})| \fallingdotseq 0$$

means the **hyperbolicity of number fields**.

cf. also the following table (cf. [IUTchIII, Fig. 3.2]):

$\boxtimes$-line bundles, MOD/LGP-labelled objects	$\boxplus$-line bundles, $\mathfrak{mod}/\mathfrak{lgp}$-labelled objects
defined only in terms of $\boxtimes$	defined in terms of both $\boxtimes$ and $\boxplus$
value group/non-coric portion "$(-)^{\Vdash\blacktriangleright}$" of $\Theta^{\times\mu}_{\mathrm{LGP}}$-link	unit group/coric portion "$(-)^{\vdash\times\mu}$" of $\Theta^{\times\mu}_{\mathrm{LGP}}/\Theta^{\times\mu}_{\mathfrak{lgp}}$-link
precise $\mathfrak{log}$-Kummer corr.	only upper semi-compatible $\mathfrak{log}$-Kummer corr.
ill-suited to log-vol. computation	suited to log-vol. computation subject to mild indeterminacies

Remark 13.13.3. In this remark, we consider the following natural questions: How about the following variants of Θ-links?

(1)
$$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{1\leq j\leq l^*} \mapsto \underline{\underline{q}}_{\underline{v}}^{\lambda} \quad (\lambda\in\mathbb{R}_{>0}),$$

(2)
$$\{(\underline{\underline{q}}_{\underline{v}}^{j^2})^N\}_{1\leq j\leq l^*} \mapsto \underline{\underline{q}}_{\underline{v}} \quad (N>1),\ \text{and}$$

(3)
$$\underline{\underline{q}}_{\underline{v}} \mapsto \underline{\underline{q}}_{\underline{v}}^{\lambda} \quad (\lambda\in\mathbb{R}_{>0}).$$

From conclusions, (1) works, and either of (2) or (3) **does not** work.

(1) ([IUTchIII, Remark 3.12.1 (ii)]) We explain the variant (1). Recall that we have $l\approx \mathrm{ht} >> |\deg(\underline{\underline{q}}_{\underline{v}})| \risingdotseq 0$. Then the resulting inequalty from "the generalised $\Theta^{\times\mu}_{\mathrm{LGP}}$-link" is
$$\lambda\cdot 0 \ \lesssim\ -(\mathrm{ht})+(\mathrm{indet.})$$
for $\lambda << l$, which gives us the *almost same inequality* of Corollary 13.13, and *weaker inequality* for $\lambda > l$ than the inequality of Corollary 13.13 (since $\deg(\underline{\underline{q}}_{\underline{v}})<0$).

(2) ([EtTh, Introduction, Remark 2.19.2, Remark 5.12.5], [IUTchII, Remark 1.12.4, Remark 3.6.4], [IUTchIII, Remark 2.1.1]) We explain the variant (2). There are several reasons that the variant (2) does not work (cf. also the **principle of Galois evaluation** of Remark 11.10.1):

(a) If we replace Θ by Θ^N $(N > 1)$, then *the crucial cyclotomic rigidity of mono-theta environments (Theorem 7.23 (1)) does not hold*, since the construction of the cyclotomic rigidity of mono-theta environments uses the quadraticity of the commutator $[\ ,\ ]$ structure of the theta group (i.e., Heisenberg group) (cf. also Remark 7.23.2). If we do not have the cyclotomic rigidity of mono-theta environments, then we have no Kummer compatibility of theta monoids (cf. Theorem 12.7).

(b) If we replace Θ by Θ^N $(N > 1)$, then *the crucial constant multiple rigidity of mono-theta environments (Theorem 7.23 (3)) does not hold either*, since, if we consider N-th power version of mono-theta environments by relating the 1-st power version of mono-theta environments (for the purpose of maintaining the cyclotomic rigidity of mono-theta environments) via N-th power map, then such N-th power map gives rise to mutually non-isomorphic line bundles, hence a constant multiple indeterminacy under inner automorphisms arising from automorphisms of corresponding tempered Frobenioid (cf. [IUTchIII, Remark 2.1.1 (ii)], [EtTh, Corollary 5.12 (iii)]).

(c) If we replace Θ by Θ^N $(N > 1)$, then, the order of zero of Θ^N at cusps is equal to $N > 1$, hence in the $\mathfrak{log}$-Kummer correspondence, one loop among the various Kummer isomorphisms between Frobenius-like cyclotomes in a column of log-theta-lattice and the vertically coric étale-like cyclotome gives us the N-power map before the loop, therefore, *the $\mathfrak{log}$-Kummer correspondence totally collapses.* cf. also Remark 12.8.1 ("vicious circles").

If it worked, then we would have

$$0 \lesssim -N(\mathrm{ht}) + (\mathrm{indet.}),$$

which gives us an inequality

$$\mathrm{ht} \lesssim \frac{1}{N}(1+\epsilon)(\text{log-diff} + \text{log-cond})$$

for $N > 1$. This contradicts Masser's lower bound in analytic number theory ([Mass2]).

(3) ([IUTchIII, Remark 2.2.2]) We explain the variant (3). In the theta function case, we have Kummer compatible splittings arisen from zero-labelled evaluation points (cf. Theorem 12.7):

$$\begin{array}{ccc}
\mathrm{id} \curvearrowright \left(O^{\times} \cdot {}_{\infty}\underline{\underline{\theta}} \curvearrowleft \Pi \overset{\text{0-labelled ev. pt.}}{\hookleftarrow} \Pi/\Delta \right) & \begin{array}{c}\longrightarrow \\ \vdots \\ \longrightarrow\end{array} & \mathrm{Aut}(G),\ \mathrm{Isomet} \curvearrowright (G \curvearrowright O^{\times\mu}) \\
{}_{\infty}\underline{\underline{\theta}} & \mapsto & 1 \in O^{\times\mu}.
\end{array}$$

Here, the crucial Kummer compatibility comes from the fact that the evaluation map relates the Kummer theory of $O^{\times}$-portion of $O^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}$ on the left to the coric $O^{\times\mu}$ on the right, via the evaluation ${}_{\infty}\underline{\underline{\theta}} \mapsto 1 \in O^{\times\mu}$. On the other hand, in the case of the variants (3) under consideration, the corresponding arrow maps $\underline{\underline{q}}^{\lambda} \mapsto 1 \in O^{\times\mu}$, hence *this is incompatible with passage to Kummer classes*, since the Kummer class of $\underline{\underline{q}}^{\lambda}$ in a suitable cohomology group of Π/Δ is never sent to the trivial element of the relavant cohomology group of G, via the full poly-isomorphism $\Pi/\Delta \overset{\text{full poly}}{\xrightarrow{\sim}} G$.

Appendix A. Motivation of the Definition of the Θ-Link.

In this section, we explain a motivation of Θ-link from a historical point of view, i.e., in the order of classical de Rham's comparison theorem, p-adic Hodge comparison theorem, Hodge-Arakelov comparison theorem, and a motivation of Θ-link. This section is an explanatory section, and we do not give proofs, or sometimes rigorous statements. cf. also [Pano, §1].

§ A.1. The Classical de Rham Comparison Theorem.

The classical de Rham's comparison theorem in the special case for $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^{\times}$ says that the pairing

$$H_1(\mathbb{G}_m(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} H^1_{\mathrm{dR}}(\mathbb{G}_m(\mathbb{C})/\mathbb{C}) \to \mathbb{C},$$

which sends $[\gamma] \otimes [\omega]$ to $\int_{\gamma} \omega$, induces a comparison isomorphism $H^1_{\mathrm{dR}}(\mathbb{G}_m(\mathbb{C})/\mathbb{C}) \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Z}} (H_1(\mathbb{G}_m(\mathbb{C}), \mathbb{Z}))^*$ (Here, we write $(-)^*$ for the $\mathbb{Z}$-dual). Note that $H_1(\mathbb{G}_m(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}\,[\gamma_0]$, $H^1_{\mathrm{dR}}(\mathbb{G}_m(\mathbb{C})/\mathbb{C}) = \mathbb{C}\left[\frac{dT}{T}\right]$, and $\int_{\gamma_0} \frac{dT}{T} = 2\pi i$, where we write γ_0 for a counter-clockwise loop around the origin, and T for a standard coordinate of $\mathbb{G}_m$.

§ A.2. p-adic Hodge-theoretic Comparison Theorem.

A p-adic analogue of the above comparison paring (in the special case for $\mathbb{G}_m$ over $\mathbb{Q}_p$) in the p-adic Hodge theory is the pairing

$$T_p\mathbb{G}_m \otimes_{\mathbb{Z}_p} H^1_{\mathrm{dR}}(\mathbb{G}_m/\mathbb{Q}_p) \to B_{\mathrm{crys}},$$

which sends $\underline{\epsilon} \otimes \left[\frac{dT}{T}\right]$ to ("$\int_{\underline{\epsilon}} \frac{dT}{T} =$") $\log[\underline{\epsilon}] = t(= t_{\underline{\epsilon}})$, where we write T_p for the p-adic Tate module, $\underline{\epsilon} = (\epsilon_n)_n$ is a system of p-power roots of unity (i.e., $\epsilon_0 = 1$, $\epsilon_1 \neq 1$, and $\epsilon_{n+1}^p = \epsilon_n$), B_{crys} is Fontaine's p-adic period ring (cf. also [Fo3]), and $t = \log[\underline{\epsilon}]$ is an element in B_{crys} defined by $\underline{\epsilon}$ (cf. also [Fo3]). The above pairing induces a comparison isomorphism $B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^1_{\mathrm{dR}}(\mathbb{G}_m/\mathbb{Q}_p) \xrightarrow{\sim} B_{\mathrm{crys}} \otimes_{\mathbb{Z}_p} (T_p\mathbb{G}_m)^*$ (Here, we write $(-)^*$ for the $\mathbb{Z}_p$-dual). Note that $\underline{\epsilon} = (\epsilon_n)_n$ is consdered as a kind of *analytic path* around the origin.

We consider the pairing in the special case for an elliptic curve E over $\mathbb{Z}_p$. We have the universal extension $0 \to (\mathrm{Lie}E^\vee_{\mathbb{Q}_p})^* \to E^\dagger_{\mathbb{Q}_p} \to E_{\mathbb{Q}_p} \to 0$ (cf. [Mess] for the universal extension) of $E_{\mathbb{Q}_p} := E \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (Here, we write $(-)^*$ for the $\mathbb{Q}_p$-dual, and $E^\vee_{\mathbb{Q}_p}(\cong E_{\mathbb{Q}_p})$ is the dual abelian variety of $E_{\mathbb{Q}_p}$). By taking the tangent space at the origin, we obtain an extension $0 \to (\mathrm{Lie}E^\vee_{\mathbb{Q}_p})^* \to \mathrm{Lie}E^\dagger_{\mathbb{Q}_p} \to \mathrm{Lie}E_{\mathbb{Q}_p} \to 0$ whose $\mathbb{Q}_p$-dual is canonically identified with the Hodge filtration of the de Rham cohomology $0 \to (\mathrm{Lie}E_{\mathbb{Q}_p})^* \to H^1_{\mathrm{dR}}(E_{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{Lie}E^\vee_{\mathbb{Q}_p} \to 0$ under a canonical isomorphism $H^1_{\mathrm{dR}}(E_{\mathbb{Q}_p}/\mathbb{Q}_p) \cong (\mathrm{Lie}E^\dagger_{\mathbb{Q}_p})^*$ (cf. also [MM] for the relation between the universal extension and the first crystalline cohomology; [BO1] and [BO2] for the isomorphism between the crystalline cohomology and the de Rham cohomology). For an element $\omega_{E^\dagger}$ of $(\mathrm{Lie}E^\dagger_{\mathbb{Q}_p})^*$, we have a natural homomorphism $\log_{\omega_{E^\dagger}} : \widehat{E^\dagger_{\mathbb{Q}_p}} \to \widehat{\mathbb{G}}_{a/\mathbb{Q}_p}$ such that the pull-back $(\log_{\omega_{E^\dagger}})^* dT$ is equal to $\omega_{E^\dagger}$, where $\widehat{E^\dagger_{\mathbb{Q}_p}}$ is the formal completion of $E^\dagger_{\mathbb{Q}_p}$ at the origin, and $\widehat{\mathbb{G}}_{a/\mathbb{Q}_p}$ is the formal additive group over $\mathbb{Q}_p$.

Now, the pairing in the p-adic Hodge theory is

$$T_pE \otimes (\mathrm{Lie}E^\dagger_{\mathbb{Q}_p})^* \to B_{\mathrm{crys}},$$

which sends $\underline{P} \otimes \omega_{E^\dagger}$ to ("$\int_{\underline{P}} \omega_{E^\dagger}$ = ") $\log_{\omega_{E^\dagger}}[\underline{P}]$, where $\underline{P} = (P_n)_n$ satisfies that $P_n \in E(\overline{\mathbb{Q}}_p)$, $P_0 = 0$, and $pP_{n+1} = P_n$. The above pairing induces a comparison isomorphism $B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^1_{\mathrm{dR}}(\mathbb{G}_m/\mathbb{Q}_p) \xrightarrow{\sim} B_{\mathrm{crys}} \otimes_{\mathbb{Z}_p} (T_p\mathbb{G}_m)^*$ (Here, we write $(-)^*$ for the $\mathbb{Z}_p$-dual). Note again that $\underline{P} = (P_n)_n$ is consdered as a kind of *analytic path* in E. cf. also [BO1] and [BO2] for the isomorphism between the de Rham cohomology and the crystalline cohomology; [MM] for the relation between the first crystalline cohomology and the universal extension; [Mess] for the relation between the universal extension and the Dieudonné module; [Fo2, Proposition 6.4] and [Fo1, Chapitre V, Proposition 1.5] for the relation between the Dieudonné module and the Tate module (the above isomorphism is a combination of these relations).

§ A.3. Hodge-Arakelov-theoretic Comparison Theorem.

Mochizuki studied a global and "discretised" analogue of the above p-adic Hodge comparison map (cf. [HASurI], [HASurII]). Let E be an elliptic curve over a number field F, $l > 2$ a prime number. Assume that we have a nontrivial 2-torsion point $P \in E(F)[2]$ (we can treat the case where $P \in E(F)$ is order $d > 0$ and d is prime to l; however, we treat the case where $d = 2$ for the simplicity). Write $\mathcal{L} = \mathcal{O}(l\,[P])$. Then roughly speaking, the main theorem of Hodge-Arakelov theory says that the evaluation map on $E^\dagger[l](= E[l])$

$$\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{\deg < l} \xrightarrow{\sim} \mathcal{L}|_{E^\dagger[l]}(= \mathcal{L}|_{E[l]} = \oplus_{E[l]} F)$$

is an isomorphism of F-vector spaces, and preserves specified integral structures (we omit the details) at non-Archimedean and Archimedean places. Here, we write $\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{\deg<l}$ for the part of $\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})$ whose relative degree is less than l (Note that Zariski locally $E^\dagger$ is isomorphic to $E \times \mathbb{A}^1 = \mathbf{Spec}\,\mathcal{O}_E[T]$). Note that $\dim_F \Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{\deg<l} = l^2$, since $\dim_F \Gamma(E, \mathcal{L}) = l$, and that $\dim_F \mathcal{L}|_{E[l]} = l^2$ since $\#E[l] = l^2$. The left-hand side is the de Rham side, and the right-hand side is the étale side. The discretasation means that we consider l-torsion points $E[l]$, not the Tate module, and in philosophy, we consider $E[l]$ as a kind of *approximation of "underling analytic manifold"* of E (like $\underline{\epsilon} = (\epsilon_n)_n$ and $\underline{P} = (P_n)_n$ were considered as a kind of analytic paths in $\mathbb{G}_m$ and E respectively). We also note that in the étale side we consider the space of functions on $E[l]$, not $E[l]$ itself, which is a common method of quantisations (like considering universal enveloping algebra of Lie algebra, not Lie algebra itself, or like considering group algebra, not group itself).

(For the purpose of the reader's easy getting the feeling of the above map, we also note that the $\mathbb{G}_m$-case (i.e., degenerated case) of the above map is the evaluation map

$$F[T]^{\deg<l} \xrightarrow{\sim} \oplus_{\zeta\in\mu_l} F$$

sending $f(T)$ to $(f(\zeta))_{\zeta\in\mu_l}$, which is an isomorphism since the Vandermonde determinant is non-vanishing.)

For $j \geq 0$, the graded quotient $\mathrm{Fil}^{-j}/\mathrm{Fil}^{-j+1}$ (in which the derivations of theta function live) with respect to the Hodge filtration given by the relative degree on the de Rham side (=theta function side) is isomorphic to $\omega_E^{\otimes(-j)}$, where ω_E is the pull-back of the cotangent bundle of E to the origin of E. On the other hand, in the étale side (=theta value side), we have a Gaussian pole $q^{j^2/8l}\mathcal{O}_F$ in the specified integral structure near the infinity (i.e., $q = 0$) of $\mathcal{M}_{\mathrm{ell}}$. This Gaussian pole comes from the values of theta functions at torsion points. We consider the degrees of the corresponding vector bundles on the moduli of elliptic curves to the both sides of the Hodge-Arakelov comparison map. The left-hand side is

$$-\sum_{j=0}^{l-1} j[\omega_E] \approx -\frac{l^2}{2}[\omega_E] = -\frac{l^2}{24}[\log q],$$

since $[\omega_E^{\otimes 2}] = [\Omega_{\mathcal{M}_{\mathrm{ell}}}] = \frac{1}{6}[\log q]$, where $\Omega_{\mathcal{M}_{\mathrm{ell}}}$ is the cotangent bundle of $\mathcal{M}_{\mathrm{ell}}$ and 6 is the degree of the λ-line over the j-line. The right-hand side is

$$-\frac{1}{8l}\sum_{j=0}^{l-1} j^2[\log q] \approx -\frac{l^2}{24}[\log q].$$

Note that these can be considered as a discrete analogue of the calculation of Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

from the point of view that $-\frac{1}{8l}\sum_{j=0}^{l-1} j^2[\log q]$ is a Gaussian distribution (i.e., $j \mapsto j^2$) in the cartesian coordinate, and $-\sum_{j=0}^{l-1} j[\omega_E] \approx -\frac{l^2}{2}[\omega_E]$ is a calculation in the polar coordinate and $[\omega_E]$ is an analogue of $\sqrt{\pi}$, since we have $\omega_E^{\otimes 2} \cong \Omega_{\mathcal{M}_{\mathrm{ell}}}$ and the integration of $\Omega_{\mathcal{M}_{\mathrm{ell}}}$ around the infinity (i.e., $q = 0$) is $2\pi i$. cf. also Remark 1.15.1

§ A.4. Motivation of the Definition of the Θ-Link.

In the situation as in the Hodge-Arakelov setting, we assume that E has everywhere stable reduction. In general, $E[l]$ does not have a global multiplicative subspace, i.e., a submodule $M \subset E[l]$ of rank 1 such that it coincides with the multiplicative subspace μ_l for each non-Archimedean bad places. However, let us *assume* such a global multiplicative subspace $M \subset E[l]$ exists in sufficiently general E in the moduli of elliptic curves. Let $M \times N \cong E[l]$ be an isomorphism as finite flat group schemes over F (not as Galois modules). Then by applying the Hodge-Arakelov comparison theorem to $E' := E/N$ over $K := F(E[l])$, we obtain an isomorphism

$$\Gamma((E')^\dagger, \mathcal{L}|_{(E')^\dagger})^{\deg < l} \xrightarrow{\sim} \bigoplus_{(-\frac{l-1}{2}=)-l^* \le j \le l^* (=\frac{l-1}{2})} (q^{\frac{j^2}{2l}} O_K) \otimes_{O_K} K,$$

where $q = (q_v)_{v:\mathrm{bad}}$ is the q-parameters of the non-Archimedean bad places. Then by the incompatibility of the Hodge filtration on the left-hand side with the direct sum decomposition in the right-hand side, the projection to the j-th factor is nontrivial for most j:

$$\mathrm{Fil}^0 = \underline{\underline{q}} O_K \hookrightarrow \underline{\underline{q}}^{j^2} O_K,$$

where we write $\underline{\underline{q}} := q^{\frac{1}{2l}}$. This morphism of arithmetic line bundles is considered as an *arithmetic analogue of Kodaira-Spencer morphism.* In the context of (Diophantine applications of) inter-universal Teichmüller theory, we take l to be a prime number in the order of the height of the elliptic curve, thus, l is very large (cf. Section 10). Hence the degree of the right-hand side in the above inclusion of the arithmetic line bundles is negative number of a very large absolute value, and the degree of the left-hand side is almost zero comparatively to the order of l. Therefore, the above inclusion implies

$$0 \lesssim -(\text{large number}) \ (\approx -\mathrm{ht}),$$

which gives us a upper bound of the height $\mathrm{ht} \lesssim 0$ in sufficiently general E in the moduli of elliptic curves.

However, there *never* exists such a global multiplicative in sufficiently general E in the moduli of elliptic curves (If it existed, then the above argument showed that the height is bounded from the above, which implies the number of isomorphism class of E is finite (cf. also Proposition C.1)). If we respect the scheme theory, then we cannot obtain the inclusion $\underline{\underline{q}}O_K \hookrightarrow \underline{\underline{q}}^{j^2}O_K$. Mochizuki's ingenious idea is: *Instead, we respect the inclusion $\underline{\underline{q}}O_K \hookrightarrow \underline{\underline{q}}^{j^2}O_K$, and we say a good-bye to the scheme theory.* The Θ-link in inter-universal Teichmüller theory is a kind of identification

$$(\Theta\text{-link}) : \qquad \{\underline{\underline{q}}^{j^2}\}_{1\leq j\leq l^*(=\frac{l-1}{2})} \qquad \mapsto \qquad \underline{\underline{q}}$$

in the outside of the scheme theory (In inter-universal Teichmüller theory, we also construct a kind of "global multiplicative subspace" in the outside of the scheme theory). So, it identifies an arithmetic line bundle of negative degree of a very large absolute value with an arithmetic line bundle of almost degree zero (in the outside of the scheme theory). This does not mean a contradiction, because both sides of the arithmetic line bundles belong to the different scheme theories, and we cannot compare their degrees. *The main theorem of the multiradial algorithm in inter-universal Teichmüller theory implies that we can compare their degrees after admitting mild indeterminacies* by using mono-anabelian reconstruction algorithms (and other techniques). We can calculate that the indeterminacies are (roughly) log-diff+log-cond by concrete calculations. Hence we obtain

$$0 \lesssim -\text{ht} + \text{log-diff} + \text{log-cond},$$

i.e., ht $\lesssim$ log-diff + log-cond. We have the following remark: We need not only to reconstruct (up to some indeterminacies) mathematical objects in the scheme theory of one side of a Θ-link from the ones in the scheme theory of the other side, but also to reduce the indeterminacies to mild ones. In order to do so, we need to control them, to reduce them by some rigidities, to kill them by some operations like taking p-adic logarithms for the roots of unity (cf. Proposition 13.7 (2c), Proposition 13.11 (2)), to estimate them by considering that some images are contained in some containers even though they are not precisely determinable (cf. Proposition 13.7 (2), Corollary 13.13), and to synchronise some indeterminacies to others (cf. Lemma 11.9, and Corollary 11.16 (1)) and so on. *This is a new kind of geometry – a geometry of controlling indeterminacies which arise from changing scheme theories i.e., changing unverses. This is Mochizuki's inter-universal geometry.*

Finally, we give some explanations on "**multiradial algorithm**" a little bit. In the classical terminology, we can consider different holomorphic structures on $\mathbb{R}^2$, i.e., $\mathbb{C} \cong \mathbb{R}^2 \cong \mathbb{C}$, where one $\mathbb{C}$ is an analytic (*not* holomorphic) dilation of another $\mathbb{C}$, and the underlying analytic structure $\mathbb{R}^2$ is shared. We can calculate the amount of the non-holomorphic dilation $\mathbb{C} \cong \mathbb{R}^2 \cong \mathbb{C}$ based on the shared underlying analytic structure $\mathbb{R}^2$ (If we consider only holomorphic structres and we do not consider the underlying analytic structure $\mathbb{R}^2$, then we cannot compare the holomorphic structures nor calculate the non-holomorphic dilation). This is a prototype of the multiradial algorithm. In philosophy, scheme theories are "arithmetically holomorphic structures" of a number field, and by going out the scheme theory, we can consider "underlying analytic structure" of the number field. The Θ-link is a kind of Teichmüller dilation of "arithmetically holomorphic structures" of the number field sharing the "underlying analytic structure". The shared "underlying analytic structure" is called *core*, and each "arithmetically holomorphic structure" is called *radial data*. The multiradial algorithm means that we can compare "arithmetically holomorphic structures" (of the both sides of Θ-link) based on the shared "underlying analytic structure" of the number field after admitting mild indeterminacies (In some sense, this is a partial (meaningful) realisation of the philosophy of "the field of one element" $\mathbb{F}_1$). Mochizuki's ideas of "underlying analytic structure" and the multiradial algorithm are really amazing discoveries.

Appendix B. Anabelian Geometry.

For a (pro-)veriety X over a field K, let Π_X (resp. Δ_X) be the arithmetic fundamental group of X (resp. the geometric fundamental group of X) for some basepoint. Let $\Delta_X^{(p)}$ be the maximal pro-p quotient of Δ_X, and write $\Pi_X^{(p)} := \Pi_X/\ker(\Delta_X \to \Delta_X^{(p)})$. For (pro-)varieties X, Y over a field K, we write $\mathrm{Hom}_K^{\mathrm{dom}}(X,Y)$ (resp. $\mathrm{Isom}_K(X,Y)$) for the set of dominant K-morphisms (resp. K-isomorphisms) from X to Y. For an algebraic closure $\overline{K}$ over K, write $G_K := \mathrm{Gal}(\overline{K}/K)$. We write $\mathrm{Hom}_{G_K}^{\mathrm{open}}(\Pi_X,\Pi_Y)$ (resp. $\mathrm{Hom}_{G_K}^{\mathrm{open}}(\Pi_X^{(p)},\Pi_Y^{(p)})$, resp. $\mathrm{Isom}_{G_K}^{\mathrm{Out}}(\Delta_X,\Delta_Y)$, resp. $\mathrm{Isom}_{G_K}^{\mathrm{Out}}(\Delta_X^{(p)},\Delta_Y^{(p)})$) for the set of open continuous homomorphisms from Π_X to Π_Y over G_K (resp. from $\Pi_X^{(p)}$ to $\Pi_Y^{(p)}$ over G_K, resp. from Δ_X to Δ_Y compatible with the outer G_K-actions up to composition with an inner automorphism arising from Δ_Y, resp. from $\Delta_X^{(p)}$ to $\Delta_Y^{(p)}$ compatible with the outer G_K-actions up to composition with an inner automorphism arising from $\Delta_Y^{(p)}$).

Theorem B.1. (Relative Version of the Grothendieck Conjecture over Sub-p-adic Fields [pGC, Theorem A]) *Let K be a sub-p-adic field (Definition 3.1 (1)). Let X be a smooth pro-variety over K. Let Y be a hyperbolic pro-curve over K. Then the natural maps*

$$\mathrm{Hom}_K^{\mathrm{dom}}(X,Y) \to \mathrm{Hom}_{G_K}^{\mathrm{open}}(\Pi_X,\Pi_Y)/\mathrm{Inn}(\Delta_Y) \to \mathrm{Hom}_{G_K}^{\mathrm{open}}(\Pi_X^{(p)},\Pi_Y^{(p)})/\mathrm{Inn}(\Delta_Y^{(p)})$$

are bijective. In particular, the natural maps

$$\mathrm{Isom}_K(X,Y) \to \mathrm{Isom}_{G_K}^{\mathrm{Out}}(\Delta_X,\Delta_Y) \to \mathrm{Isom}_{G_K}^{\mathrm{Out}}(\Delta_X^{(p)},\Delta_Y^{(p)})$$

are also bijective.

Remark B.1.1. The Isom-part of Theorem B.1 holds for a larger class of field which is called generalised sub-p-adic field ([TopAnb, Theorem 4.12]). Here, a field K

is called **generalised sub-p-adic** if there is a finitely generated extension L of the fractional field of $W(\overline{\mathbb{F}_p})$ such that we have an injective homomorphism $K \hookrightarrow L$ of fields. ([TopAnb, Definition 4.11]), where we write $W(\overline{\mathbb{F}_p})$ for the ring of Witt vectors with coefficients in $\overline{\mathbb{F}_p}$.

Appendix C. Miscellany.

§ C.1. On the Height Function.

Proposition C.1. ([GenEll, Proposition 1.4 (iv)]) *Let $\overline{\mathcal{L}} = (\mathcal{L}, ||-||_{\mathcal{L}})$ be an arithmetic line bundle such that $\mathcal{L}_{\mathbb{Q}}$ is ample. Then we have $\#\{x \in X(\overline{\mathbb{Q}})^{\leq d} \mid \mathrm{ht}_{\overline{\mathcal{L}}}(x) \leq C\} < \infty$ for any $d \in \mathbb{Z}_{\geq 1}$ and $C \in \mathbb{R}$.*

Proof. By using $\mathcal{L}_{\mathbb{Q}}^{\otimes n}$ for $n >> 0$, we have an embedding $X_{\mathbb{Q}} \hookrightarrow \mathbb{P}^N_{\mathbb{Q}}$ for some N. By taking a suitable blowing-up $f : \widetilde{X} \to X$, this embedding extends to $g : \widetilde{X} \hookrightarrow \mathbb{P}^N_{\mathbb{Z}}$ over $\mathrm{Spec}\,\mathbb{Z}$, where $\widetilde{X}$ is normal, $\mathbb{Z}$-proper, $\mathbb{Z}$-flat, and $f_{\mathbb{Q}} : \widetilde{X}_{\mathbb{Q}} \xrightarrow{\sim} X_{\mathbb{Q}}$. Then the proposition for $(X, \overline{\mathcal{L}})$ is reduced to the one for $(\widetilde{X}, f^*\overline{\mathcal{L}})$. As is shown in Section 1.1, the bounded discrepancy class of $\mathrm{ht}_{f^*\overline{\mathcal{L}}}$ depends only on $(f^*\mathcal{L})_{\mathbb{Q}}$. Thus, the proposition for $(\widetilde{X}, f^*\overline{\mathcal{L}})$ is equivalent to the one for $(\widetilde{X}, g^*\overline{\mathcal{O}_{\mathbb{P}^N_{\mathbb{Z}}}(1)})$, where $\overline{\mathcal{O}_{\mathbb{P}^N_{\mathbb{Z}}}(1)}$ is the line bundle $\mathcal{O}_{\mathbb{P}^N_{\mathbb{Z}}}(1)$ equipped with the standard Fubini-Study metric $||-||_{\mathrm{FS}}$. Then it suffices to show the proposition for $(\mathbb{P}^N_{\mathbb{Z}}, \overline{\mathcal{O}_{\mathbb{P}^N_{\mathbb{Z}}}(1)})$. For $1 \leq e \leq d$, we put $Q := (\mathbb{P}^N_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \cdots (e\text{-times}) \cdots \times_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{P}^N_{\mathbb{Z}})/(e\text{-th symmetric group})$, which is normal $\mathbb{Z}$-proper, $\mathbb{Z}$-flat. The arithmetic line bundle $\otimes_{1 \leq i \leq e} \mathrm{pr}_i^* \mathcal{O}_{\mathbb{P}^N_{\mathbb{Z}}}(1)$ on $\mathbb{P}^N \times_{\mathrm{Spec}\,\mathbb{Z}} \cdots (e\text{-times}) \cdots \times_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{P}^N$ descends to $\overline{\mathcal{L}}_Q = (\mathcal{L}_Q, ||-||_{\mathcal{L}_Q})$ on Q with $(\mathcal{L}_Q)_{\mathbb{Q}}$ ample, where pr_i is the i-th projection. For any $x \in \mathbb{P}^N(F)$ where $[F : \mathbb{Q}] = e$, the conjugates of x over $\mathbb{Q}$ determine a point $x_Q \in Q(\mathbb{Q})$, and, in turn, a point $y \in Q(\mathbb{Q})$ determines a point $x \in \mathbb{P}^N(F)$ up to a finite number of possibilities. Hence it suffices to show that $\#\{y \in Q(\mathbb{Q}) \mid \mathrm{ht}_{\overline{\mathcal{L}}_Q}(y) \leq C\} < \infty$ for any $C \in \mathbb{R}$. We embed $Q \hookrightarrow \mathbb{P}^M_{\mathbb{Z}}$ for some M by $(\mathcal{L}_Q)^{\otimes m}_{\mathbb{Q}}$ for $m >> 0$. Then by the same argument as above, it suffices to show that $\#\{x \in \mathbb{P}^M(\mathbb{Q}) \mid \mathrm{ht}_{\overline{\mathcal{O}_{\mathbb{P}^M}(1)}}(x) \leq C\} < \infty$ for any $C \in \mathbb{R}$. For $x \in \mathbb{P}^M(\mathbb{Z})(= \mathbb{P}^M(\mathbb{Q}))$, we have $\mathrm{ht}_{\overline{\mathcal{O}_{\mathbb{P}^M}(1)}}(x) = \deg_{\mathbb{Q}} x^*\overline{\mathcal{O}_{\mathbb{P}^M}(1)}$ by definition. We have $\deg_{\mathbb{Q}} : \mathrm{APic}(\mathrm{Spec}\,\mathbb{Z}) \xrightarrow{\sim} \mathbb{R}$ since any projective $\mathbb{Z}$-module is free ($\mathbb{Q}$ has class number 1), where an arithmetic line bundle $\overline{\mathcal{L}_{\mathbb{Z},C}}$ on $\mathrm{Spec}\,\mathbb{Z}$ in the isomorphism class corresponding to $C \in \mathbb{R}$ via this isomorphism is $(\mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}}, e^{-C}|-|)$ (Here $|-|$ is the usual absolute value). The set of global sections $\Gamma(\overline{\mathcal{L}_{\mathbb{Z},C}})$ is $\{a \in \mathbb{Z} \mid |a| \leq e^C\}$ which is a finite set (cf. Section 1.1 for the definition of $\Gamma(\overline{\mathcal{L}})$). We also have $\overline{\mathcal{L}_{\mathbb{Z},C_1}} \hookrightarrow \overline{\mathcal{L}_{\mathbb{Z},C_2}}$ for $C_1 \leq C_2$. Let $x_0, \ldots, x_M \in \Gamma(\mathbb{P}^M_{\mathbb{Z}}, \mathcal{O}_{\mathbb{P}^M_{\mathbb{Z}}}(1))$ be the standard generating sections ("the coordinate $(x_0 : \ldots : x_M) \in \mathbb{P}^M_{\mathbb{Z}}$") with $||x_i||_{\mathrm{FS}} \leq 1$ for $0 \leq i \leq M$ i.e., $x_0, \ldots, x_M \in \Gamma(\overline{\mathcal{O}_{\mathbb{P}^M_{\mathbb{Z}}}(1)})$. Then for $x \in \mathbb{P}^M(\mathbb{Z})(= \mathbb{P}^M(\mathbb{Q}))$ with $\mathrm{ht}_{\overline{\mathcal{O}_{\mathbb{P}^M}(1)}}(x) \leq C$, we have a map $x^*\overline{\mathcal{O}_{\mathbb{P}^M}(1)} \hookrightarrow \overline{\mathcal{L}_{\mathbb{Z},C}}$, which sends $x_0, \ldots, x_M \in \Gamma(\overline{\mathcal{O}_{\mathbb{P}^M_{\mathbb{Z}}}(1)})$ to $x^*(x_0), \ldots, x^*(x_M) \in$

$\Gamma(\overline{\mathcal{L}_{\mathbb{Z},C}})$. This map $\{x \in \mathbb{P}^M(\mathbb{Z}) \mid \mathrm{ht}_{\overline{\mathcal{O}_{\mathbb{P}^M}(1)}}(x) \leq C\} \to \Gamma(\overline{\mathcal{L}_{\mathbb{Z},C}})^{\oplus(M+1)}$, which sends x to $(x^*(x_0), \ldots, x^*(x_M))$, is injective since $x_0, \ldots, x_M \in \Gamma(\mathbb{P}^M_{\mathbb{Z}}, \mathcal{O}_{\mathbb{P}^M_{\mathbb{Z}}}(1))$ are generating sections. In short, we have $\{x \in \mathbb{P}^M(\mathbb{Q}) \mid \mathrm{ht}_{\overline{\mathcal{O}_{\mathbb{P}^M}(1)}}(x) \leq C\} \subset \{(x_0 : \ldots : x_M) \in \mathbb{P}^M(\mathbb{Q}) \mid x_i \in \mathbb{Z},\ |x_i| \leq e^C\ (0 \leq i \leq M)\}$. Now, the proposition follows from the finiteness of $\Gamma(\overline{\mathcal{L}_{\mathbb{Z},C}})^{\oplus(M+1)}$. $\square$

§ C.2. Non-critical Belyi Maps.

The following theorem, which is a refinement of a classical theorem of Belyi, is used in Proposition 1.2.

Theorem C.2. ([Belyi, Theorem 2.5], non-critical Belyi map) *Let X be a proper smooth connected curve over $\overline{\mathbb{Q}}$, and $S, T \subset X(\overline{\mathbb{Q}})$ finite sets such that $S \cap T = \emptyset$. Then there exists a morphism $\phi : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ such that (a) ϕ is unramified over $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$, (b) $\phi(S) \subset \{0, 1, \infty\}$, and (c) $\phi(T) \subset \mathbb{P}^1(\overline{\mathbb{Q}}) \setminus \{0, 1, \infty\}$.*

Proof. (Step 1): By adjoining points of $X(\overline{\mathbb{Q}})$ to T, we may assume that $\#T \geq 2g_X + 1$, where g_X is the genus of X. We consider T as a reduced effective divisor on X by abuse of notation. Let $s_0 \in \Gamma(X, \mathcal{O}_X(T))$ be such that $(s_0)_0 = T$, where we write $(s_0)_0$ for the zero divisor of s_0. We have $H^1(X, \mathcal{O}_X(T - x)) = H^0(X, \omega_X(x - T))^* = 0$ for any $x \in X(\overline{\mathbb{Q}})$ since $\deg(\omega_X(x - T)) \leq 2g_X - 2 - (2g_X + 1) + 1 = -2$. Thus, the homomorphism $\Gamma(X, \mathcal{O}_X(T)) \to \mathcal{O}_X(T) \otimes k(x)$ induced by the short exact sequnce $0 \to \mathcal{O}_X(T - x) \to \mathcal{O}_X(T) \to \mathcal{O}_X(T) \otimes k(x) \to 0$ is surjective. Hence there exists an $s_1 \in \Gamma(X, \mathcal{O}_X(T))$ such that $s_1(t) \neq 0$ for all $t \in T$ since $\overline{\mathbb{Q}}$ is infinite. Then $(s_0 : s_1)$ has no basepoints, and gives us a finite morphism $\psi : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ such that $\psi^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_X(T)$, and $\psi(t) = 0$ for all $t \in T$ since $(s_0)_0 = T$. Here, ψ is unramified over $0 \in \mathbb{P}^1_{\overline{\mathbb{Q}}}$, since $\psi^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_X(T)$ and T is reduced. We also have $0 \notin \psi(S)$ since $(s_0)_0 = T$ and $S \cap T = \emptyset$. Then by replacing X, T, and S by $\mathbb{P}^1_{\overline{\mathbb{Q}}}$, 0, and $\psi(S) \cap \{x \in \mathbb{P}^1_{\overline{\mathbb{Q}}} \mid \psi \text{ ramifies over } x\}$ respectively, the theorem is reduced to the case where $X = \mathbb{P}^1_{\overline{\mathbb{Q}}}$, $T = \{t\}$ for some $t \in \mathbb{P}^1(\mathbb{Q}) \setminus \{\infty\}$

(Step 2): Next, we reduce the theorem to the case where $X = \mathbb{P}^1_{\overline{\mathbb{Q}}}$, $S \subset \mathbb{P}^1(\mathbb{Q})$, $T = \{t\}$ for some $t \in \mathbb{P}^1(\mathbb{Q}) \setminus \{\infty\}$ as follows: We will construct a non-zero rational function $f(x) \in \mathbb{Q}(x)$ which defines a morphism $\phi : \mathbb{P}^1_{\overline{\mathbb{Q}}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ such that $\phi(S) \subset \mathbb{P}^1(\mathbb{Q})$, $\phi(t) \notin \phi(S)$, and ϕ is unramified over $\phi(t)$. By replacing S by the union of all $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates of S, we may assume that S is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable (Note that $t \notin$ (new S) since $t \in \mathbb{P}^1(\mathbb{Q})$ and $t \notin$ (old S)). Write $m(S) := \max_F([F : \mathbb{Q}] - 1)$, where F runs through the fields of definition of the points in S, and $d(S) := \sum_F([F : \mathbb{Q}] - 1)$, where F runs thrhough the fields of definition of the points in S with $[F : \mathbb{Q}] - 1 = m(S)$. Thus, $S \subset \mathbb{P}^1(\mathbb{Q})$ is equivalent to $d(S) = 0$, which holds if and only if $m(S) = 0$. We use an induction on $m(S)$, and for each fixed $m(S)$, we use an induction on $d(S)$. If $m(S), d(S) \neq 0$, take $\alpha \in S \setminus \mathbb{P}^1(\mathbb{Q})$ such that $d := [\mathbb{Q}(\alpha) : \mathbb{Q}]$ is equal to $m(S)+1$. We choose $a_1 \in \mathbb{Q}$ such that $0 < |t - a_1| < (\min_{s \in S\setminus\{\infty\}} |s - a_1|)/d(1 + d.d!)$. Then by applying an automorphism $f_1(x) := (\min_{s\in S\setminus\{\infty\}} |s - a_1|)/(x - a_1)$ of $\mathbb{P}^1_{\mathbb{Q}}$ (and replacing t and S by $f_1(t)$ and $f_1(S)$ respectively), we may assume that $|s| \leq 1$ for all $s \in S(= S\setminus\{\infty\})$ and $|t| > d(1+d.d!)$ (Note that the property (new t) $\in \mathbb{P}^1(\mathbb{Q})\setminus\{\infty\}$ still holds since $|(\text{old } t)-a_1| > 0$ and $f_1(x) \in \mathbb{Q}(x)$). Let $g(x) = x^d + c_1x^{d-1} + \cdots + c_d \in \mathbb{Q}[x]$ be the monic minimal polynomial of α over $\mathbb{Q}$. Then $|c_i| \leq d!$ for $1 \leq i \leq d$ since c_i is a summation of $\binom{d}{i}(\leq d!)$ products of i conjugates of α. Thus, $|g(s)| \leq 1+|c_1|+\cdots+|c_d| \leq 1 + d.d!$ and $|g'(s)| \leq d + d|c_1| + \cdots + d|c_d| \leq d(1 + d.d!)$ for all $s \in S(= S \setminus \{\infty\})$ since $|s| \leq 1$ (Here $g'(x)$ is the derivative of $g(x)$). Hence $t \notin g(S) \cup g(S_\alpha) =: S'$, where $S_\alpha := \{\beta \in \overline{\mathbb{Q}} \mid g'(\beta) = 0\}$. We also have $[\mathbb{Q}(\alpha') : \mathbb{Q}] < d$ for any $\alpha' \in g(S_\alpha)$ since $g(x), g'(x) \in \mathbb{Q}[x]$ and $\deg(g'(x)) < d$. Therefore, S' is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable and we have $m(S') < m(S)$ or ($m(S') = m(S)$ and $d(S') < d(S)$). This completes the induction, and we get a desired morphism ϕ by composing the constructed maps as above.

(Step 3): Now, we reduced the theorem to the case where $X = \mathbb{P}^1_{\overline{\mathbb{Q}}}$, $S \subset \mathbb{P}^1(\mathbb{Q})$, and $T = \{t\}$ for some $t \in \mathbb{P}^1(\mathbb{Q}) \setminus \{\infty\}$ with $S \cap T = \emptyset$. We choose $a_2 \in \mathbb{Q}$ such that $0 < |t - a_2| < (\min_{s\in S\setminus\{\infty\}} |s - a_2|)/4$. Then by applying an automorphism $f_2(x) := 1/(x - a_2)$ of $\mathbb{P}^1_{\mathbb{Q}}$ (and replacing t and S by $f_2(t)$ and $f_2(S)$ respectively), we may assume that $|t| \geq 4|s|$ for all $s \in S(= S \setminus \{\infty\})$. (Note that the property (new t) $\in \mathbb{P}^1(\mathbb{Q}) \setminus \{\infty\}$ still holds since $|(\text{old } t) - a_2| > 0$ and $f_2(x) \in \mathbb{Q}(x)$). New t is not equal to 0 since old t is not equal to ∞. By applying the automorphism $x \mapsto -x$ of $\mathbb{P}^1_{\mathbb{Q}}$, we may assume that $t > 0$ (still $t \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, \infty\}$). By applying an automorphism $f_3(x) := x + a_3$ of $\mathbb{P}^1_{\mathbb{Q}}$, where $a_3 := \max_{S\setminus\{\infty\}\ni s'<0} |s'|$ ($a_3 := 0$ when $\{s' \in S \setminus \{\infty\} \mid s' < 0\} = \emptyset$) and replacing t and S by $f_3(t)$ and $f_3(S)$ respectively, we may assume that $s \geq 0$ for all $s \in S(= S \setminus \{\infty\})$ and $t \geq 2s$ for all $s \in S(= S \setminus \{\infty\})$, since $(t + a_3)/(s + a_3) \geq t/(s + a_3) \geq t/2a_3 \geq 2$ where t, s are old ones (still (new t) $\in \mathbb{P}^1(\mathbb{Q})\setminus\{0, \infty\}$). By adjoining $\{0, \infty\}$ (if necessary for 0), we may assume that $S \supset \{0, \infty\}$ since $t \notin \{0, \infty\}$.

(Step 4): Thus, now we reduced the theorem to the case where $X = \mathbb{P}^1_{\overline{\mathbb{Q}}}$, $\{0, \infty\} \subset S \subset \mathbb{P}^1(\mathbb{Q})$, $T = \{t\}$ for some $t \in \mathbb{P}^1(\mathbb{Q}) \setminus \{\infty\}$ with $S \cap T = \emptyset$, and $s > 1$, $t \geq 2s$ for every $s \in S \setminus \{0, \infty\}$. We show the theorem in this case (hence the theorem in the general case) by the induction on $\#S$. If $\#S \leq 3$ then we are done. We assume that $\#S > 3$. Let $a_4 \in \mathbb{Q}$ be the second smallest $s \in S \setminus \{0, \infty\}$. By applying an automorphism $f_4(x) := x/a_4$ of $\mathbb{P}^1_{\mathbb{Q}}$ (and replacing t and S by $f_4(t)$ and $f_4(S)$ respectively), we may assume moreover that $0 < r < 1$ for some $r \in S$ and $s > 1$ for every $s \in S \setminus \{0, r, 1, \infty\}$ Write $r = m/(m + n)$ where $m, n \in \mathbb{Z}_{>0}$. We consider the function $h(x) := x^m(x - 1)^n$ and the morphisms $\psi, \psi' : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ defined by $h(x)$ and $h(x) + a_5$

respectively, where $a_5 := -\min_{s\in S\setminus\{\infty\}} h(s)$. We have $h(\{0,1,r,\infty\}) \subset \{0,h(r),\infty\}$. Thus $\#\psi(S) < \#S$ and hence $\#\psi'(S) < \#S$. Any root of the derivative $h'(x) = x^{m-1}(x-1)^{n-1}((m+n)x-m) = 0$ is in $\{0,r,1,\infty\} \subset S$. Thus ψ is unramified outside $\psi(S)$, and hence ψ' is unramified outside $\psi'(S)$. Now $h(x)$ is monotone increasing for $x > 1$ since $h'(x) > 0$ for $x > 1$. Thus we have $h(t) > h(s)$ for $s \in S \setminus \{\infty\}$ with $s > 1$ since $t \geq 2s > s$. We also have $h(t) > h(2) > 1$ since $t \geq 2$ (which comes from $t \geq 2s$ for $s = 1 \in S$). Thus, $\psi(t) \notin \psi(S)$ since $|h(x)| \leq 1$ for $0 \leq x \leq 1$. Hence we also have $\psi'(t) \notin \psi'(S)$. Now we claim that $(h(t) + a_5)/(h(s) + a_5) \geq 2$ for all $s \in S \setminus \{\infty\}$ such that $h(s) + a_5 \neq 0$. If this claim is proved, then by replacing S, t by $\psi'(S)$, $\psi'(t)$ respectively, we are in the situation with smaller $\#S$ where we can use the induction hypothesis, and we are done. We show the claim. First we observe that we have $h(t)/h(s) = (t/s)^m((t-1)/(s-1))^n \geq (t/s)^{m+n} \geq (t/s)^2$ (*) for $s \in S\setminus\{\infty\}$, since $t \geq s$ implies $(t-1)/(s-1) \geq t/s$. In the case where n is even, we have $a_5 = 0$ since $h(s) \geq 0$ for all $s \in S \setminus \{\infty\}$ and $h(0) = 0$. Thus, we have $(h(t) + a_5)/(h(s) + a_5) = h(t)/h(s) \geq (t/s)^2 \geq t/s \geq 2$ for $1 < s \in S \setminus \{\infty\}$ by (*). On the other hand, $h(s) + a_5 = h(s) = 0$ for $s = 0, 1$ and $(h(t) + a_5)/(h(r) + a_5) = h(t)/h(r) \geq h(t) = t^m(t-1)^n \geq t \geq 2$ by $0 < h(r) < 1$ and $t \geq 2$. Hence the claim holds for even n. In the case where n is odd, we have $a_5 = |h(r)| = (\frac{m}{m+n})^m(\frac{n}{m+n})^n$, since $h(x) \leq 0$ for $0 \leq x \leq 1$ and, $x = r \Leftrightarrow h'(x) = 0$ for $0 < x < 1$. We also have $0 < a_5 = (\frac{m}{m+n})^m(\frac{n}{m+n})^n \leq \frac{m}{m+n}\frac{n}{m+n} = \frac{mn}{(m-n)^2+4mn} \leq \frac{mn}{4mn} = \frac{1}{4}$. Then for $1 < s \in S \setminus \{\infty\}$ with $h(s) \geq a_5$, we have $(h(t) + a_5)/(h(s) + a_5) \geq h(t)/2h(s) \geq (t/s)^2/2 \geq 2$ by (*). For $1 < s \in S \setminus \{\infty\}$ with $h(s) \leq a_5$, we have $(h(t)+a_5)/(h(s)+a_5) \geq h(t)/2a_5 \geq 2h(t) = 2t^m(t-1)^n \geq t \geq 2$ by $0 < a_5 \leq 1/4$ and $t \geq 2$. For $s = r \in S$, we have $h(r) + a_5 = -a_5 + a_5 = 0$. For $s = 0, 1 \in S$, we have $(h(t)+a_5)/(h(s)+a_5) = (h(t)+a_5)/a_5 \geq h(t) = t^m(t-1)^n \geq t \geq 2$ by $0 < a_5 \leq 1/4$ and $t \geq 2$. Thus, we show the claim, and hence the theorem. □

§ C.3. *k*-Cores.

Lemma C.3. ([CanLift, Proposition 2.7]) *Let k be an algebraically closed field of characteristic* 0*.*

(1) If a semi-elliptic (cf. Section 3.1) orbicurve X has a nontrivial automorphism, then it does not admit k-core.

(2) There exist precisely 4 *isomorphism classes of semi-elliptic orbicurves over k which do not admit k-core.*

Proof. (Sketch) For algebraically closed fields $k \subset k'$, the natural functor from the category $\text{Ét}(X)$ of finite étale coverings over X to the category $\text{Ét}(X \times_k k')$ of finite étale coverings over $X \times_k k'$ is an equivalence of categories, and the natural map

$\mathrm{Isom}_k(Y_1, Y_2) \to \mathrm{Isom}_{k'}(Y_1 \times_k k', Y_2 \times_k k')$ is a bijection for $Y_1, Y_2 \in \mathrm{Ob}(\acute{\mathrm{E}}\mathrm{t}(X))$ by the standard arguments of algebraic geometry, i.e., For some k-variety V such that the function field $k(V)$ of V is a sub-field of k', the diagrams of finite log-étale morphisms over $(\overline{X} \times_k k', D \times_k k')$ (Here, $\overline{X}$ is a compactification and D is the complement) under consideration is the base-change of the diagrams of finite étale morphisms over V with respect to $\mathrm{Spec}\, k' \to \mathrm{Spec}\, k(V) \to V$, we specialise them to a closed point v of V, we deform them to a formal completion $\widehat{V}_v$ at v, and we algebrise them (cf. also [CanLift, Proposition 2.3], [SGA1, Exposé X, Corollaire 1.8]), and the above bijection is also shown in a similar way by noting $H^0(\overline{Y}, \omega^{\vee}_{\overline{X}/k}(-D)|_{\overline{Y}}) = 0$ for any finite morphism $\overline{Y} \to \overline{X}$ in the argments of deforming the diagrams under consideration to $\widehat{V}_v$. Thus, the natural functor $\overline{\mathrm{Loc}}_k(X) \to \overline{\mathrm{Loc}}_{k'}(X \times_k k')$ is an equivalence categories. Hence the lemma is reduced to the case where $k = \mathbb{C}$.

We assume that $k = \mathbb{C}$. Note also that the following four statements are equivalent:

(i) X does not admit k-core,

(ii) $\pi_1(X)$ is of inifinite index in the commensurator $C_{\mathrm{PSL}_2(\mathbb{R})^0}(\pi_1(X))$ in $\mathrm{PSL}_2(\mathbb{R})^0 (\cong \mathrm{Aut}(\mathcal{H}))$ (Here, we write $\mathrm{PSL}_2(\mathbb{R})^0$ for the connected component of the identity of $\mathrm{PSL}_2(\mathbb{R})$, and we write $\mathcal{H}$ for the upper half plane),

(iii) X is Margulis-arithmetic (cf. [Corr, Definition 2.2]), and

(iv) X is Shimura-arithmetic (cf. [Corr, Definition 2.3]).

The equivalence of (i) and (ii) comes from that if X admits k-core, then the morphism to k-core $X \twoheadrightarrow X_{\mathrm{core}}$ is isomorphic to $\mathcal{H}/\pi_1(X) \twoheadrightarrow \mathcal{H}/C_{\mathrm{PSL}_2(\mathbb{R})^0}(\pi_1(X))$, and that if $\pi_1(X)$ is of finite index in $C_{\mathrm{PSL}_2(\mathbb{R})^0}(\pi_1(X))$, then $\mathcal{H}/\pi_1(X) \twoheadrightarrow \mathcal{H}/C_{\mathrm{PSL}_2(\mathbb{R})^0}(\pi_1(X))$ is k-core (cf. also [CanLift, Remark 2.1.2, Remark 2.5.1]). The equivalence of (ii) and (iii) is due to Margulis ([Marg, Theorem 27 in p.337, Lemma 3.1.1 (v) in p.60], [Corr, Theorem 2.5]). The equivalence of (iii) and (iv) is [Corr, Proposition 2.4].

(1): We assume that X admits a k-core X_{core}. Let $Y \to X$ be the unique double covering such that Y is a once-punctured elliptic curve. We write $\overline{Y}$, $\overline{X_{\mathrm{core}}}$ for the smooth compactifications of Y, X_{core} respectively. Here, we have $\overline{Y} \setminus Y = \{y\}$, and a point of $\overline{Y}$ is equal to y if and only if its image is in $\overline{X_{\mathrm{core}}} \setminus X_{\mathrm{core}}$. Thus, we have $\overline{X_{\mathrm{core}}} \setminus X_{\mathrm{core}} = \{x\}$. The coarsification (or "coarse moduli space") of $\overline{X_{\mathrm{core}}}$ is the projective line $\mathbb{P}^1_k$ over k. By taking the coarification of a unique morphism $Y \twoheadrightarrow X_{\mathrm{core}}$, we obtain a finite ramified covering $\overline{Y} \twoheadrightarrow \mathbb{P}^1_k$. Since this finite ramified covering $\overline{Y} \twoheadrightarrow \mathbb{P}^1_k$ comes from a finite étale covering $Y \to X_{\mathrm{core}}$, the ramification index of $\overline{Y} \twoheadrightarrow \mathbb{P}^1_k$ is the same as all points of $\overline{Y}$ lying over a given point of $\mathbb{P}^1_k$. Thus, by the Riemann-Hurwitz formula, we obtain $-2d + \sum_i \frac{d}{e_i}(e_i - 1)$, where e_i's are the ramification indices over the ramification points of $\mathbb{P}^1_k$, and d is the degree of the morphism $\overline{Y} \twoheadrightarrow \mathbb{P}^1_k$. Hence by

$\sum_i \frac{1}{e_i}(e_i - 1) = 2$, the possibility of e_i's are $(2,2,2,2)$, $(2,3,6)$, $(2,4,4)$, and $(3,3,3)$. Since y is the unique point over x, the largest e_i is equal to d. In the case of $(2,2,2,2)$, we have $X = X_{\mathrm{core}}$, and X has no nontrivial automorphism. In other three cases, Y is a finte étale covering of the orbicurve determind by a triangle group (cf. [Take1]) of type $(2,3,\infty)$, $(2,4,\infty)$, and $(3,3,\infty)$. By [Take1, Theorem 3 (ii)], this implies that Y is Shimura-arithmetic, hence X is Shimura-arithmetic as well. This is a contradiction (cf. also [CanLift, Remark 2.1.2, Remark 2.5.1]) by the above equivalence of (i) and (iv).

(2): If X does not admit k-core, then X is Shimura-arithmetic by the above equivalence of (i) and (iv). Then by [Take2, Theorem 4.1 (i)], this implies that, in the notation of [Take2], the arithmetic Fuchsian group $\pi_1(X)$ has signature $(1;\infty)$ such that $(\mathrm{tr}(\alpha), \mathrm{tr}(\beta), \mathrm{tr}(\alpha\beta))$ is equal to $(\sqrt{5}, 2\sqrt{5}, 5)$, $(\sqrt{6}, 2\sqrt{3}, 3\sqrt{2})$, $(2\sqrt{2}, 2\sqrt{2}, 4)$, and $(3,3,3)$. This gives us precisely 4 isomorphism classes. $\square$

§ C.4. On the Prime Number Theorem.

For $x > 0$, write $\pi(x) := \#\{p \mid p : \text{prime} \leq x\}$ and $\vartheta(x) := \sum_{\text{prime}:\, p \leq x} \log p$ (Chebychev's ϑ-function). The *prime number theorem* says that

$$\pi(x) \sim \frac{x}{\log x} \quad (x \to \infty),$$

where, $\sim$ means that the ratio of the both side goes to 1. In this subsection, we show the following proposition, which is used in Proposition 1.15.

Lemma C.4. $\pi(x) \sim \frac{x}{\log x}$ $(x \to \infty)$ *if and only if* $\vartheta(x) \sim x$ $(x \to \infty)$.

This is well-known for analytic number theorists. However, we include a proof here for the convenience for arithmetic geometers.

Proof. We show the "only if" part: Note that $\vartheta(x) = \int_1^x \log t \cdot d(\pi(t)) = \pi(x)\log x - \pi(1)\log 1 - \int_1^x \frac{\pi(t)}{t}dt = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}dt$ (since $\pi(t) = 0$ for $t < 2$). Then it suffices to show that $\lim_{x\to\infty} \frac{1}{x}\int_2^x \frac{\pi(t)}{t}dt = 0$. By assumption $\frac{\pi(t)}{t} = O\left(\frac{1}{\log t}\right)$, we have $\frac{1}{x}\int_2^x \frac{\pi(t)}{t}dt = O\left(\frac{1}{x}\int_2^x \frac{dt}{\log t}\right)$. By $\int_2^x \frac{dt}{\log t} = \int_2^{\sqrt{x}} \frac{dt}{\log t} + \int_{\sqrt{x}}^x \frac{dt}{\log t} \leq \frac{\sqrt{x}}{\log 2} + \frac{x-\sqrt{x}}{\log\sqrt{x}}$, we obtain $\lim_{x\to\infty} \frac{1}{x}\int_2^x \frac{\pi(t)}{t}dt = 0$. We show the "if" part: Note that $\pi(x) = \int_{3/2}^x \frac{1}{\log t}d(\vartheta(t)) = \frac{\vartheta(x)}{\log x} - \frac{\vartheta(3/2)}{\log(3/2)} + \int_{3/2}^x \frac{\pi(t)}{t(\log t)^2}dt = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\pi(t)}{t(\log t)^2}dt$ (since $\vartheta(t) = 0$ for $t < 2$). Then it suffices to show that $\lim_{x\to\infty} \frac{\log x}{x}\int_2^x \frac{\vartheta(t)}{t(\log t)^2}dt = 0$. By assumption $\vartheta(t) = O(t)$, we have $\frac{\log x}{x}\int_2^x \frac{\vartheta(t)}{t(\log t)^2}dt = O\left(\frac{\log x}{x}\int_2^x \frac{dt}{(\log t)^2}\right)$. By $\int_2^x \frac{1}{(\log t)^2}dt = \int_2^{\sqrt{x}} \frac{dt}{(\log t)^2} + \int_{\sqrt{x}}^x \frac{dt}{(\log t)^2} \leq \frac{\sqrt{x}}{(\log 2)^2} + \frac{x-\sqrt{x}}{(\log\sqrt{x})^2}$, we obtain $\lim_{x\to\infty} \frac{\log x}{x}\int_2^x \frac{\vartheta(t)}{t(\log t)^2}dt = 0$. $\square$

§ C.5. On the Residual Finiteness of Free Groups.

Proposition C.5. (Residual Finiteness of Free Groups) *Let F be a free group. Then the natural homomorphism $F \to \widehat{F}$ to its profinite completion $\widehat{F}$ is injective.*

Proof. Let $a \in F \setminus \{1\}$. It suffices to show that there exists a normal subgroup $H \subset F$ of finite index such that $a \notin H$. Let Gen ($\subset F$) be a set of free generators of F. Write $\mathrm{Gen}^{-1} := \{a^{-1} \mid a \in \mathrm{Gen}\} \subset F$. Thus, any element of F may be written as a finite product of elements of $\mathrm{Gen} \cup \mathrm{Gen}^{-1}$. Let $a = a_N a_{N-1} \cdots a_1$, where $a_i \in \mathrm{Gen} \cup \mathrm{Gen}^{-1}$, be such a representation of a (i.e., as a finite product of elements of $\mathrm{Gen} \cup \mathrm{Gen}^{-1}$) of *minimal length*. Let $\phi : \mathrm{Gen} \to \mathfrak{S}_{N+1}$ be a map such that, for $x \in \mathrm{Gen}$, $\phi(x) \in \mathfrak{S}_{N+1}$ sends $i \mapsto i+1$ if $x = a_i$ and $j \mapsto j-1$ if $x = a_{j-1}^{-1}$. (To see that such a ϕ exists, it suffices to observe that since the representation $a = a_N a_{N-1} \cdots a_1$ is of minimal length, the equations $a_i = x$, $a_{i-1} = x^{-1}$ cannot hold simultaneously.) Since Gen is a set of free generators of F, the map $\phi : \mathrm{Gen} \to \mathfrak{S}_{N+1}$ extends to a homomorphism $\phi_F : F \to \mathfrak{S}_{N+1}$ such that, for $i = 1, \ldots, N$, the permutation $\phi_F(a_i)$ sends $i \mapsto i+1$. Write H for the kernel of ϕ_F. Since ϕ_F induces an injection of F/H into the finite group $\mathfrak{S}_{N+1}$, it follows that H is a normal subgroup of finite index in F. Then $\phi_F(a)$ sends $1 \mapsto N+1$, hence in particular, is nontrivial, i.e., $a \notin H$, as desired. □

§ C.6. Some Lists on Inter-universal Teichmüller Theory.

Model Objects
Local:

	$\underline{\mathbb{V}}^{\mathrm{bad}}$ (Example 8.8)	$\underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$ (Example 8.7)	$\underline{\mathbb{V}}^{\mathrm{arc}}$ (Example 8.11)
$\mathcal{D}_{\underline{v}}$	$\mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})^0$ $(\Pi_{\underline{v}})$	$\mathcal{B}(\underrightarrow{X}_{\underline{v}})^0$ $(\Pi_{\underline{v}})$	$\underrightarrow{\mathbb{X}}_{\underline{v}}$
$\mathcal{D}^{\vdash}_{\underline{v}}$	$\mathcal{B}(K_{\underline{v}})^0$ $(G_{\underline{v}})$	$\mathcal{B}(K_{\underline{v}})^0$ $(G_{\underline{v}})$	$(O^{\rhd}(\mathcal{C}^{\vdash}_{\underline{v}}), \mathrm{spl}^{\vdash}_{\underline{v}})$
$\mathcal{C}_{\underline{v}}$	$(\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\text{base-field}}$ $(\Pi_{\underline{v}} \curvearrowright (O^{\rhd}_{\overline{F}_{\underline{v}}})^{\mathrm{pf}})$	$\Pi_{\underline{v}} \curvearrowright (O^{\rhd}_{\overline{F}_{\underline{v}}})^{\mathrm{pf}}$	Arch. Fr'd $\mathcal{C}_{\underline{v}}$ ($\leftrightsquigarrow$ang. region)
$\underline{\underline{\mathcal{F}}}_{\underline{v}}$	temp. Fr'd $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ ($\leftrightsquigarrow\Theta$-fct.)	equal to $\mathcal{C}_v$	$(\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}})$
$\mathcal{C}^{\vdash}_{\underline{v}}$	$G_{\underline{v}} \curvearrowright O^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}}$	$G_{\underline{v}} \curvearrowright O^{\times}_{\overline{F}_{\underline{v}}} \cdot p_{\underline{v}}^{\mathbb{N}}$	equal to $\mathcal{C}_{\underline{v}}$
$\mathcal{F}^{\vdash}_{\underline{v}}$	$(\mathcal{C}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}})$	$(\mathcal{C}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}})$	$(\mathcal{C}^{\vdash}_{\underline{v}}, \mathcal{D}^{\vdash}_{\underline{v}}, \mathrm{spl}^{\vdash}_{\underline{v}})$

We use $\mathcal{C}_{\underline{v}}$ (*not* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$) with $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ and $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ with $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ for $\mathcal{F}$-prime-strips (cf. Definition 10.9 (3)), and $\underline{\underline{\mathcal{F}}}_{\underline{v}}$'s with $\underline{v} \in \underline{\mathbb{V}}$ for Θ-Hodge theatres.

Global:
$\mathcal{D}^{\circledcirc} := \mathcal{B}(\underline{C}_K)^0, \quad \mathcal{D}^{\circledcirc\pm} := \mathcal{B}(\underline{X}_K)^0,$
$\mathfrak{F}^{\Vdash}_{\mathrm{mod}} := (\mathcal{C}^{\Vdash}_{\mathrm{mod}}, \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \{\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{\rho_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v} \overset{\text{gl. to loc.}}{\xrightarrow{\sim}} \Phi^{\mathbb{R}}_{\mathcal{C}^{\vdash}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}})$
$(\ \rho_{\underline{v}} : \log^{\vdash}_{\mathrm{mod}}(p_v) \mapsto \frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]} \log_{\Phi}(p_{\underline{v}})\ ).$

Some Model Bridges, and Bridges

- (model $\mathcal{D}$-NF-bridge, Def. 10.16) $\phi^{\mathrm{NF}}_{\underline{v}} := \mathrm{Aut}_{\epsilon}(\mathcal{D}^{\circledcirc}) \circ \phi^{\mathrm{NF}}_{\bullet,\underline{v}} \circ \mathrm{Aut}(\mathcal{D}_{\underline{v}}) : \mathcal{D}_{\underline{v}} \overset{\text{poly}}{\longrightarrow} \mathcal{D}^{\circledcirc}$,
$\phi^{\mathrm{NF}}_1 := \{\phi^{\mathrm{NF}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} : \mathfrak{D}_1 \overset{\text{poly}}{\longrightarrow} \mathcal{D}^{\circledcirc}$, $\phi^{\mathrm{NF}}_j := (\text{action of } j) \circ \phi^{\mathrm{NF}}_1 : \mathfrak{D}_j \overset{\text{poly}}{\longrightarrow} \mathcal{D}^{\circledcirc}$,
$\phi^{\mathrm{NF}}_{\divideontimes} := \{\phi^{\mathrm{NF}}_j\}_{j\in\mathbb{F}^{\divideontimes}_l} : \mathfrak{D}_{\divideontimes} := \{\mathfrak{D}_j\}_{j\in\mathbb{F}^{\divideontimes}_l} \overset{\text{poly}}{\longrightarrow} \mathcal{D}^{\circledcirc}$.
- (model $\mathcal{D}$-Θ-bridge, Def. 10.17)
$$\phi^{\Theta}_{\underline{v}_j} := \mathrm{Aut}(\mathcal{D}_{>,\underline{v}}) \circ \left\{ \begin{array}{lll} \mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0 \underset{\text{labelled by } j}{\overset{\text{eval. section}}{\longrightarrow}} \mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0 & (\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}) \\ \mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0 \overset{\text{full poly}}{\xrightarrow{\sim}} \mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0 & (\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}) \end{array} \right\} \circ \mathrm{Aut}(\mathcal{D}_{\underline{v}_j})$$
$: \mathcal{D}_{\underline{v}_j} \overset{\text{poly}}{\longrightarrow} \mathcal{D}_{>,\underline{v}}, \quad \phi^{\Theta}_j := \{\phi^{\Theta}_{\underline{v}_j}\}_{\underline{v}\in\underline{\mathbb{V}}} : \mathfrak{D}_j \overset{\text{poly}}{\longrightarrow} \mathfrak{D}_>, \quad \phi^{\Theta}_{\divideontimes} := \{\phi^{\Theta}_j\}_{j\in\mathbb{F}^{\divideontimes}_l} : \mathfrak{D}_{\divideontimes} \overset{\text{poly}}{\longrightarrow} \mathfrak{D}_>.$
- (model Θ^{ell}-bridge, Def. 10.31) $\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0} := \mathrm{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm}) \circ \phi^{\Theta^{\mathrm{ell}}}_{\bullet,\underline{v}} \circ \mathrm{Aut}_+(\mathcal{D}_{\underline{v}_0}) : \mathcal{D}_{\underline{v}_0} \overset{\text{poly}}{\longrightarrow} \mathcal{D}^{\circledcirc\pm}$,
$\phi^{\Theta^{\mathrm{ell}}}_0 := \{\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0}\}_{\underline{v}\in\underline{\mathbb{V}}} : \mathfrak{D}_0 \overset{\text{poly}}{\longrightarrow} \mathcal{D}^{\circledcirc\pm}$, $\phi^{\Theta^{\mathrm{ell}}}_t := (\text{action of } t) \circ \phi^{\Theta^{\mathrm{ell}}}_0 : \mathfrak{D}_t \overset{\text{poly}}{\longrightarrow} \mathcal{D}^{\circledcirc\pm}$,
$\phi^{\Theta^{\mathrm{ell}}}_{\pm} := \{\phi^{\Theta^{\mathrm{ell}}}_t\}_{t\in\mathbb{F}_l} : \mathfrak{D}_{\pm} \overset{\text{poly}}{\longrightarrow} \mathcal{D}^{\circledcirc\pm}$.
- (model $\Theta^{\pm}$-bridge, Def. 10.30) $\phi^{\Theta^{\pm}}_{\underline{v}_t} : \mathcal{D}_{\underline{v}_t} \overset{\text{+-full poly}}{\xrightarrow{\sim}} \mathcal{D}_{\succ,\underline{v}}, \quad \phi^{\Theta^{\pm}}_t : \mathcal{D}_{\underline{v}_t} \overset{\text{+-full poly}}{\xrightarrow{\sim}} \mathcal{D}_{\succ,\underline{v}}$, $\phi^{\Theta^{\pm}}_{\pm} := \{\phi^{\Theta^{\pm}}_t\}_{t\in\mathbb{F}_l} : \mathfrak{D}_{\pm} := \{\mathfrak{D}_t\}_{t\in\mathbb{F}_l} \overset{\text{poly}}{\longrightarrow} \mathfrak{D}_{\succ}$.
- (NF-,Θ-bridge, Def. 10.24) $({}^{\ddagger}\mathfrak{F}_J \overset{{}^{\ddagger}\psi^{\mathrm{NF}}_{\divideontimes}}{\longrightarrow} {}^{\ddagger}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{\ddagger}\mathcal{F}^{\circledast}), \quad ({}^{\ddagger}\mathfrak{F}_J \overset{{}^{\ddagger}\psi^{\Theta}_{\divideontimes}}{\longrightarrow} {}^{\ddagger}\mathfrak{F}_> \dashrightarrow {}^{\ddagger}\mathcal{HT}^{\Theta})$.
- (Θ^{ell}-,$\Theta^{\pm}$-bridge, Def. 10.36) ${}^{\dagger}\psi^{\Theta^{\mathrm{ell}}}_{\pm} : {}^{\dagger}\mathfrak{F}_T \overset{\text{poly}}{\longrightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}, \quad {}^{\dagger}\psi^{\Theta^{\pm}}_{\pm} : {}^{\dagger}\mathfrak{F}_T \overset{\text{poly}}{\longrightarrow} {}^{\ddagger}\mathfrak{F}_{\succ}$.

Theatres

- (Θ-Hodge theatre, Def. 10.7) ${}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$.
- ($\mathcal{D}$-$\boxtimes$-Hodge theatre, Def. 10.18 (3)) ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes} = ({}^{\dagger}\mathcal{D}^{\circledcirc} \overset{{}^{\dagger}\phi^{\mathrm{NF}}_{\divideontimes}}{\longleftarrow} {}^{\dagger}\mathfrak{D}_J \overset{{}^{\dagger}\phi^{\Theta}_{\divideontimes}}{\longrightarrow} {}^{\dagger}\mathfrak{D}_>)$.
- ($\boxtimes$-Hodge theatre, Def. 10.24 (3)) ${}^{\ddagger}\mathcal{HT}^{\boxtimes} = ({}^{\ddagger}\mathcal{F}^{\circledast} \dashleftarrow {}^{\ddagger}\mathcal{F}^{\circledcirc} \overset{{}^{\ddagger}\psi^{\mathrm{NF}}_{\divideontimes}}{\longleftarrow} {}^{\ddagger}\mathfrak{F}_J \overset{{}^{\ddagger}\psi^{\Theta}_{\divideontimes}}{\longrightarrow} {}^{\ddagger}\mathfrak{F}_> \dashrightarrow {}^{\ddagger}\mathcal{HT}^{\Theta})$.

- ($\mathcal{D}$-$\boxplus$-Hodge theatre, Def. 10.32 (3)) ${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxplus} = ({}^\dagger\mathfrak{D}_{\succ} \overset{{}^\dagger\phi_{\pm}^{\Theta^{\pm}}}{\longleftarrow} {}^\dagger\mathfrak{D}_T \overset{{}^\dagger\phi_{\pm}^{\Theta^{\mathrm{ell}}}}{\longrightarrow} {}^\dagger\mathcal{D}^{\circledcirc\pm})$.
- ($\boxplus$-Hodge theatre, Def. 10.24 (3)) ${}^\dagger\mathcal{HT}^{\boxplus} = ({}^\dagger\mathfrak{F}_{\succ} \overset{{}^\dagger\psi_{\pm}^{\Theta^{\pm}}}{\longleftarrow} {}^\dagger\mathfrak{F}_T \overset{{}^\dagger\psi_{\pm}^{\Theta^{\mathrm{ell}}}}{\longrightarrow} {}^\dagger\mathcal{D}^{\circledcirc\pm})$.
- ($\mathcal{D}$- $\boxtimes\boxplus$-Hodge theatre, Def. 10.40 (1)) ${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} = ({}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxplus} \overset{\text{gluing}}{\dashrightarrow} {}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes})$.
- ($\boxtimes\boxplus$-Hodge theatre, Def. 10.40 (2)) ${}^\dagger\mathcal{HT}^{\boxtimes\boxplus} = ({}^\dagger\mathcal{HT}^{\boxplus} \overset{\text{gluing}}{\dashrightarrow} {}^\dagger\mathcal{HT}^{\boxtimes})$.

Properties(Proposition 10.20, Lemma 10.25, Proposition 10.34, Lemma 10.37)

- $\mathrm{Isom}({}^\dagger\phi_{\divideontimes}^{\mathrm{NF}}, {}^\ddagger\phi_{\divideontimes}^{\mathrm{NF}})$: an $\mathbb{F}_l^{\divideontimes}$-torsor.
- $\#\mathrm{Isom}({}^\dagger\phi_{\divideontimes}^{\mathrm{NF}}, {}^\ddagger\phi_{\divideontimes}^{\mathrm{NF}}) = 1$.
- $\#\mathrm{Isom}({}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}, {}^\ddagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}) = 1$.
- $\mathrm{Isom}_{\text{capsule-full poly}}({}^\dagger\mathfrak{D}_J, {}^\dagger\mathfrak{D}_{J'})^{{}^\dagger\phi_{\divideontimes}^{\mathrm{NF}}, {}^\dagger\phi_{\divideontimes}^{\Theta} \text{ form a } \mathcal{D}\text{-}\boxtimes\text{-Hodge theatre}}$: an $\mathbb{F}_l^{\divideontimes}$-torsor.
- ${}^\dagger\phi_{\divideontimes}^{\mathrm{NF}} \rightsquigarrow {}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$, up to $\mathbb{F}_l^{\divideontimes}$-indeterminacy.
- $\mathrm{Isom}({}^1\psi_{\divideontimes}^{\mathrm{NF}}, {}^2\psi_{\divideontimes}^{\mathrm{NF}}) \xrightarrow{\sim} \mathrm{Isom}({}^1\phi_{\divideontimes}^{\mathrm{NF}}, {}^2\phi_{\divideontimes}^{\mathrm{NF}})$.
- $\mathrm{Isom}({}^1\psi_{\divideontimes}^{\Theta}, {}^2\psi_{\divideontimes}^{\Theta}) \xrightarrow{\sim} \mathrm{Isom}({}^1\phi_{\divideontimes}^{\Theta}, {}^2\phi_{\divideontimes}^{\Theta})$.
- $\mathrm{Isom}({}^1\mathcal{HT}^{\Theta}, {}^2\mathcal{HT}^{\Theta}) \xrightarrow{\sim} \mathrm{Isom}({}^1\mathfrak{D}_{>}, {}^2\mathfrak{D}_{>})$.
- $\mathrm{Isom}({}^1\mathcal{HT}^{\boxtimes}, {}^2\mathcal{HT}^{\boxtimes}) \xrightarrow{\sim} \mathrm{Isom}({}^1\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}, {}^2\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes})$.
- $\mathrm{Isom}_{\text{capsule-full poly}}({}^\ddagger\mathfrak{F}_J, {}^\ddagger\mathfrak{F}_{J'})^{{}^\ddagger\psi_{\divideontimes}^{\mathrm{NF}}, {}^\ddagger\psi_{\divideontimes}^{\Theta} \text{ form a } \boxtimes\text{-Hodge theatre}}$: an $\mathbb{F}_l^{\divideontimes}$-torsor.
- $\mathrm{Isom}({}^\dagger\phi_{\pm}^{\Theta^{\pm}}, {}^\ddagger\phi_{\pm}^{\Theta^{\pm}})$: a $\{\pm1\} \times \{\pm1\}^{\underline{\mathbb{V}}}$-torsor.
- $\mathrm{Isom}({}^\dagger\phi_{\divideontimes}^{\mathrm{NF}}, {}^\ddagger\phi_{\divideontimes}^{\mathrm{NF}})$: an $\mathbb{F}_l^{\rtimes\pm}$-torsor. we have a natural isomorphism
- $\mathrm{Isom}({}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}, {}^\ddagger\mathcal{HT}^{\mathcal{D}\text{-}\boxplus})$: an $\{\pm1\}$-torsor.
- $\mathrm{Isom}_{\text{capsule-+-full poly}}({}^\dagger\mathfrak{D}_T, {}^\dagger\mathfrak{D}_{T'})^{{}^\dagger\phi_{\pm}^{\Theta^{\pm}}, {}^\dagger\phi_{\pm}^{\Theta^{\mathrm{ell}}} \text{ form a } \mathcal{D}\text{-}\boxplus\text{-Hodge theatre}}$: an $\mathbb{F}_l^{\rtimes\pm} \times \{\pm1\}^{\underline{\mathbb{V}}}$-torsor.
- ${}^\dagger\phi_{\pm}^{\Theta^{\mathrm{ell}}} \rightsquigarrow {}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$, up to $\mathbb{F}_l^{\rtimes\pm}$-indeterminacy.
- $\mathrm{Isom}({}^1\psi_{\pm}^{\Theta^{\pm}}, {}^2\psi_{\pm}^{\Theta^{\pm}}) \xrightarrow{\sim} \mathrm{Isom}({}^1\phi_{\pm}^{\Theta^{\pm}}, {}^2\phi_{\pm}^{\Theta^{\pm}})$.
- $\mathrm{Isom}({}^1\psi_{\pm}^{\Theta^{\mathrm{ell}}}, {}^2\psi_{\pm}^{\Theta^{\mathrm{ell}}}) \xrightarrow{\sim} \mathrm{Isom}({}^1\phi_{\pm}^{\Theta^{\mathrm{ell}}}, {}^2\phi_{\pm}^{\Theta^{\mathrm{ell}}})$.
- $\mathrm{Isom}({}^1\mathcal{HT}^{\boxplus}, {}^2\mathcal{HT}^{\boxplus}) \xrightarrow{\sim} \mathrm{Isom}({}^1\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}, {}^2\mathcal{HT}^{\mathcal{D}\text{-}\boxplus})$.

- $\mathrm{Isom}_{\text{capsule-}+\text{-full poly}}({}^{\ddagger}\mathfrak{F}_T, {}^{\ddagger}\mathfrak{F}_{T'})^{{}^{\ddagger}\psi_{\pm}^{\Theta^{\pm}}, {}^{\ddagger}\psi_{\pm}^{\Theta^{\mathrm{ell}}} \text{ form a } \boxplus\text{-Hodge theatre}}$: an $\mathbb{F}_l^{\rtimes\pm} \times \{\pm 1\}^{\underline{\mathbb{V}}}$-torsor.

Links

- ($\mathcal{D}$-$\boxtimes$-link, Def. 10.21) ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes} \xrightarrow{\mathcal{D}} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes}$ $({}^{\dagger}\mathfrak{D}^{\vdash}_{>} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{D}^{\vdash}_{>})$.
- ($\mathcal{D}$-$\boxplus$-link, Def. 10.35) ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus} \xrightarrow{\mathcal{D}} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxplus}$ $({}^{\dagger}\mathfrak{D}^{\vdash}_{>} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{D}^{\vdash}_{>})$.
- ($\mathcal{D}$-$\boxtimes\boxplus$-link, Cor. 11.24 (4)) ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \xrightarrow{\mathcal{D}} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus}$ $({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{D}^{\vdash}_{\Delta})$.
- (Θ-link, Def. 10.8) ${}^{\dagger}\mathcal{HT}^{\Theta} \xrightarrow{\Theta} {}^{\ddagger}\mathcal{HT}^{\Theta}$ $({}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{theta}} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$.
- ($\Theta^{\times\mu}$-link, Cor. 11.24 (3)) ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\Theta^{\times\mu}} {}^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus}$ $({}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{env}} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta})$.
- ($\Theta^{\times\mu}_{\mathrm{gau}}$-link, Cor. 11.24 (3)) ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\Theta^{\times\mu}_{\mathrm{gau}}} {}^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus}$ $({}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{gau}} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta})$.
- ($\Theta^{\times\mu}_{\mathrm{LGP}}$-link, Def. 13.9 (2)) ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\Theta^{\times\mu}_{\mathrm{LGP}}} {}^{*}\mathcal{HT}^{\boxtimes\boxplus}$ $({}^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{LGP}} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{*}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta})$.
- ($\Theta^{\times\mu}_{\mathfrak{lgp}}$-link, Def. 13.9 (2)) ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\Theta^{\times\mu}_{\mathfrak{lgp}}} {}^{*}\mathcal{HT}^{\boxtimes\boxplus}$ $({}^{(\ddagger\to)\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathfrak{lgp}} \overset{\text{full poly}}{\xrightarrow{\sim}} {}^{*}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta})$.
- ($\mathfrak{log}$-link, Def. 12.3) ${}^{\dagger}\mathcal{HT}^{\boxtimes\boxplus} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{HT}^{\boxtimes\boxplus}$
$({}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus} \overset{\sim}{\to} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\boxtimes\boxplus},\ {}^{\dagger}\mathfrak{F}_{>} \overset{\mathfrak{log}}{\to} {}^{\ddagger}\mathfrak{F}_{>},\ {}^{\dagger}\mathfrak{F}_{\succ} \overset{\mathfrak{log}}{\to} {}^{\ddagger}\mathfrak{F}_{\succ},\ \{{}^{\dagger}\mathfrak{F}_j \overset{\mathfrak{log}}{\to} {}^{\ddagger}\mathfrak{F}_j\}_{j\in J},\ \{{}^{\dagger}\mathfrak{F}_t \overset{\mathfrak{log}}{\to} {}^{\ddagger}\mathfrak{F}_t\}_{t\in T})$.

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