twisted Heilbronn

virtual characters

26/Sep/2023

Go Yamashita

§1. Introduction

F : an NF $\quad \subset \overline{F}$: alg. closure

$G_F := \mathrm{Gal}(\overline{F}/F)$

$\rho : \underset{\substack{\uparrow \\ \text{profinite top.}}}{G_F} \xrightarrow[\text{cont. hom}]{} GL_n(\underset{\substack{\uparrow \\ \text{classical top.}}}{\mathbb{C}})$ Artin rep'n

($\leadsto$ ρ factors through a finite quotient) (this is also called Artin rep'n)

Artin L-fct

$$\leadsto \quad L_F(s,\rho) := \prod_{0 \neq \mathfrak{p} \in \mathrm{Spec}\, \mathcal{O}_F} \det\left(1 - \rho^{I_\mathfrak{p}}(\mathrm{Fr}_\mathfrak{p})(N\mathfrak{p})^{-s}\right)^{-1}$$

fixed part

inertia subgp at $\mathfrak{p}$

Frobenius at $\mathfrak{p}$

($\mathrm{Re}\, s > 1$ abs. conv.)

Known Facts

$(\mathrm{Add})_L$ $\quad L_F(s, \rho_1 \oplus \rho_2) = L_F(s, \rho_1) \cdot L_F(s, \rho_2)$

$(\mathrm{Ind})_L$ $\quad L_F(s, \mathrm{Ind}_{G_{F'}}^{G_F} \lambda) = L_{F'}(s, \lambda)$

$$\Big(F \underset{\text{fin. ext.}}{\subseteq} F' \ (\subseteq \overline{F}) \Big)$$

$(\mathrm{CFT})_L$ $\quad \dim \rho = 1$

$$\underset{\substack{\uparrow \\ \text{coincidence} \\ \text{w/ a Hecke L-fct}}}{\Longrightarrow} L_F(s, \rho) \text{ has } \begin{cases} \text{hol. cont. over } \mathbb{C} \\ \qquad \text{if } \rho \not\cong \mathbb{1} \\ \text{mero. cont. over } \mathbb{C} \\ \qquad \text{w/ only a pole at } s = 1 \\ \qquad \text{if } \rho \cong \mathbb{1} \end{cases}$$

$\left(\zeta_F(s) := L_F(s, \mathbb{1}) \text{ Dedekind zeta} \right)$

$(\mathrm{CFT})_L$ + Brauer's thm $\left({}^{\forall}\rho \overset{\text{virtually}}{\cong} \sum \overset{\mathbb{Z}}{\underset{\exists}{n_i}} \mathrm{Ind}_{G_{F_i}}^{G_F} \underset{\dim=1}{{}^{\exists}\chi_i} \right)$
$+ (\mathrm{Ind})_L + (\mathrm{Add})_L$

$\Rightarrow L_F(s,\rho)$ has meromorphic cont. over $\mathbb{C}$

<u>Conj 1</u> (Artin) ρ : irreducible $\neq \mathbb{1}$

$\Rightarrow L_F(s,\rho)$ is hol. on $\mathbb{C}$

<u>Conj 2</u> (Dedekind) $K \supset F$ fin. ext.

$\Rightarrow \dfrac{\zeta_K(s)}{\zeta_F(s)}$ is hol. on $\mathbb{C}$

Remark)

1. Conj. 1 $\Rightarrow$ Conj. 2

$$\left(\frac{\zeta_K(s)}{\zeta_F(s)} = \frac{L_F(s, \mathrm{Ind}_{G_K}^{G_F} \mathbb{1}_{G_K})}{L_F(s, \mathbb{1}_{G_F})} = L_F(s, (\mathrm{Ind}_{G_K}^{G_F} \mathbb{1}_{G_K}) / \mathbb{1}_{G_F}) . \right.$$

$\mathrm{Ind}_{G_K}^{G_F} \mathbb{1}$ contains $\mathbb{1}_{G_F}$ w/ multiplicity one $: \langle \mathrm{Ind}_{G_K}^{G_F} \mathbb{1}_{G_K}, \mathbb{1}_{G_F} \rangle = \langle \mathbb{1}_{G_K}, \mathrm{Res}_{G_K}^{G_F} \mathbb{1}_{G_F} \rangle = \langle \mathbb{1}_{G_K}, \mathbb{1}_{G_K} \rangle = 1$)

2. Aramata-Brauer : Conj. 2 holds for K : Galois over F.

$$\left(\mathrm{Ind}_{G_K}^{G_F} \mathbb{1} \overset{?}{=} \sum_i \exists a_i \, \mathrm{Ind}_{G_{\exists F_i}}^{G_F} \exists \chi_i \right), \quad a_i \in \mathbb{Q}_{\geq 0}, \ \chi_i \text{ dim} = 1$$

3. Uchida - van der Waall : Conj. 2 holds, if the Galois closure $\hat{K}$ of K over F is solvable ext. over F

$$\left(\mathrm{Ind}_{\mathrm{Gal}(\hat{K}/K)}^{\mathrm{Gal}(\hat{K}/F)} \mathbb{1} \overset{?}{=} \sum \exists n_i \, \mathrm{Ind}_{G_{\exists F_i}}^{G_F} \exists \chi_i \right), \quad n_i \in \mathbb{Z}_{\geq 0}, \ \chi_i \text{ dim} = 1$$

abel $\rightarrow \mathrm{(P)} \triangleleft \mathrm{Gal}(\hat{K}/K) \leftarrow$ reduced to this case by induction; $\mathrm{(P)} \subseteq \mathrm{Gal}(\hat{K}/F)$

$\mathbb{C} \ni s_0 \neq 1$ fix $\quad$ ρ : Artin rep'n of $Gal(K/F)$

(6)

Th Let $K \supset \overline{F}^{\ker(\rho)} (\supset F)$ fin. Galois over F; ρ : irred.

① (essentially Heilbronn or Stark)

$$\operatorname{ord}_{s=s_0} \zeta_K(s) = 0 \Rightarrow \operatorname{ord}_{s=s_0} L_F(s,\rho) = 0$$

② (Stark)

$$\operatorname{ord}_{s=s_0} \zeta_K(s) = 1 \Rightarrow \operatorname{ord}_{s=s_0} L_F(s,\rho) = 0, 1$$

③ (Foote-Murty) (a generalization of Aramata-Brauer)

$$\zeta_K(s) L_F(s,\rho), \quad \frac{\zeta_K(s)}{L_F(s,\rho)} \quad \text{are hol. on } \mathbb{C} \setminus \{1\}$$

(they used the theory of Heilbronn virtual characters.)

$K \ (\supset F^{\ker(\rho)})$ $\overset{\text{fin.}}{\text{Galois}}$ over F

Dedekind zeta $\zeta_K(s)$ plays a role of a kind of "upper bound" of $L_F(s, \rho)$

Want: replace it by other $L_K(s, \lambda)$'s which plays a role of a kind of "upper bound" of $L_F(s, \rho)$'s

$\rightsquigarrow$ introduce twisted Heilbronn virtual characters

& use a theorem in the finite group theory to generalization

Main Th α : Artin rep'n of G_F, $K \supset F$ fin. Galois

Assume

(☆) $\mathrm{Im}(\alpha|_{G_K})$ is an A-group ($\to$ definition later) ($\Rightarrow L_K(s, \alpha|_{G_K})$: hol. on $\mathbb{C}\setminus\{1\}$)

$\mathbb{C} \ni s_0 \neq 1$, ρ : irred. Artin rep'n of $\mathrm{Gal}(K/F)$ ($\alpha = \mathbb{1}$ is the classical case)

Then

① $\mathrm{ord}_{s=s_0} L_K(s, \alpha|_{G_K}) = 0 \Rightarrow \mathrm{ord}_{s=s_0} L_F(s, \rho\otimes\alpha) = 0$

② $\mathrm{ord}_{s=s_0} L_K(s, \alpha|_{G_K}) = 1 \Rightarrow \mathrm{ord}_{s=s_0} L_F(s, \rho\otimes\alpha) = 0, 1$

③ $L_K(s, \alpha|_{G_K}) L_F(s, \rho\otimes\alpha)$, $\dfrac{L_K(s, \alpha|_{G_K})}{L_F(s, \rho\otimes\alpha)}$ are hol. on $\mathbb{C}\setminus\{1\}$

($\rho\otimes\alpha$: tensor product rep'n of G_F
when we regard ρ as the Artin rep'n of G_F via $G_F \xrightarrow{\text{natural}} \mathrm{Gal}(K/F)$)

§2. finite group theory

<u>Def</u> G : fin. gp

(1). G : <u>supersolvable</u> $\overset{\text{def}}{\iff}$ $\{1\} = H_0 \overset{\exists}{\leq} H_1 \leq \cdots \overset{\exists}{\leq} H_n = G$ s.t.
H_i are normal <u>in G</u> $\forall i$
& H_{i+1}/H_i are cyclic $\forall i$.

(2). (P. Hall) G : <u>A-group</u> $\overset{\text{def}}{\iff}$ $\forall$ Sylow subgroups are abelian

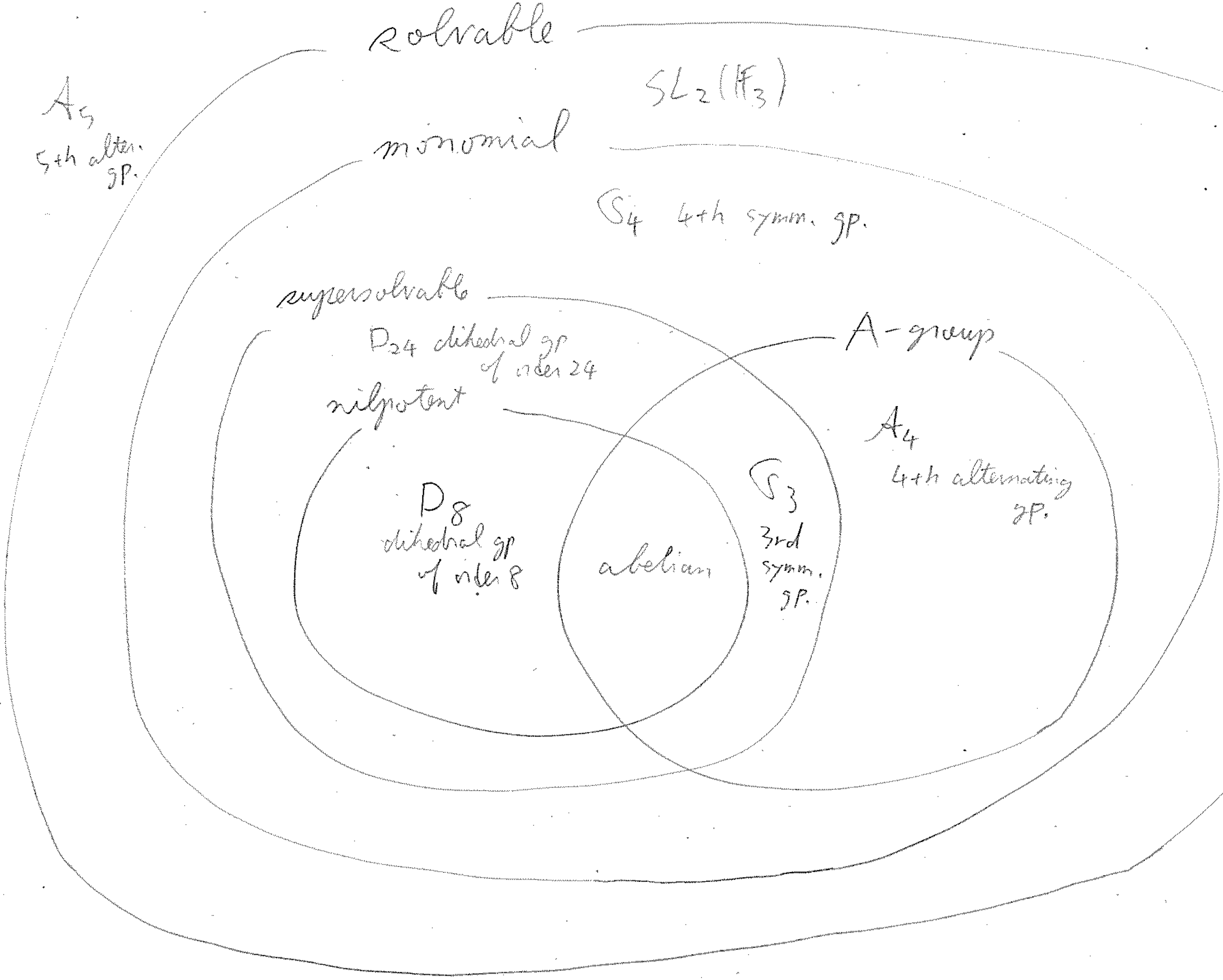
(3). G : <u>monomial</u> (or <u>M-group</u>) $\overset{\text{def}}{\iff}$ $\forall$ irred. rep'ns of G
are Ind (1-dim rep'n)

Remark

In finite groups.

$$
\begin{tikzcd}
\text{nilpotent} \arrow[r, Rightarrow] & \text{supersolvable} \arrow[r, Rightarrow] & \text{monomial} \arrow[r, Rightarrow, "\text{Taketa}"] & \text{solvable} \\
\text{abelian} \arrow[u, Rightarrow] \arrow[rr, Rightarrow] \arrow[ur, phantom, "\llcorner"] & & \text{A-group} \arrow[u, Rightarrow, "\text{Ito}"'] &
\end{tikzcd}
$$

- $\llcorner$: cartesian
- all implications are strict
- no implications between "supersolvable" an "A-group"


solvable
$SL_2(\mathbb{F}_3)$
A_5
5th alter. gp.
monomial
$\mathfrak{S}_4$ 4th symm. gp.
supersolvable
D_{24} dihedral gp of order 24
A-group
nilpotent
A_4
4th alternating gp.
D_8
dihedral gp of order 8
abelian
$\mathfrak{S}_3$
3rd symm. gp.

<u>Th</u> (Huppert's thm combined w/ Ito's thm & Taketa's thm)

G : fin. gp $\supset H$: normal subgp

Assume
$$\begin{cases} \cdot\ H : \text{an A-group} \\ \cdot\ G/H : \text{supersolvable} \end{cases}$$

$\Longrightarrow$ G : monomial

<u>e.g.</u>

$$1 \longrightarrow A_4 \longrightarrow S_4 \xrightarrow{\text{sgn}} \{\pm 1\} \longrightarrow 1$$

A-gp. $\Longrightarrow$ monomial $\Longleftarrow$ supersolvable

(Note:
$$1 \longrightarrow \{\pm 1\} \longrightarrow SL_2(\mathbb{F}_3) \longrightarrow PSL_2(\mathbb{F}_3) \longrightarrow 1$$

supersolvable   <u>not</u> monomial   A-group $\longrightarrow$ $\overset{\cong}{A_4}$.
)

§3. twisted Heilbronn virtual characters

$F \subseteq K \subseteq L \; (\subseteq \overline{F})$ fin. Gal ext'ns

$\widetilde{G} := \mathrm{Gal}(L/F) \twoheadrightarrow G := \mathrm{Gal}(K/F)$

- For a subgp $H \subset G$, put
$$\begin{array}{ccc} \widetilde{G} & \twoheadrightarrow & G \\ \cup & \llcorner & \cup \\ \widetilde{H} & \twoheadrightarrow & H \end{array}$$
(e.g. $\widetilde{\{1\}} = \ker(\widetilde{G} \to G) = \mathrm{Gal}(L/K)$)

- For an Artin rep'n ρ of G, $\rho^{\widetilde{G}} :=$ the composition of ρ & $\widetilde{G} \xrightarrow{\text{natural}} G$

- Recall projection formula
$$(\mathrm{Ind}_H^G \lambda) \otimes \rho \cong \mathrm{Ind}_H^G (\lambda \otimes \mathrm{Res}_H^G \rho)$$
(ρ : rep'n of G, λ : rep'n of H)

fix $1 \neq s_0 \in \mathbb{C}$, for an Artin rep'n α of ~~G_F~~ $\tilde{G}$

$$\boxed{n_{\tilde{G}}(\alpha) := \mathrm{ord}_{s=s_0} L_F(s, \alpha)}$$

$\left(\begin{array}{c}(CFT)_L \\ +(Ind)_L\end{array} \Rightarrow\right)$ (CFT) : $n_{\tilde{G}}(\alpha) \geq 0$ if $\alpha = \mathrm{Ind}\,(1\text{-dim})$

$((Add)_L \Rightarrow)$ (Add) : $n_{\tilde{G}}(\alpha_1 \oplus \alpha_2) = n_{\tilde{G}}(\alpha_1) + n_{\tilde{G}}(\alpha_2)$

$((Ind)_L \Rightarrow)$ (Ind) : $n_{\tilde{G}}\left(\mathrm{Ind}_{\tilde{H}}^{\tilde{G}} \beta\right) = n_{\tilde{H}}(\beta)$ (β: Artin rep'n of $\tilde{H}$)

$\left(\begin{array}{c}\text{projection} \\ \text{formula}\end{array} \Rightarrow\right)$ $\begin{array}{c}\text{(Proj)} \\ +\text{(Ind)}\end{array}$: $n_{\tilde{G}}\left((\mathrm{Ind}_{H}^{G} \lambda)^{\tilde{G}} \otimes \alpha\right) = n_{\tilde{G}}\left(\mathrm{Ind}_{\tilde{H}}^{\tilde{G}} (\lambda^{\tilde{H}} \otimes (\alpha|_{\tilde{H}}))\right)$

$$\underset{(Ind)_L}{=} n_{\tilde{H}}\left(\lambda^{\tilde{H}} \otimes (\alpha|_{\tilde{H}})\right)$$

Def (twisted Heilbronn virtual characters)

$$\theta_G^{\alpha} := \sum_{\rho \in \mathrm{Irr}(G)} n_{\tilde{G}}(\rho^{\tilde{G}} \otimes \alpha)\, \mathrm{Tr}(\rho)$$

virtual character of G

$\{\text{irred. rep's of } G\}/\cong$

not $\tilde{G}$!

put $r_\alpha := n_{\widetilde{\{1\}}}(\alpha|_{\widetilde{\{1\}}}) = n_{\widetilde{\{1\}}}(\alpha|_{\ker(\tilde{G} \twoheadrightarrow G)})$

$= \mathrm{ord}_{s=s_0} L_K(s, \alpha|_{G_K})$

• (Add) $\Longrightarrow$ $\vartheta_G^{\alpha_1} + \vartheta_G^{\alpha_2} = \vartheta_G^{\alpha_1 \oplus \alpha_2}$

$$\left(\langle \rho_1, \rho_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} (\mathrm{Tr}\, \rho_1)(g)(\mathrm{Tr}\, \rho_2)(g^{-1}) \right)$$

• $\binom{\text{Inn}}{\text{Prod}}$: $\displaystyle \langle \vartheta_G^{\alpha}, \lambda \rangle_G = \sum_{\rho \in \mathrm{Irr}(G)} n_{\widehat{G}}(\rho^{\widehat{G}} \otimes \alpha) \langle \rho, \lambda \rangle_G$

$$\overset{(\mathrm{Add})}{=} n_{\widetilde{G}} \left(\Bigg(\underbrace{\bigoplus_{\rho \in \mathrm{Irr}(G)} \rho^{\oplus \langle \rho, \lambda \rangle_G}}_{\cong \lambda} \Bigg)^{\widetilde{G}} \otimes \alpha \right) = n_{\widetilde{G}}(\lambda^{\widetilde{G}} \otimes \alpha)$$

• (Tak) : $\displaystyle r_\alpha = n_{\widetilde{\{1\}}}(\mathbb{1} \otimes (\alpha|_{\widetilde{\{1\}}})) \underset{(\mathrm{Proj})+(\mathrm{Ind})}{=} n_{\widehat{G}}\left((\mathrm{Ind}_{\cancel{H}\,\{1\}}^{G} \mathbb{1})^{\widehat{G}} \otimes \alpha \right)$

$$= n_{\widetilde{G}} \left(\Bigg(\bigoplus_{\rho \in \mathrm{Irr}(G)} \rho^{\oplus \dim \rho} \Bigg)^{\widetilde{G}} \otimes \alpha \right) = n_{\widetilde{G}} \left(\Bigg(\bigoplus_{\rho \in \mathrm{Irr}(G)} (\rho^{\widetilde{G}})^{\oplus \dim \rho} \Bigg) \otimes \alpha \right)$$

$$\underset{(\mathrm{Add})}{=} \sum_{\rho \in \mathrm{Irr}(G)} (\dim \rho)\, n_{\widetilde{G}}(\rho^{\widetilde{G}} \otimes \alpha)$$

cf.

$$\zeta_K(s) = L_F(s, \mathrm{Ind}_{G_K}^{G_F} \mathbb{1}) = \prod_{\rho \in \mathrm{Irr}(\mathrm{Gal}(K/F))} L_F(s, \rho)^{\dim \rho}$$

Takagi's equality

Important property

$$(\mathrm{Res}) : \mathrm{Res}^G_H\left(\vartheta^\alpha_G\right) = \vartheta_H^{\mathrm{Res}^{\tilde G}_{\tilde H}(\alpha)}.$$

proof) $\langle \vartheta^\alpha_G|_H , {}^\forall\lambda \rangle_H = \langle \vartheta^\alpha_G , \mathrm{Ind}^G_H \lambda \rangle_G \underset{}{\overset{(\mathrm{InnProd})}{=}} n_{\tilde G}\left((\mathrm{Ind}^G_H \lambda)^{\tilde G} \otimes \alpha \right)$

$$\underset{(\mathrm{Proj})+(\mathrm{Ind})}{=} n_{\tilde H}\left(\lambda^{\tilde H} \otimes (\alpha|_{\tilde H}) \right) \underset{(\mathrm{InnProd})}{=} \langle \vartheta_H^{\alpha|_{\tilde H}} , \lambda \rangle_H$$

//

In particular,

$$(\mathrm{A}\#1) : \vartheta^\alpha_G(1) = \vartheta^\alpha_G|_{\{1\}}(1) \underset{(\mathrm{Res})}{=} \vartheta^{\alpha|_{\widetilde{\{1\}}}}_{\{1\}}(1) = r_\alpha$$

From now on, assume $L = K \cdot \bar{F}^{\ker(\alpha)}$ $(\subseteq \bar{F})$

and

$\mathrm{Im}(\alpha|_{G_K})$ $(\cong \mathrm{Gal}(L/K) = \ker(\tilde{G} \to G))$

is an A-group

∘ (Gen): $\forall H \subset G$: supersolvable subgp,

i.e. not virtual

$\vartheta^{\alpha}_{G}|_{H}$ is a (genuine) character of G

of degree r_α

proof) By (Res), it suffices to show it for $\vartheta^{\alpha|_{\tilde{H}}}_{H}$.

$$1 \to \ker(\tilde{G} \to G) \to \tilde{H} \to H \to 1$$

$$\begin{array}{ccccc} \uparrow & & \nearrow & & \uparrow \\ \text{A-gp} & \Rightarrow & \text{monomial} & \Leftarrow & \text{supersolvable} \end{array}$$

$\Downarrow$

$n_{\tilde{H}}(\lambda^{\tilde{H}} \otimes (\alpha|_{\tilde{H}})) \geq 0$ for $\forall \lambda$: Artin repn of H //

- $\forall g \in G$, $|\vartheta_G^\alpha(g)| \leq r_\alpha$

(proof) $\vartheta_G^\alpha|_{\langle g\rangle}$ is a genuine char. of degree r_α

subgp gen'd by g by (Gen).)

- (Weil) : $\langle \vartheta_G^\alpha, \vartheta_G^\alpha \rangle_G = \left| \langle \vartheta_G^\alpha, \vartheta_G^\alpha \rangle_G \right| \leq \frac{1}{|G|} \sum_{g \in G} |\vartheta_G^\alpha(g)| \, |\vartheta_G^\alpha(g^{-1})|$

$$\leq \frac{1}{|G|} \sum_{g \in G} r_\alpha^2 = r_\alpha^2$$

the above inequality

$$\langle \vartheta_G^\alpha, \vartheta_G^\alpha \rangle_G = \sum_{\rho \in \mathrm{Irr}(G)} n_{\widehat{G}}(\rho^{\widehat{G}} \otimes \alpha)^2$$

(Tak) : $$\sum_{\rho \in \mathrm{Irr}(G)} (\dim \rho)\, n_{\tilde{G}}(\rho^{\tilde{G}} \otimes \alpha) = r_\alpha$$

(Heil) : $$\sum_{\rho \in \mathrm{Irr}(G)} n_{\tilde{G}}(\rho^{\tilde{G}} \otimes \alpha)^2 \leqq r_\alpha^2$$

<u>proof of Main Th.</u>

① $r_\alpha = 0 \underset{\text{(Heil)}}{\Longrightarrow} \forall \rho \in \mathrm{Irr}(G),\ n_{\tilde{G}}(\rho^{\hat{G}} \otimes \alpha) = 0$

② $r_\alpha = 1 \underset{\text{(Heil)}}{\Longrightarrow}$ all except <u>possibly</u> one $\rho \in \mathrm{Irr}(G)$, $n_{\hat{G}}(\rho^{\hat{G}} \otimes \alpha) = 0$

$\hookrightarrow n_{\hat{G}}(\rho^{\hat{G}} \otimes \alpha) = \pm 1$

$\underset{\text{(Tak)}}{\Longrightarrow} \forall \rho \in \mathrm{Irr}(G),\ n_{\hat{G}}(\rho^{\hat{G}} \otimes \alpha) = 0, 1$

③ By (Heil), $\forall \rho \in \mathrm{Irr}(G)$, $|n_{\hat{G}}(\rho^{\hat{G}} \otimes \alpha)| \leqq r_\alpha$ //

Other applications

Th (a generalization of Hu-Kaneko-Martin-Schildkraut's theorem)

$\forall K \supset F$ non-abelian fin. Gal. ext.

$\forall \chi$: Artin rep'n of G_K s.t. $\mathrm{Im}(\chi)$: an A-gp & χ : extendable to G_F

$\Rightarrow$ $L_K(s,\chi)$ has ∞-ly many zeros of order > 1

in the critical strip $0 < \mathrm{Re}(s) < 1$.

Th (a generalization of Browkin's theorem)

$\forall F$: an NF, $\forall n \geq 1$, $\exists K \supset F$ fin. Gal. ext. s.t.

$\forall \chi$: Artin rep'n of G_K s.t. $\mathrm{Im}(\chi)$: an A-gp & χ : extendable to G_F.

$L_K(s,\chi)$ has ∞-ly many zeros of order $\geq n$

in the critical strip $0 < \mathrm{Re}(s) < 1$

Thank you very much.