p-ADIC ÉTALE COHOMOLOGY AND CRYSTALLINE COHOMOLOGY FOR OPEN VARIETIES

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This text is a report of a talk "p-adic étale cohomology and crystalline cohomology for open varieties" in the symposium "Algebraic Number Theory and Related Topics" (2-6/Dec/2002 at RIMS).

The aim of the talk was, roughly speaking, "to extend the main theorems of p-adic Hodge theory for open or non-smooth varieties" by the method of Fontaine-Messing-Kato-Tsuji, which do not use Faltings' almost étale theory. (see [FM],[Ka2], and [Tsu1]). Here, the main theorems of p-adic Hodge theory are: the Hodge-Tate conjecture (C_{HT} for short), the de Rham conjecture (C_{dR}), the crystalline conjecture (C_{crys}), the semi-stabele conjecture (C_{st}), and the potentially semi-stable conjecture (C_{pst}). The theorems C_{dR}, C_{crys}, and C_{st} are called the "comparison theorems".

In the section 1, we review the main theorems of the p-adic Hodge theory. In the section 2, we state the main results. In the section 3. we see the idea of the proof

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Notations

Let K be a complete discrete valuation field of characteristic 0, k the residue field of K, perfect, characteristic $p > 0$, and O_K the valuation ring of K. Denote $\overline{K}$ be the algebraic closure of K, $\overline{k}$ the algebraic closure of k, G_K the absolute Galois group of K, and $\mathbb{C}_p$ the p-adic completion of $\overline{K}$. (Note that it is an abuse of the notation. If $[K : \mathbb{Q}_p] < \infty$, it coincide the usual notations.) Let W be the ring of Witt vectors with coefficient in k, and K_0 the fractional field of W. It is the maximum absolutely unramified (i.e., p is a uniformizer in K_0) subfield of K. The word "log-structure" means Fontaine-Illusie-Kato's log-structure (see. [Ka1]). We do not review the notion of log-structure in this report.

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1. The main theorems of p-adic Hodge theory

The p-adic Hodge theory compares cohomology theories with additional structures, that is, Galois actions, Hodge filtrations, Frobenius endmorphisms, Monodoromy operators:

(1) étale cohomology $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ —topological:
$\mathbb{Q}_p$-vector space +Galois action

(2) (algebraic) de Rham cohomology $H^m_{\text{dR}}(X_K/K)$ —analytic:
K-vector space +Hodge filtration

(3) (log-)crystalline cohomology $K_0 \otimes_W H^m_{\text{crys}}(Y/W)$ —analytic:
K_0-vector space +Frobenius endmorphism (+ Monodromy operator).

In the p-adic Hodge theory, we use Fontaine's p-adic period rings B_{dR}, B_{crys}, and B_{st}. We do not review the definitions and fundamental properties of these rings. (see. [Fo])

In the proof of the comparison theorems, we use the "syntomic cohomology". This is a vector space endowed with the Galois action. However, being different from the étale cohomology it is an analytic cohomology defined by differential forms. It is the theoritical heart of the p-adic Hodge theory by the method of Fontaine-Messing-Kato-Tsuji that the syntomic cohomology is isomorphic to the étale cohomology compatible with Galois action.

In this section, we state the main theorems of p-adic Hodge theory: C_{HT}, C_{dR}, C_{crys}, C_{st}, and C_{pst}. Roughly spealing, we can state the main theorems as the following way:

- the Hodge-Tate conjecture (C_{HT}):
  There exists a Hodge-Tate decomposition on the p-adic étale cohomology.
- the de Rham conjecture (C_{dR}):
  There exists a comparison isomorphism between the p-adic étale cohomology and the de Rham cohomology.
- the crystalline conjecture (C_{crys}):
  In the good reduction case, we have stronger result than C_{dR}, that is, there exists a comparison isomorphism between the p-adic étale cohomology and the crystalline cohomology.
- the semi-stable conjecture (C_{st}):
  In the semi-stable reduction case, we have stronger result than C_{dR}, that is, there exists a comparison isomorphism between the p-adic étale cohomology and the log-crystalline cohomology.
- the potentially semi-stable conjecture (C_{pst}):
  The p-adic étale cohomology has "only a finite monodromy".

The following theorems were formulated by Tate, Fontaine, Jannsen, proved by Tate, Faltings, Fontaine-Messing, Kato under various assumptions, and proved by

Tsuji under no assumptions (1999 [Tsu1]). Later, Faltings and Niziol got alternative proofs (see. [Fa],[Ni]).

Theorem 1.1 (the Hodge-Tate conjecture (C_{HT}))**.** *Let X_K be a proper smooth variety over K. Then, there exists the following canonical isomorphism, which is compatible with the Galois action.*

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \bigoplus_{0 \le i \le m} \mathbb{C}_p(-i) \otimes_K H^{m-i}(X_K, \Omega^i_{X_K/K}).$$

Here, G_K acts by $g \otimes g$ on LHS, by $g \otimes 1$ on RHS.

remark . This is an analogue of the Hodge decomopositon. In this isomorphism, the following fact is remarkable: In general, it seems very difficult to know the action of Galois group on the étale cohomology. However, afer tensoring $\mathbb{C}_p$, the Galois action is very easy:

$$\bigoplus_{0 \le i \le m} \mathbb{C}_p(-i)^{\oplus h^{i,m-i}}$$

($h^{i,m-i} := \dim_K H^{m-i}(X, \Omega^i_{X/K})$.)

Theorem 1.2 (the de Rham conjecture (C_{dR}))**.** *Let X_K be a proper smooth variety over K. Then, there exists the following canonical isomorphism, which is compatible with the Galois action and filtrations.*

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K).$$

Here, G_K acts by $g \otimes g$ on LHS, by $g \otimes 1$ on RHS. We endow filtrations by $\mathrm{Fil}^i \otimes H^m_{\text{ét}}$ on LHS, by $\mathrm{Fil}^i = \Sigma_{i=j+k} \mathrm{Fil}^j \otimes \mathrm{Fil}^k$ on RHS.

remark . By takin graded quotient, we get $C_{\mathrm{dR}} \Rightarrow C_{\mathrm{HT}}$.

Theorem 1.3 (the crystalline conjecture (C_{crys}))**.** *Let X_K be a proper smooth variety over K, X be a proper smooth model of X_K over O_K. Y be the special fiber of X.*

Then, there exists the following canonical isomorphism, which is compatible with the Galois action, and Frobenius endmorphism.

$$B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{crys}} \otimes_W H^m_{\mathrm{crys}}(Y/W)$$

Moreover, after tensoring B_{dR} over B_{crys}, and using the Berthelo-Ogus isomorphism (see. [Be]*):*

$$K \otimes_W H^m_{\mathrm{crys}}(Y/W) \cong H^m_{\mathrm{dR}}(X_K/K),$$

we get an isomorphism:

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K),$$

which is compatible with filtrations. Here, G_K acts by $g \otimes g$ on LHS, by $g \otimes 1$ on RHS, Frobenius endmorphism acts by $\varphi \otimes \varphi$ on LHS, by $\varphi \otimes 1$ on RHS. We endow filtrations by $\mathrm{Fil}^i \otimes H^m_{\text{ét}}$ on LHS, by $\mathrm{Fil}^i = \Sigma_{i=j+k} \mathrm{Fil}^j \otimes \mathrm{Fil}^k$ on RHS.

remark . By taking the Galois invariant part of the comparison isomorphism:

$$B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{crys}} \otimes_W H^m_{\mathrm{crys}}(Y/W),$$

we get:

$$(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))^{G_K} \cong K_0 \otimes_W H^m_{\mathrm{crys}}(Y/W).$$

By taking $\mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_{B_{\mathrm{crys}}} \bullet) \cap (\bullet)^{\varphi=1}$ of the comparison isomorphism, we get:

$$H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)) \cap (B_{\mathrm{crys}} \otimes_W H^m_{\mathrm{crys}}(Y/W))^{\varphi=1}.$$

We can, that is, recover the crystalline cohomology & de Rham cohomology from the étale cohomology and vice versa with all additional strucuture. (Grothendieck's mysterious functor.)

Theorem 1.4 (the semi-stable conjecture (C_{st}))**.** *Let X_K be a proper smooth variety over K, X be a proper semi-stable model of X_K over O_K. (i.e., X is regular and proper flat over O_K, its general fiber is X_K and its special fiber is normal crossing divisor.) Let Y be the special fiber of X, and M_Y be a natural log-structure on Y.*

Then, there exists the following canonical isomorphism, which is compatible with the Galois action, and Frobenius endmorphism, monodromy operator.

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{st}} \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times))$$

Moreover, after tensoring B_{dR} over B_{st}, and using the Hyodo-Kato isomorphism (see. [HKa]*) (it depens on the choice of the uniformizer pi of K):*

$$K \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times)) \cong H^m_{\mathrm{dR}}(X_K/K)$$

we get an isomorphism:

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)$$

which is compatible with filtrations. Here, G_K acts by $g \otimes g$ on LHS, by $g \otimes 1$ on RHS, Frobenius endmorphism acts by $\varphi \otimes \varphi$ on LHS, by $\varphi \otimes 1$ on RHS, monodromy operator acts by $N \otimes 1$ on LHS, by $N \otimes 1 + 1 \otimes N$ on RHS. We endow filtrations by $\mathrm{Fil}^i \otimes H^m_{\text{ét}}$ on LHS, by $\mathrm{Fil}^i = \Sigma_{i=j+k} \mathrm{Fil}^j \otimes \mathrm{Fil}^k$ on RHS.

remark . By taking the Galois invariant part of the comparison isomorphism:

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{st}} \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times))$$

we get:

$$(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))^{G_K} \cong K_0 \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times))$$

By taking $\mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_{B_{\mathrm{st}}} \bullet) \cap (\bullet)^{\varphi=1, N=0}$ of the comparison isomorphism, we get:

$$H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)) \cap (B_{\mathrm{st}} \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times)))^{\varphi=1, N=0}$$

We can, that is, recover the log-crystalline cohomology & de Rham cohomology from the étale cohomology and vice versa with all additional strucuture. (Grothendieck's mysterious functor.)

remark . From $B_{\mathrm{st}}^{N=0} = B_{\mathrm{crys}}$, we get $C_{\mathrm{st}} \Rightarrow C_{\mathrm{crys}}$.

remark . By using de Jong's alteration(see. [dJ]), we get $C_{\mathrm{st}} \Rightarrow C_{\mathrm{dR}}$. We need a slight argument to showing that it is compatible not only with the action of $\mathrm{Gal}(\overline{K}/L)$ for a suitable finite extention L of K, but also with the aciton of G_K. (see. [Tsu4])

In the following theorem, we do not review the definition of the potentially semi-stable representation.

Theorem 1.5 (the potentially semi-stable conjecture (C_{pst}))**.** *Let X_K be a proper variety over K. Then, the p-adic étale cohomology $H^m_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p)$ is a potentially semi-stable representation of G_K.*

remark . By using de Jong's alteration (see. [dJ]) and truncated simplicial schemes, we get $C_{\mathrm{st}} \Rightarrow C_{\mathrm{pst}}$. (see. [Tsu3])

The logical dependence is the following:

$$C_{\mathrm{pst}} \Leftarrow C_{\mathrm{st}} \Rightarrow C_{\mathrm{crys}},\ C_{\mathrm{st}} \Rightarrow C_{\mathrm{dR}} \Rightarrow C_{\mathrm{HT}}.$$

$C_{\mathrm{st}} \Rightarrow C_{\mathrm{crys}}$ and $C_{\mathrm{dR}} \Rightarrow C_{\mathrm{HT}}$ are trivial. For $C_{\mathrm{st}} \Rightarrow C_{\mathrm{dR}}$, we use de Jong's alteration. For $C_{\mathrm{st}} \Rightarrow C_{\mathrm{pst}}$, we use de Jong's alteration and truncated simplicial scheme. i.e., C_{st} is the deepest theorem.

2. The main results

In this section, we state the main results without proof (see. [Y]). In this report, we do not mention weight filtrations.

We call C_{HT}(resp. C_{dR}, C_{crys}, C_{st}, C_{pst}) in the previous section proper smooth C_{HT}(resp. proper smooth C_{dR}, proper C_{crys}, proper C_{st}, proper C_{pst}). Roughly speaking, we remove conditions of the main theorems in the following way.

	former	results
C_{HT}	proper smooth	separated finite type
C_{dR}	proper smooth	separated finite type
C_{crys}	proper good reduction model	"open" good reduction model
C_{st}	proper semi-stable reduction model	"open" semi-stable reduction model
C_{pst}	proper	separated finite type

In the above, the word "open" means "proper minus normal crossing divisor". In C_{dR} case, we use Hartshorne's algebraic de Rham cohomology for open non-smooth varieties. In C_{HT} case, the Hodge-Tate decomposition of the open non-smooth C_{HT} is a formal decomposition, and it relates cohomologies of the sheaf of differential forms only in the "open" smooth case.

We consider cohomologies with proper support H^m_c and cohomologies without proper support H^m. Moreover, we can consider "partially proper support cohomologies" in "open" smooth cases: If we decompose the normal crossing divisor D into $D = D^1 \cup D^2$, "partially proper support cohomologies" are cohomologies with support only on D^1, that is,

$$H^m_{\text{ét}}(X_{\overline{K}}, D^1_{\overline{K}}, D^2_{\overline{K}}) := H^m_{\text{ét}}(X_{\overline{K}}, Rj_{2*}j_{1!}\mathbb{Q}_p),$$

$$H^m_{\mathrm{dR}}(X_K, D^1_K, D^2_K) := H^m(X_K, I(D^1)\Omega_{X_K/K}(\log D_K)),$$

$$H^m_{\text{log-crys}}(Y, C^1, C^2) := K_0 \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times), K(C^1)\mathcal{O}_{(Y,M_Y)/(W,\mathcal{O}^\times)}),$$

Here, $j_1 : (X \setminus D)_{\overline{K}} \hookrightarrow (X \setminus D^2)_{\overline{K}}$, $j_2 : (X \setminus D^2)_{\overline{K}} \hookrightarrow X_{\overline{K}}$, Y(resp. C, C^i) are the special fiber of X(resp. D, D^i), and $I(D^1)$(resp. $K(D^1)$) are the ideal sheaf of $\mathcal{O}_X$(resp. $\mathcal{O}_{(Y,M_Y)/(W,\mathcal{O}^\times)}$) defined by D^1(resp. C^1) (see. [Tsu2]). They are called the "minus log". Naturally, we have $H^m(X, \emptyset, D) = H^m(X \setminus D)$ and $H^m(X, D, \emptyset) = H^m_c(X \setminus D)$ for étale, de Rham, and log-crystalline cohomologies.

For example, the diagonal class $[\Delta]$ of a open variety belongs to a cohomology with partially proper support on $D \times X (\subset (D \times X) \cup (X \times D))$, that is, in $H^{2d}(X \times X, D \times X, X \times D)$. When we consider algebraic correspondences on open varieties, we need to consider partially proper support cohomologies. Thus, in a sense, when we consider not only a comparison between varieties but also a comparison of Hom, we have to consider partially proper support cohomologies. In this way, it is important to show comparison isomorphisms for partially proper support cohomologies.

First, we prove a extended version of Hyodo-Kato isomorphism:

Proposition 2.1. *Let X be a proper semi-stable model over O_K, D be a horizontal normal crossing divisor of X, which is also normal crossing to the special fiber. We decompose D into $D = D^1 \cup D^2$. Put Y (resp. C) to be the special fiber of X (resp. D). Fix a uniformizer pi of K. Then, we have the following isomorphism:*

$$K \otimes_{K_0} H^m_{\text{log-crys}}(Y, C^1, C^2) \cong H^m_{\mathrm{dR}}(X_K, D^1_K, D^2_K).$$

Thus, the pair

$$(H^m_{\text{log-crys}}(Y, C^1, C^2), H^m_{\mathrm{dR}}(X_K, D^1_K, D^2_K))$$

has a filtered (φ, N)-module structure.

The main result is the following:

Theorem 2.2 ("open" C_{st})**.** *Let X be a proper semi-stable model over O_K, D be a horizontal normal crossing divisor of X, which is also normal crossing to the special* *fiber. We decompose D into $D = D^1 \cup D^2$. Put Y (resp. C) to be the special fiber of X (resp. D). Then, we have the following canonical B_{st}-linear isomorphism:*

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, D^1_{\overline{K}}, D^2_{\overline{K}}) \cong B_{\mathrm{st}} \otimes_{K_0} H^m_{\text{log-crys}}(Y, C^1, C^2)$$

Here, that is compatible the additional structures equipped by the following table:

	B_{st}	$\otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, D^1_{\overline{K}}, D^2_{\overline{K}})$	$\cong$	B_{st}	$\otimes_{K_0} H^m_{\text{log-crys}}(Y, C^1, C^2)$
Gal	g	$\otimes g$		g	$\otimes 1$
Frob	φ	$\otimes 1$		φ	$\otimes \varphi$
Monodromy	N	$\otimes 1$		$N \otimes 1$	$+1 \otimes N$
Fil^i after $B_{\mathrm{dR}} \otimes_{B_{\mathrm{st}}}$ }	Fil^i	$\otimes H^m_{\text{ét}}$		$\sum_{i=j+k} \mathrm{Fil}^j$	$\otimes \mathrm{Fil}^k$

Moreover, this is compatible with product structures.

In particular, if $D^1 = \phi$, then we get

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}((X \setminus D)_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{st}} \otimes_{K_0} H^m_{\text{log-crys}}(Y \setminus C),$$

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét},c}((X \setminus D)_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{st}} \otimes_{K_0} H^m_{\text{log-crys},c}(Y \setminus C).$$

remark . A proof for cohomologies with proper support (H_c) in the case of $D^2 = \emptyset$ and D is simple normal crossing was given by T. Tsuji in [Tsu8]. That proof asserts there exist a comparison isomorphism of H_c's. Taking dual, we get the comparison isomorphism of H's, but we can not verify that the isomorphism is the one which has constructed in [Tsu2], because the proof neglects product structures. Later, he also gave an alternative proof for cohomologies without support (H) in the case of $D^2 = \emptyset$ and D is simple normal crossing, by removing smooth divisors one by one (see. [Tsu5]). That proof asserts there exist a comparison isomorphism of H's. Taking dual, we get the comparison isomorphism of H_c's, but we can not verify that the isomorphism is the one which has constructed in the above personal conversations, because the proof neglects product structures. In that method, we cannot treat normal crossing divisors, and partially proper support cohomologies.

Anyway, we want to construct comparison maps of H and H_c (more generally, H_1 and H_2), *which is compatible with product structures,* and to show the comparison maps are isomorphism.

From this "open"C_{st}, by the similar argument of

$$C_{\mathrm{pst}} \Leftarrow C_{\mathrm{st}} \Rightarrow C_{\mathrm{crys}}, \; C_{\mathrm{st}} \Rightarrow C_{\mathrm{dR}} \Rightarrow C_{\mathrm{HT}}$$

in the previous section, we can extend C_{HT}, C_{dR}, C_{crys}, and C_{pst}.

The "open"C_{crys} is immediately deduced from the "open"C_{st}.

Theorem 2.3 ("open"C_{crys})**.** *Let X be a proper smooth model over O_K, D be a horizontal normal crossing divisor of X, which is also normal crossing to the special fiber.*

We decompose D into $D = D^1 \cup D^2$. Put Y (resp. C) to be the special fiber of X (resp. D). Then, we have the following canonical B_{st}-linear isomorphism, which is compatible with the Galois actions, the Frobenius endmorphisms, the filtrations after tensoring B_{dR} over B_{crys}:

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, D^1_{\overline{K}}, D^2_{\overline{K}}) \cong B_{\mathrm{st}} \otimes_{K_0} H^m_{\text{log-crys}}(Y, C^1, C^2)$$

By de Jong's alteration and truncated simplicial scheme argument (see. [Tsu3]), we can deduce the open non-smooth $C_{\mathrm{dR}}$from the "open"C_{st}. Here, in the case of open non-smooth, we use the de Rham cohomology of (Deligne-)Hartshorne. (see. [Ha1][Ha2])

Theorem 2.4 (open non-smooth C_{dR})**.** *Let U_K be a separated variety of finite type over K. Then, we have the following canonical isomorphism, which is compatible with the Galois actions and filtrations:*

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(U_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(U_K/K)$$

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét},c}(U_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR},c}(U_K/K).$$

In the case of "open" smooth, we can consider partially proper support cohomologies by de Jong's alteration and diagonal class argument (see. [Tsu4]).

Theorem 2.5 ("open" C_{dR})**.** *Let X_K be a proper smooth variety over K, and D_K be a normal crossing divisor of X_K. We decompose D into $D_K = D^1_K \cup D^2_K$. Then, we have the following canonical isomorphism, which is compatible with the Galois actions and filtrations:*

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, D^1_{\overline{K}}, D^2_{\overline{K}}) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR},i}(X_K, D^1_K, D^2_K)$$

By taking graded quotient, we can deduce the open non-smooth C_{HT} from the open non-smooth C_{dR}. However, the Hodge-Tate decomposition of the open non-smooth C_{HT} is a formal decomposition, and it relates cohomologies of the sheaf of differential forms only in the "open" smooth case.

Theorem 2.6 (open non-smooth C_{HT})**.** *Let U_K be a separated variety of finite type over K. Then, we have the following canonical isomorphism, which is compatible with the Galois actions:*

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(U_{\overline{K}}, \mathbb{Q}_p) \cong \bigoplus_{-\infty \ll i \ll \infty} \mathbb{C}_p(-i) \otimes_K \mathrm{gr}^i H^m_{\mathrm{dR}}(U_K/K)$$

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét},c}(U_{\overline{K}}, \mathbb{Q}_p) \cong \bigoplus_{-\infty \ll i \ll \infty} \mathbb{C}_p(-i) \otimes_K \mathrm{gr}^i H^m_{\mathrm{dR},c}(U_K/K).$$

Theorem 2.7 ("open" C_{HT})**.** *Let X_K be a proper smooth variety over K. and D_K be a normal crossing divisor of X_K. We decompose D into $D_K = D_K^1 \cup D_K^2$. Then, we have the following canonical isomorphism, which is compatible with the Galois actions:*

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, D^1_{\overline{K}}, D^2_{\overline{K}}) \cong \bigoplus_{0\leq j\leq m} \mathbb{C}_p(-j) \otimes_K H^{m-j}(X_K, I(D^1)\Omega^j_{X_K/K}(\log D_K)).$$

By de Jong's alteration and truncated simplicial scheme argument (see. [Tsu3]), we can deduce the open non-smooth C_{pst} from the "open"C_{st}:

Theorem 2.8 (open non-smooth C_{pst})**.** *Let U_K be a separated variety of finite type over K. Then, the p-adic étale cohomologies $H^m_{\text{ét}}(U_{\overline{K}}, \mathbb{Q}_p)$, $H^m_{\text{ét},c}(U_{\overline{K}}, \mathbb{Q}_p)$ are potentially semi-stable representations.*

3. The idea of the proof

In this section, we see how difficulties arise, and the idea of the proof of the main result ("open" C_{st}). We use the idea of "hollow-log" schemes in the proof, however, we do not deeply see them in this report. In the proof, we do not use Faltings' almost étale theory. In the method of Fontaine-Messing-Kato-Tsuji, we use the intermediate cohomology "syntomic cohomology" (see. [FM][Ka2][Tsu1]):

$$H^m_{\mathrm{syn}}(\overline{X}, \overline{D^1}, \overline{D^2}) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^m_{\mathrm{syn}}((\overline{X}, \overline{M}), \widetilde{\mathcal{S}}_n(r)(-\log D^1)).$$

Here, $\widetilde{\mathcal{S}}_n(r)(-\log D^1)$ is the minus-log syntomic complex, which is defined by differential forms.

Roughly speaking, we construct tha maps

$$H^m_{\text{ét}} \longleftarrow H^m_{\mathrm{syn}} \longrightarrow B_{\mathrm{st}} \otimes_{K_0} H^m_{\text{log-crys}},$$

and show the left homomorphism is an ismorphism. Then, we get the map

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}} \longrightarrow B_{\mathrm{st}} \otimes_{K_0} H^m_{\text{log-crys}}.$$

By using product structures, we show that the comparison map is an isomorphism. In the method of Fontaine-Messing-Kato-Tsuji, it is the technical heart to show the map $H^m_{\mathrm{syn}} \to H^m_{\text{ét}}$ is an isomorphism. In the proper case, by calculating the structure of the syntomic complex $\mathcal{S}'_n(r)$ and the p-adic vanishing cycle $i^*Rj_*\mathbb{Z}/p^n\mathbb{Z}(r)$ using symbol maps, we got the theorem, which says the map

$$i_*\mathcal{S}'_n(r) \longrightarrow i_*i^*Rj_*\mathbb{Z}/p^n\mathbb{Z}(r)'$$

is an isomorphism up to bounded torsion for n. Here, $j : X_{\overline{K}} \hookrightarrow X_{O_{\overline{K}}}$, $i : Y_{\overline{k}} \hookrightarrow X_{O_{\overline{K}}}$.

By showing the Bloch-Kato conjecture about Milnor K-groups and Galois cohomologies for henselian discrete valuation field, Bloch-Kato calculated the p-adic vanishing cycle $i^*Rj_*\mathbb{Z}/p^n\mathbb{Z}(r)$ in the good reduction case (see. [BK]). By extending the method, Hyodo calculated the p-adic vanishing cycle $i^*Rj_*\mathbb{Z}/p^n\mathbb{Z}(r)$ in the semi-stable reduction case (see. [H]). The Bloch-Kato conjecture arises from Kato's higher dimensional class field theory by Milnor K-groups.

On the other hand, the cohomology of syntomic complex $\mathcal{S}'_n(r)$ can be considered to be the p-adic Hodge cohomology,(see. [Ba]) that is, it calculates the Ext^i in the category of "family of filtered φ-modules". (In the comparison theorem, we change the base field. Thus, the Galois group acts on the syntomic cohomology in the use of the comparison.) The structure of syntomic complexes was calculated and applied to the comparison theorem by Kurihara, Kato, Messing, Tsuji. (see. [Ka2][Ka3][KM][Ku][Tsu1][Tsu6][Tsu7]) It is highly non-trivial that the map

$$i_*\mathcal{S}'_n(r) \longrightarrow i_*i^*Rj_*\mathbb{Z}/p^n\mathbb{Z}(r)'$$

is an isomorphism up to bounded torsion for n.

In the open case, we do not touch the calculations of the structures. We have difficulties in other places.

First, we find difficulties in the method of reducing to proper case by "weight" spectral sequences. Thus we do not use the method of "weight" spectral sequences. More precisely, it seems difficult to show that the map in the case $D^1 = \emptyset$

$$i_*\mathcal{S}'_n(r) \longrightarrow i_*i^*Rj_*Rj^\circ_*\mathbb{Z}/p^n\mathbb{Z}(r)'$$

sends the μ-th filtration on $i_*\mathcal{S}'_n(r)$, which is defined by the number of log-poles, to the μ-th filtration $i_*i^*Rj_*\tau_{\leq\mu}Rj^\circ_*\mathbb{Z}/p^n\mathbb{Z}(r)'$ on $i_*i^*Rj_*Rj^\circ_*\mathbb{Z}/p^n\mathbb{Z}(r)'$. Here, $j^\circ : (X \setminus D)_{\overline{K}} \hookrightarrow X_{\overline{K}}$. It seems that it will need a more ring theory for

$$\mathscr{A}_{\mathrm{crys}}(\overline{A^h}, Z, F_Z).$$

Especially, a behavior of the functor $\mathscr{A}_{\mathrm{crys}}(-)$ under a closed immersion:

(1) a regularness of the sequence $\{T_1, \ldots, T_a\}$ in $\mathscr{A}_{\mathrm{crys}}(\overline{A^h}, Z, F_Z)$,
(2) a definition of Fil^r_p on $\mathscr{A}_{\mathrm{crys}}(\overline{A^h}, Z, F_Z)/(T_1, \ldots, T_k)$,
(3) a fundamental exact sequence for $\mathscr{A}_{\mathrm{crys}}(\overline{A^h}, Z, F_Z)/(T_1, \ldots, T_k)$.

Here, $\overline{A^h}$ and Z is as usual, $F_Z = \{F_{Z_n}\}_n$ is a compatible sequence of a lift of Frobenius on Z_n, $\{\mathrm{dlog}\, T_1, \ldots, \mathrm{dlog}\, T_a\}$ is a basis of $\omega^1_{Z_n/W_n}$, and $\mathscr{A}_{\mathrm{crys}}(\overline{A^h}, Z, F_Z)$ is the ring defined by $\overline{A^h}$,Z,and F_Z, which is larger than $A_{\mathrm{crys}}(\overline{A^h})$. (In [Tsu1], he denote $\mathrm{Spec}\, \mathscr{A}_{\mathrm{crys}}(\overline{A^h}, Z, F_Z)/p^n$ to be $\overline{E_n}$.) It seems difficult to show the regularness of the sequence $\{T_1, \ldots, T_a\}$ in $\mathscr{A}_{\mathrm{crys}}(\overline{A^h}, Z, F_Z)$ without the almost étale theory. It is not ever proved that

$$i_*\mathcal{S}'_n(r) \longrightarrow i_*i^*Rj_*Rj^\circ_*\mathbb{Z}/p^n\mathbb{Z}(r)'$$

is compatible with the filtrations,

Even if we could show the above map is compatible with the filtrations, it seems difficult to show that its graded quotients are also comparison maps constructed in the proper case: In the straight thinking, we have to look how differential forms arise in Galois cohomologies –that needs the almost étale theory. However, we can show that its graded quotients are also comparison maps constructed in the proper case by using the method of "hollow-log" schemes. In that method, we can avoid the calculation of

$$H^*(\mathrm{Gal}(\overline{A^h}/A^h), \mathscr{A}_{\mathrm{crys}}(\overline{A^h}, Z, F_Z)).$$

This fact is not used for the proof of the main theorem, since we do not use the method of "weight" spectral sequences.

Second, when we do not use the method of "weight" spectral sequences, we need product structures, because we use product structures to show the map

$$\gamma_m : B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}} \overset{\cong}{\longleftarrow} B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\mathrm{syn}} \longrightarrow B_{\mathrm{st}} \otimes_{K_0} H^m_{\text{log-crys}}$$

is an isomorphism. We find difficulties in making product structures. To make product structures, we consider "hollow-log" schemes. For the simplicity, we assume that the divisor is simple normal crossing and $D^1 = \emptyset$. For $D = \cup_{1 \leq i \leq s} D_i$ (D_i is irreducible) and $n \geq 0$, put

$$D^{(n)} := \coprod_{I \subset \{1, \ldots, s\}} \bigcap_{j \in I} D_j.$$

Let $M_{D^{(n)}}$ be the pull back of the log structure M on X. Then, $(D^{(n)}, M_{D^{(n)}})$ are "hollow-log" schemes. It can be considered a kind of "tube" around $D^{(n)}$.

However, log-crystalline cohomologies for these "hollow-log" schemes are in general infinite dimensional. Thus, we overcome difficulties by finding a modified crystalline sheaf, whose log-crystalline cohomology is finite dimensional. By using these ingredients, we finish the proof.

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