p-ADIC ÉTALE COHOMOLOGY AND CRYSTALLINE COHOMOLOGY FOR OPEN VARIETIES

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This text is a report of a talk "p-adic étale cohomology and crystalline cohomology for open varieties" in a symposium at Waseda University (13-15/March/2003).

The aim of the talk was, roughly speaking, "to extend the main theorems of p-adic Hodge theory for open or non-smooth varieties" by the method of Fontaine-Messing-Kato-Tsuji, which do not use Faltings' almost étale theory. (see [FM],[Ka2], and [Tsu1]). Here, the main theorems of p-adic Hodge theory are: the Hodge-Tate conjecture (C_{HT} for short), the de Rham conjecture (C_{dR}), the crystalline conjecture (C_{crys}), the semi-stabele conjecture (C_{st}), and the potentially semi-stable conjecture (C_{pst}). The theorems C_{dR}, C_{crys}, and C_{st} are called the "comparison theorems".

In the section 1, we review the main theorems of the p-adic Hodge theory. In the section 2, we state the main results. In this report, the auther only states the results without the proof.

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Notations

Let K be a complete discrete valuation field of characteristic 0, k the residue field of K, perfect, characteristic $p > 0$, and O_K the valuation ring of K. Denote $\overline{K}$ be the algebraic closure of K, $\overline{k}$ the algebraic closure of k, G_K the absolute Galois group of K, and $\mathbb{C}_p$ the p-adic completion of $\overline{K}$. (Note that it is an abuse of the notation. If $[K : \mathbb{Q}_p] < \infty$, it coincide the usual notations.) Let W be the ring of Witt vectors with coefficient in k, and K_0 the fractional field of W. It is the maximum absolutely unramified (i.e., p is a uniformizer in K_0) subfield of K. The word "log-structure" means Fontaine-Illusie-Kato's log-structure (see. [Ka1]). We do not review the notion of log-structure in this report.

1. The main theorems of p-adic Hodge theory

The p-adic Hodge theory compares cohomology theories with additional structures, that is, Galois actions, Hodge filtrations, Frobenius endmorphisms, Monodoromy operators:

(1) étale cohomology $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ —topological:
$\mathbb{Q}_p$-vector space +Galois action

(2) (algebraic) de Rham cohomology $H^m_{\mathrm{dR}}(X_K/K)$ —analytic:
K-vector space +Hodge filtration

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(3) (log-)crystalline cohomology $K_0 \otimes_W H^m_{\mathrm{crys}}(Y/W)$ —analytic:
K_0-vector space +Frobenius endmorphism (+ Monodromy operator).

In the p-adic Hodge theory, we use Fontaine's p-adic period rings B_{dR}, B_{crys}, and B_{st}. We do not review the definitions and fundamental properties of these rings. (see. [Fo])

In the proof of the comparison theorems, we use the "syntomic cohomology". This is a vector space endowed with the Galois action. However, being different from the étale cohomology it is an analytic cohomology defined by differential forms. It is the theoritical heart of the p-adic Hodge theory by the method of Fontaine-Messing-Kato-Tsuji that the syntomic cohomology is isomorphic to the étale cohomology compatible with Galois action.

In this section, we state the main theorems of p-adic Hodge theory: C_{HT}, C_{dR}, C_{crys}, C_{st}, and C_{pst}. Roughly spealing, we can state the main theorems as the following way:

- the Hodge-Tate conjecture (C_{HT}):
  There exists a Hodge-Tate decomposition on the p-adic étale cohomology.
- the de Rham conjecture (C_{dR}):
  There exists a comparison isomorphism between the p-adic étale cohomology and the de Rham cohomology.
- the crystalline conjecture (C_{crys}):
  In the good reduction case, we have stronger result than C_{dR}, that is, there exists a comparison isomorphism between the p-adic étale cohomology and the crystalline cohomology.
- the semi-stable conjecture (C_{st}):
  In the semi-stable reduction case, we have stronger result than C_{dR}, that is, there exists a comparison isomorphism between the p-adic étale cohomology and the log-crystalline cohomology.
- the potentially semi-stable conjecture (C_{pst}):
  The p-adic étale cohomology has "only a finite monodromy".

The following theorems were formulated by Tate, Fontaine, Jannsen, proved by Tate, Faltings, Fontaine-Messing, Kato under various assumptions, and proved by Tsuji under no assumptions (1999 [Tsu1]). Later, Faltings and Niziol got alternative proofs (see. [Fa],[Ni]).

Theorem 1.1 (the Hodge-Tate conjecture (C_{HT}))**.** *Let X_K be a proper smooth variety over K. Then, there exists the following canonical isomorphism, which is compatible with the Galois action.*

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \bigoplus_{0\le i\le m} \mathbb{C}_p(-i) \otimes_K H^{m-i}(X_K, \Omega^i_{X_K/K}).$$

Here, G_K acts by $g \otimes g$ on LHS, by $g \otimes 1$ on RHS.

remark . This is an analogue of the Hodge decomopositon. In this isomorphism, the following fact is remarkable: In general, it seems very difficult to know the action of Galois group on the étale cohomology. However, afer tensoring $\mathbb{C}_p$, the Galois action is very easy:

$$\bigoplus_{0\le i\le m} \mathbb{C}_p(-i)^{\oplus h^{i,m-i}}$$

($h^{i,m-i} := \dim_K H^{m-i}(X, \Omega^i_{X/K})$.)

Theorem 1.2 (the de Rham conjecture (C_{dR})). *Let X_K be a proper smooth variety over K. Then, there exists the following canonical isomorphism, which is compatible with the Galois action and filtrations.*

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K).$$

Here, G_K acts by $g \otimes g$ on LHS, by $g \otimes 1$ on RHS. We endow filtrations by $\mathrm{Fil}^i \otimes H^m_{\text{ét}}$ *on LHS, by* $\mathrm{Fil}^i = \Sigma_{i=j+k}\mathrm{Fil}^j \otimes \mathrm{Fil}^k$ *on RHS.*

remark . By takin graded quotient, we get $C_{\mathrm{dR}} \Rightarrow C_{\mathrm{HT}}$.

Theorem 1.3 (the crystalline conjecture (C_{crys})). *Let X_K be a proper smooth variety over K, X be a proper smooth model of X_K over O_K. Y be the special fiber of X.*

Then, there exists the following canonical isomorphism, which is compatible with the Galois action, and Frobenius endmorphism.

$$B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{crys}} \otimes_W H^m_{\mathrm{crys}}(Y/W)$$

Moreover, after tensoring B_{dR} over B_{crys}, and using the Berthelo-Ogus isomorphism (see. [Be]*):*

$$K \otimes_W H^m_{\mathrm{crys}}(Y/W) \cong H^m_{\mathrm{dR}}(X_K/K),$$

we get an isomorphism:

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K),$$

which is compatible with filtrations. Here, G_K acts by $g \otimes g$ on LHS, by $g \otimes 1$ on RHS, Frobenius endmorphism acts by $\varphi \otimes \varphi$ on LHS, by $\varphi \otimes 1$ on RHS. We endow filtrations by $\mathrm{Fil}^i \otimes H^m_{\text{ét}}$ *on LHS, by* $\mathrm{Fil}^i = \Sigma_{i=j+k}\mathrm{Fil}^j \otimes \mathrm{Fil}^k$ *on RHS.*

remark . By taking the Galois invariant part of the comparison isomorphism:

$$B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{crys}} \otimes_W H^m_{\mathrm{crys}}(Y/W),$$

we get:

$$(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))^{G_K} \cong K_0 \otimes_W H^m_{\mathrm{crys}}(Y/W).$$

By taking $\mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_{B_{\mathrm{crys}}} \bullet) \cap (\bullet)^{\varphi=1}$ of the comparison isomorphism, we get:

$$H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)) \cap (B_{\mathrm{crys}} \otimes_W H^m_{\mathrm{crys}}(Y/W))^{\varphi=1}.$$

We can, that is, recover the crystalline cohomology & de Rham cohomology from the étale cohomology and vice versa with all additional strucuture. (Grothendieck's mysterious functor.)

Theorem 1.4 (the semi-stable conjecture (C_{st})). *Let X_K be a proper smooth variety over K, X be a proper semi-stable model of X_K over O_K. (i.e., X is regular and proper flat over O_K, its general fiber is X_K and its special fiber is normal crossing divisor.) Let Y be the special fiber of X, and M_Y be a natural log-structure on Y.*

Then, there exists the following canonical isomorphism, which is compatible with the Galois action, and Frobenius endmorphism, monodromy operator.

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{st}} \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times))$$

Moreover, after tensoring B_{dR} over B_{st}, and using the Hyodo-Kato isomorphism (see. [HKa]*) (it depens on the choice of the uniformizer pi of K):*

$$K \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times)) \cong H^m_{\mathrm{dR}}(X_K/K)$$

we get an isomorphism:

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)$$

which is compatible with filtrations. Here, G_K acts by $g \otimes g$ on LHS, by $g \otimes 1$ on RHS, Frobenius endmorphism acts by $\varphi \otimes \varphi$ on LHS, by $\varphi \otimes 1$ on RHS, monodromy operator acts by $N \otimes 1$ on LHS, by $N \otimes 1 + 1 \otimes N$ on RHS. We endow filtrations by $\mathrm{Fil}^i \otimes H^m_{\text{ét}}$ on LHS, by $\mathrm{Fil}^i = \Sigma_{i=j+k} \mathrm{Fil}^j \otimes \mathrm{Fil}^k$ on RHS.

remark . By taking the Galois invariant part of the comparison isomorphism:

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{st}} \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times))$$

we get:

$$(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))^{G_K} \cong K_0 \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times))$$

By taking $\mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_{B_{\mathrm{st}}} \bullet) \cap (\bullet)^{\varphi=1, N=0}$ of the comparison isomorphism, we get:

$$H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)) \cap (B_{\mathrm{st}} \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times)))^{\varphi=1, N=0}$$

We can, that is, recover the log-crystalline cohomology & de Rham cohomology from the étale cohomology and vice versa with all additional strucuture. (Grothendieck's mysterious functor.)

remark . From $B_{\mathrm{st}}^{N=0} = B_{\mathrm{crys}}$, we get $C_{\mathrm{st}} \Rightarrow C_{\mathrm{crys}}$.

remark . By using de Jong's alteration(see. [dJ]), we get $C_{\mathrm{st}} \Rightarrow C_{\mathrm{dR}}$. We need a slight argument to showing that it is compatible not only with the action of $\mathrm{Gal}(\overline{K}/L)$ for a suitable finite extention L of K, but also with the aciton of G_K. (see. [Tsu4])

In the following theorem, we do not review the definition of the potentially semi-stable representation.

Theorem 1.5 (the potentially semi-stable conjecture (C_{pst})). *Let X_K be a proper variety over K. Then, the p-adic étale cohomology $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is a potentially semi-stable representation of G_K.*

remark . By using de Jong's alteration (see. [dJ]) and truncated simplicial schemes, we get $C_{\mathrm{st}} \Rightarrow C_{\mathrm{pst}}$. (see. [Tsu3])

The logical dependence is the following:

$$C_{\mathrm{pst}} \Leftarrow C_{\mathrm{st}} \Rightarrow C_{\mathrm{crys}}, \; C_{\mathrm{st}} \Rightarrow C_{\mathrm{dR}} \Rightarrow C_{\mathrm{HT}}.$$

$C_{\mathrm{st}} \Rightarrow C_{\mathrm{crys}}$ and $C_{\mathrm{dR}} \Rightarrow C_{\mathrm{HT}}$ are trivial. For $C_{\mathrm{st}} \Rightarrow C_{\mathrm{dR}}$, we use de Jong's alteration. For $C_{\mathrm{st}} \Rightarrow C_{\mathrm{pst}}$, we use de Jong's alteration and truncated simplicial scheme. i.e., C_{st} is the deepest theorem.

2. The main results

In this section, we state the main results without proof (see. [Y]). In this report, we do not mention "weight" filtrations.

We call C_{HT}(resp. C_{dR}, C_{crys}, C_{st}, C_{pst}) in the previous section proper smooth C_{HT}(resp. proper smooth C_{dR}, proper C_{crys}, proper C_{st}, proper C_{pst}). Roughly speaking, we remove conditions of the main theorems in the following way.

	former	results
C_{HT}	proper smooth	separated finite type
C_{dR}	proper smooth	separated finite type
C_{crys}	proper good reduction model	"open" good reduction model
C_{st}	proper semi-stable reduction model	"open" semi-stable reduction model
C_{pst}	proper	separated finite type

In the above, the word "open" means "proper minus normal crossing divisor". In C_{dR} case, we use Hartshorne's algebraic de Rham cohomology for open non-smooth varieties. In C_{HT} case, the Hodge-Tate decomposition of the open non-smooth C_{HT} is a formal decomposition, and it relates cohomologies of the sheaf of differential forms only in the "open" smooth case.

We consider cohomologies with proper support H_c^m and cohomologies without proper support H^m. Moreover, we can consider "partially proper support cohomologies" in "open" smooth cases: If we decompose the normal crossing divisor D into $D = D^1 \cup D^2$, "partially proper support cohomologies" are cohomologies with support only on D^1, that is,

$$H^m_{\text{ét}}(X_{\overline{K}}, D^1_{\overline{K}}, D^2_{\overline{K}}) := H^m_{\text{ét}}(X_{\overline{K}}, Rj_{2*}j_{1!}\mathbb{Q}_p),$$

$$H^m_{\mathrm{dR}}(X_K, D^1_K, D^2_K) := H^m(X_K, I(D^1)\Omega_{X_K/K}(\log D_K)),$$

$$H^m_{\text{log-crys}}(Y, C^1, C^2) := K_0 \otimes_W H^m_{\text{log-crys}}((Y, M_Y)/(W, \mathcal{O}^\times), K(C^1)\mathcal{O}_{(Y,M_Y)/(W,\mathcal{O}^\times)}),$$

Here, $j_1 : (X \setminus D)_{\overline{K}} \hookrightarrow (X \setminus D^2)_{\overline{K}}$, $j_2 : (X \setminus D^2)_{\overline{K}} \hookrightarrow X_{\overline{K}}$, Y(resp. C, C^i) are the special fiber of X(resp. D, D^i), and $I(D^1)$(resp. $K(D^1)$) are the ideal sheaf of $\mathcal{O}_X$(resp. $\mathcal{O}_{(Y,M_Y)/(W,\mathcal{O}^\times)}$) defined by D^1(resp. C^1) (see. [Tsu2]). They are called the "minus log". Naturally, we have $H^m(X, \emptyset, D) = H^m(X \setminus D)$ and $H^m(X, D, \emptyset) = H^m_c(X \setminus D)$ for étale, de Rham, and log-crystalline cohomologies.

For example, the diagonal class $[\Delta]$ of a open variety belongs to a cohomology with partially proper support on $D \times X (\subset (D \times X) \cup (X \times D))$, that is, in $H^{2d}(X \times X, D \times X, X \times D)$. When we consider algebraic correspondences on open varieties, we need to consider partially proper support cohomologies. Thus, in a sense, when we consider not only a comparison between varieties but also a comparison of Hom, we have to consider partially proper support cohomologies. In this way, it is important to show comparison isomorphisms for partially proper support cohomologies.

First, we prove a extended version of Hyodo-Kato isomorphism:

Proposition 2.1. *Let X be a proper semi-stable model over O_K, D be a horizontal normal crossing divisor of X, which is also normal crossing to the special fiber. We decompose D into $D = D^1 \cup D^2$. Put Y (resp. C) to be the special fiber of X (resp. D). Fix a uniformizer* *pi of K. Then, we have the following isomorphism:*

$$K \otimes_{K_0} H^m_{\text{log-crys}}(Y, C^1, C^2) \cong H^m_{\text{dR}}(X_K, D^1_K, D^2_K).$$

Thus, the pair

$$(H^m_{\text{log-crys}}(Y, C^1, C^2), H^m_{\text{dR}}(X_K, D^1_K, D^2_K))$$

has a filtered (φ, N)-module structure.

The main result is the following:

Theorem 2.2 ("open" C_{st})**.** *Let X be a proper semi-stable model over O_K, D be a horizontal normal crossing divisor of X, which is also normal crossing to the special fiber. We decompose D into $D = D^1 \cup D^2$. Put Y (resp. C) to be the special fiber of X (resp. D). Then, we have the following canonical B_{st}-linear isomorphism:*

$$B_{\text{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, D^1_{\overline{K}}, D^2_{\overline{K}}) \cong B_{\text{st}} \otimes_{K_0} H^m_{\text{log-crys}}(Y, C^1, C^2)$$

Here, that is compatible the additional structures equipped by the following table:

	B_{st}	$\otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, D^1_{\overline{K}}, D^2_{\overline{K}})$	$\cong$	B_{st}	$\otimes_{K_0} H^m_{\text{log-crys}}(Y, C^1, C^2)$
Gal	g	$\otimes g$		g	$\otimes 1$
Frob	φ	$\otimes 1$		φ	$\otimes \varphi$
Monodromy	N	$\otimes 1$		$N \otimes 1$	$+1 \otimes N$
Fil^i after $B_{\text{dR}} \otimes_{B_{\text{st}}}$	Fil^i	$\otimes H^m_{\text{ét}}$		$\sum_{i=j+k} \text{Fil}^j$	$\otimes \text{Fil}^k$

Moreover, this is compatible with product structures.

In particular, if $D^1 = \phi$, then we get

$$B_{\text{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}((X \setminus D)_{\overline{K}}, \mathbb{Q}_p) \cong B_{\text{st}} \otimes_{K_0} H^m_{\text{log-crys}}(Y \setminus C),$$

$$B_{\text{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét},c}((X \setminus D)_{\overline{K}}, \mathbb{Q}_p) \cong B_{\text{st}} \otimes_{K_0} H^m_{\text{log-crys},c}(Y \setminus C).$$

remark . It seems difficult to show the compatibility of Leray spectral sequences, so it seems that we cannot reduce to the proper case without the almost étale theory.

remark . A proof for cohomologies with proper support (H_c) in the case of $D^2 = \emptyset$ and D is simple normal crossing was given by T. Tsuji in [Tsu8]. That proof asserts there exist a comparison isomorphism of H_c's. Taking dual, we get the comparison isomorphism of H's, but we can not verify that the isomorphism is the one which has constructed in [Tsu2], because the proof neglects product structures. Later, he also gave an alternative proof for cohomologies without support (H) in the case of $D^2 = \emptyset$ and D is simple normal crossing, by removing smooth divisors one by one (see. [Tsu5]). That proof asserts there exist a comparison isomorphism of H's. Taking dual, we get the comparison isomorphism of H_c's, but we can not verify that the isomorphism is the one which has constructed in the above personal conversations, because the proof neglects product structures. In that method, we cannot treat normal crossing divisors, and partially proper support cohomologies.

Anyway, we want to construct comparison maps of H and H_c (more generally, for partially proper support cohomologies), *which is compatible with product structures,* and to show the comparison maps are isomorphism.

From this "open"C_{st}, by the similar argument of

$$C_{\mathrm{pst}} \Leftarrow C_{\mathrm{st}} \Rightarrow C_{\mathrm{crys}}, \ C_{\mathrm{st}} \Rightarrow C_{\mathrm{dR}} \Rightarrow C_{\mathrm{HT}}$$

in the previous section, we can extend C_{HT}, C_{dR}, C_{crys}, and C_{pst}.

The "open"C_{crys} is immediately deduced from the "open"C_{st}.

Theorem 2.3 ("open"C_{crys})**.** *Let X be a proper smooth model over O_K, D be a horizontal normal crossing divisor of X, which is also normal crossing to the special fiber. We decompose D into $D = D^1 \cup D^2$. Put Y (resp. C) to be the special fiber of X (resp. D). Then, we have the following canonical B_{st}-linear isomorphism, which is compatible with the Galois actions, the Frobenius endmorphisms, the filtrations after tensoring B_{dR} over B_{crys}:*

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, D^1_{\overline{K}}, D^2_{\overline{K}}) \cong B_{\mathrm{st}} \otimes_{K_0} H^m_{\text{log-crys}}(Y, C^1, C^2)$$

By de Jong's alteration and truncated simplicial scheme argument (see. [Tsu3]), we can deduce the open non-smooth $C_{\mathrm{dR}}$from the "open"C_{st}. Here, in the case of open non-smooth, we use the de Rham cohomology of (Deligne-)Hartshorne. (see. [Ha1][Ha2])

Theorem 2.4 (open non-smooth C_{dR})**.** *Let U_K be a separated variety of finite type over K. Then, we have the following canonical isomorphism, which is compatible with the Galois actions and filtrations:*

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(U_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(U_K/K)$$

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét},c}(U_{\overline{K}}, \mathbb{Q}_p) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR},c}(U_K/K).$$

In the case of "open" smooth, we can consider partially proper support cohomologies by de Jong's alteration and diagonal class argument (see. [Tsu4]).

Theorem 2.5 ("open" C_{dR})**.** *Let X_K be a proper smooth variety over K, and D_K be a normal crossing divisor of X_K. We decompose D into $D_K = D^1_K \cup D^2_K$. Then, we have the following canonical isomorphism, which is compatible with the Galois actions and filtrations:*

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, D^1_{\overline{K}}, D^2_{\overline{K}}) \cong B_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR},i}(X_K, D^1_K, D^2_K)$$

By taking graded quotient, we can deduce the open non-smooth C_{HT} from the open non-smooth C_{dR}. However, the Hodge-Tate decomposition of the open non-smooth C_{HT} is a formal decomposition, and it relates cohomologies of the sheaf of differential forms only in the "open" smooth case.

Theorem 2.6 (open non-smooth C_{HT})**.** *Let U_K be a separated variety of finite type over K. Then, we have the following canonical isomorphism, which is compatible with the Galois actions:*

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(U_{\overline{K}}, \mathbb{Q}_p) \cong \bigoplus_{-\infty \ll i \ll \infty} \mathbb{C}_p(-i) \otimes_K \mathrm{gr}^i H^m_{\mathrm{dR}}(U_K/K)$$

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét},c}(U_{\overline{K}}, \mathbb{Q}_p) \cong \bigoplus_{-\infty \ll i \ll \infty} \mathbb{C}_p(-i) \otimes_K \mathrm{gr}^i H^m_{\mathrm{dR},c}(U_K/K).$$

Theorem 2.7 (“open” C_{HT}). *Let X_K be a proper smooth variety over K. and D_K be a normal crossing divisor of X_K. We decompose D into $D_K = D^1_K \cup D^2_K$. Then, we have the following canonical isomorphism, which is compatible with the Galois actions:*

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, D^1_{\overline{K}}, D^2_{\overline{K}}) \cong \bigoplus_{0\leq j\leq m} \mathbb{C}_p(-j) \otimes_K H^{m-j}(X_K, I(D^1)\Omega^j_{X_K/K}(\log D_K)).$$

By de Jong’s alteration and truncated simplicial scheme argument (see. [Tsu3]), we can deduce the open non-smooth C_{pst} from the “open”C_{st}:

Theorem 2.8 (open non-smooth C_{pst}). *Let U_K be a separated variety of finite type over K. Then, the p-adic étale cohomologies $H^m_{\text{ét}}(U_{\overline{K}}, \mathbb{Q}_p)$, $H^m_{\text{ét},c}(U_{\overline{K}}, \mathbb{Q}_p)$ are potentially semi-stable representations.*

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