p-ADIC ÉTALE COHOMOLOGY AND CRYSTALLINE COHOMOLOGY FOR OPEN VARIETIES

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Contents

This text is a report of a talk "p-adic étale cohomology and crystalline cohomology for open varieties" in the symposium "Hodge Theory and Algebraic Geometry" (7-11/Oct/2002 at Hokkaido University). It seemed that more algebraic geometers rather than arithmeticians participated in this symposium. Thus, in that talk, the auther began with an introduction to the p-adic Hodge theory in the view of the theory of p-adic representations. However, in this report, we do not treat the theory of p-adic representations, and we treat only a review of the main theorems and the main results.

The aim of the talk was, roughly speaking, "to extend the main theorems of p-adic Hodge theory for open varieties" by the method of Fontaine-Messing-Kato-Tsuji (see [FM],[Ka2], and [Tsu1]). Here, the main theorems of p-adic Hodge theory are: the Hodge-Tate conjecture (C_{HT} for short), the de Rham conjecture (C_{dR}), the crystalline conjecture (C_{crys}), the semi-stabele conjecture (C_{st}), and the potentially semi-stable conjecture (C_{pst}). The theorems C_{dR}, C_{crys}, and C_{st} are called the "comparison theorems".

The section 1 is an introduction to the p-adic Hodge theory. In the section 2, we review the main theorems of the p-adic Hodge theory. In the section 3, we state the main results. In this report, we only announce the main results.

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Notations

Let K be a complete discrete valuation field of characteristic 0, k the residue field of K, perfect, characteristic $p > 0$, and O_K the valuation ring of K. Denote $\overline{K}$ be the algebraic closure of K, $\overline{k}$ the algebraic closure of k, G_K the absolute Galois group of K, and $\mathbb{C}_p$ the p-adic completion of $\overline{K}$. (Note that it is an abuse of the notation. If $[K : \mathbb{Q}_p] < \infty$,

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it coincide the usual notations.) Let W be the ring of Witt vectors with coefficient in k, and K_0 the fractional field of W. It is the maximum absolutely unramified (i.e., p is a uniformizer in K_0) subfield of K. Let P_0 be the fractional field of the ring of Witt vectors with coefficient of $\overline{k}$, and σ the Frobenius endmorphism on W, K_0, $W(\overline{k})$, and P_0, indeced by the absolute Frobenius on k, and $\overline{k}$. The word "log-structure" means Fontaine-Illusie-Kato's log-structure (see [Ka1]). We do not review the notion of log-structure in this report.

1. Introduction

The p-adic Hodge theory is called to be a p-adic analogue of the Hodge theory over $\mathbb{C}$. This means that for p-adic étale cohomologies of varieties over p-adic field, there exist similar decompositions, which is called the Hodge-Tate decomopositon to the Hodge decompositions for singular cohomologies of varieties over $\mathbb{C}$. However, we can say that the p-adic Hodge theory is not just a p-adic analogue of the Hodge decomposition.

The p-adic Hodge theory has no logical relations with the Hodge theory over $\mathbb{C}$. One can learn the p-adic Hodge theory without knowing the Hodge theory over $\mathbb{C}$, however, we begin with comparing with the Hodge theory over $\mathbb{C}$ to see the conceptual relation.

The following theorem is classical.

Theorem 1.1 (Hodge decomposition (Kodaira-Hodge)). *For compact Kähler manifold X, there exists a canonical isomorphism* :

$$\mathbb{C} \otimes_{\mathbb{Q}} H^m_{\mathrm{sing}}(X, \mathbb{Q}) \cong H^m_{\mathrm{dR}}(X/\mathbb{C}) \cong H^m(X, \mathcal{O}_X) \oplus H^{m-1}(X, \Omega^1_{X/\mathbb{C}}) \oplus \cdots \oplus H^0(X, \Omega^m_{X/\mathbb{C}})$$

(H^m_{sing} means singular cohomology.)

On the other hand, one of the conclusion of p-adic Hodge theory is : There exists the following decomposition, which is called Hodge-Tate decomposition for a p-adic étale cohomology of a variety X, which is proper smooth over K. Former, it was called the Hodge-Tate conjecture, C_{HT} for short.

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong$$
$$\mathbb{C}_p \otimes_K H^m(X, \mathcal{O}_X) \oplus \mathbb{C}_p(-1) \otimes_K H^{m-1}(X, \Omega^1_{X/K}) \oplus \cdots \oplus \mathbb{C}_p(-m) \otimes_K H^0(X, \Omega^m_{X/K}).$$

Moreover, this is compatible with the action of the Galois group G_K. $\mathbb{C}_p(-i)$ means the $(-i)$-th Tate twist of $\mathbb{C}_p$.

In the classical Hodge theory, we relate topological cohomologies, that is, singular cohomologies with analytic cohomologies, that is, de Rham cohomologies (it is only the holomorphic Poincaré lemma, not "Hodge theory"). The singular cohomology has $\mathbb{Q}$-structure (or $\mathbb{Z}$-structure), and the de Rham cohomology has Hodge filtration, which comes from Hodge decomposition (this is the "Hodge theory").

For example, we can not distinguish elliptic curves by using only singular cohomologies (they are topologically homeomorphic), and by using only de Rham cohomologies ($\mathrm{Fil}^0 =$ whole space, $\mathrm{Fil}^1 =$ 1-dimensional, $\mathrm{Fil}^2 = 0$). However, we can recover an elliptic curve by using the both cohomologies:

$$0 \to H_1(E, \mathbb{Z}) \to \mathrm{Lie}(E) \to E(\mathbb{C}) \to 0,$$
$$0 \to \mathrm{coLie}(E^*) \to H^1_{\mathrm{dR}}(X/\mathbb{C})^* \to \mathrm{Lie}(E) \to 0.$$

By such a way, we can get deeper information from a comparison isomorphism of two cohomology theories and additional structures of cohomology theories.

The p-adic Hodge theory treats varieties over p-adic field, and it compares topological cohomologies, that is, étale cohomologies and analytic cohomologies, that is, de Rham cohomologies and (log-)crystalline cohomologies. By using p-adic Hodge theory, which relates étale cohomologies with differential forms, we can formulate a conjecture (Tamagawa number conjecture of Bloch-Kato), which precisely predicts special values of Hasse-Weil L-functions of varieties (or, motives). That conjecture is not the theme of this report, thus we do not further mention that.

The p-adic Hodge theory compares cohomology theories with additional structures, that is, Galois actions, Hodge filtrations, Frobenius endmorphisms, Monodoromy operators:

(1) the Hodge theory over $\mathbb{C}$
- singular cohomology $H^m_{\mathrm{sing}}(X, \mathbb{Q})$ —topological:
  $\mathbb{Q}$-vector space (+$\mathbb{Z}$-structure)
- de Rham cohomology $H^m_{\mathrm{dR}}(X/\mathbb{C})$ —analytic:
  $\mathbb{C}$-vector space +Hodge filtration

(2) the p-adic Hodge theory
- étale cohomology $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ —topological:
  $\mathbb{Q}_p$-vector space +Galois action
- (algebraic) de Rham cohomology $H^m_{\mathrm{dR}}(X_K/K)$ —analytic:
  K-vector space +Hodge filtration
- (log-)crystalline cohomology $K_0 \otimes_W H^m_{\mathrm{crys}}(Y/W)$ —analytic:
  K_0-vector space +Frobenius endmorphism (+ Monodromy operator).

For arithmetic geometers, the singular cohomology is called the Betti cohomology. In the proof of the comparison theorems, we use the "syntomic cohomology". This is a vector space endowed with the Galois action. However, being different from the étale cohomology it is an analytic cohomology defined by differential forms. It is the theoritical heart of the p-adic Hodge theory by the method of Fontaine-Messing-Kato-Tsuji that the syntomic cohomology is isomorphic to the étale cohomology compatible with Galois action.

2. The main theorems of p-adic Hodge theory

In this section, we state the main theorems of p-adic Hodge theory: C_{HT}, C_{dR}, C_{crys}, C_{st}, and C_{pst}. Roughly spealing, we can state the main theorems as the following way:

- the Hodge-Tate conjecture (C_{HT}):
  There exists a Hodge-Tate decomposition on the p-adic étale cohomology.
- the de Rham conjecture (C_{dR}):
  There exists a comparison isomorphism between the p-adic étale cohomology and the de Rham cohomology.
- the crystalline conjecture (C_{crys}):
  In the good reduction case, we have stronger result than C_{dR}, that is, there exists a comparison isomorphism between the p-adic étale cohomology and the crystalline cohomology.
- the semi-stable conjecture (C_{st}):
  In the semi-stable reduction case, we have stronger result than C_{dR}, that is, there

exists a comparison isomorphism between the p-adic étale cohomology and the log-crystalline cohomology.

- the potentially semi-stable conjecture (C_{pst}):
  The p-adic étale cohomology has "only a finite monodromy".

We will state precisely in the following.

In the p-adic Hodge theory, we use Fontaine's p-adic period rings: B_{dR}, B_{crys}, B_{st} (see [Fo]). In the Hodge theory over $\mathbb{C}$, we can compare the singular cohomology and de Rham cohomology after tensoring $\mathbb{C}$. On the other hand, in the p-adic Hodge theory, we can compare the étale cohomology and the de Rham cohomoloy after tensoring B_{dR}. We pick up the fundamental properties of them.

(1) B_{dR} : a complete discrete valuation field over K with residue field $\mathbb{C}_p$. It contains $\overline{K}$. (It does not contain $\mathbb{C}_p$.) The Galois group G_K acts on B_{dR}. It has the filtration by its valuation, and its graded quotient $\mathrm{gr}^i B_{\mathrm{dR}}$ is $\mathbb{C}_p(i)$. And, $\mathbb{Q}_p(1) \subset \mathrm{Fil}^1 B_{\mathrm{dR}}$.

$$B_{\mathrm{dR}}^{G_K} = K.$$

(2) B_{crys} : It is an algebra over K_0, and G_K-stable subring of B_{dR}. It contains P_0.(It does not contain $\overline{K}$.) $K \otimes_{K_0} B_{\mathrm{crys}} \to B_{\mathrm{dR}}$ is also injective. For the filtration, which comes from B_{dR}, we get $\mathrm{gr}^i B_{\mathrm{crys}} = \mathbb{C}_p(i)$. And, $\mathbb{Q}_p(1) \subset \mathrm{Fil}^1 B_{\mathrm{dR}} \cap B_{\mathrm{crys}}$. There exists a σ-semi-linear injective endomorphism φ, which commutes with the action of G_K. (Frobenius endmorphism)

$$B_{\mathrm{crys}}^{G_K} = K_0,\ \mathrm{Fil}^0 B_{\mathrm{dR}} \cap B_{\mathrm{crys}}^{\varphi=1} = \mathbb{Q}_p.$$

(3) B_{st} : It is an algebra over K_0, and has G_K-action. It contains B_{crys}. It contains P_0.(It does not contain $\overline{K}$.) After fixing an uniformizer π of K, we can regard it as a subring of B_{dR}. $K \otimes_{K_0} B_{\mathrm{st}} \to B_{\mathrm{dR}}$ is also injective. The Frobenius endmorphism on B_{crys} is extended to B_{st}. The ring B_{st} has a B_{crys}-derivation $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$, which commutes with the G_K-action and satisfies $N\varphi = p\varphi N$.

$$B_{\mathrm{st}}^{G_K} = K_0,\ B_{\mathrm{st}}^{N=0} = B_{\mathrm{crys}},\ \mathrm{Fil}^0 B_{\mathrm{dR}} \cap B_{\mathrm{st}}^{\varphi=1,N=0} = \mathbb{Q}_p.$$

The following theorems were formulated by Tate, Fontaine, Jannsen, proved by Tate, Faltings, Fontaine-Messing, Kato under various assumptions, and proved by Tsuji under no assumptions (1999 [Tsu1]). Later, Faltings and Niziol got alternative proofs (see [Fa],[Ni]).

Theorem 2.1 (the Hodge-Tate conjecture (C_{HT}))**.** *Let X_K be a proper smooth variety over K. Then, there exists the following canonical isomorphism, which is compatible with the Galois action.*

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \bigoplus_{0 \le i \le m} \mathbb{C}_p(-i) \otimes_K H^{m-i}(X_K, \Omega^i_{X_K/K}).$$

Here, G_K acts by $g \otimes g$ on LHS, by $g \otimes 1$ on RHS.

remark . This is an analogue of the Hodge decomopositon. In this isomorphism, the following fact is remarkable: In general, it is very difficult to know the action of Galois group on the étale cohomology. However, afer tensoring $\mathbb{C}_p$, the Galois action is very easy:

$$\bigoplus_{0 \le i \le m} \mathbb{C}_p(-i)^{\oplus h^{i,m-i}}$$

$(h^{i,m-i} := \dim_K H^{m-i}(X, \Omega^i_{X/K}).)$

Theorem 2.2 (the de Rham conjecture (C_{dR}))**.** *Let X_K be a proper smooth variety over K. Then, there exists the following canonical isomorphism, which is compatible with the Galois action and filtrations.*

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K).$$

Here, G_K acts by $g \otimes g$ on LHS, by $g \otimes 1$ on RHS. We endow filtrations by $\mathrm{Fil}^i \otimes H^m_{\text{ét}}$ on LHS, by $\mathrm{Fil}^i = \Sigma_{i=j+k}\mathrm{Fil}^j \otimes \mathrm{Fil}^k$ on RHS.

remark . By takin graded quotient, we get $C_{\mathrm{dR}} \Rightarrow C_{\mathrm{HT}}$.

Theorem 2.3 (the crystalline conjecture (C_{crys}))**.** *Let X_K be a proper smooth variety over K, X be a proper smooth model of X_K over O_K. Y be the special fiber of X.*

Then, there exists the following canonical isomorphism, which is compatible with the Galois action, and Frobenius endmorphism.

$$B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{crys}} \otimes_W H^m_{\mathrm{crys}}(Y/W)$$

Moreover, after tensoring B_{dR} over B_{crys}, and using the Berthelo-Ogus isomorphism (see [Be]*):*

$$K \otimes_W H^m_{\mathrm{crys}}(Y/W) \cong H^m_{\mathrm{dR}}(X_K/K),$$

we get an isomorphism:

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K),$$

which is compatible with filtrations. Here, G_K acts by $g \otimes g$ on LHS, by $g \otimes 1$ on RHS, Frobenius endmorphism acts by $\varphi \otimes \varphi$ on LHS, by $\varphi \otimes 1$ on RHS. We endow filtrations by $\mathrm{Fil}^i \otimes H^m_{\text{ét}}$ on LHS, by $\mathrm{Fil}^i = \Sigma_{i=j+k}\mathrm{Fil}^j \otimes \mathrm{Fil}^k$ on RHS.

remark . By taking the Galois invariant part of the comparison isomorphism:

$$B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{crys}} \otimes_W H^m_{\mathrm{crys}}(Y/W),$$

we get:

$$(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))^{G_K} \cong K_0 \otimes_W H^m_{\mathrm{crys}}(Y/W).$$

By taking $\mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_{B_{\mathrm{crys}}} \bullet) \cap (\bullet)^{\varphi=1}$ of the comparison isomorphism, we get:

$$H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)) \cap (B_{\mathrm{crys}} \otimes_W H^m_{\mathrm{crys}}(Y/W))^{\varphi=1}.$$

We can, that is, recover the crystalline cohomology & de Rham cohomology from the étale cohomology and vice versa with all additional strucuture. (Grothendieck's mysterious functor.)

Theorem 2.4 (the semi-stable conjecture (C_{st}))**.** *Let X_K be a proper smooth variety over K, X be a proper semi-stable model of X_K over O_K. (i.e., X is regular and proper flat over O_K, its general fiber is X_K and its special fiber is normal crossing divisor.) Let Y be the special fiber of X, and M_Y be a natural log-structure on Y.*

Then, there exists the following canonical isomorphism, which is compatible with the Galois action, and Frobenius endmorphism, monodromy operator.

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{st}} \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times))$$

Moreover, after tensoring B_{dR} *over* B_{st}*, and using the Hyodo-Kato isomorphism (see* [H-Ka]*) (it depens on the choice of the uniformizer pi of* K*):*

$$K \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times)) \cong H^m_{\mathrm{dR}}(X_K/K)$$

we get an isomorphism:

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)$$

which is compatible with filtrations. Here, G_K *acts by* $g \otimes g$ *on LHS, by* $g \otimes 1$ *on RHS, Frobenius endmorphism acts by* $\varphi \otimes \varphi$ *on LHS, by* $\varphi \otimes 1$ *on RHS, monodromy operator acts by* $N \otimes 1$ *on LHS, by* $N \otimes 1 + 1 \otimes N$ *on RHS. We endow filtrations by* $\mathrm{Fil}^i \otimes H^m_{\text{ét}}$ *on LHS, by* $\mathrm{Fil}^i = \Sigma_{i=j+k} \mathrm{Fil}^j \otimes \mathrm{Fil}^k$ *on RHS.*

remark . By taking the Galois invariant part of the comparison isomorphism:

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{st}} \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times))$$

we get:

$$(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))^{G_K} \cong K_0 \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times))$$

By taking $\mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_{B_{\mathrm{st}}} \bullet) \cap (\bullet)^{\varphi=1, N=0}$ of the comparison isomorphism, we get:

$$H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)) \cap (B_{\mathrm{st}} \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times)))^{\varphi=1, N=0}$$

We can, that is, recover the log-crystalline cohomology & de Rham cohomology from the étale cohomology and vice versa with all additional strucuture. (Grothendieck's mysterious functor.)

remark . From $B_{\mathrm{st}}^{N=0} = B_{\mathrm{crys}}$, we get $C_{\mathrm{st}} \Rightarrow C_{\mathrm{crys}}$.

remark . By using de Jong's alteration(see [dJ]), we get $C_{\mathrm{st}} \Rightarrow C_{\mathrm{dR}}$. We need a slight argument to showing that it is compatible not only with the action of $\mathrm{Gal}(\overline{K}/L)$ for a suitable finite extention L of K, but also with the aciton of G_K. (see [Tsu4])

In the following theorem, we do not review the definition of the potentially semi-stable representation.

Theorem 2.5 (the potentially semi-stable conjecture (C_{pst}))**.** *Let* X_K *be a proper variety over* K*. Then, the p-adic étale cohomology* $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ *is a potentially semi-stable representation of* G_K*.*

remark . By using de Jong's alteration (see [dJ]) and truncated simplicial schemes, we get $C_{\mathrm{st}} \Rightarrow C_{\mathrm{pst}}$. (see [Tsu3])

The logical dependence is the following:

$$C_{\mathrm{pst}} \Leftarrow C_{\mathrm{st}} \Rightarrow C_{\mathrm{crys}},\ C_{\mathrm{st}} \Rightarrow C_{\mathrm{dR}} \Rightarrow C_{\mathrm{HT}}.$$

$C_{\mathrm{st}} \Rightarrow C_{\mathrm{crys}}$ and $C_{\mathrm{dR}} \Rightarrow C_{\mathrm{HT}}$ are trivial. For $C_{\mathrm{st}} \Rightarrow C_{\mathrm{dR}}$, we use de Jong's alteration. For $C_{\mathrm{st}} \Rightarrow C_{\mathrm{pst}}$, we use de Jong's alteration and truncated simplicial scheme. i.e., C_{st} is the deepest theorem.

3. The main results

In this section, we state the main results without proof (see [Y]). In this report, we do not mention weight filtrations and functoriarities.

We call C_{HT}(resp. C_{dR}, C_{crys}, C_{st}, C_{pst}) in the previous section proper smooth C_{HT}(resp. proper smooth C_{dR}, proper C_{crys}, proper C_{st}, proper C_{pst}). Roughly speaking, we remove conditions of the main theorems in the following way.

(1) proper smooth $C_{\mathrm{HT}} \rightsquigarrow$ open non-smooth C_{HT}
X_K is separated of finite type over K.
Or, "open" smooth C_{HT}
X_K can be compactified into a proper smooth variety over K, such that its complement is a normal crossing divisor.

(2) proper smooth $C_{\mathrm{dR}} \rightsquigarrow$ open non-smooth C_{dR}
X_K is separated of finite type over K.
Or, "open" smooth C_{dR}
X_K can be compactified into a proper smooth variety over K, such that its complement is a normal crossing divisor.

(3) proper $C_{\mathrm{crys}} \rightsquigarrow$ "open" C_{crys}
X can be compactified into a proper smooth variety over O_K, such that its complement is a horizontal normal crossing divisor, which is also normal crossing to the special fiber.

(4) proper $C_{\mathrm{st}} \rightsquigarrow$ "open" C_{st}
X can be compactified into a proper semi-stable family over O_K, such that its complement is a horizontal normal crossing divisor, which is also normal crossing to the special fiber.

(5) proper $C_{\mathrm{pst}} \rightsquigarrow$ open non-smooth C_{pst}
X_K is separated of finite type over K.

In the above, the word open means arbitrary open, on the other hand, the word "open" means "proper minus normal crossing divisor".

We consider cohomologies with proper support H_c^m and cohomologies without proper support H^m. Moreover, we can consider "partially proper support cohomologies" in "open" smooth cases: If we decompose the normal crossing divisor D into $D = D^1 \cup D^2$, "partially proper support cohomologies" are cohomologies with support only on D^i $(i = 1, 2)$. We denote them H_i^m $(i = 1, 2)$, that is,

$$H^m_{\text{ét},1}((X \setminus D)_{\overline{K}}, \mathbb{Q}_p) := H^m_{\text{ét}}(X_{\overline{K}}, Rj_{2*}j_{1!}\mathbb{Q}_p),$$

$$H^m_{\text{ét},1}((X \setminus D)_{\overline{K}}, \mathbb{Q}_p) := H^m_{\text{ét}}(X_{\overline{K}}, Rk_{1*}k_{2!}\mathbb{Q}_p).$$

(Here, $j_1 : (X \setminus D)_{\overline{K}} \hookrightarrow (X \setminus D^2)_{\overline{K}}$, $j_2 : (X \setminus D^2)_{\overline{K}} \hookrightarrow X_{\overline{K}}$, $k_1 : (X \setminus D^1)_{\overline{K}} \hookrightarrow X_{\overline{K}}$, and $k_2 : (X \setminus D)_{\overline{K}} \hookrightarrow (X \setminus D^1)_{\overline{K}}$.)

$$H^m_{\mathrm{dR},i}((X \setminus D)_K/K) := H^m(X_K, I(D^i)\Omega_{X_K/K}(\log D)).$$

$$H^m_{\text{log-crys},i}(Y \setminus C) := K_0 \otimes_W H^m_{\text{log-crys},i}((Y, M_Y)/(W, \mathcal{O}^\times), K(C^i)\mathcal{O}_{(Y,M_Y)/(W,\mathcal{O}^\times)}).$$

Here, Y(resp. C, C^i) are the special fiber of X(resp. D, D^i), and $I(D^i)$(resp. $K(D^i)$) are the ideal sheaf of $\mathcal{O}_X$(resp. $\mathcal{O}_{(Y,M_Y)/(W,\mathcal{O}^\times)}$) defined by D^i(resp. C^i) (see [Tsu2]). They are called the "minus log".

When we consider algebraic correspondences on open varieties, we need to consider partially proper support cohomologies. Thus, in a sense, when we consider not only a comparison between varieties but also a comparison of Hom, we have to consider partially proper support cohomologies. In this way, it is impotant to show comparison isomorphisms for partially proper support cohomologies.

We state the main result.

First, we prove a extended version of Hyodo-Kato isomorphism:

Proposition 3.1. *Let X be a proper semi-stable model over O_K, D be a horizontal normal crossing divisor of X, which is also normal crossing to the special fiber. We decompose D into $D = D^1 \cup D^2$. Put Y (resp. C) to be the special fiber of X (resp. D). Fix a uniformizer pi of K. Then, we have the following isomorphism:*

$$K \otimes_{K_0} H^m_{\text{log-crys},i}(Y \setminus C) \cong H^m_{\mathrm{dR},i}((X \setminus D)_K/K).$$

Thus, the pair

$$(H^m_{\text{log-crys},i}(Y \setminus C), H^m_{\mathrm{dR},i}((X \setminus D)_K/K))$$

has a filtered (φ, N)-module structure.

The main result is the following:

Theorem 3.2 ("open" C_{st})**.** *Let X be a proper semi-stable model over O_K, D be a horizontal normal crossing divisor of X, which is also normal crossing to the special fiber. We decompose D into $D = D^1 \cup D^2$. Put Y (resp. C) to be the special fiber of X (resp. D). Then, for $i = 1, 2$, we have the following canonical B_{st}-linear isomorphism:*

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét},i}((X \setminus D)_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{st}} \otimes_{K_0} H^m_{\text{log-crys},i}(Y \setminus C)$$

Here, that is compatible the additional structures equipped by the following table:

	B_{st}	$\otimes_{\mathbb{Q}_p} H^m_{\text{ét},i}((X \setminus D)_{\overline{K}}, \mathbb{Q}_p)$	$\cong$	B_{st}	$\otimes_{K_0} H^m_{\text{log-crys},i}(Y \setminus C)$
Gal	g	$\otimes g$		g	$\otimes 1$
Frob	φ	$\otimes 1$		φ	$\otimes \varphi$
Monodromy	N	$\otimes 1$		$N \otimes 1$	$+1 \otimes N$
Fil^i after $B_{\mathrm{dR}} \otimes_{B_{\mathrm{st}}}$ }	Fil^i	$\otimes H^m_{\text{ét}}$		$\sum_{i=j+k} \mathrm{Fil}^j$	$\otimes \mathrm{Fil}^k$

Moreover, this is compatible with product structures.

In particular, if $D^1 = \phi$, then we get

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}((X \setminus D)_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{st}} \otimes_{K_0} H^m_{\text{log-crys}}(Y \setminus C),$$
$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét},c}((X \setminus D)_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{st}} \otimes_{K_0} H^m_{\text{log-crys},c}(Y \setminus C).$$

remark . A proof for cohomologies with proper support (H_c) in the case of $D^2 = \emptyset$ and D is simple normal crossing was given by T. Tsuji in personal conversations. That proof asserts there exist a comparison isomorphism of H_c's. Taking dual, we get the comparison isomorphism of H's, but we can not verify that the isomorphism is the one which has constructed in [Tsu2], because the proof neglects product structures. Later, he also gave an alternative proof for cohomologies without support (H) in the case of $D^2 = \emptyset$ and D is simple normal crossing, by removing smooth divisors one by one (see [Tsu5]). That proof asserts there exist a comparison isomorphism of H's. Taking dual, we get the comparison isomorphism of H_c's, but we can not verify that the isomorphism is the one which has constructed in the above personal conversations, because the proof neglects product structures.

Anyway, we want to construct comparison maps of H and H_c (more generally, H_1 and H_2), *which is compatible with product structures,* and to show the comparison maps are isomorphism.

From this "open"C_{st}, by the similar argument of

$$C_{\mathrm{pst}} \Leftarrow C_{\mathrm{st}} \Rightarrow C_{\mathrm{crys}},\ C_{\mathrm{st}} \Rightarrow C_{\mathrm{dR}} \Rightarrow C_{\mathrm{HT}}$$

in the previous section, we can extend C_{HT}, C_{dR}, C_{crys}, and C_{pst}.

The "open"$C_{\mathrm{crys}}$is immediately deduced from the "open"C_{st}.

Theorem 3.3 ("open"C_{crys})**.** *Let X be a proper smooth model over O_K, D be a horizontal normal crossing divisor of X, which is also normal crossing to the special fiber. We decompose D into $D = D^1 \cup D^2$. Put Y (resp. C) to be the special fiber of X (resp. D). Then, for $i = 1, 2$, we have the following canonical B_{st}-linear isomorphism, which is compatible with the Galois actions, the Frobenius endmorphisms, the filtrations after tensoring B_{dR} over B_{crys}:*

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\mathrm{\acute{e}t},i}((X \setminus D)_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{st}} \otimes_{K_0} H^m_{\text{log-crys},i}(Y \setminus C)$$

By de Jong's alteration and truncated simplicial scheme argument (see [Tsu3]), we can deduce the open non-smooth $C_{\mathrm{dR}}$from the "open"C_{st}. Here, in the case of open non-smooth, we use the de Rham cohomology of (Deligne-)Hartshorne. (see [Ha1][Ha2])

Theorem 3.4 (open non-smooth C_{dR})**.** *Let X_K be a separated variety of finite type over K. Then, we have the following canonical isomorphism, which is compatible with the Galois actions and filtrations:*

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)$$

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\mathrm{\acute{e}t},c}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR},c}(X_K/K).$$

In the case of "open" smooth, we can consider partially proper support cohomologies by de Jong's alteration and diagonal class argument (see [Tsu4]).

Theorem 3.5 ("open" C_{dR})**.** *Let X_K be a proper smooth variety over K, and D_K be a normal crossing divisor of X_K. We decompose D into $D_K = D^1_K \cup D^2_K$. Then, for $i = 1, 2$, we have the following canonical isomorphism, which is compatible with the Galois actions and filtrations:*

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\mathrm{\acute{e}t},i}((X \setminus D)_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR},i}((X \setminus D)_K/K)$$

By taking graded quotient, we can deduce the open non-smooth C_{HT} from the open non-smooth C_{dR}. However, the Hodge-Tate decomposition of the open non-smooth C_{HT} is a formal decomposition, and it relates cohomologies of the sheaf of differential forms only in the "open" smooth case.

Theorem 3.6 (open non-smooth C_{HT}). *Let X_K be a separated variety of finite type over K. Then, we have the following canonical isomorphism, which is compatible with the Galois actions:*

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \bigoplus_{-\infty \ll i \ll \infty} \mathbb{C}_p(-i) \otimes_K \mathrm{gr}^i H^m_{\mathrm{dR}}(X_K/K)$$
$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét},c}(X_{\overline{K}}, \mathbb{Q}_p) \cong \bigoplus_{-\infty \ll i \ll \infty} \mathbb{C}_p(-i) \otimes_K \mathrm{gr}^i H^m_{\mathrm{dR},c}(X_K/K).$$

Theorem 3.7 ("open" C_{HT}). *Let X_K be a proper smooth variety over K. and D_K be a normal crossing divisor of X_K. We decompose D into $D_K = D^1_K \cup D^2_K$. Then, for $i = 1, 2$, we have the following canonical isomorphism, which is compatible with the Galois actions:*

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét},i}(X_{\overline{K}}, \mathbb{Q}_p) \cong \bigoplus_{0 \le j \le m} \mathbb{C}_p(-j) \otimes_K H^{m-j}(X_K, I(D^i)\Omega^j_{X_K/K}(\log D)).$$

By de Jong's alteration and truncated simplicial scheme argument (see [Tsu3]), we can deduce the open non-smooth C_{pst} from the "open"C_{st}:

Theorem 3.8 (open non-smooth C_{pst}). *Let X_K be a separated variety of finite type over K. Then, the p-adic étale cohomologies $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, $H^m_{\text{ét},c}(X_{\overline{K}}, \mathbb{Q}_p)$ are potentially semi-stable representations.*

Finally, we mention with a few words about the proof of the main result ("open" C_{st}). In the method of Fontaine-Messing-Kato-Tsuji, we use the intermediate cohomology "syntomic cohomology" (see [FM][Ka2][Tsu1]). In the open case, we find difficulties in making product structures. To make product structures, we consider "bettari-log" schemes. (By the Japanese word "bettari", we image that the log-structure is spread on the whole scheme.) However, log-crystalline cohomologies for these "bettari-log" schemes are in general infinite dimensional. Thus, we overcome difficulties by finding a modified crystalline sheaf, whose log-crystalline cohomology is finite dimensional. We construct a spectral sequence relating (étale, log-crystalline, and syntomic) cohomologies of the open variety with (étale, log-crystalline, and syntomic) cohomologies of the "bettari-log" schemes, and a spectral sequence relating (étale, log-crystalline, and syntomic) cohomologies of the "bettari-log" schemes with (étale, log-crystalline, and syntomic) cohomologies of log-smooth schemes. By using these ingredients, we finish the proof.

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