BOUNDS FOR THE DIMENSIONS OF p-ADIC MULTIPLE L-VALUE SPACES

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This text is a report of a talk "bounds for the dimensions of p-adic multiple L-value spaces" (23/Aug/2005) in Number Theory Summer School 2005.

For natural numbers $k_1, \ldots, k_{d-1} \geq 1$, $k_d \geq 2$, the following infinite sum

$$\zeta(k_1, \ldots, k_d) := \sum_{n_1 < \ldots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}} (= \lim_{\mathbb{C} \ni z \to 1} \mathrm{Li}_{k_1, \ldots, k_d}(z)) \in \mathbb{R}$$

absolutely converges, and is called the **multiple zeta value** (MZV). Here, $\mathrm{Li}_{k_1,\ldots,k_d}(z) := \sum_{n_1<\ldots<n_d} \frac{z^{n_d}}{n_1^{k_1}\cdots n_d^{k_d}}$ is the **multiple polylogarithm** function. The study of MZV's is started from Euler. After Zagier made the study of MZV's revive in the modern times, MZV's are studied actively by many mathematicians now.

For natural number $k_1, \ldots, k_d \geq 1$ and N-th roots of unity $\zeta_1, \ldots, \zeta_d$ satisfying $(k_d, \zeta_d) \neq (1, 1)$, the multiple L-value (MLV) is defined by the following absolutely converging infinite sum:

$$L(k_1, \ldots, k_d; \zeta_1, \ldots, \zeta_d) := \sum_{n_1 < \ldots < n_d} \frac{\zeta_1^{-n_1} \zeta_2^{n_1 - n_2} \cdots \zeta_d^{n_{d-1} - n_d}}{n_1^{k_1} \cdots n_d^{k_d}} (= \lim_{\mathbb{C} \ni z \to 1} \mathrm{Li}_{k_1, \ldots, k_d; \zeta_1, \ldots, \zeta_d}(z)) \in \mathbb{C}.$$

Here, $\mathrm{Li}_{k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d}(z) := \sum_{n_1<\ldots<n_d} \frac{\zeta_1^{-n_1}\zeta_2^{n_1-n_2}\ldots\zeta_d^{n_{d-1}-n_d} z^{n_d}}{n_1^{k_1}\cdots n_d^{k_d}}$ is the twisted multiple polylogarithm function.

Now, we want to consider a p-adic analogue of MZV's and MLV's. The infinite sum

$$\sum_{n_1 < \ldots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}$$

does not converges in the p-adic topology. On the other hand, the multiple polylogarithm function $\mathrm{Li}_{k_1,\ldots,k_d}(z)$ has the iterated integral representation:

$$\frac{d\mathrm{Li}_{k_1,\ldots,k_d}(z)}{dz} = \begin{cases} \frac{1}{z} \mathrm{Li}_{k_1,\ldots,k_d - 1}(z) & \text{if } k_d > 1, \\ \frac{1}{1-z} \mathrm{Li}_{k_1,\ldots,k_{d-1}}(z) & \text{if } k_d = 1, \text{and } d > 1, \\ \frac{1}{1-z} & \text{if } k_d = 1, \text{and } d = 1. \end{cases}$$

Considering a p-adic analogue of this iterated integral representation, Furusho defined the **p-adic multiple polylogarithm** functions $\mathrm{Li}^a_{k_1,\ldots,k_d}(z)$ by using Coleman's p-adic iterated integral theory ([C]) (Here, a is a branching parameter. We do not explain it in this article), defined the **p-adic multiple zeta values** (p-adic MZV's) to be the limit values of the p-adic multiple polylogarithm functions (cf. [Fu1]):

$$\zeta_p(k_1, \ldots, k_d) := \lim_{\mathbb{C}_p \ni z \to 1}{}' \mathrm{Li}_{k_1,\ldots,k_d}(z) \in \mathbb{Q}_p,$$

and studied their properties and relations (cf. [Fu1][Fu2]). Here, $\mathbb{C}_p$ is the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. We do not explain the meaning of $\lim'$ in this article.

Example 1.1. (Coleman) For $n > 1$, we have

$$\zeta_p(n) = \frac{p^n}{p^n - 1} L_p(n, \omega^{1-n}).$$

Here, L_p is the p-adic L-function of Kubota-Leopoldt, and ω is the Teichmuüller character. In particular, $\zeta_p(2n) = 0$ for $n \geq 1$. We get the p-adic L-function of Kubota-Leopoldt by p-adically interpolating values at **negative** integer. Note that this proof of $\zeta_p(2n) = 0$ is somewhat indirect, since the above formula is a comparison between the p-adic polylogarithms and the p-adic L-function at **positive** integer. (Furusho also shows $\zeta_p(2n) = 0$ from 2-,3-cycle relations. This comes from the fact that "the angles of the triangle in the 3-cycle relation are 0" in the p-adic world. We also say "π^2 is 0 in the p-adic world" from the fact $\zeta_p(2n) = 0$.)

On the other hand, the values $\zeta_p(2n+1)$ are difficult. For $n \geq 1$, we have the following equivalences: $\zeta_p(2n+1) \neq 0 \Leftrightarrow L_p(2n+1, \omega^{-2n}) \neq 0 \Leftrightarrow H^2(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(-n)) = 0$ (higher Leopoldt conjecture). This holds in the case where p is a regular prime or the case where $p - 1$ devides n. However, it is not known whether this holds or not in general. $\square$

Analogously, we can define twisted p-adic multiple polylogarithms $\mathrm{Li}^a_{k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d}(z)$ and p-adic multiple L-values

$$L_p(k_1, \ldots, k_d; \zeta_1, \ldots, \zeta_d) := \lim_{\mathbb{C}_p \ni z \to 1}{}' \mathrm{Li}^a_{k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d}(z) \in \mathbb{Q}_p(\mu_N)$$

for $p \nmid N$ (cf. [Y]). For $w > 0$, we define $Z^p_w[N] \subset \mathbb{Q}_p$ to be the following:

$$Z^p_w[N] := \left\langle L_p(k_1, \ldots, k_d; \zeta_1, \ldots, \zeta_d) \,\middle|\, \begin{array}{l} d \geq 1, k_i \geq 1, \zeta_i \in \mu_N \text{ for } i = 1, \ldots, d, \\ k_1 + \cdots + k_d = w, (k_d, \zeta_d) \neq (1, 1) \end{array} \right\rangle_{\mathbb{Q}},$$

(the $\mathbb{Q}$-vector space generated by $L_p(k_1, \ldots, k_d; \zeta_1, \ldots, \zeta_d)$'s), and $Z^p_0[N] := \mathbb{Q}$. Put $Z^p_\bullet[N] := \oplus_w Z^p_w[N]$ (formal direct sum), $Z^p_w := Z^p_w[1]$, and $Z^p_\bullet := Z^p_\bullet[1]$. We call Z^p_w (resp. $Z^p_w[N]$) **the space of p-adic multiple zeta values of weight w** (resp. **the space of p-adic multiple L-values of weight w**).

It is known that there are many relations among p-adic MZV's and p-adic MLV's as the usual MZV's and MLV's (cf. [Fu1][Fu2][Y]). For the relations of p-adic MZV's and p-adic MLV's, we have some conjectures, which are analogous to the complex case.

Conjecture 1.2. *(Furusho) All linear relations among p-adic MZV's are derived from* 2-,3-,5-*cycle relations.*

We do not explain 2-,3-,5-cycle relations in this article.

Conjecture 1.3. *(isobar conjecture, Furusho) All linear relations among p-adic MZV's are linear combinations of relations among p-adic MZV's of the same weights. In particular, the formal direct sum* $Z^p_\bullet := \oplus_w Z^p_w$ *has the natural embedding into* $\mathbb{Q}_p$.

For the p-adic MLV's, we also conjecture that all linear relations among p-adic MZV's are linear combinations of relations among p-adic MZV's of the same weights ([Y]).

The main result of the talk at Number Theory Summer School 2005 concerns the dimensions of the space of p-adic MLV's. First, we review the complex case. Zagier conjectures the dimensions of the space of MZV's as follows:

Conjecture 1.4. *(dimension conjecture, Zagier) We define a sequence $\{D_n\}_n$ to be $D_0 = 1$, $D_1 = 0$, $D_2 = 1$, $D_{n+3} = D_{n+1} + D_n$ $(n \geq 0)$. (The generating function is $\sum_{n=0}^{\infty} D_n t^n = 1/(1-t^2-t^3)$.) Then, we have $\dim_{\mathbb{Q}} Z_w = D_w$ for $w \geq 0$. Here, using MZV's and MLV's, we define Z_w, $Z_w[N]$ by the same way of Z_w^p, $Z_w^p[N]$.*

Theorem 1.5. *(Goncharov, Terasoma, Deligne-Goncharov* [G1][T][DG]*) For $w \geq 0$, we have $\dim_{\mathbb{Q}} Z_w \leq D_w$.*

This theorem says that there are enormous relations among MZV's. The opposite inequality seems to be a trancedental number theoric problem, and that we cannot prove it by the present algebraic geometical methods. For MLV's, we have the following.

Theorem 1.6. *(Deligne-Goncharov*[DG]*) For $N = 2$ (resp. $N > 2$), we define a sequence $\{D_n[N]\}_n$ by a generating function $\sum_{n=0}^{\infty} D_n[2]t^n = 1/(1-t-t^2)$ (resp. $\sum_{n=0}^{\infty} D_n[N]t^n = 1/(1-(\frac{\varphi(N)}{2}+\nu)t+(\nu-1)t^2))$. Here, φ is the Euler function, and ν is the number of prime numbers dividing N. Then, we have $\dim_{\mathbb{Q}} Z_w[N] \leq D_w[N]$ for $w \geq 0$ and $N \geq 1$.*

Remark . For $N > 4$, it is known that the equality does not hold in general (Goncharov[G2]). The gap is related to the space of cusp forms for $\Gamma_1(N)$ of weight 2 when N is a prime number (loc. cit.).

Now, we return to the p-adic case. The following is the p-adic analogoue of Zagier's conjecture.

Conjecture 1.7. *(dimension conjecture, Furusho-Y.) We define a sequence $\{d_n\}_n$ to be $d_0 = 1$, $d_1 = 0$, $d_2 = 0$, $d_{n+3} = d_{n+1} + d_n$ $(n \geq 0)$. (The generating function is $\sum_{n=0}^{\infty} d_n t^n = (1-t^2)/(1-t^2-t^3)$.) Then, we have $\dim_{\mathbb{Q}} Z_w^p = d_w$ for $w \geq 0$.*

The main result is the following:

Theorem 1.8. *(Y.*[Y]*) For $N = 2$ (resp. $N > 2$), we define a sequence $\{d_n[N]\}_n$ by a generating function $\sum_{n=0}^{\infty} d_n[2]t^n = (1-t^2)/(1-t-t^2)$ (resp. $\sum_{n=0}^{\infty} d_n[N]t^n = (1-t)/(1-(\frac{\varphi(N)}{2}+\nu)t+(\nu-1)t^2))$. Here, φ is the Euler function, and ν is the number of prime numbers dividing N. Then, we have $\dim_{\mathbb{Q}} Z_w^p[N] \leq d_w[N]$ for $w \geq 0$ and $N \geq 1$.*

This theorem also says that there are enormous relations among p-adic MLV's. The opposite inequality seems to be a p-adic trancedental number theoric problem, and that we cannot prove it by the present algebraic geometical methods.

Remark . It is **not** known that $\dim_{\mathbb{Q}} Z_w^p[N]$ does not depend on p. It seems to be a difficult problem (cf. the higher Leopoldt conjecture in Example 1.1).

Remark . For $N > 4$, it is known that the equality does not hold in general by the same reason. The gap is related to the space of cusp forms for $\Gamma_1(N)$ of weight 2 when N is a prime number.

These sequences have K-theoric meanings, and we prove the upper bounds by relating the K-theory. For example, we have

$$\frac{1}{1-t^2-t^3} = \frac{1}{1-t^2}\frac{1}{1-\frac{t^3}{1-t^2}} = \frac{1}{1-t^2}\frac{1}{1-(t^3+t^5+t^7+\cdots)}.$$

The term $1/(1-t^2)$ corresponds to π^2 in the weight 2, and $t^3+t^5+t^7+\cdots$ corresponds to

$$\operatorname{rank} K_{2n-1}(\mathbb{Z}) = \begin{cases} 0 & \text{for } n\text{: even or } n=1, \\ 1 & \text{for } n\text{: odd and } n \neq 1. \end{cases}$$

In the p-adic case, the generating function $(1-t^2)/(1-t^2-t^3)$ loses the factor $1/(1-t^2)$. It corresponds to the fact that "$\pi^2 = 0$ in the p-adic world". The difference between the complex case and p-adic case of the generating functions is $1/(1-t)$, not $1/(1-t^2)$ for $N > 2$. It corresponds to the fact that in the complex case, we have $-\log(1-\zeta) + \log(1-\zeta^{-1}) = -\log(-\zeta) = (\text{rational number}) \cdot \pi$ in the weight 1, and that it vanishes in the p-adic case, since "$\pi = 0$ in the p-adic world".

The ingredients of the main theorem is Deligne-Goncharov's category of mixed Tate motives over $\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}]$ ([DG]), Deligne-Goncharov's motivic pro-unipotent fundamental groupoids of $\mathbb{U}_N := \mathbb{P}^1 - \{0,\infty\} \cup \mu_N$ ([DG]), and Tannakian interpretations ([Fu2]) using Besser's Frobenius invariant path ([B]).

We briefly explain the proof of the main theorem. We construct an element φ_p deeply related to the p-adic MLV's in the $\mathbb{Q}_p(\mu_N)$-valued point of a pro-unipotent group U_ω deeply related to the K-theory. (Roughly speaking, φ_p is an element representing "the difference between de Rham and rigid".) By $\varphi_p \in U_\omega(\mathbb{Q}_p(\mu_N))$, φ_p satisfies the defining equations of U_ω. (The author does not know the concrete defining equations). The scheme U_ω is "small enough" by the relation to K-theory. Thus, we have enormous relations among p-adic MLV's from the fact that φ_p satisfies the defining equations. From this, we get the upper bounds. This proof is the p-adic analogue of Deligne-Goncharov's proof in the complex case. They construct an element a_σ^0 deeply related to MLV's in the $\mathbb{C}$-valued point of a pro-unipotent group U_ω (the same one in the above) deeply related to the K-theory. (Roughly speaking, a_σ^0 is an element representing " the diference between de Rham and Betti".) They prove the upper bounds of the space of MLV's from this $a_\sigma^0 \in U_\omega(\mathbb{C})$.

For this element a_σ^0, we have the following conjecture:

Conjecture 1.9. *(Grothendieck,* [DG]*) The element $a_\sigma^0 \in U_\omega(\mathbb{C})$ is $\mathbb{Q}$-Zariski dense.*

In the case where $N = 1$, this conjecture $(+\alpha)$ induces Zagier's dimension conjecture and the isobar conjecture (cf. isobar conjecture 1.3 in the p-adic case) that all linear relations among MZV's are linear combinations of relations among MZV's of the same weights. In the p-adic case, we have the following conjecture:

Conjecture 1.10. *(Y.,* [Y]*) The element $\varphi_p \in U_\omega(\mathbb{Q}(\mu_N))$ is $\mathbb{Q}$-Zariski dense.*

In the case where $N = 1$, this conjectrue $(+\alpha)$ induces the dimension conjecture 1.7 in the p-adic case and the isobar conjecture 1.3 in the p-adic case.

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